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Research article

Completeness of metric spaces and existence of best proximity points

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Abstract: In this paper, we discuss the existence of best proximity points of new generalized proximal contractions of metric spaces. Moreover, we obtain a completeness characterization of underlying metric space via the best proximity points. Some new best proximity point theorems have been derived as consequences of main results in (partially ordered) metric spaces.

Keywords: Suzuki-type; best proximity point; proximal contractions; α_p-proximal admissible; partial order
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1. Introduction and preliminaries

Fixed point theory has provided various tools to solve nonlinear functional equations in mathematics and many other related disciplines. In the context of metric spaces, one of the earlier fixed point theorems is the famous Banach contraction principle (shortly as BCP) [4] which has been applied to solve nonlinear operator equations (see [3] and references therein). BCP states that "if (Ω, ρ) is a complete metric space (shortly as C-MS) and a mapping $\mathcal{F} : \Omega \to \Omega$ satisfies $\rho(\mathcal{F}u, \mathcal{F}v) \leq k\rho(u, v)$ for all $u, v \in \Omega$, and for some $k \in [0, 1)$, then there is a unique point u in Ω such that $u = \mathcal{F}u$, that is, $\mathcal{F}$ has a unique fixed point". There may arise situations where fixed points of mappings do not exist, for example, if A, B are non-empty subsets of Ω and $\mathcal{F} : A \to B$ a nonself mapping, then $u = \mathcal{F}u$ may not have any solution. In such a situation, it is very useful to have a point u in A satisfying

$$\rho(u, \mathcal{F}u) = \rho(A, B) \tag{1.1}$$

where

$$\rho(A, B) = \inf_{z\in A, w\in B} \rho(z, w)$$

and if a point u in A exists that satisfies (1.1) is termed as a "best proximity point (BPP)" of $\mathcal{F}$. Fan [15] discussed best approximation theorems in the context of normed spaces. For further generalizations of Fan's results, we direct the interested reader to [22, 25]. Basha [6] generalized BCP for nonself mappings by proving BPP results for a new proximal contractions. Basha and Shahzad [8] extended these contractions and introduced proximal contractions of two different types and obtained best proximity points (BPPs). Samet et al. [24] initiated $\alpha - \psi -$contractive type self-mappings which were further extended by Jleli and Samet [18] to the nonself contractive mappings along with the provision of results concerning the existence of singleton set of BPPs. Hussain et al. [16] initiated "modified $(\alpha, \psi)-$proximal rational contractions" and some other useful results in this direction appeared in [20]. For more on the problem of existence of BPPs in various directions, we refer the readers to [1, 7, 12–14, 21].

In 2008, Suzuki [26] introduced a useful extension of BCP that characterized metric completeness as well (also compare [27]). Abkar and Gabeleh [2] developed the existence of BPPs of Suzuki-type mappings and Hussain et al. [17] introduced "modified Suzuki $\alpha - \psi$-proximal contractions".

We introduce a new set of proximal contractions of metric spaces and prove the existence of BPPs. Moreover, we also obtain metric completeness characterization via BPPs. As consequences of main results, we derive some important results in the literature as corollaries. We provide examples and show that some previous results are not applicable. Further, as applications, we obtain results corresponding to the main findings in the setup of "metric spaces equipped with a partial order".

Let (Ω, ρ) be a metric space and A and B non-empty subsets in Ω. Throughout this article, we use the following notations.

1) $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{N}_0$ for the set of nonnegative reals, reals, positive integers and nonnegative integers, respectively,
2) $C(\Omega)$ for the class of non-empty and closed subsets of (Ω, ρ),
3) $\rho^*(z, w)$ for $\rho(z, w) - \rho(A, B)$ where $z \in A$ and $w \in B$,
4) $BPP(\mathcal{F})$ for the set of BPPs of the mapping $\mathcal{F} : A \to B$ and
5) $\widetilde{\mathcal{F}}_{(\mathcal{F})}$ for the set of fixed points of the mapping $\mathcal{F} : A \to B$.

Define

$$A_0 = \{z \in A : \rho(z, w) = \rho(A, B) \text{ for some } w \in B\}, \text{ and}$$
$$B_0 = \{w \in B : \rho(z, w) = \rho(A, B) \text{ for some } z \in A\}.$$

Note that if A_0 is non-empty then so is B_0. If A_0 is non-empty, then "pair (A, B) has P-property (shortly as P_P)" if

$$\begin{cases} \rho(z_1, w_1) = \rho(A, B) \\ \rho(z_2, w_2) = \rho(A, B) \end{cases} \Longrightarrow \rho(z_1, z_2) = \rho(w_1, w_2)$$

for all $z_1, z_2 \in A$ and $w_1, w_2 \in B$. Now we introduce α-admissible mappings via a function p of $\Omega \times \Omega$.

Definition 1.1. *Let $\alpha : \Omega \times \Omega \to [0, +\infty)$ and $p : \Omega \times \Omega \to [1, +\infty)$ be functions. Then mapping $\mathcal{F} : \Omega \to \Omega$ is α_p-admissible if*

$$\alpha(z, w) \geq p(z, w) \Longrightarrow \alpha(\mathcal{F}z, \mathcal{F}w) \geq p(\mathcal{F}z, \mathcal{F}w)$$

for all z, w *in* Ω. *If* $p(z, w) = 1$ *for all* $z, w \in A$, *then* $\mathcal{F}$ *becomes* α*-admissible mapping introduced in [24]. If* $p : \Omega \times \Omega \to [1, +\infty)$ *is replaced by a function* $\eta : \Omega \times \Omega \to [0, \infty)$, *then* $\mathcal{F}$ *becomes* α_η*−admissible introduced in [16].*

We introduce α_p-proximal admissible mappings via a function p of $A \times A$.

Definition 1.2. *Let* $\alpha : A \times A \to [0, +\infty)$, $p : A \times A \to [1, +\infty)$ *be functions. Then* $\mathcal{F} : A \to B$ *is* α_p*-proximal admissible if*

$$\begin{cases} \alpha(z_1, z_2) \geq p(z_1, z_2), \\ \rho(w_1, \mathcal{F}z_1) = \rho(A, B), \\ \rho(w_2, \mathcal{F}z_2) = \rho(A, B), \end{cases} \implies \alpha(w_1, w_2) \geq p(w_1, w_2)$$

for all $z_1, z_2, w_1, w_2 \in A$. *If* $p(z, w) = 1$ *for all* $z, w \in A$, *then* $\mathcal{F}$ *becomes* α*-proximal admissible given in [18]. If* $p : A \times A \to [1, +\infty)$ *is replaced by a function* $\eta : A \times A \to [0, \infty)$, *then* $\mathcal{F}$ *becomes* α_η*-proximal admissible given in [16].*

A function $\psi : [0, \infty) \to [0, \infty)$ is a "Bianchini-Grandolfi gauge function (also known as (c)-comparison function)" if ψ is non-decreasing, and there is $l_0 \in \mathbb{N}$, and $s \in (0, 1)$ implies

$$\psi^{l+1}(\tau) \leq s\psi^{l}(\tau) + v_l$$

for $l_0 \leq l$ and $\tau \in \mathbb{R}^+$, where $v_l \geq 0$ for $l \in \mathbb{N}$ and $\sum_{l=1}^{\infty} v_l < \infty$. We denote the set of such functions by Θ. The next lemma provides some useful characterizations of such functions.

Lemma 1.3. *[9] If* $\psi \in \Theta$, *then*

(i) *for all* $\tau \in \mathbb{R}^+$, $\lim_{n\to\infty} \psi^n(\tau) = 0$,

(ii) ψ *is continuous at* 0,

(iii) *for each* $\tau \in (0, \infty)$, $\psi(\tau) < \tau$,

(iv) *for any* $\tau \in \mathbb{R}^+$, $\sum_{k=1}^{\infty} \psi^k(\tau) < \infty$.

The next BPP result is due to Abkar and Gabeleh [2] for Suzuki-type contractions.

Theorem 1.4. *[2] Let* (Ω, ρ) *be a C-MS such that* $A, B \in C(\Omega)$, $A_0 \neq \emptyset$ *and* $\theta : [0, 1) \to \left(2^{-1}, 1\right]$ *a function defined as* $\theta(r) = \dfrac{1}{1 + r}$. *Further,* $\mathcal{F} : A \to B$ *is a mapping with* $\mathcal{F}(A_0) \subseteq B_0$ *and there is a* $t \in [0, 1)$ *so that*

$$\theta(t)\rho^*(w, \mathcal{F}w) \leq \rho(w, z) \implies \rho(\mathcal{F}w, \mathcal{F}z) \leq t\rho(w, z) ,$$

for all $w, z \in A$. *Further, if* (A, B) *has* P_P, *then BPP(*$\mathcal{F}$*) is singleton.*

The next result is due to Jleli et al. [19].

Theorem 1.5. *[19] Let* (Ω, ρ) *be a C-MS,* $A, B \in C(\Omega)$, *and* $\alpha : A \times A \to [0, \infty)$. *Suppose that* $\mathcal{F} : A \to B$ *is continuous and*

1) A_0 *is non-empty,* $\mathcal{F}(A_0) \subseteq B_0$ *and* (A, B) *has* P_P,

2) $\mathcal{F}$ is α-proximal admissible,
3) there exist u_0, u_1 in A_0 with $\rho(u_1, \mathcal{F}u_0) = \rho(A, B)$ and $\alpha(u_0, u_1) \geq 1$ and

$$\begin{aligned}&\alpha(w,z)\rho(\mathcal{F}w,\mathcal{F}z)\\ &\leq \psi\left[\max\left\{\begin{array}{l}\rho(w,z), \left(\dfrac{\rho(w,\mathcal{F}w)+\rho(z,\mathcal{F}z)}{2}\right)-\rho(A,B),\\ \left(\dfrac{\rho(z,\mathcal{F}w)+\rho(w,\mathcal{F}z)}{2}\right)-\rho(A,B)\end{array}\right\}\right]\end{aligned}$$

for all $w, z \in A$, and for some $\psi \in \Theta$.

Then BPP($\mathcal{F}$) is non-empty.

Hussain et al. [17] presented the following result.

Theorem 1.6. *[17] Let (Ω, ρ) be a C-MS, $A, B \in C(\Omega)$, $\alpha : A \times A \to [0, \infty)$ and $\mathcal{F} : A \to B$ a continuous mapping. Further*

1) A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$ and (A, B) has P_P,
2) $\mathcal{F}$ is α_η-proximal admissible and $\eta(u, v) = 2$,
3) $\alpha(u_0, u_1) \geq 2$ and $\rho(u_1, \mathcal{F}u_0) = \rho(A, B)$ for some u_0 and u_1 in A_0 and

$$\rho^*(w, \mathcal{F}w) \leq \alpha(w, z)\rho(w, z) \Longrightarrow \rho(\mathcal{F}w, \mathcal{F}z) \leq \psi(\rho(w, z)),$$

for all $w, z \in A$, and for some $\psi \in \Theta$.

Then BPP($\mathcal{F}$) is singleton.

We introduce Suzuki-type generalized α-ψ-proximal contraction.

Definition 1.7. *Let (Ω, ρ) be a metric space, $A, B \in C(\Omega)$ and $\alpha : A \times A \to [0, \infty)$. A nonself-mapping $\mathcal{F} : A \to B$ is a Suzuki-type generalized α-ψ-proximal contraction if*

$$\rho^*(w, \mathcal{F}w) \leq \alpha(w, z)\rho(w, z) \implies \rho(\mathcal{F}w, \mathcal{F}z) \leq \psi(M(w, z)) \tag{1.2}$$

for all $w, z \in A$, where $\psi \in \Theta$ and

$$M(w,z) = \max\left\{\begin{array}{l}\rho(w,z), \rho(w,\mathcal{F}w)-\rho(A,B), \rho(z,\mathcal{F}z)-\rho(A,B),\\ \rho(z,\mathcal{F}w)-\rho(A,B), \dfrac{\rho(w,\mathcal{F}z)-\rho(A,B)}{2},\\ \dfrac{(\rho(w,\mathcal{F}w)-\rho(A,B))(\rho(z,\mathcal{F}z)-\rho(A,B))}{1+(\rho(w,z))}\end{array}\right\}.$$

2. Main results

The following is the first main result.

Theorem 2.1. *Let (Ω, ρ) be a C-MS, $A, B \in C(\Omega)$ and $\mathcal{F} : A \to B$ a Suzuki-type generalized α-ψ-proximal contraction. Further*

a$_1$ *A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$ and (A, B) has P_P,*

a$_2$ *$\mathcal{F}$ is α_p-proximal admissible,*

a$_3$ *$\alpha(u_0, u_1) \geq p(u_0, u_1)$ and $\rho(u_1, \mathcal{F}u_0) = \rho(A, B)$ for some u_0 and u_1 in A_0 and*

a$_4$ *$\mathcal{F}$ is continuous.*

Then $BPP(\mathcal{F})$ is singleton.

Proof. From the assumption (a_3), we have

$$\rho(u_1, \mathcal{F}u_0) = \rho(A, B), \text{ and } \alpha(u_0, u_1) \geq p(u_0, u_1)$$

for some u_0 and u_1 in A_0. As $\mathcal{F}(A_0) \subseteq B_0$, so

$$\rho(u_2, \mathcal{F}u_1) = \rho(A, B)$$

for some $u_2 \in A_0$. Consequently

$$\begin{aligned} &\alpha(u_0, u_1) \geq p(u_0, u_1), \\ &\rho(u_1, \mathcal{F}u_0) = \rho(A, B), \text{ and } \rho(u_2, \mathcal{F}u_1) = \rho(A, B). \end{aligned}$$

From (a_2), we obtain $\alpha(u_1, u_2) \geq p(u_1, u_2)$. As $\mathcal{F}(A_0) \subseteq B_0$, so there is $u_3 \in A_0$ such that

$$\rho(u_3, \mathcal{F}u_2) = \rho(A, B).$$

Continuing the process, a sequence $\{u_n\}$ in A_0 is obtained that satisfies

$$\begin{aligned} &\alpha(u_n, u_{n+1}) \geq p(u_n, u_{n+1}), \\ &\rho(u_n, \mathcal{F}u_{n-1}) = \rho(A, B), \text{ and } \rho(u_{n+1}, \mathcal{F}u_n) = \rho(A, B), \end{aligned} \tag{2.1}$$

for all $n \in \mathbb{N}_0$. By P_P

$$\rho(u_n, u_{n+1}) = \rho(\mathcal{F}u_{n-1}, \mathcal{F}u_n)$$

for all $n \in \mathbb{N}$. Further

$$\rho(u_{n-1}, \mathcal{F}u_{n-1}) \leq \rho(u_{n-1}, u_n) + \rho(u_n, \mathcal{F}u_{n-1}) = \rho(u_n, u_{n-1}) + \rho(A, B).$$

That is

$$\begin{aligned} \rho^*(u_{n-1}, \mathcal{F}u_{n-1}) &= \rho(u_{n-1}, \mathcal{F}u_{n-1}) - \rho(A, B) \\ &\leq \rho(u_{n-1}, u_n) \leq p(u_n, u_{n-1})\rho(u_{n-1}, u_n) \\ &\quad \leq \alpha(u_n, u_{n-1})\rho(u_{n-1}, u_n). \end{aligned}$$

From (1.2), we get

$$\rho(u_n, u_{n+1}) = \rho(\mathcal{F}u_{n-1}, \mathcal{F}u_n) \leq \psi(M(u_{n-1}, u_n)) \tag{2.2}$$

for all $n \in \mathbb{N}$, where

$$M(u_{n-1}, u_n) = \max\left\{\begin{array}{l} \rho(u_{n-1}, u_n), \rho(u_{n-1}, \mathcal{F}u_{n-1}) - \rho(A, B), \\ \rho(u_n, \mathcal{F}u_n) - \rho(A, B), \rho(u_n, \mathcal{F}u_{n-1}) - \rho(A, B), \\ \dfrac{\rho(u_{n-1}, \mathcal{F}u_n) - \rho(A, B)}{2}, \\ \dfrac{(\rho(u_{n-1}, \mathcal{F}u_{n-1}) - \rho(A, B)(\rho(u_n, \mathcal{F}u_n) - \rho(A, B))}{1 + \rho(u_{n-1}, u_n)} \end{array}\right\} \tag{2.3}$$
$$\leq \max\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1})\}.$$

Hence from (2.2) and (2.3), we get

$$\rho(u_n, u_{n+1}) \leq \psi(\max\{\rho(u_{n-1}, u_n), \rho(u_n, u_{n+1})\}) \tag{2.4}$$

for all $n \in \mathbb{N}$. If for some $N \in \mathbb{N}_0$, $u_{N+1} = u_N$, then from (2.1) $\rho(u_N, \mathcal{F}u_N) = \rho(A, B)$, that is, u_N is a BPP of $\mathcal{F}$. So assume $u_{n+1} \neq u_n$ for all $n \in \mathbb{N}_0$. If

$$\rho(u_{n-1}, u_n) \leq \rho(u_n, u_{n+1}),$$

then (2.4) implies

$$\rho(u_n, u_{n+1}) \leq \psi(\rho(u_n, u_{n+1})) < \rho(u_n, u_{n+1})$$

a contradiction as $\psi \in \Theta$. Thus

$$\rho(u_n, u_{n+1}) \leq \psi(\rho(u_n, u_{n-1}) \leq \psi^n(\rho(u_1, u_0) \tag{2.5}$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} \psi^n(\rho(u_1, u_0) < \infty$, therefore

$$\sum_{k \geq h}^{\infty} \psi^k(\rho(u_1, u_0) < \varepsilon$$

for some $h \in \mathbb{N}$. Hence

$$\rho(u_n, u_m) \leq \sum_{k=n}^{m-1} \rho(u_k, u_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(\rho(u_1, u_0)) \leq \sum_{k \geq h}^{\infty} \psi^k(\rho(u_1, u_0)) < \varepsilon$$

for all $m > n > h$. Thus $\{u_n\}$ is a Cauchy sequence in A. As A is a closed subset of (Ω, ρ) which is complete, so we get a $u^* \in A$ with $u_n \to u^*$ as $n \to \infty$. By (a_4), $\mathcal{F}u_n \to \mathcal{F}u^*$ as $n \to \infty$. This gives

$$\rho(A, B) = \lim_{n \to \infty} \rho(u_{n+1}, \mathcal{F}u_n) = \rho(u^*, \mathcal{F}u^*).$$

Hence $\rho(u^*, \mathcal{F}u^*) = \rho(A, B)$. Now for the uniqueness of BPP of $\mathcal{F}$. If $v, z \in A_0$ are BPPs of $\mathcal{F}$ with $v \neq z$, then

$$\rho(v, \mathcal{F}v) = \rho(z, \mathcal{F}z) = \rho(A, B). \tag{2.6}$$

By P_P, we get

$$\rho(v, z) = \rho(\mathcal{F}v, \mathcal{F}z). \tag{2.7}$$

Now $\rho^*(v, \mathcal{F}v) = \rho(v, \mathcal{F}v) - \rho(A, B) = 0 \leq \alpha(v, z)\rho(v, z)$. By (1.2) $\rho(\mathcal{F}v, \mathcal{F}z) \leq \psi(M(v, z))$. From (2.7)

$$\rho(v, z) \leq \psi(M(v, z)) \leq \psi(\rho(v, z)) < \rho(v, z)$$

a contradiction. This proves the uniqueness. □

We prove the next result without the assumption of continuity on $\mathcal{F}$.

Theorem 2.2. *Let (Ω, ρ) be a C-MS, $A, B \in C(\Omega)$ and $\mathcal{F} : A \to B$ a Suzuki-type generalized α-ψ-proximal contraction satisfying;*

a$_1$ *A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$ and (A, B) has P_P,*

a$_2$ *$\mathcal{F}$ is α_p-proximal admissible,*

a$_3$ *$\alpha(u_0, u_1) \geq p(u_0, u_1)$ and $\rho(u_1, \mathcal{F}u_0) = \rho(A, B)$ for some u_0 and u_1 in A_0, and*

a$_4$ *for any sequence $\{u_n\}$ in A, $\alpha(u_n, u_{n+1}) \geq p(u_n, u_{n+1})$ and $u_n \to u \in A$ as $n \to \infty$, implies $\alpha(u_n, u) \geq 2$.*

Then $BPP(\mathcal{F})$ is singleton.

Proof. On the similar lines as in Theorem 2.1, a sequence $\{u_n\}$ is obtained in A_0 that converges to $u^* \in A$, and satisfies

$$\begin{aligned}
&\rho(u_n, u_{n+1}) = \rho(\mathcal{F}u_{n-1}, \mathcal{F}u_n) \text{ for all } n \in \mathbb{N},\\
&\rho(u_n, u_{n+1}) < \rho(u_{n-1}, u_n) \text{ for all } n \in \mathbb{N},\\
&\alpha(u_n, u_{n+1}) \geq p(u_n, u_{n+1}) \text{ for all } n \in \mathbb{N}_0 \text{ and}\\
&\rho(u_{n+1}, \mathcal{F}u_n) = \rho(A, B).
\end{aligned}$$

By (a$_4$), $\alpha(u_n, u^*) \geq 2$ for all $n \in \mathbb{N}$. Note that

$$\begin{aligned}
\rho^*(u_n, \mathcal{F}u_n) &= \rho(u_n, \mathcal{F}u_n) - \rho(A, B)\\
&\leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, \mathcal{F}u_n) - \rho(A, B) = \rho(u_n, u_{n+1}),
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\rho^*(u_{n+1}, \mathcal{F}u_{n+1}) &= \rho(u_{n+1}, \mathcal{F}u_{n+1}) - \rho(A, B)\\
&\leq \rho(\mathcal{F}u_n, \mathcal{F}u_{n+1}) + \rho(u_{n+1}, \mathcal{F}u_n) - \rho(A, B)\\
&= \rho(\mathcal{F}u_n, \mathcal{F}u_{n+1}) = \rho(u_{n+1}, u_{n+2}) < \rho(u_n, u_{n+1}).
\end{aligned} \tag{2.9}$$

Hence (2.8) and (2.9) imply that

$$\rho^*(u_n, \mathcal{F}u_n) + \rho^*(u_{n+1}, \mathcal{F}u_{n+1}) < 2\rho(u_n, u_{n+1}). \tag{2.10}$$

If

$$\rho^*(u_n, \mathcal{F}u_n) > \alpha(u_n, u^*)\rho(u_n, u^*),$$

and

$$\rho^*(u_{n+1}, \mathcal{F}u_{n+1}) > \alpha(u_{n+1}, u^*)\rho(u_{n+1}, u^*),$$

hold for some $n \in \mathbb{N}$, then we obtain

$$\rho^*(u_n, \mathcal{F}u_n) > \alpha(u_n, u^*)\rho(u_n, u^*) \geq 2\rho(u_n, u^*),$$

and

$$\rho^*(u_{n+1}, \mathcal{F}u_{n+1}) > \alpha(u_{n+1}, u^*)\rho(u_{n+1}, u^*) \geq 2\rho(u_{n+1}, u^*),$$

hold for some $n \in \mathbb{N}$. Further, by (2.10) we get

$$\begin{aligned} 2\rho(u_n, u_{n+1}) &\leq 2\rho(u_n, u^*) + 2\rho(u_{n+1}, u^*) \\ &< \rho^*(u_n, \mathcal{F}u_n) + \rho^*(u_{n+1}, \mathcal{F}u_{n+1}) < 2\rho(u_n, u_{n+1}), \end{aligned}$$

a contradiction. That is, for all $n \in \mathbb{N}$, either

$$\rho^*(u_n, \mathcal{F}u_n) \leq \alpha(u_n, u^*)\rho(u_n, u^*), \tag{2.11}$$

or

$$\rho^*(u_{n+1}, \mathcal{F}u_{n+1}) \leq \alpha(u_{n+1}, u^*)\rho(u_{n+1}, u^*), \tag{2.12}$$

hold. If (2.11) holds for infinite many $n \in \mathbb{N}_0$, then using (1.2), we obtain

$$\begin{aligned} &\rho(\mathcal{F}u_n, \mathcal{F}u^*) \leq \psi(M(u_n, u^*) \\ &= \psi\left[\max\left\{\begin{array}{l} \rho(u_n, u^*), \rho(u_n, \mathcal{F}u_n) - \rho(A, B), \\ \rho(u^*, \mathcal{F}u^*) - \rho(A, B), \rho(u^*, \mathcal{F}u_n) - \rho(A, B), \\ \dfrac{\rho(u_n, \mathcal{F}u^*) - \rho(A, B)}{2}, \\ \dfrac{(\rho(u_n, \mathcal{F}u_n) - \rho(A, B))(\rho(u^*, \mathcal{F}u^*) - \rho(A, B))}{1 + \rho(u_n, u^*)} \end{array}\right\}\right] \\ &\leq \psi(N^*), \end{aligned}$$

where

$$N^* = \max\left\{\begin{array}{c} \rho(u_n, u^*), \rho(u_n, u_{n+1}), \rho(u^*, \mathcal{F}u^*) - \rho(A, B), \rho(u^*, u_{n+1}), \\ \dfrac{\rho(u_n, \mathcal{F}u^*) - \rho(A, B)}{2}, (\rho(u_n, u_{n+1}))(\rho(u^*, \mathcal{F}u^*) - \rho(A, B)) \end{array}\right\}.$$

If $N^* = \rho(u_n, u_{n+1})$ then

$$\begin{aligned} \rho(u_{n+1}, \mathcal{F}u^*) - \rho(A, B) &= \rho(u_{n+1}, \mathcal{F}u^*) - \rho(\mathcal{F}u_n, u_{n+1}) \\ &\leq \rho(\mathcal{F}u_n, \mathcal{F}u^*) \leq \psi(\rho(u_n, u_{n+1})), \end{aligned}$$

as n tends to ∞, so

$$\rho(u^*, \mathcal{F}u^*) - \rho(A, B) \leq 0,$$

that is $\rho(u^*, \mathcal{F}u^*) - \rho(A, B)) = 0$, hence u^* is a BPP of $\mathcal{F}$. If $N^* = \rho(u^*, \mathcal{F}u^*) - \rho(A, B)$ then

$$\begin{aligned} \rho(u_{n+1}, \mathcal{F}u^*) - \rho(A, B) &= \rho(u_{n+1}, \mathcal{F}u^*) - \rho(\mathcal{F}u_n, u_{n+1}) \\ &\leq \rho(\mathcal{F}u_n, \mathcal{F}u^*) \leq \psi(\rho(u^*, \mathcal{F}u^*) - \rho(A, B)), \end{aligned}$$

as n tends to ∞, so we obtain

$$\rho(u^*, \mathcal{F}u^*) - \rho(A, B) \leq \psi(\rho(u^*, \mathcal{F}u^*) - \rho(A, B)),$$

if $\rho(u^*, \mathcal{F}u^*) - \rho(A, B)) = 0$, then u^* is the BPP of $\mathcal{F}$, if

$$\rho(u^*, \mathcal{F}u^*) - \rho(A, B)) > 0,$$

then

$$\rho(u^*, \mathcal{F}u^*) - \rho(A, B)) < \rho(u^*, \mathcal{F}u^*) - \rho(A, B))$$

a contradiction as $\psi \in \Theta$. If

$$N^* = \frac{\rho(u_n, \mathcal{F}u^*) - \rho(A, B)}{2}$$

then

$$\rho(\mathcal{F}u_n, \mathcal{F}u^*) \leq \psi\left(\frac{\rho(u_n, \mathcal{F}u^*) - \rho(A, B)}{2}\right) \leq \frac{\rho(u_n, \mathcal{F}u^*) - \rho(A, B)}{2},$$

implies

$$\begin{aligned} \rho(u_{n+1}, \mathcal{F}u^*) &\leq \rho(\mathcal{F}u_n, \mathcal{F}u^*) + \rho(u_{n+1}, \mathcal{F}u_n) \\ &\leq \frac{\rho(u_n, \mathcal{F}u^*) - \rho(A, B)}{2} + \rho(A, B). \end{aligned}$$

That yields

$$2\rho(u_{n+1}, \mathcal{F}u^*) \leq \rho(u_n, \mathcal{F}u^*) + \rho(A, B).$$

On considering limit as $n \to \infty$, we have

$$2\rho(u^*, \mathcal{F}u^*) \leq \rho(u^*, \mathcal{F}u^*) + \rho(A, B).$$

Hence $\rho(u^*, \mathcal{F}u^*) \leq \rho(A, B)$ implies u^* is a BPP of $\mathcal{F}$. If

$$N^* = (\rho(u_n, u_{n+1}))(\rho(u^*, \mathcal{F}u^*) - \rho(A, B)),$$

then

$$\begin{aligned} \rho(u_{n+1}, \mathcal{F}u^*) - \rho(A, B) &= \rho(u_{n+1}, \mathcal{F}u^*) - \rho(\mathcal{F}u_n, u_{n+1}) \leq \rho(\mathcal{F}u_n, \mathcal{F}u^*) \\ &\leq \psi\left((\rho(u_n, u_{n+1}))(\rho(u^*, \mathcal{F}u^*) - \rho(A, B))\right) \\ &\leq (\rho(u_n, u_{n+1}))(\rho(u^*, \mathcal{F}u^*) - \rho(A, B)). \end{aligned}$$

On considering limit as $n \to \infty$,we get

$$\rho(u^*, \mathcal{F}u^*) - \rho(A, B) \leq 0$$

implies u^* is a BPP of $\mathcal{F}$ and the proof is complete. Consequently

$$\rho(\mathcal{F}u_n, \mathcal{F}u^*) \leq \psi(\max\{\rho(u_n, u^*), \rho(u_{n+1}, u^*)\}). \tag{2.13}$$

Similarly if (2.12) holds for infinite many $n \in \mathbb{N}_0$, then via (1.2), it follows that

$$\rho(\mathcal{F}u_{n+1}, \mathcal{F}u^*) \leq \psi(\max\{\rho(u_{n+1}, u^*), \rho(u_{n+2}, u^*)\}). \tag{2.14}$$

Hence either (2.13) or (2.14) holds for all $n \in \mathbb{N}_0$. If we consider limit as $n \to +\infty$ in (2.13) and (2.14), we have

$$\text{either } \mathcal{F}u_n \to \mathcal{F}u^* \text{ or } \mathcal{F}u_{n+1} \to \mathcal{F}u^* \text{ as } n \to \infty,$$

that is, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $\mathcal{F}u_{n_k} \to \mathcal{F}u^*$ as $k \to \infty$. Since $u_{n_k} \to u^*$ as $k \to \infty$,

$$\rho(A, B) = \lim_{k\to\infty} \rho(u_{n_k+1}, \mathcal{F}u_{n_k}) = \rho(u^*, \mathcal{F}u^*).$$

The uniqueness of the BPP of $\mathcal{F}$ is followed on the similar lines as in Theorem 2.1. □

Example 2.3. *Let* $\Omega = \mathbb{R}$*,* $\rho(u,v) = |u-v|$*,* $A = (-\infty,-8]$*, and* $B = [2,+\infty)$*. Define* $\mathcal{F} : A \to B$*,* $\alpha : \Omega\times\Omega \to [0,\infty)$*,* $\psi : [0,\infty) \to [0,\infty)$ *and* $p : \Omega\times\Omega \to [1,\infty)$ *as*

$$\mathcal{F}u = \begin{cases} -\dfrac{3u}{16} + |u+16|\, e^{\frac{1}{u}}, \ \text{if } u \in (-\infty,-16), \\ \\ -\dfrac{u}{8} + 1, \ \text{if } u \in [-16,-8], \end{cases}$$

$$\alpha(u,v) = \begin{cases} |u| + |v|\,, \ \text{if } u,v \in [-16,-8], \\ 0, \ \text{otherwise,} \end{cases}$$

$$\psi(t) = \frac{t}{8}, \ \text{and } p(u,v) = 2.$$

Note that $\rho(A,B) = 10$*. Further, if* $u,v \in (-\infty,-16]$*, then*

$$\rho^*(u,\mathcal{F}u) > \alpha(u,v)\rho(u,v).$$

If $u,v \in [-16,-8]$*, then*

$$\rho(\mathcal{F}u,\mathcal{F}v) = |\mathcal{F}u - \mathcal{F}v| = \frac{1}{8}\rho(u,v) \le \psi(M(u,v)),$$

this implies that $\mathcal{F}$ *is Suzuki-type generalized* $\alpha-\psi$*-proximal contraction. Moreover,*

$$A_0 = \{-8\}, \ \text{and } B_0 = \{2\},$$

and $\mathcal{F}(A_0) \subseteq B_0$*. Further* (A,B) *has* P_P *and*

$$\begin{aligned} &\alpha(u_1,u_2) \ge p(u_1,u_2), \\ &\rho(w_1,\mathcal{F}u_1) = \rho(A,B) = 10, \ \text{and } \rho(w_2,\mathcal{F}u_2) = \rho(A,B) = 10, \end{aligned}$$

implies $u_1,u_2 \in [-16,-8]$ *and* $\mathcal{F}u_1,\mathcal{F}u_2 \in [2,3]$ *for all* $u_1,u_2 \in [-16,-8]$*. Hence* $w_1 = w_2 = -8$*, that is* $\alpha(w_1,w_2) = 16$ *and* $\alpha(w_1,w_2) \ge p(w_1,w_2)$*. That is,* $\mathcal{F}$ *is* α_p*-proximal admissible. Note that* $\mathcal{F}$ *is continuous. All axioms of Theorem 2.1 hold. So* $\mathcal{F}$ *has a BPP which is* -8*.*

Following example shows that Theorems 2.1 and 2.2 generalize some results properly in the literature.

Example 2.4. *Let* $\Omega = \{a,b,c,e,f\}$ *and a metric* ρ *on* Ω *defined as*

$$\begin{aligned} \rho(a,b) &= 1, \ \rho(a,c) = 4, \ \rho(a,e) = 5, \ \rho(a,f) = 8, \ \rho(b,c) = 3, \\ \rho(b,e) &= 6, \ \rho(b,f) = 9, \ \rho(c,e) = 7, \ \rho(c,f) = 10, \ \rho(e,f) = 12, \\ \rho(u,v) &= \rho(v,u) \text{ and } \rho(u,u) = 0 \text{ for all } u,v \text{ in } \Omega. \end{aligned}$$

Suppose $A = \{b,e\}$ *and* $B = \{a,f\}$*. Define* $\mathcal{F} : A \to B$ *by*

$$\mathcal{F}(b) = a, \ \mathcal{F}(e) = f.$$

Also define $\alpha : \Omega \times \Omega \to [0, \infty)$ $\psi : [0, \infty) \to [0, \infty)$ *and* $p : \Omega \times \Omega \to [1, \infty)$ *as*

$$\alpha(u, v) = \begin{cases} \rho(u, v), & \text{if } u \neq v, \\ 0, & \text{otherwise,} \end{cases}$$
$$\psi(t) = \frac{4}{5}t, \text{ and } p(u, v) = 2.$$

As

$$\rho(A, B) = 1, \; A_0 = \{b\}, \text{ and } B_0 = \{a\}.$$

Now we check that $\mathcal{F}$ *is Suzuki-type generalized* $\alpha - \psi$*-proximal contraction. Since*

$$\begin{aligned} &\rho(b, \mathcal{F}b) - \rho(A, B) = 0 \leq 36 = \alpha(b, e)\rho(b, e), \\ &\rho(e, \mathcal{F}e) - \rho(A, B) = 11 \leq 36 = \alpha(e, b)\rho(e, b), \text{ and} \\ &\rho(\mathcal{F}b, \mathcal{F}e) = 8 \leq \frac{44}{5} = \psi\left(M(b, e)\right), \end{aligned}$$

therefore $\mathcal{F}$ *is Suzuki-type generalized* $\alpha - \psi$*-proximal contraction. Note that* $\mathcal{F}(A_0) \subseteq B_0$ *and* (A, B) *has* P_P*. Further,* $\mathcal{F}$ *is clearly* α_p*-proximal admissible. All axioms of Theorems 2.1 and 2.2 hold. So* $\mathcal{F}$ *has a unique BPP* b*. Now we illustrate that some results in the literature are not applicable in this Example.*

Remark 2.5. *1) Note that Theorems 1.4 and 1.6 are not applicable in this example as*

$$\rho(b, \mathcal{F}b) - \rho(A, B) = 0 \leq 36 = \alpha(b, e)\rho(b, e)$$

but

$$\rho(\mathcal{F}b, \mathcal{F}e) = 8 \nleq \frac{24}{5} = \frac{4}{5}(6) = \psi(6) = \psi(\rho(b, e).$$

2) It can be seen that Theorem 1.5 is not applicable here as

$$\begin{aligned} \alpha(b, e)\rho(\mathcal{F}b, \mathcal{F}e) = (6)(8) = 48 \nleq \frac{52}{10} &= \frac{4}{5}\left(\max\left\{6, 6, \frac{13}{2}\right\}\right) \\ &= \psi\left[\max\left\{\begin{array}{l} \rho(b, e), \dfrac{\rho(b, \mathcal{F}b) + \rho(e, \mathcal{F}e)}{2} - \rho(A, B), \\ \dfrac{\rho(e, \mathcal{F}b) + \rho(b, \mathcal{F}e)}{2} - \rho(A, B) \end{array}\right\}\right]. \end{aligned}$$

Now we present important consequences of the Theorems 2.1 and 2.2.

Corollary 2.6. *Let* (Ω, ρ) *be a C-MS,* $A, B \in C(\Omega)$ *and a mapping* $\mathcal{F} : A \to B$*. Further*

(i) A_0 *is non-empty,* $\mathcal{F}(A_0) \subseteq B_0$ *and* (A, B) *has* P_P*,*

(ii) *for* $\delta : [0, 1) \to \left(0, 2^{-1}\right]$*,* $\mathcal{F}$ *satisfies*

$$\delta(r)\rho^*(u, \mathcal{F}u) \leq \rho(u, v) \Longrightarrow \rho(\mathcal{F}u, \mathcal{F}v) \leq \psi(M(u, v)) \tag{2.15}$$

for all $u, v \in A$*, and for some* $r \in [0, 1)$ *and* $\psi \in \Theta$*.*

Then $BPP(\mathcal{F})$ is singleton.

Proof. For a fixed $r \in [0,1)$ define $\alpha^r : A \times A \to [0,\infty)$ by $\alpha^r(u,v) = \dfrac{1}{\delta(r)}$ for all $u,v \in A$. Define $p : A \times A \to [1,\infty)$ as $p(u,v) = 2$ for all $u,v \in A$. Since $\dfrac{1}{\delta(r)} \geq 2$ for $r \in [0,1)$, $\alpha^r(w_1,w_2) \geq p(w_1,w_2)$ for all $w_1,w_2 \in A$, $\mathcal{F}$ is an α^r_p-proximal admissible. Also if

$$\rho^*(u,\mathcal{F}u) \leq \alpha^r(u,v)\rho(u,v),$$

then $\delta(r)\rho^*(u,\mathcal{F}u) \leq \rho(u,v)$, and by (2.15), we get $\rho(\mathcal{F}u,\mathcal{F}v) \leq \psi(M(u,v))$. Hence all the conditions in Theorem 2.2 are met. Consequently conclusion holds true by Theorem 2.2. □

If $\psi(x) = tx$ in Corollary 2.6, for $0 \leq t < 1$, we get the next result.

Corollary 2.7. *Let (Ω,ρ) be a C-MS, $A,B \in C(\Omega)$ and a mapping $\mathcal{F} : A \to B$. Further,*

(i) *A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$ and (A,B) has P_P,*

(ii) *for $\delta : [0,1) \to \left(0,2^{-1}\right]$, $\mathcal{F}$ satisfies*

$$\delta(t)\rho^*(u,\mathcal{F}u) \leq \rho(u,v) \Longrightarrow \rho(\mathcal{F}u,\mathcal{F}v) \leq tM(u,v), \tag{2.16}$$

for all $u,v \in A$ and for some $t \in [0,1)$.

Then $BPP(\mathcal{F})$ is singleton.

Above corollary yields another important result.

Corollary 2.8. *Let (Ω,ρ) be a C-MS, $A,B \in C(\Omega)$ and a mapping $\mathcal{F} : A \to B$. Further,*

(i) *A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$ and (A,B) has P_P, and*

$$2^{-1}\theta(k)\rho^*(u,\mathcal{F}u) \leq \rho(u,v) \Longrightarrow \rho(\mathcal{F}u,\mathcal{F}v) \leq kM(u,v)$$

for all $u,v \in A$ for some $k \in [0,1)$, where $\theta : [0,1) \to \left(2^{-1},1\right]$ is defined as

$$\theta(t) = \begin{cases} 1 \text{ if } 0 \leq t \leq 2^{-1}\left(\sqrt{5}-1\right), \\ (1-t)t^{-2} \text{ if } 2^{-1}\left(\sqrt{5}-1\right) < t < \sqrt{2}, \\ (1+t)^{-1} \text{ if } \sqrt{2} \leq t < 1. \end{cases} \tag{2.17}$$

Then $BPP(\mathcal{F})$ is singleton.

Proof. If $\delta(t) = \dfrac{\theta(t)}{2}$, then the result follows from Corollary 2.7. □

From Corollary 2.8, we fetch a result given in [2].

Corollary 2.9. *Let (Ω,ρ) be a C-MS, $A,B \in C(\Omega)$ and a mapping $\mathcal{F} : A \to B$. Further,*

(i) *A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$ and (A,B) has P_P,*

(ii) *for* $\beta : [0,1) \to (1/2, 1]$, $\mathcal{F}$ *satisfies*

$$\beta(t)\rho^*(u, \mathcal{F}u) \leq \rho(u,v) \Longrightarrow \rho(\mathcal{F}u, \mathcal{F}v) \leq tM(u,v),$$

for all $u, v \in A$ *and for some* $t \in [0,1)$, *where*

$$\beta(t) = \frac{1}{2+2t}.$$

Then $BPP(\mathcal{F})$ *is singleton.*

If $\delta(t) = 2^{-1}$ in Corollary 2.7, then the following corollary emerges.

Corollary 2.10. *Let* (Ω, ρ) *be a C-MS,* $A, B \in C(\Omega)$ *and a mapping* $\mathcal{F} : A \to B$ *satisfying*

$$2^{-1}\rho^*(u, \mathcal{F}u) \leq \rho(u,v) \Longrightarrow \rho(\mathcal{F}u, \mathcal{F}v) \leq tM(u,v)$$

for all $u, v \in A$ *and for some* $t \in [0,1)$. *Further* A_0 *is non-empty,* $\mathcal{F}(A_0) \subseteq B_0$ *and* (A, B) *has* P_P. *Then* $BPP(\mathcal{F})$ *is singleton.*

3. Completeness of metric spaces via the best proximity points

Completeness is an important property of metric spaces which is related to "end problem" in behavioral science (compare [5]). In a complete metric spaces, every Banach contraction has a fixed point but converse does not hold true. That means, there are incomplete metric spaces where every Banach contraction has a fixed point (see [11]). For more on completeness, we refer to [10]. In this section, we obtain completeness characterization via the existence of best proximity points.

If we set $M(u,v) = \rho(u,v)$ in Corollary 2.8, we obtain the following corollary.

Corollary 3.1. *Let* (Ω, ρ) *be a C-MS,* $A, B \in C(\Omega)$ *and a mapping* $\mathcal{F} : A \to B$. *Further if* A_0 *is non-empty,* $\mathcal{F}(A_0) \subseteq B_0$, (A, B) *has* P_P, *and*

$$2^{-1}\theta(k)\rho^*(u, \mathcal{F}u) \leq \rho(u,v) \Longrightarrow \rho(\mathcal{F}u, \mathcal{F}v) \leq k\rho(u,v)$$

for all $u, v \in A$ *for some* $k \in [0,1)$, *where* $\theta : [0,1) \to \left(2^{-1}, 1\right]$ *is same as defined in Corollary 2.8. Then* $BPP(\mathcal{F})$ *is singleton.*

If we set $A = B = \Omega$ in Corollary 3.1, we get the following result.

Corollary 3.2. *Let* (Ω, ρ) *be a C-MS and a mapping* $\mathcal{F} : \Omega \to \Omega$. *Further if*

$$2^{-1}\theta(k)\rho(u, \mathcal{F}u) \leq \rho(u,v) \Longrightarrow \rho(\mathcal{F}u, \mathcal{F}v) \leq k\rho(u,v)$$

for all $u, v \in \Omega$ *for some* $k \in [0,1)$, *where* $\theta : [0,1) \to \left(2^{-1}, 1\right]$ *is same as defined in Corollary 2.8. Then* $\widetilde{\mathcal{F}}_{(\mathcal{F})}$ *is singleton.*

In the following theorem, we obtain completeness of metric space via the best proximity point theorem.

Theorem 3.3. *Let (Ω,ρ) be a metric space, $\theta : [0,1) \to \left(2^{-1},1\right]$ is same as defined in Corollary 2.8. For $k \in (0,1)$ and $\eta \in \left(0, \frac{\theta(k)}{2}\right]$, let $A_{k,\eta}$ be a class of mappings $\mathcal{F} : A \to B$ that satisfies (a) and (b) given below.*

(a) *for $x, y \in A$,*

$$\eta\rho^*(u,\mathcal{F}u) \le \rho(u,v) \text{ implies } \rho(\mathcal{F}u,\mathcal{F}v) \le k\rho(u,v). \tag{3.1}$$

(b) *A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$ and (A,B) has P_P,*

Let $A^*_{k,\eta}$ be a class of mappings $\mathcal{F} : \Omega \to \Omega$ that satisfies

(d) *for $x, y \in \Omega$,*

$$\eta\rho(u,\mathcal{F}u) \le \rho(u,v) \text{ implies } \rho(\mathcal{F}u,\mathcal{F}v) \le k\rho(u,v). \tag{3.2}$$

Let $B_{k,\eta}$ be a class of mappings $\mathcal{F}$ that satisfies (d) and

(e) *$\mathcal{F}(\Omega)$ is denumerable,*

(f) *every $M \subseteq \mathcal{F}(\Omega)$ is closed.*

Then the statements 1–4 are equivalent:

1) *The metric space (Ω,ρ) is complete.*
2) *Every mapping $\mathcal{F} \in A_{k,\frac{\theta(k)}{2}}$ has a best proximity point for all $k \in [0,1)$.*
3) *Every mapping $\mathcal{F} \in A^*_{k,\frac{\theta(k)}{2}}$ has a fixed point for all $k \in [0,1)$.*
4) *Every mapping $\mathcal{F} \in B_{k,\eta}$ has a fixed point for some $k \in [0,1)$ and $\eta \in \left(0, \frac{\theta(k)}{2}\right]$.*

Proof. By Corollary 3.1, (1) implies (2). For $A = B = \Omega$, $A^*_{k,\frac{\theta(k)}{2}} \subseteq A_{k,\frac{\theta(k)}{2}}$. Hence (2) implies (3). Since $B_{r,\eta} \subseteq A^*_{r,\frac{\theta(k)}{2}}$, therefore (3) implies (4). For (4) implies (1), assume on contrary that (4) holds but (Ω,ρ) is incomplete. That is, there is a sequence $\{u_n\}$, which is Cauchy but does not converge. Define a function $g : \Omega \to [0,\infty)$ as

$$g(x) = \lim_{n\to\infty} \rho(x,u_n)$$

for $x \in \Omega$. As $\{u_n\}$ is a Cauchy sequence in (Ω,ρ), so $\rho(x,u_n)$ is Cauchy in $\mathbb{R}$. Hence g is well defined. Further, $g(x) > 0$ for all x in Ω. For $\epsilon > 0$, there exists $K_\epsilon \in \mathbb{N}$ such that

$$\rho(u_m,u_n) < \frac{\epsilon}{2}$$

for all $m, n \ge K_\epsilon$. Hence we get

$$0 \le g(u_m) = \lim_{n\to\infty} \rho(u_m,u_n) \le \frac{\epsilon}{2} < \epsilon$$

for all $m \ge K_\epsilon$. That is

$$\lim_{m\to\infty} g(u_m) = 0. \tag{3.3}$$

From (3.3), for every $x \in \Omega$, there exists a $\upsilon \in \mathbb{N}$ such that

$$g(u_\upsilon) \leq \left(\frac{r\eta}{3 + r\eta}\right) g(x). \tag{3.4}$$

If $\mathcal{F}(x) = u_\upsilon$, then

$$g(\mathcal{F}x) \leq \left(\frac{r\eta}{3 + r\eta}\right) g(x) \text{ and } \mathcal{F}x \in \{u_n : n \in \mathbb{N}\} \tag{3.5}$$

for all $x \in \Omega$. From (3.5), we have $g(\mathcal{F}x) < g(x)$, hence $\mathcal{F}x \neq x$ for all $x \in \Omega$. That is, $\widetilde{\mathcal{F}}_{(\mathcal{F})}$ is empty. As $\mathcal{F}(\Omega) \subset \{u_n : n \in \mathbb{N}\}$, so (e) holds. Note that (f) holds as well. Further, g satisfies

$$g(x) - g(y) \leq \rho(x, y) \leq g(x) + g(y)$$

for all $x, y \in \Omega$. Now fix $x, y \in \Omega$ such that

$$\eta\rho(x, \mathcal{F}x) \leq \rho(x, y).$$

We need to show that (3.2) holds. Observe that

$$\begin{cases} \rho(x, y) \geq \eta\rho(x, \mathcal{F}x) \geq \eta(g(x) - g(\mathcal{F}x)) \\ \geq \eta\left(1 - \dfrac{r\eta}{3 + r\eta}\right) g(x) = \dfrac{3\eta}{3 + r\eta} g(x). \end{cases} \tag{3.6}$$

We have two cases. Case (1) Suppose $g(y) \geq 2g(x)$, then

$$\begin{aligned} \rho(\mathcal{F}x, \mathcal{F}y) &\leq g(\mathcal{F}x) + g(\mathcal{F}y) \\ &\leq \frac{r\eta}{3 + r\eta} g(x) + \frac{r\eta}{3 + r\eta} g(y) \\ &\leq \frac{r}{3}(g(x) + g(y)) + \frac{2r}{3}(g(y) - 2kg(x)) \\ &= \frac{r}{3}(g(x) + g(y) + 2g(y) - 4g(x)) \\ &\leq \frac{r}{3}(3g(y) - 3g(x)) \leq r\rho(x, y). \end{aligned}$$

Case (2) whenever $g(y) < 2g(x)$, from (3.6)

$$\begin{aligned} \rho(\mathcal{F}x, \mathcal{F}y) &\leq g(\mathcal{F}x) + g(\mathcal{F}y) \leq \frac{r\eta}{3 + r\eta} g(x) + \frac{r\eta}{3 + r\eta} g(y) \\ &\leq \frac{r\eta}{3 + r\eta} g(x) + \frac{2r\eta}{3 + r\eta} g(x) = \frac{3r\eta}{3 + r\eta} g(x) \\ &\leq r\rho(x, y). \end{aligned}$$

Hence

$$\eta\rho(x, \mathcal{F}x) \leq \rho(x, y) \text{ implies } \rho(\mathcal{F}x, \mathcal{F}y) \leq r\rho(x, y)$$

for all $x, y \in \Omega$. From (4) $\widetilde{\mathcal{F}}_{(\mathcal{F})}$ is non-empty, a contradiction. Hence Ω is complete. □

4. Best proximity point results in partially ordered metric spaces

Order structure is very important in connection with domain of words problem in computer sciences, equilibrium problems in economics (compare [23, 28] and references therein) and many other related disciplines. In this section, with the help of the function α (used in the last section), and the main results in the last section, we deduce some important consequences related to the BPPs of nonself mappings of ordered metric spaces. Denote $(\Omega, \rho, \sqsubseteq)$ by "partially ordered metric spaces" where ρ is a metric on Ω and $\sqsubseteq$ a partial order on Ω.

Definition 4.1. *[8] A mapping $\mathcal{F} : A \to B$ is proximally order preserving (POP) if*

$$\begin{cases} u \sqsubseteq v, \\ \rho(w, \mathcal{F}u) = \rho(A, B), \\ \rho(z, \mathcal{F}v) = \rho(A, B), \end{cases} \implies w \sqsubseteq z$$

for all $u, v, w, z \in A$.

If $A = B$, then $\mathcal{F}$ becomes nondecreasing mapping.

Theorem 4.2. *Let $(\Omega, \rho, \sqsubseteq)$ be a partially ordered C-MS, $A, B \in C(\Omega)$ and a continuous mapping $\mathcal{F} : A \to B$ satisfying*

$$\rho^*(u, \mathcal{F}u) \leq \rho(u, v) \Longrightarrow \rho(\mathcal{F}u, \mathcal{F}v) \leq \psi(M(u, v))$$

for all $u, v \in A$ with $u \sqsubseteq v$, and $\psi \in \Theta$. Further A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$, (A, B) has P_P, and $\mathcal{F}$ is POP. Moreover $u_0 \sqsubseteq u_1$ and $\rho(u_1, \mathcal{F}u_0) = \rho(A, B)$ for some u_0 and u_1 in A_0. Then $BPP(\mathcal{F})$ is singleton.

Proof. Define $\alpha : A \times A \to [0, +\infty)$ and $p : A \times A \to [1, +\infty)$ as

$$\begin{aligned} \alpha(u, v) &= \begin{cases} 1, & \text{iff } u \sqsubseteq v, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \\ p(u, v) &= 1 \text{ for all } u, v \text{ in } A. \end{aligned}$$

As

$$\begin{cases} \alpha(u, v) = p(u, v) = 1, \\ \rho(z, \mathcal{F}u) = \rho(A, B), \\ \rho(w, \mathcal{F}v) = \rho(A, B). \end{cases} \text{ is equivalent to } \begin{cases} u \sqsubseteq v, \\ \rho(z, \mathcal{F}u) = \rho(A, B), \\ \rho(w, \mathcal{F}v) = \rho(A, B), \end{cases}$$

implies $z \sqsubseteq w$. Thus $\alpha(z, w) = 1 = p(z, w)$. Consequently $\mathcal{F}$ is α_p-proximal admissible. Furthermore, the elements u_0 and u_1 in A_0 with $u_0 \sqsubseteq u_1$ and $\rho(u_1, \mathcal{F}u_0) = \rho(A, B)$ implies

$$\rho(u_1, \mathcal{F}u_0) = \rho(A, B) \text{ and } \alpha(u_0, u_1) \geq 1.$$

Let

$$\rho^*(u, \mathcal{F}u) \leq \alpha(u, v)\rho(u, v).$$

As $\alpha(u, v) = 1$ for all $u, v \in A$ with $u \sqsubseteq v$, so $\rho^*(u, \mathcal{F}u) \leq \rho(u, v)$. We get $\rho(\mathcal{F}u, \mathcal{F}v) \leq \psi(M(u, v))$. Thus by Theorem 2.1, $BPP(\mathcal{F})$ is singleton. □

For a particular choice of the function ψ in Theorem 4.2, we get an important corollary as given below.

Corollary 4.3. *Let $(\Omega, \rho, \sqsubseteq)$ be a partially ordered C-MS, $A, B \in C(\Omega)$ and a continuous mapping $\mathcal{F} : A \to B$ satisfying*

$$\rho^*(u, \mathcal{F}u) \leq \rho(u, v) \Longrightarrow \rho(\mathcal{F}u, \mathcal{F}v) \leq rM(u, v)$$

for all $u, v \in A$ with $u \sqsubseteq v$, and for some $r \in (0, 1)$. Further A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$, (A, B) has P_P and $\mathcal{F}$ is POP. Moreover $u_0 \sqsubseteq u_1$ and $\rho(u_1, \mathcal{F}u_0) = \rho(A, B)$ for some u_0 and u_1 in A_0. Then $BPP(\mathcal{F})$ is singleton.

Theorem 4.4. *Let $(\Omega, \rho, \sqsubseteq)$ be a partially ordered C-MS, $A, B \in C(\Omega)$ and a mapping $\mathcal{F} : A \to B$ satisfying*

$$2^{-1}\rho^*(u, \mathcal{F}u) \leq \rho(u, v) \Longrightarrow \rho(\mathcal{F}u, \mathcal{F}v) \leq \psi(M(u, v)), \tag{4.1}$$

for all $u, v \in A$ with $u \sqsubseteq v$, and $\psi \in \Theta$. Further A_0 is non-empty, $\mathcal{F}(A_0) \subseteq B_0$, (A, B) has P_P and $\mathcal{F}$ is POP. Moreover $u_0 \sqsubseteq u_1$ and $\rho(u_1, \mathcal{F}u_0) = \rho(A, B)$ for some u_0 and u_1 in A_0 and for any non-decreasing sequence $\{u_n\}$ in A such that $u_n \to u \in A$ as $n \to \infty$, implies $u_n \sqsubseteq u$ for all $n \in \mathbb{N}$. Then $BPP(\mathcal{F})$ is singleton.

Proof. Defining $\alpha : \Omega \times \Omega \to [0, \infty)$ by

$$\alpha(u, v) = \begin{cases} 2, & \text{iff } u \sqsubseteq v, \\ 0, & \text{otherwise,} \end{cases}$$

and $p : A \times A \to [1, +\infty)$ by $p(u, v) = 2$ for all u, v in A. Using the similar lines as in the proof of Theorem 4.2, we show that $\mathcal{F}$ is Suzuki-type generalized $(\alpha, \psi)-$proximal contraction. Assume $\alpha(u_n, u_{n+1}) \geq 2$ for all $n \in \mathbb{N}$ such that $u_n \to u \in A$ as $n \to \infty$. Then $u_n \sqsubseteq u_{n+1}$ for all $n \in \mathbb{N}$ and so $\alpha(u_n, u) = 2$ for all $n \in \mathbb{N}$. Consequently by Theorem 2.2, $BPP(\mathcal{F})$ is singleton. □

Remark 4.5. *By Considering $A = B = \Omega$ in Theorems 2.1, 2.2, 4.2–4.4 we get the corresponding new fixed point theorems in the context of (partially ordered) metric spaces.*

5. Conclusions

This article dealt with the existence of best proximity points of generalized proximal contractions of complete metric spaces. We provided some examples to explain the main results and to show that the obtained results are proper generalizations of some existing results in the literature. Moreover, in this paper, a completeness characterization has been linked with the existence of best proximity points of mappings of metric spaces. One can consider the results in this paper for further study in the setup of more general spaces like b-metric spaces and quasi metric spaces. In quasi metric spaces, the problem of Smyth completeness via the existence of best proximity points of certain mapping would be worth doing.

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Conflict of interest

The authors declare that they do not have any conflict of interests regarding this paper.

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