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Research article

An improved upper bound for the dynamic list coloring of 1-planar graphs

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Abstract: A graph is 1-planar if it can be drawn in the plane such that each of its edges is crossed at most once. A dynamic coloring of a graph G is a proper vertex coloring such that for each vertex of degree at least 2, its neighbors receive at least two different colors. The list dynamic chromatic number $ch_d(G)$ of G is the least number k such that for any assignment of k-element lists to the vertices of G, there is a dynamic coloring of G where the color on each vertex is chosen from its list. In this paper, we show that if G is a 1-planar graph, then $ch_d(G) \leq 10$. This improves a result by Zhang and Li [16], which says that every 1-planar graph G has $ch_d(G) \leq 11$.

Keywords: 1-planar graph; dynamic coloring; list coloring
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1. Introduction

Graphs in this paper are simple and finite. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the neighborhood of v in G is $N_G(v) = \{u \in V(G) : u$ is adjacent to v in $G\}$. Vertices in $N_G(v)$ are called neighbors of v, and $d_G(v) = |N_G(v)|$ is the degree of v in G. A proper k-coloring is a mapping $\phi : V(G) \to \{1, 2, \ldots, k\}$ such that any adjacent vertices receive different colors. A proper vertex coloring is called a dynamic coloring if for every vertex v of degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a proper (resp. dynamic) k-coloring is the chromatic number (resp. dynamic chromatic number) of G, denoted by $\chi(G)$ (resp. $\chi_d(G)$). The concept of dynamic coloring was first introduced in [12], which is a generalization of the classical graph coloring.

A graph is said to be planar, if it can be drawn in the plane so that its edges intersect only at their ends. The well-known Four-Color Theorem states that $\chi(G) \leq 4$ for every planar graph G. Chen et al. [5] showed that $\chi_d(G) \leq 5$ if G is a planar graph, and it is conjectured that $\chi_d(G) \leq 4$ if G is a planar graph other than C_5. In 2013, Kim, Lee and Park [8] proved this conjecture. Furthermore, Kim, Lee and Oum [9] proved the same conclusion for K_5-minor-free graphs. The dynamic coloring of graphs has been extensively investigated in past decades, we refer to [1–5,7,10–14].

For each integer $k \geq 3$, let SK_k denote the graph obtained from complete graph K_k by inserting a new vertex to each of the edges in K_k. Thus for a fixed $k \geq 3, SK_k$ is a bipartite graph with a bipartition (X, Y) where $|X| = k$ and $|Y| = |E(K_k)|$, such that each vertex in Y is adjacent to exactly two vertices in X, and distinct vertices in X are adjacent to $k-1$ vertices in Y as do in K_k. Thus, $\chi(SK_k) = 2$ and $\chi_d(SK_k) = k$. So it is an example showing that the gap $\chi_d(G) - \chi(G)$ can be arbitrarily big. There is a vast literature dealing with the relationship between $\chi(G)$ and $\chi_d(G)$, see [2,10,12].

For every vertex $v \in V(G)$, let $L(v)$ denote a list of colors available at v. An L-coloring is a proper coloring φ such that $\varphi(v) \in L(v)$ for every vertex $v \in V(G)$. A graph G is k-choosable if it has an L-coloring whenever all lists have size at least k. The list chromatic number $ch(G)$ of G is the least integer k such that G is k-choosable. A dynamic L-coloring is a dynamic coloring of G such that each vertex is colored by a color from its list. A graph G is called dynamically k-choosable if it has a dynamic L-coloring whenever all lists have size at least k. The dynamic list chromatic number $ch_d(G)$ of G is the least integer k such that G is dynamically k-choosable.

Note that $\chi(G) \leq \chi_d(G) \leq ch_d(G)$ for every graph G. Esperet [6] showed that there is a planar bipartite graph G with $ch(G) = \chi_d(G) = 3$ and $ch_d(G) = 4$ and moreover, there exists for every $k \geq 5$ a bipartite graph G_k with $ch(G_k) = \chi_d(G_k) = 3$ and $ch_d(G_k) \geq k$. Hence the gap between $\chi_d(G)$ and $ch_d(G)$ can be any large. For further information on the dynamic list coloring of graphs, we refer the reader to [2] and [9].

A 1-planar graph is a graph that can be drawn in the plane so that each edge has at most one crossing. Recently, Zhang and Li [16] considered the dynamic list coloring of 1-planar graphs and proved $7 \leq \chi_d(G) \leq ch_d(G) \leq 11$ for every 1-planar graph G. Hence a natural problem is proposed.

Problem 1. (*Zhang and Li [16]*) Determine the minimum integers l_1 and l_2 so that every 1-planar graph is dynamically l_1-colorable and dynamically l_2-choosable, respectively.

The purpose of this paper is to close the gap between the lower and upper bound by proving the following theorem.

Theorem 1. Every 1-planar graph is dynamically 10-choosable.

2. Notations and terminology

A plane graph is a particular drawing in the Euclidean plane of a certain planar graph. Let G be a plane graph. We use $F(G)$ to denote the set of faces in G. For a face $f \in F(G)$, we use $\partial(f)$ to denote the boundary walk of f and write $f = [u_1 u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $\partial(f)$ in clockwise order. The degree of a face is the number of edge-steps in its boundary walk. For $x \in V(G) \cup F(G)$, let $d_G(x)$ denote the degree of x in G. A vertex of degree k (at most k, at least k, respectively) is called a k-vertex (k^--vertex, k^+-vertex, respectively). Similarly, we can define k-face, k^--face, and k^+-face. If X is the set of vertices and edges deleted, the resulting subgraph is denoted by $G - X$.

Let G be a plane drawing of a 1-planar graph such that each edge has at most one crossing and the number of such crossings are as few as possible. Let $C(G)$ denote the set of crossings in G. The associated plane graph, denoted $G^\times$, of G is a plane graph with

$$V(G^\times) = V(G) \cup C(G), E(G^\times) = E_0(G) \cup E_1(G),$$

where $E_0(G)$ is the set of non-crossed edges in G and

$E_1(G) = \{xz, zy \mid xy \in E(G) \setminus E_0(G)$ and z is a crossing point on $xy\}$.

Vertices in $V(G)$ are said to be true vertices of $G^\times$, and vertices in $C(G)$ are false vertices of $G^\times$. It is easy to observe that $d_{G^\times}(v) = d_G(v)$ for each $v \in V(G)$, and $d_{G^\times}(v) = 4$ for each $v \in C(G)$. A 3-face is false if it is incident to a false vertex in $G^\times$, and is true otherwise.

A 4-face $f = [uxvy]$ in $G^\times$ is called a special 4-face if $d_{G^\times}(u) \geq 10$, $d_{G^\times}(v) = 2$, x and y are false vertices, in this case, the vertex v is called a special 2-vertex. And non-special 2-vertex otherwise. A 5-face $f = [uxvyw]$ in $G^\times$ is called a special 5-face if $d_{G^\times}(v) = 2$, $d_{G^\times}(u), d_{G^\times}(w) \geq 10$, x and y are false vertices. A 10-vertex is bad if, which is incident with three special 4-faces and seven 3-faces in $G^\times$, and non-bad otherwise.

In the figure of this paper, black (white) bullets represent vertices whose degrees are exactly (at least) the one shown in the figure.

3. Proof of Theorem 1

We shall argue by contradiction to prove Theorem 1. Throughout the rest of this section, we assume that G is a counterexample to Theorem 1 such that $|V(G)| + |E(G)|$ is minimized, which is called a dynamically minimal graph. Specifically, there exists a 10-list assignment L to the vertices of G such that G is not dynamically L-choosable. By the minimality of G, for any 1-planar graph H with $|V(H)| + |E(H)| < |V(G)| + |E(G)|$ is dynamically L-choosable.

In the following two subsections, we first exhibit the structure of this minimum counterexample G. Secondly, relying on these properties, we use the Discharging Method to obtain a contradiction.

3.1. Structure and properties of a counterexample to Theorem 1

Zhang and Li [16] investigated the propositions of the dynamically minimal graphs. They gave the following lemma.

Lemma 1. (*Zhang and Li [16]*) Let G be a dynamically minimal graph. Then the following assertions hold.

(1) $\delta(G) \geq 2$.

(2) Each edge of G is incident with at least one 10^+-vertex.

(3) If u is a vertex incident with a triangle in G, then $d_G(u) \geq 10$.

(4) If u is a true vertex incident with a false 3-face of $G^\times$, then $d_G(u) \geq 8$.

(5) Let $f = [wuvx_1 \cdots x_s]$ be a 4^+-face of $G^\times$ with $d_G(u) \leq 7$, then both w and v are false.

(6) Each 6-face in $G^\times$ is incident with at most two special 2-vertices.

Lemma 2. G does not contain k-vertices, where $3 \leq k \leq 7$.

Proof. Suppose not, let v be a k-vertex with $3 \leq k \leq 7$. Let $N_G(v) = \{u, w, x_1, \ldots, x_t, y_1, \ldots, y_s\}$, where $d_G(x_i) = 2$ for each $1 \leq i \leq t$ and $d_G(y_j) \geq 3$ for each $1 \leq j \leq s$. Let $x_i^{'} = N_G(x_i) \backslash \{v\}$ for $1 \leq i \leq t$, $u^{'} \in N_G(u) \backslash \{v\}$, $w^{'} \in N_G(w) \backslash \{v\}$ and $y_j^{'} \in N_G(y_j) \backslash \{v\}$ for $1 \leq j \leq s$. Note that t or s may be 0, in which case $N_G(v) = \{u, w, y_1, \ldots, y_k\}$ or $N_G(v) = \{u, w, x_1, \ldots, x_k\}$, respectively. Let $H = G - \{x_1, \ldots, x_t\} - \{vy_1, \ldots, vy_s\}$, which is a 1-plane graph. By the minimality of G, H has a dynamic L-coloring ϕ such that $\phi(u) \neq \phi(w)$. Firstly, we recolor v with a color form $L(v) \backslash \{\phi(u), \phi(w), \phi(u^{'}), \phi(w^{'}), \phi(y_1), \ldots, \phi(y_s)\}$. Next, for each $1 \leq i \leq t$, we color x_i by a color from $L(x_i) \setminus \{\phi(v), \phi(x_i^{'}), \phi(x_i^{''})\}$, where $x_i^{''} \in N_G(x_i^{'}) \setminus \{x_i\}$. So we get a dynamic L-coloring of G, a contradiction.

Lemma 3. G dose not contain two 2-vertices u and v such that $N_G(u) = N_G(v)$.

Proof. Suppose, to the contrary, that G has 2-vertices u and v with $N_G(u) = N_G(v) = \{x, y\}$. By Lemma 1(2), x and y are 10^+-vertices. Let $H = G - \{u\}$, which is still 1-planar. By the minimality of G, H has a dynamic L-coloring ϕ. It follows that $\phi(x) \neq \phi(y)$, as $d_G(v) = 2$. We obtain a dynamic L-coloring of G by coloring u with a color from $L(u) \setminus \{\phi(x), \phi(y)\}$, a contradiction.

3.2. Discharging

We will complete the proof of Theorem 1 in this subsection. Let $G^\times$ be the associated plane graph of G corresponding to a plane embedding of G with the following properties:

(P1) Every edge is crossed by at most one other edge.

(P2) The number of crossing points is as small as possible.

For a k-vertex $v \in V(G^\times)$, we denote the neighbors of v in $G^\times$ by $v_0, v_1, \ldots, v_{k-1}$ in clockwise order, and the faces of $G^\times$ incident to v by $f_0, f_1, \ldots, f_{k-1}$ with $vv_i, vv_{i+1} \in \partial(f_i)$ for $i = 0, 1, \ldots, k-1$, where the indices are taken as modulo k. For a fixed face $f \in F(G^\times)$ and an edge $e \in E(f)$, we use f_e to denote the other face adjacent to f and incident to e. In particular, $f = f_e$ if e is a cut edge.

We first define an initial weight function $\omega(x) = d_{G^\times}(x) - 4$ for each $x \in V(G^\times) \cup F(G^\times)$. Since $G^\times$ is a connected plane graph, by Euler's formula $|V(G^\times)| - |E(G^\times)| + |F(G^\times)| = 2$ and the relation

$$\sum_{v \in V(G^\times)} d_{G^\times}(v) = \sum_{f \in F(G^\times)} d_{G^\times}(f) = 2|E(G^\times)|,$$

we obtain the following identity:

$$\sum_{v \in V(G^\times)} (d_{G^\times}(v) - 4) + \sum_{f \in F(G^\times)} (d_{G^\times}(f) - 4) = -8.$$

Next, we design some discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω' is produced. However, the total sum of weights is kept fixed when the discharging is in process. Nevertheless, we will show that $\omega'(x) \geq 0$ for all $x \in V(G^\times) \cup F(G^\times)$. This leads to the following contradiction

$$0 \leq \sum_{x \in V(G^\times) \cup F(G^\times)} \omega'(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} \omega(x) = -8,$$

which completes the proof.

For x, y and $z \in V(G^\times) \cup F(G^\times)$, let $\tau(x \to y)$ and $\tau(x \xrightarrow{z} y)$ denote the amount of weight that x transfers to y directly and across z, respectively. Our discharging rules are defined in $G^\times$ as follows.

(R1) Every true 3-face in $G^\times$ receives $\frac{1}{3}$ from each of its incident 10^+-vertices. Every false 3-face in $G^\times$ receives $\frac{1}{2}$ from each of its incident 8^+-vertices.

(R2) Every 10^+-vertices incident with a special 4-face f sends 1 to special 2-vertex through f.

(R3) Every 5^+-face in $G^\times$ sends 1 to each of its incident special 2-vertices if there are some ones.

(R4) Suppose that $f = [v_0 v_1 \cdots v_m]$ is a 5^+-face in $G^\times$ and v_i is a non-special 2-vertex.

(R4.1) If both of v_{i-2} and v_{i+2} are 10^+-vertices, then $\tau(f \to v_i) = 2$;

(R4.2) If exactly one of v_{i-2} and v_{i+2} is a 10^+-vertex, then $\tau(f \to v_i) = 1$.

(R5) Every 10^+-vertex sends $\frac{1}{2}$ to its each incident special 5-face.

(R6) Suppose v is a bad 10-vertex and $f = [vxuy]$ is a special 4-face with u is a special 2-vertex, x and y are false vertices. Let vv_1 (resp. vv_2) cross v_0u (resp. v_3u) in G at the crossing x (resp. y). Say the other face incident with u is f^*. Assume that $d(v_1) \le d(v_2)$ by symmetry, we carry out the following subrules (see Figure 1):

(R6.1) If $d(v_1) = 2$ and $d(v_2) \ge 10$, then $\tau(f^* \xrightarrow{u} v) = \frac{1}{9}$ provided that $d_{G^\times}(f^*) = 7$ and $\tau(v_2 \xrightarrow{f^* \text{ and } u} v) = \frac{1}{9}$ provided that $d_{G^\times}(f^*) = 6$;

(R6.2) If $8 \le d(v_1) \le 9$, then $\tau(v_1 \xrightarrow{x} v) = \frac{1}{9}$;

(R6.3) If $d(v_1) \ge 10$ and $5 \le d_{G^\times}(f^*) \le 7$, then $\tau(f^* \xrightarrow{u} v) = \frac{1}{9}$;

(R6.4) If $d_{G^\times}(f^*) \ge 8$, then $\tau(f^* \xrightarrow{u} v) = \frac{1}{9}$.

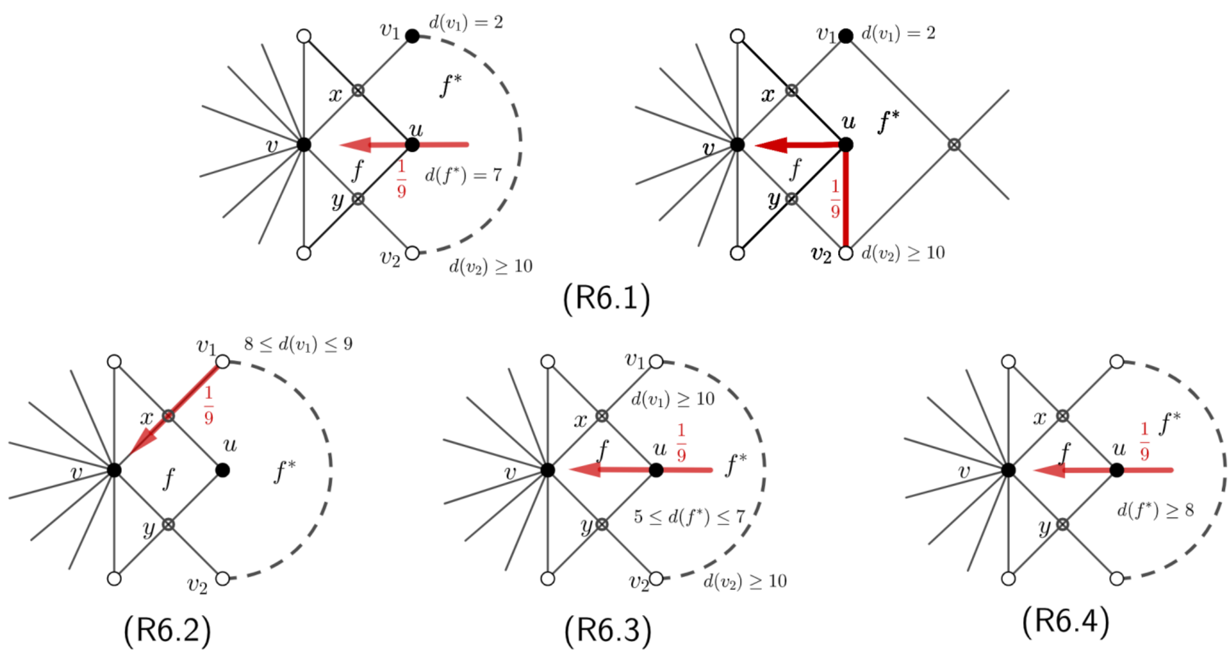

Figure 1. The discharging rule (R6).

Let $f = [v_0 v_1 \cdots v_m]$ be a 5^+-face in $G^\times$. For $0 \le i \le m$, we define some notations as follows.

$V_2^s(f) = \{v_i \in V(G^\times) | v_i$ is a special 2-vertex incident with f $\}$ and $n_2^s(f) = |V_2^s(f)|$;

$V_2^{'}(f) = \{v_i \in V(G^\times) | v_i$ is a non-special 2-vertex incident with f and both of v_{i-2} and v_{i+2} are 10^+-vertices$\}$ and $n_2^{'}(f) = |V_2^{'}(f)|$;

$V_2^{''}(f) = \{v_i \in V(G^\times) | v_i$ is a non-special 2-vertex incident with f and exactly one of v_{i-2} and v_{i+2} is a 10^+-vertex $\}$ and $n_2^{''}(f) = |V_2^{''}(f)|$.

Claim 1. $2(n_2^s(f) + n_2^{''}(f)) + 4n_2^{'}(f) \le d(f)$.

Proof. By Lemma 1(5), every 2-vertex is adjacent to false vertices in $G^\times$. If $n_2^{'}(f) = 0$, then $n_2^s(f) + n_2^{''}(f) \le \frac{d(f)}{2}$. It follows that $2(n_2^s(f) + n_2^{''}(f)) + 4n_2^{'}(f) \le d(f)$. So assume that $n_2^{'}(f) \ge 1$. Let $v_{i_0}, v_{i_1}, \ldots, v_{i_{t-1}} \in V_2^{'}(f)$ incident to f in clockwise order, where $t = n_2^{'}(f)$. For $0 \le j \le t-1$, let $v_{i_j-2}, v_{i_j-1}, v_{i_j}, v_{i_j+1}, v_{i_j+2}$ be five corresponding vertices incident to f, where $v_{i_j} \in V_2^{'}(f)$, and v_{i_j-1}, v_{i_j+1} false, and v_{i_j-2}, v_{i_j+2} are 10^+-vertices. It follows that the following vertices

$$v_{i_0-1}, v_{i_0}, v_{i_0+1}, v_{i_1-1}, v_{i_1}, v_{i_1+1}, \ldots, v_{i_{t-1}-1}, v_{i_t}, v_{i_{t-1}+1}$$

are mutually distinct, and $n_2^{'}(f) \leq n_{10^+}(f)$, where $n_{10^+}(f)$ denote the number of 10^+-vertices incident with f. Thus, $n_2^s(f) + n_2^{''}(f) \leq \frac{d(f)-3n_2^{'}(f)-n_{10^+}(f)}{2}$. Consequently, $2(n_2^s(f) + n_2^{''}(f)) + 4n_2^{'}(f) \leq d(f)$.

Claim 2. If $f = [uxvyw]$ is a special 5-face in $G^\times$ such that v is a 2-vertex, x, y are false vertices and $d_{G^\times}(u), d_{G^\times}(w) \geq 10$, then f_{xu}, f_{uw} and f_{wy} are not special 4-faces.

Proof. Assume vv_1 crosses uu_1 in G at the point x. Since $d_{G^\times}(v) = 2$, $d_{G^\times}(v_1) \geq 10$ by Lemma 1(2). It follows that f_{xu} is not special 4-face according to the definition of special 4-face. Similarly, f_{uw} and f_{wy} are not special 4-faces.

Lemma 4. Every face in $G^\times$ has a nonnegative final charge.

Proof. Let $f = [v_0v_1 \cdots v_{k-1}]$ be a k-face in $G^\times$, where $k \geq 3$.

Case 1. $d_{G^\times}(f) = 3$.

Then $\omega(f) = -1$. If f is a true 3-face, then every vertex incident with f is a 10^+-vertex by Lemma 1(3), and thus $\omega'(f) \geq -1 + \frac{1}{3} \times 3 = 0$ by (R1). If f is a false 3-face, then f is incident to two 8^+-vertices by Lemma 1(4). It follows that $\omega'(f) = -1 + \frac{1}{2} \times 2 = 0$ by (R1).

Case 2. $d_{G^\times}(f) = 4$.

No rule is valid for f and thus $\omega'(f) = \omega(f) = 0$.

Case 3. $d_{G^\times}(f) = 5$.

Then f is incident to at most one 2-vertex. If not, then there exists an edge is crossed two times by Lemma 1(5). Assume that f is incident to a 2-vertex, say v_0, then v_1 and v_4 are false by Lemma 1(5). Moreover, at least one of v_2 and v_3 is a 10^+-vertex by Lemma 1(2). If exactly one of v_2 and v_3 is a 10^+-vertex, then $\omega'(f) \geq 5 - 4 - 1 = 0$ by (R3) and (R4.2). So assume that v_2 and v_3 are 10^+-vertices, then f is a special 5-face. If v_0 is a non-special 2-vertex, then $\omega'(f) \geq 5 - 4 - 2 + \frac{1}{2} + \frac{1}{2} = 0$ by (R4.1) and (R5). Otherwise, v_0 is a special 2-vertex. By (R3), (R5) and (R6.3), $\omega'(f) \geq 5-4-1+\frac{1}{2}+\frac{1}{2}-\frac{1}{9} = \frac{8}{9}$.

Case 4. $d_{G^\times}(f) = 6$.

By Lemma 1(6) and Claim 1, $n_2^s(f) \leq 2$ and $n_2'(f) \leq 1$, respectively. If $n_2'(f) = 1$, assume $v_0 \in V_2'(f)$, then v_0 is a non-special 2-vertex, v_1 and v_5 are false vertices, v_2 and v_4 are 10^+-vertices. It follows that $n_2^s(f) + n_2^{''}(f) = 0$. Hence, $\omega'(f) = 6 - 4 - 2 = 0$ by (R4.1). Now, assume that $n_2'(f) = 0$.

Suppose $n_2^s(f) = 2$. Without loss of generality, assume that v_0 is a special 2-vertex. Then v_1 and v_5 are false vertices. If v_3 is also a special 2-vertex, then v_2 and v_4 are false vertices. There exists an edge is crossed two times, this is impossible. So assume v_2 is another special 2-vertex. Then f sends 1 to v_0 and v_2 by (R3), respectively, and (R6.3) is not applied. Thus, $\omega'(f) = 6 - 4 - 2 = 0$.

Suppose $n_2^s(f) = 1$. Similarly, we may assume that v_0 is a special 2-vertex. Then v_1 and v_5 are false vertices. If v_2 and v_4 are 10^+-vertices, then v_3 is a 8^+-vertex by Lemma 1(5). This implies that f sends 1 to v_0 by (R3), and $\frac{1}{9}$ through v_0 by (R6.3), respectively. Therefore, $\omega'(f) = 6 - 4 - 1 - \frac{1}{9} = \frac{8}{9}$. Otherwise, (R6.3) is not applied, and $\omega'(f) \geq 6 - 4 - 1 - 1 = 0$ by (R3) and (R4.2).

Suppose $n_2^s(f) = 0$. Then $n_2^{''}(f) \leq 3$ by Claim 1. If $n_2^{''}(f) \leq 2$, then $\omega'(f) \geq 6 - 4 - 1 - 1 = 0$ by (R4.2). Otherwise, $n_2^{''}(f) = 3$, this is impossible.

Case 5. $d_{G^\times}(f) = 7$.

Then $n_2^s(f) + n_2^{''}(f) \leq 3$ and $n_2^{'}(f) \leq 1$ by Claim 1. If $n_2^{'}(f) = 1$, then $n_2^s(f) + n_2^{''}(f) \leq 1$ by Claim 1. Assume $v_0 \in V_2'(f)$, then v_0 is a non-special 2-vertex, v_1 and v_6 are false vertices, v_2 and v_5 are 10^+-vertices. This implies that $n_2^s(f) = 0$. Hence, $\omega'(f) = 7 - 4 - 2 - 1 = 0$ by (R3) and (R4). Next, assume that $n_2^{'}(f) = 0$. If $n_2^s(f)+n_2^{''}(f) \leq 2$, then $\omega'(f) \geq 7-4-2\times 1-2\times\frac{1}{9} = \frac{7}{9}$ by (R3), (R4.2), (R6.4) and (R6.3). Now, we have $n_2^s(f) + n_2^{''}(f) = 3$. This implies that f is incident with three 2-vertices, and four false vertices by Lemma 1(5). Hence, there exists an edge is crossed two times, which contradicts the property **(P1)**.

Case 6. $d_{G^\times}(f) = 8$.

Then $n_2^s(f) + n_2^{''}(f) \leq 4$ and $n_2^{'}(f) \leq 2$ by Claim 1. If $n_2^{'}(f) = 2$, then $n_2^s(f) + n_2^{''}(f) = 0$ by Claim 1, and hence $\omega'(f) = 8 - 4 - 2 \times 2 = 0$ by (R4.1). If $n_2^{'}(f) = 1$, then $n_2^s(f) + n_2^{''}(f) \leq 1$ by the definition of $V_2^{'}(f)$, and hence $\omega'(f) = 8 - 4 - 2 - \frac{10}{9} = \frac{8}{9}$ by (R3) and (R4). Now assume that $n_2^{'}(f) = 0$. By the definition of $V_2^{''}(f)$ and Lemma 3, we may derive that $n_2^s(f) + n_2^{''}(f) \leq 3$. Hence, $\omega'(f) \geq 8 - 4 - 3 \times \frac{10}{9} = \frac{2}{3}$ by (R3) and (R4.2).

Case 7. $d_{G^\times}(f) \geq 9$.

By Claim 1, (R3), (R4) and (R6.4), we have the following inequality.

$$\begin{aligned}
\omega'(f) &\geq d(f) - 4 - 2n_2^{'}(f) - (1 + \frac{1}{9})(n_2^s(f) + n_2^{''}(f)) \\
&\geq d(f) - 4 - 2(\frac{1}{4}d(f) - \frac{1}{2}(n_2^s(f) + n_2^{''}(f))) - \frac{10}{9}(n_2^s(f) + n_2^{''}(f))) \\
&= \frac{d(f)}{2} - 4 - \frac{1}{9}(n_2^s(f) + n_2^{''}(f)) \\
&\geq \frac{d(f)}{2} - 4 - \frac{1}{9} \times \frac{d(f)}{2} \\
&= \frac{4d(f) - 36}{9} \geq 0.
\end{aligned}$$

Lemma 5. Every 2-vertex in $G^\times$ has a nonnegative final charge.

Proof. By Lemma 1(3) and (4), we derive that v is not incident with a triangle in $G^\times$. By Lemma 1(5), the neighbors of v in $G^\times$, say x and y, are both false vertices.

Assume that v is a special 2-vertex. Let $f = [vxuy]$ be a special 4-face, and f^* be the other face incident to v. Then v receives 1 from f by (R2). Since G is a simple graph, $d_{G^\times}(f^*) \geq 5$, then v receives 1 from f^* by (R3). Hence, $\omega'(v) \geq 2 - 4 + 2 \times 1 = 0$. We may assume that v is a non-special 2-vertex.

If v is incident with a 4-face, say $f = [uxvy]$, then $d(u) \leq 9$. Let u_1 (resp. u_2) be the vertices in G such that uu_1 (resp. uu_2) passes through the crossing x (resp. y). Since G is a simple graph, $u_1 \neq u_2$, and u_1 and u_2 are 10^+-vertices by Lemma 1(2). Hence, v is incident with a 5^+-face, which sends 2 to v by (R4.1). It follows that $\omega'(v) = 2 - 4 + 2 = 0$.

If v is incident with two 5^+-faces f_1 and f_2, then let u_1u_2 (resp. w_1w_2) be edge of G that pass through the crossing x (resp. y), such that u_1 and w_1 (resp. u_2 and w_2) are vertices on f_1 (resp. f_2). By Lemma 1(2), there are at least two 10^+-vertices among u_1, u_2, w_1 and w_2. Therefore, either $v \in V_2^{'}(f_1)$ or $v \in V_2^{'}(f_2)$, or both $v \in V_2^{''}(f_1)$ and $v \in V_2^{''}(f_2)$. In each case v receives at least 2 from f_1 and f_2 by (R4.1) and (R4.2), and thus $\omega'(v) \geq 2 - 4 + 2 = 0$.

Remark 1. Let v be a 10^+-vertex, which is incident with a 6-face $f^* = [vv_1v_2v_3v_4v_5]$. If v sends out weight through f^* by (R6.1), then v_1, v_3 and v_5 are false vertices, v_2 and v_4 are 2-vertices. We call such face a special 6-face, and denote $f_6^s(v)$ by the number of special 6-faces incident with v.

Claim 3. Let $f = [vxuy]$ be a special 4-face with $d(v) \geq 10$. Then the faces f_{vx} and f_{vy} are neither special 4-faces nor special 6-faces.

Proof. Let vv_1 (resp. vv_2) cross uu_1 (resp. uu_2) in G at the crossing x (resp. y). The definition of the special 4-face implies that u is a 2-vertex. Therefore, u_1 and u_2 are 10^+-vertices by Lemma 1(2), and the Claim holds.

Claim 4. Let v be a 10^+-vertex, and let f_1, f_2 and f_3 be three consecutive faces that are incident with v in $G^\times$.

(1) If f_1, f_2 and f_3 are not special 4-faces, then v totally sends to f_1, f_2 and f_3 or to bad 10-vertex through these faces at most $\frac{3}{2}$;

(2) If at least one of f_1, f_2 and f_3 is a special 4-faces, then v totally sends to f_1, f_2 and f_3 or to bad 10-vertex through these faces at most 2.

Proof. By (R1), (R2) and (R5), v sends 1 to its incident special 4-face and at most $\frac{1}{2}$ to its incident 3-face or special 5-face. In addition, v sends $2 \times \frac{1}{9}$ to bad 10-vertex through special 6-face by (R6.4).

1) Suppose f_1, f_2 and f_3 are not special 4-faces. It follows that v sends at most $\frac{1}{2} \times 3 = \frac{3}{2}$ to f_1, f_2 and f_3 or to bad 10-vertex through these faces.

2) Suppose that at least one of f_1, f_2 and f_3 is a special 4-faces. Furthermore, there are at most two special 4-faces among f_1, f_2 and f_3 by Claim 3. This implies that either exactly one of f_1, f_2 and f_3 is a special 4-faces, or f_1 and f_3 are special 4-faces. In the latter case, f_2 is not 3-face, nor special 5-face by Claim 2, nor special 6-face by Claim 3. Hence, in each case, v sends to f_1, f_2 and f_3 or to bad 10-vertex through these faces at most 2.

Remark 2. Let v be a 10^+-vertex and $f_0, f_1, \dots, f_{d-1}$ be the faces in clockwise order around v, where $d = d(v)$. For $0 \le i \le d-1$, let a_i be the weight that v sends to f_i or to bad 10-vertex through f_i, and $\mu_i = a_{i-1} + a_i + a_{i+1}$, where the subscripts are taken modular d. By Claim 4, we conclude that $\sum_{i=0}^{d-1} a_i = \frac{1}{3} \sum_{i=0}^{d-1} \mu_i \le \frac{2}{3} d$.

For a true vertex v, denote by $f^3(v)$ and $n^c(v)$ the number of 3-faces incident with v and and the number of crossing vertices that are adjacent to v in $G^\times$, respectively.

Lemma 6. [15] Let G be a 1-plane graph. If $d_G(v) \ge 5$, then $f^3(v) + n^c(v) \le \lfloor \frac{3d_G(v)}{2} \rfloor$.

Claim 5. Let $v \in V(G^\times)$ with $8 \le d(v) \le 9$. If v is adjacent to bad 10-vertices in G, then $f^3(v) \le d(v) - 1$.

Proof. Suppose that v is adjacent to a bad 10-vertex u. By the definition of bad 10-vertex and Lemma 1(2), uv passes through a crossing, say x. Let zw be the other edge in G passes through x, and let f_1, f_2, f_3 and f_4 be the face that is incident with the path vxw, wxu, zxu and zxv in $G^\times$. Then one of f_2 and f_3 is a 3-face and the other is a special 4-face. Without loss of generality, assume that f_2 is a triangle and f_3 is a special 4-face. It follows that z is a 2-vertex, and so f_4 is not a 3-face. Hence, $f^3(v) \le d(v) - 1$.

Lemma 7. Every vertex in $G^\times$ with $8 \le d_{G^\times}(v) \le 9$ or $d_{G^\times}(v) \ge 11$ has a nonnegative final charge.

Proof. Assume that $8 \le d(v) \le 9$. If v is not incident with any bad 10-vertex, then $\omega'(v) \ge d(v) - 4 - \frac{1}{2}d(v) \ge 0$ by (R1). Otherwise, v is incident with bad 10-vertices. By (R1) and (R6.2), we have

$$\begin{aligned}
\omega'(v) &\ge d(v) - 4 - \frac{1}{2}f^3(v) - \frac{1}{9}n^c(v) \\
&\ge d(v) - \frac{1}{2}f^3(v) - \frac{1}{9}(\lfloor \frac{3d(v)}{2} \rfloor - f^3(v)) - 4 \\
&\ge \frac{5}{6}d(v) - \frac{7}{18}f^3(v) - 4 \\
&\ge \frac{5}{6}d(v) - \frac{7}{18}(d(v) - 1) - 4 \\
&= \frac{8d(v) - 65}{18}.
\end{aligned}$$

Hence, if $d(v) = 9$ or $d(v) = 8$ and $f^3(v) \leq 6$, then $\omega'(v) \geq 0$. We may assume that $d(v) = 8$ and $f^3(v) = 7$ by Claim 5. Since false vertices are not adjacent in $G^\times$, $n^c(v) \leq 4$. Thus, $\omega'(v) \geq 8 - 4 - 7 \times \frac{1}{2} - 4 \times \frac{1}{9} = \frac{1}{18}$.

Assume that $d(v) \geq 12$. According to Remark 2, we have $\omega'(v) \geq d(v) - 4 - \sum_{i=1}^{d} a_i \geq \frac{1}{3}d - 4 \geq 0$.

Now suppose that $d(v) = 11$. If v is incident with at most three special 4-faces, then $\omega'(v) \geq 11 - 4 - 3 \times 1 - 8 \times \frac{1}{2} = 0$ by (R1) and (R2). Otherwise, v is incident with at least four special 4-faces. This implies that there are two faces f_i and f_{i+2} are special 4-faces, where the subscripts are taken modular 11. Without loss of generality, we may suppose that $i = 1$. In this case, by Claim 2, f_2 is not 3-face, nor special 5-face, nor special 6-face, and therefore $a_2 = 0$. Furthermore, f_4 and f_0 are not special 4-faces by Claim 3. Thus, $\mu_1 = \frac{1}{2} + 1 + 0 = \frac{3}{2}$ and $\mu_3 = 0 + 1 + \frac{1}{2} = \frac{3}{2}$. Therefore, by Remark 2, we have:

$$\sum_{i=0}^{10} a_i = \frac{1}{3}\sum_{i=0}^{10} \mu_i = \frac{1}{3}(\mu_1 + \mu_3 + \sum_{\substack{0 \leq i \leq 10 \\ i \neq 1,3}} \mu_i) \leq \frac{1}{3} \times (\frac{3}{2} + \frac{3}{2} + 2 \times 9) = 7.$$

It follows that $\omega'(v) = d(v) - 4 - \sum_{i=0}^{10} a_i \geq 11 - 4 - 7 = 0$.

Lemma 8. Every 10-vertex in $G^\times$ has a nonnegative final charge.

Proof. Assume that v is a bad 10-vertex. Let $f = [vxuy]$ be a special 4-face with u is a special 2-vertex, x and y are false vertices. Let vv_1 cross v_0u in G at the crossing x. We denote f^* by the other face incident with u. Since G is a simple graph, $d_{G^\times}(f^*) \geq 5$. Lemma 2 implies that v_1 and v_2 are both 2-vertices or 8^+-vertices. Assume that $d(v_1) \leq d(v_2)$ by symmetry. Note that if $8 \leq d(v_1) \leq 9$, then $\tau(v_1 \xrightarrow{x} v) = \frac{1}{9}$ by (R6.2). Thus it suffices to suppose that $d(v_1) = d(v_2) = 2$, or $d(v_1) = 2$ and $d(v_2) \geq 10$, or $d(v_1) \geq 10$. Assume that $d(v_1) = d(v_2) = 2$. It is easy to see that $d_{G^\times}(f^*) \geq 8$ by Lemma 1(5). By (R6.4), $\tau(u \xrightarrow{f} v) = \frac{1}{9}$. Assume that $d(v_1) = 2$ and $d(v_2) \geq 10$, then $d_{G^\times}(f^*) \geq 6$ by Lemma 1(5). If $d_{G^\times}(f^*) \geq 8$, then $\tau(u \xrightarrow{f} v) = \frac{1}{9}$ as above. By (R6.1), if $d_{G^\times}(f^*) = 7$, then $\tau(f^* \xrightarrow{u} v) = \frac{1}{9}$; if $d_{G^\times}(f^*) = 6$, then $\tau(v_2 \xrightarrow{f^* \text{ and } u} v) = \frac{1}{9}$. Assume that $d(v_1) \geq 10$. If $d_{G^\times}(f^*) \geq 8$, then $\tau(u \xrightarrow{f} v) = \frac{1}{9}$ as above. If $5 \leq d_{G^\times}(f^*) \leq 7$, then $\tau(f^* \xrightarrow{u} v) = \frac{1}{9}$ by (R6.3). In summary, v receives at least $\frac{1}{9}$ from element according to special 4-face. Since v is incident with three special 4-faces, $\omega'(v) \geq 10 - 4 - 3 \times 1 - 6 \times \frac{1}{2} - \frac{1}{3} + 3 \times \frac{1}{9} = 0$ by (R1) and (R2).

Assume that v is a non-bad 10-vertex. We denote the number of special 4-faces incident with v by $f_4^s(v)$. Thus, $f_4^s(v) \leq 5$ by Claim 3. We need to consider two cases:

Case 1. $f_6^s(v) = 0$

If $f_4^s(v) \leq 2$, then $\omega'(v) \geq 10 - 4 - 2 \times 1 - 8 \times \frac{1}{2} = 0$ by (R1), (R2) and (R5).

If $f_4^s(v) = 3$, then v is incident with at most six 3-faces in $G^\times$. Otherwise, v is a bad 10-vetrex. Furthermore, if v is incident with six 3-faces, then the remaining face is not a special 5-face. Thus, v sends at most $6 \times \frac{1}{2} = 3$ to 3-faces and special 5-faces by (R1) and (R5). This implies that $\omega'(v) \geq 10 - 4 - 3 \times 1 - 3 = 0$.

If $f_4^s(v) = 4$, then there are three faces f_{i-2}, f_i and f_{i+2} are special 4-faces, where the subscripts are taken modular 10. Assume, without loss of generality, $i = 1$. If so, f_0 and f_2 are neither 3-faces nor special 5-faces, which implies that v sends out at most $4 \times \frac{1}{2} = 2$ by (R1) and (R5). Consequently, $\omega'(v) \geq 10 - 4 - 4 \times 1 - 2 = 0$.

If $f_4^s(v) = 5$, without loss of generality, assume that f_i are special 4-faces by Claim 3, where $i = 1, 3, 5, 7, 9$. Then f_j are neither 3-faces nor special 5-faces by Claim 2, where $j = 0, 2, 4, 6, 8$. Therefore, $\omega'(v) \geq 10 - 4 - 5 \times 1 = 1$.

Case 2. $f_6^s(v) \geq 1$

Without loss of generality, assume that f_0 is a special 6-face. Let $f_0 = [vwxyzu]$ with w, y and u are false vertices, x and z are 2-vertices. Let zz_1 be another edge of G that passes through the crossings u, where $z_1 \in \partial(f_1)$. It follows from Lemma 1(2) that z_1 is a 10^+-vertices. This implies that f_1 is neither special 4-face nor special 5-face. By symmetry, f_9 is neither special 4-face nor special 5-face. Thus, $\mu_0 \leq 2 \times \frac{1}{2} + 2 \times \frac{1}{9} = \frac{11}{9}$ by (R1) and (R6.1). If f_2 is a special 4-face, then f_1 is not 3-face, nor special k-face, where $k \in \{4, 5, 6\}$. Otherwise, $a_2 \leq \frac{1}{2}$. In each case, $\mu_1 \leq 1 + 2 \times \frac{1}{9} = \frac{11}{9}$. Similarly, $\mu_9 \leq \frac{11}{9}$. Therefore, by Claim 4 and Remark 2, we have:

$$\sum_{i=0}^{9} a_i = \frac{1}{3}\sum_{i=0}^{9} \mu_i = \frac{1}{3}(\mu_0 + \mu_1 + \mu_9 + \sum_{\substack{0 \leq i \leq 9 \\ i \neq 0,1,9}} \mu_i) \leq \frac{1}{3} \times (3 \times \frac{11}{9} + 2 \times 7) = \frac{53}{9}.$$

This yields $\omega'(v) \geq 10 - 4 - \frac{53}{9} = \frac{1}{9}$.

4. Conclusions and future works

In this paper, we closed the gap between the lower and upper bound of Problem 1 by proving that 1-planar graphs are dynamically 10-choosable. It is interesting to determine the smallest integer c, where $7 \leq c \leq 10$, such that every 1-planar graph is dynamically c-choosable.

A graph is IC-planar (independent-crossing-planar) if it has a 1-planar drawing so that each vertex is incident with at most one crossing edge. A graph is NIC-planar (near-independent-crossing-planar) if it admits a drawing in the plane with at most one crossing per edge and such that two pairs of crossing edges share at most one common end vertex. Both of them specialize 1-planarity, but generalize planarity. Thus, the following is a natural problem:

Problem 2. What is the smallest integers l_1 and l_2 such that every IC-planar (or NIC-planar graph) graph is dynamically l_1-colorable and dynamically l_2-choosable, respectively.

Recently, Hu and Kong proved that IC-planar is dynamically 7-choosable (in preparation). One can see Figure 2, which is an IC-planar graph with dynamic chromatic number is 6. Hence, we have $6 \leq l_1 \leq l_2 \leq 7$ for IC-planar graph.

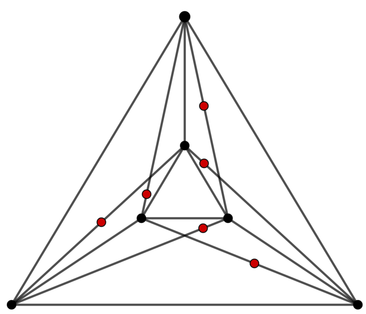

Figure 2. An IC-planar graph.

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Conflict of interest

The authors have contributed to this work equally and declare that they have no conflict of interest.

References

1. A. Ahadi, S. Akbari, A. Dehghan, M. Ghanbari, On the difference between chromatic number and dynamic chromatic number of graphs, *Discrete Math.*, **312** (2012), 2579–2583. https://doi.org/10.1016/j.disc.2011.09.006
2. M. Alishahi, On the dynamic coloring of graphs, *Discrete Appl. Math.*, **159** (2011), 152–156. https://doi.org/10.1016/j.dam.2010.10.012
3. P. Borowiecki, E. Sidorowicz, Dynamic coloring of graphs, *Fund. Inform.*, **114** (2012), 105–128. https://doi.org/10.3233/FI-2012-620
4. N. Bowler, J. Erde, F. Lehner, M. Merker, M. Pitz, K. Stavropoulos, A counterexample to montgomery's conjecture on dynamic colourings of regular graphs, *Discrete Appl. Math.*, **229** (2017), 151–153. https://doi.org/10.1016/j.dam.2017.05.004
5. Y. Chen, S. Fan, H. J. Lai, H. Song, L. Sun, On dynamic coloring for planar graphs and graphs of higher genus, *Discrete Appl. Math.*, **160** (2012), 1064–1071. https://doi.org/10.1016/j.dam.2012.01.012
6. L. Esperet, Dynamic list coloring of bipartite graphs, *Discrete Appl. Math.*, **158** (2010), 1963–1965. https://doi.org/10.1016/j.dam.2010.08.007
7. D. Karpov, Dynamic proper colorings of a graph, *J. Math. Sci.*, **179** (2011), 601–615. https://doi.org/10.1007/s10958-011-0612-3
8. Y. Kim, S. Lee, S. Oum, Dynamic coloring of graphs having no K_5 minor, *Discrete Appl. Math.*, **206** (2016), 81–89. https://doi.org/10.1016/j.dam.2016.01.022
9. S. J. Kim, S. Lee, W. J. Park, Dynamic coloring and list dynamic coloring of planar graphs, *Discrete Appl. Math.*, **161** (2013), 2207–2212. https://doi.org/10.1016/j.dam.2013.03.005
10. H. J. Lai, J. Lin, B. Montgomery, T. Shuib, S. Fan, Conditional colorings of graphs, *Discrete Math.*, **306** (2006), 1997–2004. https://doi.org/10.1016/j.disc.2006.03.052
11. S. Loeb, T. Mahoney, B. Reiniger, J. Wise, Dynamic coloring parameters for graphs with given genus, *Discrete Appl. Math.*, **235** (2018), 129–141. https://doi.org/10.1016/j.dam.2017.09.013
12. B. Montgomery, *Dynamic coloring of graphs*, West Virginia University, 2001.
13. S. Saqaeeyan, E. Mollaahamdi, Dynamic chromatic number of bipartite graphs, *Sci. Ann. Comput. Sci.*, **26** (2016), 249–261. https://doi.org/10.7561/SACS.2016.2.249

14. N. Vlasova, D. Karpov, Bounds on the dynamic chromatic number of a graph in terms of its chromatic number, *J. Math. Sci.*, *232* (2018), 21–24. https://doi.org/10.1007/s10958-018-3855-4

15. X. Zhang, J. Wu, On edge coloring of 1-planar graphs, *Inform. Process. Lett.*, **111** (2011), 124–128. https://doi.org/10.1016/j.ipl.2010.11.001

16. X. Zhang, Y. Li, Dynamic list coloring of 1-planar graphs, *Discrete Math.*, **344** (2021), 112333. https://doi.org/10.1016/j.disc.2021.112333