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*Research article*

# One step proximal point schemes for monotone vector field inclusion problems

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**Abstract:** In this paper, we propose one step convex combination of proximal point algorithms for countable collection of monotone vector fields in CAT(0) spaces. We establish $\Delta$-convergence and strong convergence theorems for approximating a common solution of a countable family of monotone vector field inclusion problems. Furthermore, we apply our methods to solve a family of minimization problems, compute Frechét mean and geometric median in CAT(0) spaces, and solve a kinematic problem in robotic motion control. Finally, we give a numerical example to show the efficiency and robustness of the proposed scheme in comparison to a known scheme in the literature.

**Keywords:** CAT(0) space; monotone vector field; proximal point algorithm; resolvent operator; tangent space; $\Delta$-convergence
**Mathematics Subject Classification:** 47H05, 47J25, 49J40, 65K10, 65K15

## 1. Introduction

Let $H$ be a real Hilbert space. Suppose $A : H \to 2^H$ be a monotone mapping, that is,

$$\langle u - w, x - y\rangle \geq 0, \;\; \forall u, w \in \mathbb{D}(A) := \{z \in H : Az \neq \emptyset\}, x \in Au, y \in Aw.$$

The problem of finding an element

$$w \in \mathbb{D}(A) \text{ such that } 0 \in Aw \tag{1.1}$$

is known as monotone inclusion problem (MIP for short). This problem has vast applications in several fields of sciences and engineering such as in inverse problems, image recovery, fuzzy theory, game theory, signal processing, robotic control, etc. Due to its applications, several iterative schemes for finding a solution of (1.1) were developed. The most popular developed scheme is proximal point algorithm (PPA for short) which can be traced to Martinet [25] and Rockerfellar [30]. With some control conditions, Rockerfeller [30] proved weak convergence of a scheme generated by the PPA. Since then many researchers have studied and developed different modifications of PPA in consideration of large range of applications of MIP (see, e.g., [16, 18, 27, 33] and the references therein). It has been observed recently that several problems can be tackled in a structural metric space called a CAT(0) space. As a result, several concepts including that of monotone mappings have been extended from linear spaces and Hadamard manifolds to CAT(0) spaces (see, e.g., [2, 4, 5, 9, 24] and the references therein).

Given a CAT(0) space $(X, d)$, Berg and Nokolaev [3] denoted $(u, w) \in X \times X$ by $\overrightarrow{uw}$ and defined a quasilinearization map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ by

$$\langle \overrightarrow{uw}, \overrightarrow{vy} \rangle = \frac{1}{2}\left(d^2(u, y) + d^2(w, v) - d^2(u, v) - d^2(w, y)\right), \quad (u, v, w, y \in X).$$

Using this notion of quasilinearization, Kakavandi and Amini [17] introduced the dual space of a CAT(0) space $(X, d)$ as follows. Let $\phi : X \to \mathbb{R}$ be a function and let $L$ be the function defined by $L(\phi) := \sup\left\{\frac{\phi(w) - \phi(v)}{d(w, v)} : w, v \in X, w \neq v\right\}$. Consider $\Theta : \mathbb{R} \times (X \times X) \to C(X, \mathbb{R})$ defined by $\Theta(t, u, w)(x) = t\langle \overrightarrow{uw}, \overrightarrow{ux} \rangle$ for all $t \in \mathbb{R},\ u, w, x \in X$, where $C(X, \mathbb{R})$ denotes the space of continuous real-valued functions on $X$. Then it is known that the map $D$ on $\mathbb{R} \times X \times X$ defined by

$$D((t, u, w), (s, x, y)) = L(\Theta(t, u, w) - \Theta(s, x, y))$$

is a pseudometric on $\mathbb{R} \times X \times X$. Moreover, $D$ forms an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of $(t, u, w)$ is $[t\overrightarrow{uw}] := \left\{s\overrightarrow{xy} : D((t, u, w), (s, x, y)) = 0\right\}$. Let $X^* := \left\{t\overrightarrow{uw} : (t, u, w) \in \mathbb{R} \times X \times X\right\}$. Then $(X^*, D)$ is the dual space of $(X, d)$ as defined in [17]. Moreover, $X^*$ acts on $X \times X$ by $\langle x^*, \overrightarrow{uw} \rangle = t\langle \overrightarrow{xy}, \overrightarrow{uw} \rangle$, for $x^* = [t\overrightarrow{xy}] \in X^*$, $u, w \in X$. In addition, the authors observed that if $X$ is a Hilbert space, then $[t\overrightarrow{xy}] = t(y - x)$.

Recently, Khatibzadeh and Ranjbar [21] introduced the concept of monotonicity in a CAT(0) space $X$ with dual space $X^*$ as follows. A mapping $A : X \to 2^{X^*}$ is called *monotone* if

$$\langle u^* - w^*, \overrightarrow{uw} \rangle \geq 0,\ \forall u, w \in \{x \in X : Ax \neq \emptyset\}, u^* \in Au, w^* \in Aw.$$

The authors approximate a solution of MIP by establishing a $\Delta$-convergence theorem for the following PPA:

$$v_{n+1} = J^A_{\mu_n} v_n,\ v_1 \in X, \tag{1.2}$$

where $\mu_n \in (0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_n = \infty$ and $J^A_\mu$ is the resolvent operator of monotone map $A$ of order $\mu > 0$, which is defined by

$$J^A_\mu z := \left\{w \in X : \left[\frac{1}{\mu}\overrightarrow{wz}\right] \in Aw\right\}. \tag{1.3}$$

Thereafter, Ranjbar and Khatibsadeh [28] proposed the following scheme

$$v_{n+1} = \sigma_n v_n \oplus (1-\sigma_n) J^A_{\mu_n} v_n, \;\; v_1 \in X, \tag{1.4}$$

where $\{\mu_n\} \subset (0,\infty)$ and $\{\sigma_n\} \subset [0,1]$. The authors established $\Delta$-convergence of the sequence generated therefrom to a solution of MIP. For more details on PPA and MIP see [8, 19, 20, 31] and the references therein.

In 2020, Dehghan et al. [9] introduced a new method of approximating a common solution of a finite collection of MIP in a complete CAT(0) space and established that the method is faster than that of Takahashi and Shimoji [33]. For $j \in \{1,2,\cdots,m\}$, let each $A_j : X \to 2^{X^*}$ be a monotone mapping with resolvent operator $J^{(j)}_{\mu_n}$. Then the authors scheme is as follows:

$$v_{n+1} = \sigma_n V \oplus (1-\sigma_n) T_n v_n, \tag{1.5}$$

where $V$ is fixed in $X$, $\{\sigma_n\}$ is a sequence in $[0,1]$ such that $\lim\limits_{n\to\infty} \sigma_n = 0$, $\sum\limits_{n=1}^{\infty} \sigma_n = \infty$, $\sum\limits_{n=2}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty$, and $T_n$ is a map defined by

$$\begin{cases} U_n^{(0)} v = v \\ U_n^{(1)} v = a_n^{(1)} J^{(1)}_{\mu_n} v \oplus b_n^{(1)} U_n^{(0)} v \oplus c_n^{(1)} U_n^{(0)} v \\ U_n^{(2)} v = a_n^{(2)} J^{(2)}_{\mu_n} v \oplus b_n^{(2)} U_n^{(1)} v \oplus c_n^{(2)} U_n^{(1)} v \\ U_n^{(3)} v = a_n^{(3)} J^{(3)}_{\mu_n} v \oplus b_n^{(3)} U_n^{(2)} v \oplus c_n^{(3)} U_n^{(2)} v \\ \vdots \\ U_n^{(m-1)} v = a_n^{(m-1)} J^{(m-1)}_{\mu_n} v \oplus b_n^{(m-1)} U_n^{(m-2)} v \oplus c_n^{(m-1)} U_n^{(m-2)} v \\ T_n v = a_n^{(m)} J^{(m)}_{\mu_n} v \oplus b_n^{(m)} U_n^{(m-1)} v \oplus c_n^{(m)} U_n^{(m-1)} v, \end{cases}$$

with $\{a_n^{(j)}\}$, $\{b_n^{(j)}\}$ and $\{c_n^{(j)}\}$ in $[0,1]$ satisfying

$$a_n^{(j)} + b_n^{(j)} + c_n^{(j)} = 1, \;\; j = 1,2,\cdots,m.$$

In 2021, Chaipunya et al. [7] observed that although CAT(0) spaces generalize Hilbert spaces and Hadamard Manifolds, the known concept of monotonicity that resulted in the establishment of the resolvent operator in (1.3) barely has a relationship with the Hadamard manifolds. To incorporate the monotonicity of Hadamard manifolds, the authors introduced a new concept of monotonicity called monotone vector field (MVF for short) using tangent spaces not the dual spaces. They analyzed that this notion coincides with the notion of monotonicity found in both Hilbert spaces and Hadamard manifolds unlike the monotonicity structure in [21].

Inspired by the work of Chaipunya et al. [7], Dehghan et al. [9], Ranjbar and Khatibzadeh [28], Khatibzadeh and Ranjbar [21] and motivated by the research on this direction, we establish some iterative schemes for approximating a common solution of a countable family of monotone vector field inclusion problems (MVFIP) in CAT(0) setting. We propose a one step convex combination scheme of proximal point algorithms and establish convergence theorems for the sequence generated therefrom. We apply our results to solve a countable family of minimization problems, compute Frechét mean and geometric median in CAT(0) spaces, and also solve a kinematic problem in robotic motion control.

Moreover, we give a numerical example to show the applicability and robustness of the proposed scheme by comparing it with the scheme of Dehghan et al. [9].

The paper is outlined as follows. In Section 2, we recall preliminaries consisting of definitions, lemmas and some known results that are essential in the subsequent sections. We present the proposed scheme and its convergence analysis in Section 3. Finally, we illustrate the applications and the example in Section 4.

## 2. Preliminaries

In this section, we state some basic facts and known results that will be useful in the subsequent sections. The details can be found in [4, 24, 26], unless otherwise specified.

Let $(X, d)$ be a metric space and $u$, $v$ be two points in $X$. A map $\tau_u^v : [0, \ell] \subset \mathbb{R} \to X$ is called a *geodesic path* from $u$ to $v$ if $\tau_u^v(0) = u$, $\tau_u^v(\ell) = v$ and $d(\tau_u^v(t_1), \tau_u^v(t_2)) = |t_1 - t_2|$, for every $t_1, t_2 \in [0, \ell]$. The image of $\tau_u^v$ is a *geodesic segment* joining $u$ and $v$, and denoted by $[u, v]$ if unique. A metric space $(X, d)$ is a *geodesic space* if every two elements $u$, $v$ in $X$ are joined by a geodesic segment, and is said to be *uniquely geodesic space* if every two points $u$, $v$ are joined by a unique geodesic segment $[u, v]$ in $X$. For $u$, $v$ having unique geodesic segment and for $t \in [0, 1]$, we denote by $(1 - t)u \oplus tv$ the unique point $w \in [u, v]$ such that

$$d(u, w) = td(u, v) \quad \text{and} \quad d(v, w) = (1 - t)d(u, v). \tag{2.1}$$

Moreover, for finite elements $\{v_j\}_1^m \subset X$ and $\{t_j\}_1^m \subset (0, 1)$, the notation $\bigoplus_{j=1}^{m} t_j v_j$ is adopted from [10, p.460], as follows:

$$\bigoplus_{j=1}^{m} t_j v_j := (1 - t_m)\left(\frac{t_1}{1 - t_m} v_1 \oplus \frac{t_2}{1 - t_m} v_2 \oplus \cdots \oplus \frac{t_{m-1}}{1 - t_m}\right) \oplus t_m v_m.$$

A geodesic space $(X, d)$ is called a CAT(0) space if and only if it satisfies the CN-inequality of Bruhat and Tits [6] as follows. Let $w, v \in X$ and $z$ be a midpoint of a geodesic segment connecting $w$ and $v$, then

$$d^2(z, x) \leq \frac{1}{2} d^2(w, x) + \frac{1}{2} d^2(v, x) - \frac{1}{4} d^2(w, v), \tag{2.2}$$

for every $x \in X$. CAT(0) spaces include pre-Hilbert spaces, Hilbert balls, Euclidean buildings, $\mathbb{R}$-trees and Hadamard manifolds. For more details on basics of CAT(0) spaces and their examples see [4, 5, 15, 22, 24, 29]. As direct consequences of (2.1) and (2.2), the following inequalities hold for $u_1, u_2, u_3 \in X$ and $t \in [0, 1]$ (see also [13]):

$$d((1 - t)u_1 \oplus tu_2, u_3) \leq (1 - t)d(u_1, u_3) + td(u_2, u_3); \tag{2.3}$$

$$d^2((1 - t)u_1 \oplus tu_2, u_3) \leq (1 - t)d^2(u_1, u_3) + td^2(u_2, u_3) - t(1 - t)d^2(u_1, u_2). \tag{2.4}$$

A subset $E$ of a CAT(0) space $X$ is *convex* if all geodesic segments connecting any two points of $E$ are contained in $E$. A geodesic triangle $\Delta(u, v, w)$ in $X$ is a set of three points $u$, $v$, $w$ together with three geodesic segments connecting each pair. For a uniquely geodesic space the triangle $\Delta$ is simply

$$\Delta(v, u, w) := [v, w] \cup [w, u] \cup [u, v], \tag{2.5}$$

where $u, v, w \in X$. A comparison triangle of a geodesic triangle $\Delta(u, v, w)$ is a triangle in the Euclidean space $\left(\mathbb{R}^2, \|\cdot\|_2\right)$ denoted by $\bar{\Delta}(\bar{u}, \bar{v}, \bar{w})$ satisfying

$$d(u, v) = \|\bar{u} - \bar{v}\|_2, \qquad d(v, w) = \|\bar{v} - \bar{w}\|_2, \qquad d(u, w) = \|\bar{u} - \bar{w}\|_2.$$

For any point $z \in \Delta(u, v, w)$, if $z$ lies in the segment connecting $u$ and $v$, then a comparison point of $z$ in a comparison triangle $\bar{\Delta}(\bar{u}, \bar{v}, \bar{w})$ is the point $\bar{z} \in [\bar{u}, \bar{v}] \subset \bar{\Delta}(\bar{u}, \bar{v}, \bar{w})$ with $d(u, z) = \|\bar{u} - \bar{z}\|_2$. For points in the other segments of $\Delta(u, v, w)$, their comparison points are defined in a similar way.

Let $\bar{\angle}_x(u, w)$ denotes the comparison angle between $u$ and $w$ at $x$, i.e,

$$\bar{\angle}_x(x, x) := 0,\ \bar{\angle}_x(x, w) = \bar{\angle}_x(w, x) := \frac{\pi}{2},\ \cos\bar{\angle}_x(u, w) := \frac{\langle \bar{u} - \bar{x}, \bar{w} - \bar{x}\rangle}{\|\bar{u} - \bar{x}\|_2\|\bar{w} - \bar{x}\|_2},$$

where $u, w \in X\backslash\{x\}$ and $\bar{\Delta}(\bar{u}, \bar{w}, \bar{x})$ is a comparison triangle of $\triangle(u, w, x)$. Then the *Alexandrov angle* between two geodesic issuing from a common point $x \in X$ is defined by

$$\alpha_x(\tau_1, \tau_2) = \lim_{s,t\to 0^+} \bar{\angle}_x(\tau_1(t), \tau_2(s)).$$

The Alexandrov angle $\alpha_x$ defines a pseudometric on the set of all geodesic issuing from $x$. We denote the set of all geodesic issuing from $x$ by $S_x$ and the metric identification of the pseudometric space by $(S_x, \bar{\angle}_x)$. In this work, the elements of $S_x$ are denoted by $\tau \equiv [\tau]$. As in [7], $\sim$ forms an equivalence relation on $[0, \infty) \times S_x$ in the sense that $(t, \tau_1) \sim (s, \tau_2)$ if and only if

$$t\eta(\tau_1) = s\eta(\tau_2) = 0 \text{ or } t\eta(\tau_1) = s\eta(\tau_2) > 0 \text{ with } \tau_1 = \tau_2,$$

where $\eta(\tau) := 0$ if $\tau$ is a geodesic connecting only one point and $\eta(\tau) := 1$ otherwise. Then $T_xX := ([0, \infty) \times S_x)/\sim$ together with the metric $d_x$ defined by

$$d_x(t\tau_1, s\tau_2) := \sqrt{t^2\eta(\tau_1) + s^2\eta(\tau_2) - 2st\eta(\tau_1)\eta(\tau_2)cos\angle_x(\tau_1, \tau_2)}$$

form a metric space $(T_xX, d_x)$ known as the *tangent space* of $X$.

In the sequel, we denote a complete CAT(0) space by $(X, d)$, the tangent space of $X$ at $u$ by $(T_uX, d_u)$ and a nonempty convex closed subset of $X$ by $E$. we shall denote the tangent bundle of $X$, $\bigcup_{u\in X} T_uX$ by $TX$, and adopt the notation $\mathbf{0} := \{0_u : u \in X\}$, where $0_u := 0\tau = s\tau_u^u$ for which $s > 0$ and $\tau \in S_u$. We shall say that a vector field $A : X \to TX$ satisfies condition $(S)$ if for any $s > 0$ and $x \in X$, there exists $u \in X$ such that $sd(u, x)\tau_u^x \in Au$.

**Definition 2.1.** *[7] A vector field $A : X \to TX$ is said to be monotone if*

$$L_x(\xi, \tau_u^w) + L_x(\phi, \tau_w^u) \leq 0,$$

*for every $(u, \xi), (w, \phi) \in \{(x, u) \in X \times TX : u \in Ax\}$, where*

$$L_x(t\tau_1, s\tau_2) = st\eta(\tau_1)\eta(\tau_2)\cos\bar{\angle}_x(\tau_1, \tau_2).$$

In the sequel, $A^{-1}(\mathbf{0})$ denotes the solution set of MVFIP and $J_\mu$ denotes the $\mu$-resolvent of $A$ defined by

$$J_\mu(z) := \left\{w \in X : \frac{1}{\mu}d(w, z)\tau_w^z \in Aw\right\},\ \forall z \in X.$$

**Lemma 2.2.** *[7, p.15] Let $A : X \to TX$ be a monotone vector field satisfying condition $(S)$ and let $J_\mu$ be the resolvent of $A$. Then the following statements hold*

*(i)* $J_\mu$ *is well-defined and singlevalued on* $X$.
*(ii)* $d\left(J_\mu(x), J_\mu(y)\right) \leq d(x,y)$ *for every* $x, y$ *in* $X$.
*(iii)* $A^{-1}(\mathbf{0}) = \{x \in X : x = J_\mu(x)\}$.

**Definition 2.3.** *Let $\{v_n\}$ be a bounded sequence in $X$. Then the asymptotic center $A(\{v_n\})$ of $\{v_n\}$ is defined by*

$$A(\{v_n\}) := \{u \in Y : \limsup_{n\to\infty} d(u, v_n) = \inf_{u\in X} \limsup_{n\to\infty} d(u, v_n)\}.$$

**Remark 2.4.** *It is shown in [12, Proposition 7] that $A(\{v_n\})$ has only one element.*

**Definition 2.5.** *A bounded sequence $\{v_n\}$ $\Delta$-converges to a point $v$ in $X$ if $\{v\}$ is the unique asymptotic centre for every subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and strongly converges to $v$ if $\lim_{n\to\infty} d(v_n, v) = 0$.*

**Lemma 2.6.** *[23, Proposition 3.7] Let $T : E \to X$ be a map such that $d(Tu, Tw) \leq d(u, w)$ for every $u, w \in E$. If $\{w_n\}$ $\Delta$-converges to $w$ and $d(w_n, Tw_n) \to 0$, then $w = Tw$.*

**Lemma 2.7.** *[11] The asymptotic centre of any bounded sequence in $E$ is contained in $E$.*

**Lemma 2.8.** *[23, Proposition 3.6] Every bounded sequence $\{w_n\}$ in $E$ has a $\Delta$-convergent subsequence $\{w_{n_k}\}$.*

**Lemma 2.9.** *[13, Lemma 2.8] Let $\{v_n\}$ be a sequence in $X$ with $A(\{v_n\}) = \{v\}$. Suppose that $\{v_{n_k}\}$ is a subsequence of $\{v_n\}$ with $A(\{v_{n_k}\}) = \{w\}$ and the sequence $\{d(v_n, w)\}$ converges. Then $v = w$.*

## 3. Main results

In this section, we maintain the notation $(X, d)$ for a complete CAT(0) space, $E$ for a nonempty closed convex subset of $X$, $TE$ for the tangent bundle of $E$ and $J_\mu^0$ for identity map.

**Theorem 3.1.** *For $m \in \mathbb{N}$, let $A_j : E \to TE$, $j = 1, 2, \cdots, m$ be monotone vector fields satisfying condition $(S)$ and $\Gamma := \bigcap_{j=1}^{m} A_j^{-1}(0) \neq \emptyset$. For each $j \in \{1, 2, \cdots, m\}$, let $J_\mu^j$ be the $\mu$-resolvent operator of $A_j$. Let $v_1$ be chosen arbitrarily in $E$ and $\{v_n\}$ be defined iteratively by*

$$v_{n+1} = \bigoplus_{j=0}^{m} \gamma_n^j J_\mu^j(v_n), \; n \geq 1, \tag{3.1}$$

*where $\{\gamma_n^j\} \subset (0, 1)$ such that $\sum_{j=0}^{m} \gamma_n^j = 1$ and $\liminf_{n\to\infty} \gamma_n^j > 0$ for every $j \in \{0, 1, \cdots, m\}$. Then $\{v_n\}$ $\Delta$-converges to a point in $\Gamma$.*

*Proof.* Let $v^* \in \Gamma$. By (3.1), (2.3) and Lemma 2.2, we have

$$d(v_{n+1}, v^*) = d\left(\bigoplus_{j=0}^{m} \gamma_n^j J_\mu^j(v_n), v^*\right)$$

$$\leq \sum_{j=0}^{m} \gamma_n^j d(J_\mu^j(v_n), v^*)$$
$$= \sum_{j=0}^{m} \gamma_n^j d(J_\mu^j(v_n), J_\mu^j(v^*))$$
$$\leq d(v_n, v^*). \tag{3.2}$$

This implies that for every $v^* \in \Gamma$, $\{d(v_n, v^*)\}$ is a nonincreasing sequence in $\mathbb{R}$ and since it is bounded, the limit exists.

Now, let $n \in \mathbb{N}$ and set $\delta_k^{(n)} = \bigoplus_{j=0}^{k} \frac{\gamma_n^j}{\beta_k} J_\mu^j(v_n)$, where $\beta_k := \sum_{j=0}^{k} \gamma_n^j$, $k \in \{0, 2, \cdots, m\}$. Then $\beta_k \in (0, 1)$, $\beta_0 = \gamma_n^0, \beta_m = 1$ and $\delta_0^{(n)} = J_0(v_n) = v_n$. Moreover,

$$\frac{\beta_{k-1}}{\beta_k} \geq \gamma_n^0, \text{ and} \tag{3.3}$$

$$\delta_k^{(n)} = \frac{\beta_{k-1}}{\beta_k} \delta_{k-1}^{n)} \bigoplus \frac{\gamma_n^k}{\beta_k} J_\mu^k(v_n), \tag{3.4}$$

for every $k \in \{1, 2, \cdots, m\}$. Let $v^* \in \Gamma$. Then from (3.4), (2.4) and (3.3), we have the following for every $k \in \{1, 2, \cdots, m\}$

$$\begin{aligned}
d^2\left(\delta_k^{(n)}, v^*\right) &= d^2\left(\frac{\beta_{k-1}}{\beta_k} \delta_{k-1}^{n)} \bigoplus \frac{\gamma_n^k}{\beta_k} J_\mu^k(v_n), v^*\right) \\
&\leq \frac{1}{\beta_k}\left[\beta_{k-1} d^2\left(\delta_{k-1}^{(n)}, v^*\right) + \gamma_n^k d^2\left(J_\mu^k(v_n), v^*\right) - \frac{\beta_{k-1}\gamma_n^k}{\beta_k} d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right)\right] \\
&\leq \frac{1}{\beta_k}\left[\beta_{k-1} d^2\left(\delta_{k-1}^{(n)}, v^*\right) + \gamma_n^k d^2\left(J_\mu^k(v_n), v^*\right) - \gamma_n^0 \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right)\right].
\end{aligned} \tag{3.5}$$

From (3.1) and (3.5), we have

$$\begin{aligned}
d^2(v_{n+1}, v^*) &= d^2\left(\delta_m^{(n)}, v^*\right) \\
&\leq \frac{1}{\beta_m}\left[\beta_{m-1} d^2\left(\delta_{m-1}^{(n)}, v^*\right) + \gamma_n^m d^2\left(J_\mu^m(v_n), v^*\right) - \gamma_n^0 \gamma_n^m d^2\left(\delta_{m-1}^{(n)}, J_\mu^m(v_n)\right)\right] \\
&\leq \beta_{m-1} d^2\left(\delta_{m-1}^{(n)}, v^*\right) + \gamma_n^m d^2\left(J_\mu^m(v_n), v^*\right) - \gamma_n^0 \gamma_n^m d^2\left(\delta_{m-1}^{(n)}, J_\mu^m(v_n)\right) \\
&\leq \left[\beta_{m-2} d^2\left(\delta_{m-2}^{(n)}, v^*\right) + \gamma_n^{m-1} d^2\left(J_\mu^{m-1}(v_n), v^*\right) - \gamma_n^0 \gamma_n^{m-1} d^2\left(\delta_{m-2}^{(n)}, J_\mu^{m-1}(v_n)\right)\right] \\
&\quad + \gamma_n^m d^2\left(J_\mu^m(v_n), v^*\right) - \gamma_n^0 \gamma_n^m d^2\left(\delta_{m-1}^{(n)}, J_\mu^m(v_n)\right) \\
&= \beta_{m-2} d^2\left(\delta_{m-2}^{(n)}, v^*\right) + \sum_{k=m-1}^{m} \gamma_n^k d^2\left(J_\mu^k(v_n), v^*\right) - \gamma_n^0 \sum_{k=m-1}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right).
\end{aligned}$$

Continuing in this pattern, we get that

$$d^2(v_{n+1}, v^*) \leq \beta_{m-3} d^2\left(\delta_{m-3}^{(n)}, v^*\right) + \sum_{k=m-2}^{m} \gamma_n^k d^2\left(J_\mu^k(v_n), v^*\right) - \gamma_n^0 \sum_{k=m-2}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right)$$

$$
\begin{aligned}
&\vdots \\
&\le \beta_1 d^2\left(\delta_1^{(n)}, v^*\right) + \sum_{k=2}^{m} \gamma_n^k d^2\left(J_\mu^k(v_n), v^*\right) - \gamma_n^0 \sum_{k=2}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right) \\
&\le \beta_0 d^2\left(\delta_0^{(n)}, v^*\right) + \sum_{k=1}^{m} \gamma_n^k d^2\left(J_\mu^k(v_n), v^*\right) - \gamma_n^0 \sum_{k=1}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right) \\
&= \sum_{k=0}^{m} \gamma_n^k d^2\left(J_\mu^k(v_n), v^*\right) - \gamma_n^0 \sum_{k=1}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right).
\end{aligned}
$$

This and Lemma 2.2 imply that

$$
\begin{aligned}
d^2\left(v_{n+1}, v^*\right) &\le \sum_{k=0}^{m} \gamma_n^k d^2\left(J_\mu^k(v_n), v^*\right) - \gamma_n^0 \sum_{k=1}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right) \\
&= \sum_{k=0}^{m} \gamma_n^k d^2\left(J_\mu^k(v_n), J_\mu^k(v^*)\right) - \gamma_n^0 \sum_{k=1}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right) \\
&\le d^2\left(v_n, v^*\right) - \gamma_n^0 \sum_{k=1}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right).
\end{aligned} \tag{3.6}
$$

Consequently, we have that for every $j \in \{1, 2, \cdots, m\}$,

$$
\begin{aligned}
\gamma_n^0 \gamma_n^j d^2\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right) &\le \gamma_n^0 \sum_{k=1}^{m} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right) \\
&\le d^2\left(v_n, v^*\right) - d^2\left(v_{n+1}, v^*\right).
\end{aligned} \tag{3.7}
$$

Since $\liminf_{n\to\infty} \gamma_n^0 \gamma_n^j \ge \liminf_{n\to\infty} \gamma_n^0 \cdot \liminf_{n\to\infty} \gamma_n^j > 0$ for each $j$, it follows from (3.2) and (3.7) that

$$
\lim_{n\to\infty} d^2\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right) = 0,\ \forall j \in \{1, \cdots, m\}.
$$

Thus,

$$
\lim_{n\to\infty} d\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right) = 0,\ \forall j \in \{1, \cdots, m\}. \tag{3.8}
$$

From (3.4), (2.1) and (3.8), we get that for every $j \in \{1, \cdots, m\}$,

$$
d\left(\delta_{j-1}^{(n)}, \delta_j^{(n)}\right) \le d\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right) \to 0,\ \text{as } n \to \infty. \tag{3.9}
$$

Furthermore, for any $j \in \{1, \cdots, m\}$, we have

$$
\begin{aligned}
d\left(v_n, J_\mu^j(v_n)\right) &= d\left(\delta_0^{(n)}, J_\mu^j(v_n)\right) \\
&\le d\left(\delta_0^{(n)}, \delta_1^{(n)}\right) + d\left(\delta_1^{(n)}, \delta_2^{(n)}\right) + \cdots + d\left(\delta_{j-2}^{(n)}, \delta_{j-1}^{(n)}\right) + d\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right) \\
&= \sum_{k=1}^{j-1} d\left(\delta_{k-1}^{(n)}, \delta_k^{(n)}\right) + d\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right).
\end{aligned} \tag{3.10}
$$

It follows from (3.10), (3.8) and (3.9) that

$$\lim_{n\to\infty} d\left(v_n, J_\mu^j(v_n)\right) = 0,\ \forall j \in \{1, \cdots, m\}. \tag{3.11}$$

From (3.2), $\{v_n\}$ is bounded. Suppose that $x$ is an arbitrary point in the union of asymptotic centers of all subsequences of $\{v_n\}$ and let $\{v_{n_k}\}$ be a subsequence of $\{v_n\}$ with $A(\{v_{n_k}\}) = \{x\}$. Since $\{v_{n_k}\}$ is bounded, Lemma 2.8 gives the existence of a subsequence $\left\{v_{n_{k_j}}\right\}$ of $\{v_{n_k}\}$ that $\Delta$-converges to $v$ and Lemma 2.7 implies that $v \in E$. Now by Lemma 2.6 and (3.11), we have that $v = J_\mu^j(v)$ for every $j \in \{1, 2, \cdots, m\}$. Thus $v \in \Gamma$, by Lemma 2.2(iii). By (3.2), $\{d(v_n, v)\}$ converges and by Lemma 2.9, $x = v$. Thus, the union of asymptotic centers of all subsequences of $\{v_n\}$ (denoted by $\omega_A(\{v_n\})$) is in $\Gamma$. To complete the proof, it suffices to show that $\omega_A(\{v_n\})$ consists of only one element. Let $A(\{v_n\}) = \{v\}$ and suppose that there exists $w \in \omega_A(\{v_n\})$ with $v \neq w$. Now, let $\{v_{n_k}\}$ be the subsequence of $\{v_n\}$ with $A(\{v_{n_k}\}) = \{w\}$. Then by (3.2) and the definition of asymptotic center, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, w) \\
&= \lim_{n\to\infty} d(v_n, w) \\
&= \limsup_{k\to\infty} d(v_{n_k}, w) \\
&< \limsup_{k\to\infty} d(v_{n_k}, v) \\
&\leq \limsup_{n\to\infty} d(v_n, v),
\end{aligned}$$

which contradicts $v \neq w$. Therefore, $\omega_A(\{v_n\}) \subseteq \Gamma$ consists of exactly one element. ■

**Corollary 3.2.** *Let $E$, $A_j$, $\Gamma$, $J_\mu^j$ and $\{v_n\}$ be as in Theorem 3.1. Suppose $E$ is compact. Then $\{v_n\}$ converges strongly to a member of $\Gamma$.*

*Proof.* We have seen in (3.2) that $\{v_n\}$ is a bounded sequence in $E$. Since $E$ is compact, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that converges strongly to some point $v$ in $E$. Thus, $\{v_{n_k}\}$ $\Delta$-converges to $v \in E$. Then $v \in \omega_A(\{y_n\}) \subset \Gamma$. Consequently, by (3.2), we have

$$\lim_{n\to\infty} d(v_n, v) = \lim_{k\to\infty} d(v_{n_k}, v) = 0,$$

which completes the proof. ■

**Theorem 3.3.** *Let $A_j : E \to TE$, $j \geq 1$ be monotone vector fields satisfying condition $(S)$ and $\Gamma := \bigcap_{j=1}^{\infty} A_j^{-1}(0) \neq \emptyset$. For each $j \in \mathbb{N}$, let $J_\mu^j$ be the $\mu$-resolvent operator of $A_j$. Let $v_1$ be chosen arbitrarily in $E$ and $\{v_n\}$ be defined iteratively by*

$$v_{n+1} = \bigoplus_{j=0}^{n} \gamma_n^j J_\mu^j(v_n),\ n \geq 1, \tag{3.12}$$

*where $\{\gamma_n^j\} \subset (0, 1)$ such that $\sum_{j=0}^{n} \gamma_n^j = 1$ and $\liminf_{n\to\infty} \gamma_n^j > 0$ for every $j \in \mathbb{N}$. Then $\{v_n\}$ $\Delta$-converges to a point in $\Gamma$.*

The proof of Theorem 3.3 follows similar lines as in the proof of Theorem 3.1. But for completeness, we sketch the proof as follows.

*Proof.* Let $v^* \in \Gamma$. By (3.1), (2.3) and Lemma 2.2, we have

$$\begin{aligned} d(v_{n+1}, v^*) &= d\left(\bigoplus_{j=0}^{n} \gamma_n^j J_\mu^j(v_n), v^*\right) \\ &\leq \sum_{j=0}^{n} \gamma_n^j d(J_\mu^j(v_n), v^*) \\ &= \sum_{j=0}^{n} \gamma_n^j d(J_\mu^j(v_n), J_\mu^j(v^*)) \\ &\leq d(v_n, v^*). \end{aligned} \tag{3.13}$$

This implies that for every $v^* \in \Gamma$, $\{d(v_n, v^*)\}$ is a nonincreasing sequence in $\mathbb{R}$ and since it is bounded, the limit exists.

Now let $n \in \mathbb{N}$ and set $\delta_k^{(n)} = \bigoplus_{j=0}^{k} \frac{\gamma_n^j}{\beta_k} J_\mu^j(v_n)$, where $\beta_k := \sum_{j=0}^{k} \gamma_n^j$, $k \in \{0, 2, \cdots, n\}$. Then $\beta_k \in (0, 1)$, $\beta_0 = \gamma_n^0, \beta_n = 1$ and $\delta_0^{(n)} = J_0(v_n) = v_n$. Moreover,

$$\frac{\beta_{k-1}}{\beta_k} \geq \gamma_n^0, \text{ and} \tag{3.14}$$

$$\delta_k^{(n)} = \frac{\beta_{k-1}}{\beta_k}\delta_{k-1}^{n)} \bigoplus \frac{\gamma_n^k}{\beta_k} J_\mu^k(v_n), \tag{3.15}$$

for every $k \in \{1, 2, \cdots, n\}$. Let $v^* \in \Gamma$. Following similar arguments as in obtaining (3.6) with (3.15), we get

$$d^2(v_{n+1}, v^*) \leq d^2(v_n, v^*) - \gamma_n^0 \sum_{k=1}^{n} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_k(v_n)\right). \tag{3.16}$$

Let $j \geq 1$, we have from (3.16) that for any $n \geq j$,

$$\begin{aligned} \gamma_n^0 \gamma_n^j d^2\left(\delta_{j-1}^{(n)}, J_j(v_n)\right) &\leq \gamma_n^0 \sum_{k=1}^{n} \gamma_n^k d^2\left(\delta_{k-1}^{(n)}, J_\mu^k(v_n)\right) \\ &\leq d^2(v_n, v^*) - d^2(v_{n+1}, v^*). \end{aligned} \tag{3.17}$$

Moreover, since $\liminf_{n\to\infty} \gamma_n^0 \gamma_n^j \geq \liminf_{n\to\infty} \gamma_n^0 \cdot \liminf_{n\to\infty} \gamma_n^j > 0$ for each $j \in \mathbb{N}$, we have from (3.13) and (3.17) that

$$\lim_{n\to\infty} d\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right) = 0, \ \forall j \in \mathbb{N}. \tag{3.18}$$

From (3.15), (2.1) and (3.18), we get that for every $j \in \mathbb{N}$,

$$d\left(\delta_{j-1}^{(n)}, \delta_j^{(n)}\right) \leq d\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right) \to 0, \text{ as } n \to \infty. \tag{3.19}$$

Furthermore, for any $j \in \mathbb{N}$, we have that

$$\begin{aligned} d\left(v_n, J_\mu^j(v_n)\right) &= d\left(\delta_0^{(n)}, J_\mu^j(v_n)\right) \\ &\leq d\left(\delta_0^{(n)}, \delta_1^{(n)}\right) + d\left(\delta_1^{(n)}, \delta_2^{(n)}\right) + \cdots + d\left(\delta_{j-2}^{(n)}, \delta_{j-1}^{(n)}\right) + d\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right) \\ &= \sum_{k=1}^{j-1} d\left(\delta_{k-1}^{(n)}, \delta_k^{(n)}\right) + d\left(\delta_{j-1}^{(n)}, J_\mu^j(v_n)\right). \end{aligned} \tag{3.20}$$

It follows from (3.18), (3.19) and (3.20), that

$$\lim_{n\to\infty} d\left(v_n, J_\mu^j(v_n)\right) = 0, \ \forall j \in \mathbb{N}. \tag{3.21}$$

The proof is completed as in similar arguments of proof of Theorem 3.1 from (3.11) to end. ∎

**Remark 3.4.** *It is important to note that the result in Theorem 3.1 can be obtain from Theorem 3.3 by taking each $A_j$ to be the set containing only the zero of the tangent bundle $TE$ for any $j \in \{m+1, m+2, m+3, \cdots\}$, and* (3.1) *coincides with* (3.12) *for $n \geq m$.*

**Corollary 3.5.** *Let $E$, $A_j$, $\Gamma$ and $\{v_n\}$ be as in Theorem 3.3. Suppose that $E$ is compact. Then $\{v_n\}$ converges strongly to a member of $\Gamma$.*

*Proof.* The proof follows similar arguments as in the proof of Corollary 3.2 with (3.13) in place of (3.2). ∎

## 4. Applications and numerical example

In the following discussion, we apply the proposed scheme to solve a countable family of minimization problems, to compute Frechét mean and geometric median in CAT(0) spaces and also to solve a kinematic problem in robotic motion control. Furthermore, we give a numerical example to show the computational overview of our scheme by comparing it with the scheme (1.5) of Dehghan et al. [9].

### *4.1. Application to minimization problems*

A function $g : X \to (-\infty, +\infty]$ is called *convex* if

$$g(tu \oplus (1-t)v) \leq tg(x) + (1-t)g(y) \text{ for all } t \in (0,1) \text{ and } x, y \in X.$$

If the set $\mathbb{D}(g) := \{x \in X : g(x) < +\infty\} \neq \emptyset$, then $g$ is proper. The function $g$ is said to be *lower semi-continuous* at a point $u \in \mathbb{D}(g)$ if $g(w) \leq \liminf_{n\to\infty} g(w_n)$ for any convergent sequence $\{w_n\}$ with limit $w \in \mathbb{D}(g)$. If $g$ is lower semi-continuous at every point in $\mathbb{D}(g)$, then it is lower semi-continuous on $X$. The problem of finding

$$u \in X \qquad \text{such that} \qquad g(u) \leq g(v), \quad \forall v \in X \tag{4.1}$$

is a well-known minimization problem (MP). The subdifferential of $g$ (as given in [7, p.11]) is a vector field $\partial g : X \to TX$ defined by

$$\partial g(x) = \{x^* \in T_xX : g(y) \geq g(x) + d(x,y)L_x(x^*, \gamma_x^y), (y \in X)\},$$

where $L_x(t\gamma_1, s\gamma_2) = ts\zeta(\gamma_1)\zeta(\gamma_2)\cos\measuredangle_x(\gamma_1, \gamma_2)$ for any $t\gamma_1, s\gamma_2 \in T_xX$.

It has been shown in [7] that:

(a) $\partial g$ is a monotone vector field that satisfies condition $(S)$,
(b) the solution of (4.1) coincides with the set $(\partial g)^{-1}(\mathbf{0})$.

In view of $(a)$ and $(b)$ above, we obtain the following results.

**Corollary 4.1.** *Let $g_j : X \to (-\infty, +\infty]$, $j = 1, 2, \cdots, m$ be a family of convex, proper and lower semi-continuous functions with $\Gamma := \bigcap_{j=1}^{m} \arg\min_{x\in X} g_j(x) \neq \emptyset$. For each $j \in \{1, 2, \cdots, m\}$, let $J_\mu^j$ be the $\mu$-resolvent operator of $\partial g_j$. For $v_1 \in X$, let $\{v_n\}$ be generated by*

$$v_{n+1} = \bigoplus_{j=0}^{m} \gamma_n^j J_\mu^j(v_n),\ n \geq 1, \tag{4.2}$$

*where $\{\gamma_n^j\} \subset (0, 1)$ such that $\sum_{j=0}^{m} \gamma_n^j = 1$ and $\liminf_{n\to\infty} \gamma_n^j > 0$ for every $j \in \{0, 1, \cdots, m\}$. Then $\{v_n\}$ $\Delta$-converges to a point in $\Gamma$.*

**Corollary 4.2.** *Let $g_j : X \to (-\infty, +\infty]$, $j \in \mathbb{N}$ be a family of convex, proper and lower semi-continuous functions with $\Gamma := \bigcap_{j\geq 1} \arg\min_{x\in X} g_j(x) \neq \emptyset$. For each $j \in \mathbb{N}$, let $J_\mu^j$ be the $\mu$-resolvent operator of $\partial g_j$. For $v_1 \in X$, let $\{v_n\}$ be generated by*

$$v_{n+1} = \bigoplus_{j=0}^{n} \gamma_n^j J_\mu^j(v_n),\ n \geq 1, \tag{4.3}$$

*where $\{\gamma_n^j\} \subset (0, 1)$ such that $\sum_{j=0}^{n} \gamma_n^j = 1$ and $\liminf_{n\to\infty} \gamma_n^j > 0$ for every $j \in \mathbb{N}$. Then $\{v_n\}$ $\Delta$-converges to a point in $\Gamma$.*

*4.2. Application in computing mean and median:*

Frechét mean and geometric median in CAT(0) spaces have many applications toward real-life setting. Some of the applications come from diffusion tensor imaging, consensus algorithms, computational phylogenetics and modeling of airway systems in human lungs and blood vessels [1, 2, 14].

Let $\{v_i\}_{i=1}^{p}$ be elements of $X$ and let $\{\alpha_i\}_{i=1}^{p}$ be positive weights satisfying $\sum_{i=1}^{p} \alpha_i = 1$. The Frechét mean of $\{v_i\}_{i=1}^{p}$ is given by

$$\underset{w\in X}{argmin} \left\{ \sum_{i=1}^{p} \alpha_i d^2(w, v_i) \right\} \tag{4.4}$$

and the geometric median of $\{v_i\}_{i=1}^{p}$ is

$$\underset{w\in X}{argmin} \left\{ \sum_{i=1}^{p} \alpha_i d(w, v_i) \right\}. \tag{4.5}$$

Let $f$ and $g$ be two real-valued functions on $X$ defined by

$$f(w) = \sum_{k=1}^{p} \alpha_i d^2(w, v_k),\ g(w) = \sum_{k=1}^{p} \alpha_i d(w, v_k) \text{ for every } w \in X.$$

Then by properties of metric $d$, $f$ and $g$ are convex proper and lower semi-continuous functions on $X$. Consequently we have the following results from Theorem 4.1.

**Corollary 4.3.** *Let $J_\mu$ be the $\mu$-resolvent operator of $\partial f$. Then, for $v_1 \in X$, the sequence $\{v_n\}$ generated by*

$$v_{n+1} = \gamma_n^0 v_n \oplus (1 - \gamma_n^0) J_\mu(v_n), \ n \geq 1, \tag{4.6}$$

*where $\{\gamma_n^0\} \subset (0, 1)$ such that $\liminf_{n\to\infty} \gamma_n^0 > 0$, approximates the Frechét mean of $\{v_i\}_{i=1}^p$.*

**Corollary 4.4.** *Let $J_\mu$ be the $\mu$-resolvent operator of $\partial g$. Then, for $v_1 \in X$, the sequence $\{v_n\}$ generated by*

$$v_{n+1} = \gamma_n^0 v_n \oplus (1 - \gamma_n^0) J(v_n), \ n \geq 1, \tag{4.7}$$

*where $\{\gamma_n^0\} \subset (0, 1)$ such that $\liminf_{n\to\infty} \gamma_n^0 > 0$, approximates the geometric median of $\{v_i\}_{i=1}^p$.*

### *4.3. Two-arm robotic motion control*

Let $l \in \mathbb{N}$. For $k \in \{1, 2, \cdots, l\}$, consider $g_k : X \to \mathbb{R}$ be defined by $g_k(w) = d^2(w, \sigma_k)$, for every $w \in X$ and some $\sigma_k \in X$. Then by properties of metric $d$, each $g_k$ is convex proper and lower semi-continuous function. The problem of finding the minimizers of $g_k$ at each $k$ takes many forms and has been of great interest in optimization problems such as least square, time variant, total variations and so on. In this context, we consider a special case, when $X = \mathbb{R}^2$ equipped with the Euclidean distance $d$ and apply our method to solve discrete-time kinematics problem of two-arm robotic manipulator. This problem is to solve the following at each instant of time:

$$\min g_k(\delta_k), \tag{4.8}$$

where $\delta_k = f(\phi_k)$ is the end effector and $f$ is the kinematic mapping define by

$$f(\phi) = f(\phi_1, \phi_2) = \begin{bmatrix} r_1 \cos(\phi_1) + r_2 \cos(\phi_1 + \phi_2) \\ \\ r_1 \sin(\phi_1) + r_2 \sin(\phi_1 + \phi_2) \end{bmatrix},$$

in which $r_1, r_2$ are lengths of the arms (see, e.g. [32] for more details).

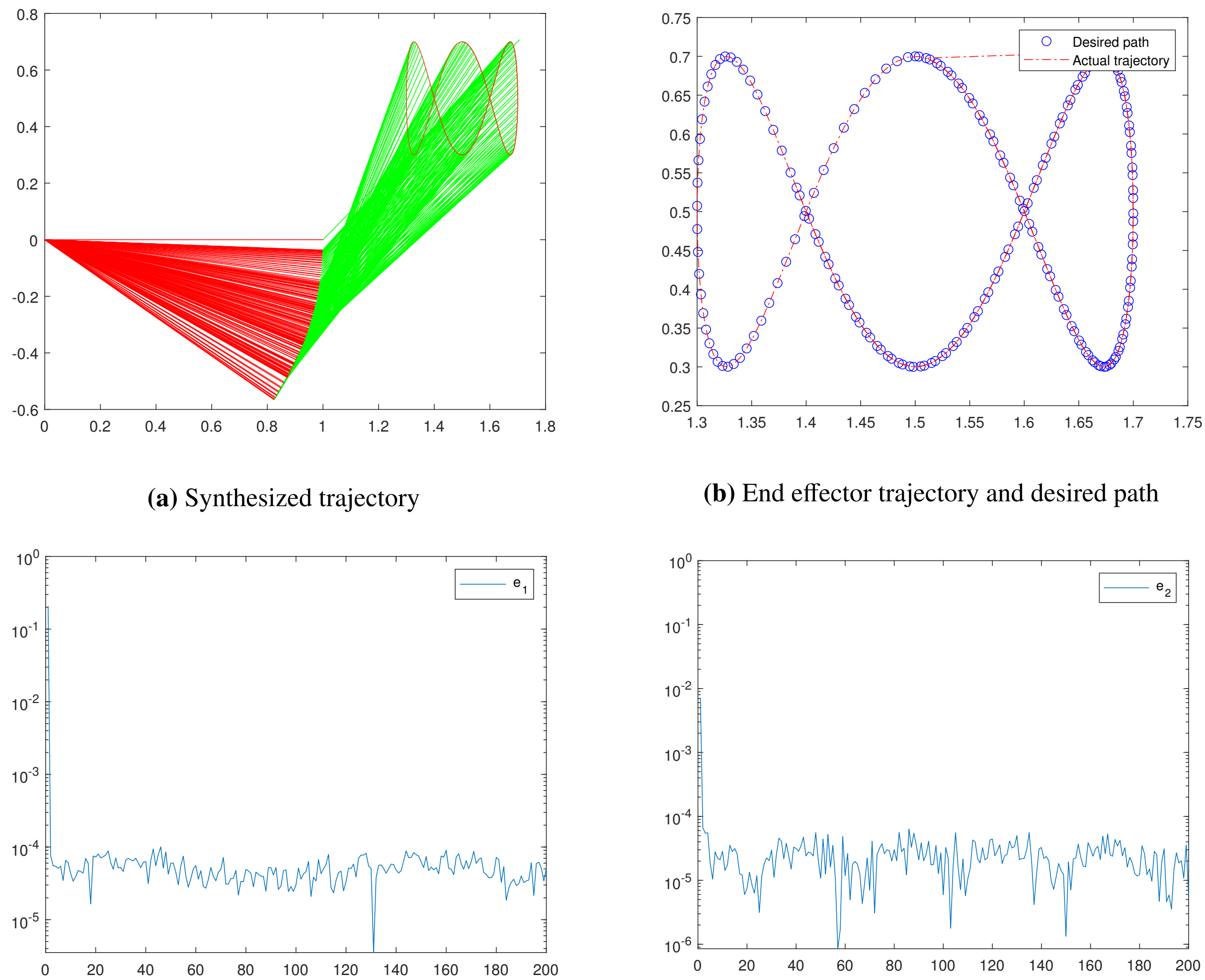
In this work, we track the curve

$$\sigma_k = \begin{bmatrix} \frac{3}{2} + \frac{1}{5}\sin(t_k) \\ \\ \frac{1}{2} + \frac{1}{5}\sin\left(3t_k + \frac{\pi}{2}\right) \end{bmatrix}.$$

We split the time frame of ten seconds into 200 instants, making $l = 200$ and take the unit arms lengths, i.e., $r_1 = r_2 = 1$. We set the starting point $v_1 = \left(0, \frac{\pi}{4}\right)^T$, $\gamma_n^0 = \dfrac{n}{3n+1}$, $\mu = 1$, $J_\mu^k$ is the subdifferential of $g_k$ and observe that the proposed scheme in (3.1), which reduces to

$$v_{n+1} = \gamma_n^0 v_n + (1 - \gamma_n^0) J(v_n), \ n \geq 1 \tag{4.9}$$

solves the problem in (4.8) for each $k$. The generated results are plotted in Figure 1. Specifically, Figure 1a shows synthesized trajectories generated through (4.9), Figure 1b displays end effector trajectory and desired path. Figure 1c and 1d shows the tracking of the residual error on horizontal and vertical axes, respectively. It can be observed from Figure 1a and 1b that the process is completed successfully. Moreover, Figure 1c and 1d show that the residual error is about $10^{-5}$.




**(a)** Synthesized trajectory

**(b)** End effector trajectory and desired path

**(c)** Tracking error on horizontal axis

**(d)** Tracking error on vertical axis

**Figure 1.** Numerical display for two-arm robotic motion control.

### *4.4. Numerical example*

Consider $\mathbb{R}^2$ equipped with the Euclidean distance $d$. For $j \in \{1, 2\}$, we defined

$$A_j x = A_j(x_1, x_2) = \Big\{\big((1 - j)x_1 + x_2, (1 - j)x_2 - x_1\big)\Big\}.$$

It follows from [34] that, each $A_j$ is monotone on $X$. Thus they are vector fields since the known monotone mappings on Hilbert spaces are monotone vector fields. Moreover, their corresponding resolvent operators are:

$$J^1_\mu x = J^1_\mu(x_1, x_2) = \left(\frac{x_1 - \mu x_2}{1 + \mu^2}, \frac{x_2 + \mu x_1}{1 + \mu^2}\right) \text{ and}$$
$$J^2_\mu x = J^2_\mu(x_1, x_2) = \left(\frac{(1 + \mu)x_1 - \mu x_2}{2\mu^2 + 2\mu + 1}, \frac{(1 + \mu)x_2 + \mu x_1}{2\mu^2 + 2\mu + 1}\right).$$

It can be observed that the proposed iterative scheme in (3.1) becomes

$$v_{n+1} = \gamma_n^0 v_n + \gamma_n^1 J_\mu^1(v_n) + \gamma_n^2 J_\mu^2(v_n),\ n \geq 1, \tag{4.10}$$

and the scheme in (1.5) becomes

$$\begin{cases} U_n^{(1)} v_n = a_n^{(1)} J_{\mu_n}^{(1)}(v_n) + b_n^{(1)} v_n + c_n^{(1)} v_n, \\ U_n^{(2)} v_n = a_n^{(2)} J_{\mu_n}^{(2)}(v_n) + b_n^{(2)} U_n^{(1)} v_n + c_n^{(2)} U_n^{(1)} v_n, \\ v_{n+1} = \sigma_n V + (1-\sigma_n) U_n^{(2)} v_n,\ n \geq 1. \end{cases} \tag{4.11}$$

In this example, we use $V = (1, 0.2)$, $\sigma_n = \frac{1}{2(n+1)}$, $\gamma_n^0 = a_n^{(1)} = a_n^{(2)} = \frac{1}{2}$, $\gamma_n^1 = b_n^{(1)} = b_n^{(2)} = \frac{1}{5}$ and $\gamma_n^2 = c_n^{(1)} = c_n^{(2)} = \frac{3}{10}$. The comparison results between the proposed scheme (3.1) and the scheme (1.5) of [9] is shown in Table 1. For each algorithms, we record the number of iteration (Iter.) of which $E_n := \|v_{n+1} - v_n\| < 10^{-6}$, the execution time (Time) and the value of the sequence $\{v_n\}$ at the final state.

**Table 1.** Convergence analysis of the proposed scheme in comparison to the scheme of [9].

| | | $\{v_n\}$ generated by (3.1) | | | $\{v_n\}$ generated by (1.5) | | |
|---|---|---|---|---|---|---|---|
| $v_1$ | $\mu$ | Iter. | Time | value of $v_n$ | Iter. | Time | value of $v_n$ |
| (21, 16) | 0.2 | 236 | 0.0022 | (-1.0179e-06, 9.014e-06) | 1811 | 0.00923 | (0.00092203, 0.0015568) |
| | 3 | 31 | 0.0007 | (8.9575e-07, -4.2359e-07) | 880 | 0.00285 | (0.00080785, 0.00034527) |
| | 12 | 26 | 0.0006 | (-3.9751e-07, 7.3048e-07) | 837 | 0.00267 | (0.00080889, 0.00020842) |
| | 50 | 25 | 0.0031 | (5e-07, 7.717e-07) | 827 | 0.00525 | (0.00080853, 0.00017292) |
| (-11, -23) | 0.2 | 236 | 0.0008 | (4.845e-06, -7.2983e-06) | 1811 | 0.00448 | (0.00092203, 0.0015568) |
| | 3 | 31 | 0.0002 | (-9.5638e-07, -3.051e-08) | 880 | 0.00198 | (0.00080785, 0.00034527) |
| | 12 | 26 | 0.0001 | (6.6336e-07, -4.5269e-07) | 837 | 0.00187 | (0.00080889, 0.00020842) |
| | 50 | 25 | 0.0001 | (-8.9799e-08, -8.8343e-07) | 827 | 0.00206 | (0.00080853, 0.00017292) |
| (-31, 25) | 0.2 | 243 | 0.0008 | (-5.1672e-06, -7.1231e-06) | 1811 | 0.004 | (0.00092203, 0.0015568) |
| | 3 | 32 | 0.0005 | (2.861e-08, 8.6054e-07) | 880 | 0.00207 | (0.00080785, 0.00034527) |
| | 12 | 27 | 0.0002 | (-4.5256e-07, -4.6037e-07) | 837 | 0.00187 | (0.00080889, 0.00020842) |
| | 50 | 26 | 0.0001 | (-6.6243e-07, 2.1957e-07) | 827 | 0.00185 | (0.00080853, 0.00017292) |
| (6, -12) | 0.2 | 226 | 0.0009 | (5.2649e-06, -6.8775e-06) | 1811 | 0.00403 | (0.00092203, 0.0015568) |
| | 3 | 30 | 0.0003 | (-6.2497e-07, -6.1114e-07) | 880 | 0.00201 | (0.00080785, 0.00034527) |
| | 12 | 25 | 0.0001 | (7.9441e-07, 2.0826e-07) | 837 | 0.00191 | (0.00080889, 0.00020842) |
| | 50 | 24 | 0.0002 | (6.6279e-07, -6.5062e-07) | 827 | 0.00203 | (0.00080853, 0.00017292) |
| (-2, 100) | 0.2 | 257 | 0.0009 | (-2.8778e-06, -8.6754e-06) | 1811 | 0.00634 | (0.00092203, 0.0015568) |
| | 3 | 33 | 0.0002 | (8.3806e-07, 9.2163e-07) | 880 | 0.00205 | (0.00080785, 0.00034527) |
| | 12 | 28 | 0.0002 | (-8.3412e-07, 1.9723e-08) | 837 | 0.00239 | (0.00080889, 0.00020842) |
| | 50 | 27 | 0.0002 | (-3.3721e-07, 8.1482e-07) | 827 | 0.0024 | (0.00080853, 0.00017292) |

## 5. Conclusions

In this work, we introduced one step schemes involving convex combination of proximal point algorithms in CAT(0) spaces and established $\Delta$-convergence theorems of the generated sequence to a common solution of countable family of monotone vector field inclusion problems. We also deduced strong convergence results. Furthermore, we apply the proposed scheme to solve a countable family of minimization problems, to compute Frechét mean and geometric median in CAT(0) spaces and also to solve a kinematic problem in robotic motion control. We finally give a numerical example to demonstrate the efficiency of the proposed scheme in comparison to that of Dehghan et al. [9]. Our results herein generalised some results in the literature. For example, Theorem 4.3 of [21] and Theorem 4.1 of [28] are generalized from one monotone to a countable number of MVF, Theorem 3 of [18] is generalised from Hilbert spaces to CAT(0) spaces and from one monotone mapping to a number of MVF, and the results of [9] is generalized from finite family to countable family.

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## Conflict of interest

The authors declare that they have no competing interests.

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