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Research article

Eigenvalues of fourth-order boundary value problems with distributional potentials

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Abstract: This paper aims to investigate the fourth-order boundary value problems with distributional potentials. We first prove that the operators associated with the problems are self-adjoint and the corresponding eigenvalues are real. Then we obtain that the eigenvalues of the problems depend not only continuously but also smoothly on the parameters of the problems: the boundary conditions, the coefficient functions and the endpoints. Moreover, we find the differential expressions for each parameter.

Keywords: fourth-order boundary value problems; distributional potentials; quasi-derivatives; eigenvalues
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1. Introduction

In the long history of the development on the studies of the differential operators, the dependence of eigenvalues on the problems plays an important role in the fundamental theory of differential operators and numerical computation of the spectrum. In the classical Sturm-Liouville(S–L) problems, the dependence of eigenvalues on the problems is widely studied by many authors [5, 14–16]. The detailed explanation of the dependence of eigenvalues on the Sturm-Liouville problems can be found in [31] and [32].

Recent years, the research on dependence of eigenvalues of a differential operator or boundary value problem on the problem has been extended in various aspects. For example, in 2015, Zhang et al. studied the eigenvalues of the Sturm-Liouville problems with interface conditions and obtained that the eigenvalues depend not only continuously but also smoothly on the coefficient functions, the boundary conditions and the interface conditions [35]. In 2016 and 2017, Zhu et al. generalized the problems to discrete case and singular case of Sturm-Liouville problems in [36, 37], respectively. They showed the continuous dependence of eigenvalues on the problems and the discontinuous dependence of the n-th eigenvalue on the boundary conditions in detail. In 2019, Hu et al. investigated the continuity and discontinuity of the n-th eigenvalue of high dimensional Sturm-Liouville problems in detail [11]. In 2020, Zhang and Li [34] extended the eigenvalue dependent problems to Sturm-Liouville problems with eigenparameter-dependent boundary conditions. The analogous results can also be found from the references in these literature.

Higher order boundary value problems always appear in the physical problems and engineering problems such as the vibration beams and fluid mechanics and so on, see [8] and [23]. For higher order boundary value problems, there are still several literature consisting on the dependence of eigenvalues on the problems, for example in [7, 17, 19–21, 26, 28, 33].

The Sturm-Liouville problems with distributional potentials are another research topic for the past few years. In quantum mechanics, the Schrödinger equation with generalized potential functions is widely used to describe the interaction between individual particles [1, 18]. The integrability condition of potential function has to be weakened in the classical Sturm-Liouville theory to find the way for the solution. Such a problem not only generalizes the classical Sturm-Liouville theory, but also gives a list of new characters on physical problems. These years, scholars studied this topic from different aspects and some important results have been made [1, 3, 18, 27, 29, 30]. In 2020, Uğurlu and Bairamov considered the fourth-order differential operators with distributional potentials, they not only proved the existence and uniqueness of solutions of the fourth-order differential equation, but also showed the deficiency indices theory of the corresponding minimal symmetric operator [29].

Recently, we considered the eigenvalues of the second order Sturm-Liouville problems with distributional potentials and eigenparameter-dependent boundary conditions, and found the continuity and differential properties on these problems [2]. However, for higher order boundary value problems with distributional potentials, there are few studies, and the dependence of eigenvalues on the problems has not been studied yet. Motivated by this way, in this paper, we will consider the fourth-order boundary value problems with distributional potentials and show the eigenvalue dependence of these kind of problems. We show the continuity and differential properties of the eigenvalues on the data, including the boundary conditions, the coefficient functions and the endpoints. The main novelty of this paper is to extend eigenvalue dependence results to higher order boundary value problems with distributional potentials. Compare to previous known results, we not only show the self-adjointness and some eigenvalue properties of the problems, but also list some derivative formulas of several distributional potential functions, which have not been discussed before. It is hard to display the differential expressions of eigenvalues with respect to the coefficient functions, due to the complicated relations between the much more coefficient functions appeared in the differential equation. In order to overcome this difficulty and solve the problem better, we give the differential expressions of some coefficient functions under the vanishing condition of certain coefficient functions.

This paper is organized as follows. Following this Introduction, in Section 2 we introduce the problems studied here and show some basic results related to the problems. Section 3 shows the continuous dependence of eigenvalues on the problems. In Section 4, the differential properties of eigenvalues with respect to the data of the problems are given, in particular, the derivative formulas of the boundary conditions, the coefficient functions and the endpoints are listed respectively. At last, a brief conclusion is listed in Section 5.

2. Notation

Consider the general fourth-order differential equation with distributional potentials

$$\begin{aligned}&\left\{\left[q_2(y^{(2)}-s_1y^{(1)}-s_2y)\right]^{(1)}+q_2s_1(y^{(2)}-s_1y^{(1)})-q_1(y^{(1)}+s_4y)+s_3y\right\}^{(1)}\\&+q_2s_2y^{(2)}-s_3y^{(1)}+q_1s_4(y^{(1)}+s_4y)+q_0y=\lambda wy \text{ on } J'=(a',b')\subseteq\mathbb{R},\ \lambda\in\mathbb{C},\end{aligned}\tag{2.1}$$

by introducing the quasi-derivatives

$$\begin{aligned}y^{[0]}&=y,\\y^{[1]}&=-y^{(1)},\\y^{[2]}&=q_2(y^{(2)}-s_1y^{(1)}-s_2y),\\y^{[3]}&=-\left[q_2(y^{(2)}-s_1y^{(1)}-s_2y)\right]^{(1)}-q_2s_1(y^{(2)}-s_1y^{(1)})+q_1(y^{(1)}+s_4y)-s_3y,\end{aligned}$$

the differential equation (2.1) can be written as the following form:

$$-(y^{[3]})'+s_2y^{[2]}-(q_2s_1s_2-s_3+q_1s_4)y^{[1]}+(q_2s_2^2+q_1s_4^2+q_0)y=\lambda wy \text{ on } J'.\tag{2.2}$$

Let

$$J=[a,b]\subseteq J'=(a',b'),\ \ -\infty\le a'<a<b<b'\le\infty,\tag{2.3}$$

and assume the coefficients satisfying:

$$\begin{aligned}&q_0,\ q_1,\ q_2,\ s_1,\ s_2,\ s_3,\ s_4:(a',b')\to\mathbb{R},\ q_0,\ q_1,\ q_2^{-1},\ s_1,\ s_2,\ s_3,\ s_4,\ q_1s_4,\ q_2s_1,\\&q_2s_1^2,\ q_2s_2,\ q_1s_4^2,\ q_2s_1s_2,\ q_2s_2^2,\ w\in L_{loc}(J'),\ q_2>0 \text{ on } J \text{ and } w>0 \text{ a.e. on } J'.\end{aligned}\tag{2.4}$$

Consider the boundary conditions (BCs)

$$AY(a)+BY(b)=0,\quad Y=\left(y^{[0]},y^{[1]},y^{[2]},y^{[3]}\right)^T,\tag{2.5}$$

where the complex $4*4$ matrices A and B satisfy:

$$\mathbf{rank}(A|B)=4;\ \ AEA^*=BEB^*,\ \ with\ E=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}.\tag{2.6}$$

It is well known the BCs (2.5) are self-adjoint BCs under the conditions (2.6) [9,19]. Here the equation (2.1) or (2.2) together with the BCs (2.5) is called as boundary value problem (BVP) with distributional potentials. The fourth-order self-adjoint BCs are much complicated and have many canonical forms compared to the second order case. In this paper we will consider certain types of special self-adjoint BCs. For more general self-adjoint BCs and corresponding eigenvalue dependence results of the fourth-order case without distributional potentials the reader may refer to [19]. From [26], we know that there are three types of self-adjoint BCs and each type of them can be introduced as follows

- (i) Separated self-adjoint BCs:

$$\begin{aligned} &y^{[0]}(a)\cos\alpha - y^{[1]}(a)\sin\alpha = 0,\\ &y^{[2]}(a)\cos\alpha - y^{[3]}(a)\sin\alpha = 0, \quad 0 \le \alpha < \pi;\\ &y^{[0]}(b)\cos\beta - y^{[1]}(b)\sin\beta = 0,\\ &y^{[2]}(b)\cos\beta - y^{[3]}(b)\sin\beta = 0, \quad 0 < \beta \le \pi. \end{aligned} \tag{2.7}$$

- (ii) Real coupled self-adjoint BCs:

$$Y(b) = KY(a), \tag{2.8}$$

where $K \in SL_4(\mathbb{R})$, i.e., K satisfies

$$K = (k_{ij})_{4\times4}, \quad k_{ij} \in \mathbb{R}, \ \ \mathbf{det}K = 1, \ KEK^* = E. \tag{2.9}$$

- (iii) Complex coupled self-adjoint BCs:

$$Y(b) = e^{i\theta}KY(a), \tag{2.10}$$

where K satisfies (2.9) and $-\pi < \theta < 0$ or $0 < \theta < \pi$.

Let $l(y) = -(y^{[3]})' + s_2y^{[2]} - (q_2s_2s_1 - s_3 + q_1s_4)y^{[1]} + (q_2s_2^2 + q_1s_4^2 + q_0)y$ and $\tau[y] = w^{-1}l(y)$ on $J' = (a', b')$, then we can introduce the following definition.

Definition 1. *A linearly independent system of solutions $y_1, \cdots, y_4$ of the equation*

$$l(y) = \lambda wy \ \ or \ \ \tau[y] = \lambda y, \ \ x \in (a', b')$$

is called a fundamental system.

Let the weighted space be defined as

$$\mathcal{H} = L^2_w(J) = \left\{ y : \int_a^b |y(x)|^2 w(x)\mathrm{d}x < \infty \right\},$$

with the inner product $(f, g)_{\mathcal{H}} = \int_a^b f\bar{g}w\mathrm{d}x$ for any $f, g \in \mathcal{H}$.

For any $y, \chi \in \mathcal{H}$, the Lagrange form $[y, \chi]$ of the functions y and χ is defined as

$$[y, \chi] = y^{[0]}\bar{\chi}^{[3]} - y^{[3]}\bar{\chi}^{[0]} - y^{[1]}\bar{\chi}^{[2]} + y^{[2]}\bar{\chi}^{[1]}. \tag{2.11}$$

Definition 2. *Let*

$$\mathcal{D} = \left\{ y \in \mathcal{H} : y^{[0]}, y^{[1]}, y^{[2]}, y^{[3]} \in AC_{loc}(a', b'), \ \ \tau[y] \in \mathcal{H} \right\},$$

then the operators related to the problems studied in this paper can be listed as follows:

-

$$D = \{y \in \mathcal{D} : (2.5) \ and \ (2.6) \ hold\},$$

$$Ty = \tau[y], \ y \in D.$$

- $$D_s = \{y \in \mathcal{D} : (2.7)\ holds\},$$

$$T_s y = \tau[y],\ y \in D_s.$$

- $$D_{rc} = \{y \in \mathcal{D} : (2.8)\ holds\},$$

$$T_{rc} y = \tau[y],\ y \in D_{rc}.$$

- $$D_c = \{y \in \mathcal{D} : (2.10)\ holds\},$$

$$T_c y = \tau[y],\ y \in D_c.$$

Lemma 1. *Let $\lambda = \mu$ and $\lambda = \nu$ be the eigenvalues of the BVP (2.2), (2.5) and (2.6), u and v are the eigenfunctions corresponding to μ and ν, then*

$$\begin{aligned}
&(Tu, v) - (u, Tv)\\
&= (\mu - \nu)\int_a^b u\bar{v}w = [u, v]_a^b\\
&= [u^{[0]}\overline{v}^{[3]} - u^{[3]}\overline{v}^{[0]} - u^{[1]}\overline{v}^{[2]} + u^{[2]}\overline{v}^{[1]}](b) - [u^{[0]}\overline{v}^{[3]} - u^{[3]}\overline{v}^{[0]} - u^{[1]}\overline{v}^{[2]} + u^{[2]}\overline{v}^{[1]}](a).
\end{aligned}$$

Proof. This follows from integration by parts. □

Theorem 1. *The fourth-order differential expression $l(y)$ is formally symmetric on J'.*

Proof. Note that the differential expression $l(y)$ can be transformed into

$$l(y) = (q_2 y^{(2)})^{(2)} - \left\{\left[(q_2 s_1)^{(1)} + q_2 s_1^2 + q_1\right] y^{(1)}\right\}^{(1)} + \left[q_1 s_4^2 - (q_2 s_2)^{(2)} - (q_1 s_4)^{(1)} + s_3^{(1)} + q_0\right] y.$$

Now let

$$\begin{aligned}
P_2 &= q_2;\\
P_1 &= -\left[(q_2 s_1)^{(1)} + q_2 s_1^2 + q_1\right];\\
P_0 &= q_1 s_4^2 - (q_2 s_2)^{(2)} - (q_1 s_4)^{(1)} + s_3^{(1)} + q_0.
\end{aligned}$$

Then $l(y)$ can be written as

$$l(y) = (P_2 y^{(2)})^{(2)} + (P_1 y^{(1)})^{(1)} + P_0 y,$$

from the definition of basic formally symmetric differential expression, it is easy to see that $l(y)$ is symmetric on J'. □

Lemma 2. *[10] A linear submanifold D of $\mathcal{D}$ is the self-adjoint domain of the operator T if and only if*

- $\mathbf{rank}(A|B) = 4$;
- $AEA^* = BEB^*$;
- $D = \left\{ y \in \mathcal{D} : AY(a) + BY(b) = 0, \quad Y = \left(y^{[0]}, y^{[1]}, y^{[2]}, y^{[3]}\right)^T \right\}$.

From the well-known theory of ordinary differential operators and by Theorem 1 and Lemma 2, we have the following conclusion immediately

Theorem 2. *T is self-adjoint operator in $\mathcal{H}$.*

Corollary 1. *T_s, T_{rc} and T_c are self-adjoint operators in $\mathcal{H}$.*

Since the eigenvalues and eigenfunctions of the BVP (2.2), (2.5) and (2.6) coincide with the eigenvalues and eigenfunctions of the operator T, we have the following conclusions immediately.

Theorem 3. *The fourth-order BVP (2.2), (2.5) and (2.6) has only a discrete spectrum consisting of an infinite but countable number of real eigenvalues, each eigenvalue has a geometric multiplicity at most 4 and the multiplicity may be different for different eigenvalue. Moreover, the eigenvalues are bounded below and form a sequence accumulating to $+\infty$, and can be ordered as:*

$$-\infty < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots.$$

Proof. The proof please see [17] and its references. □

Corollary 2. *Assume that λ_1 and λ_2 are eigenvalues of the BVP (2.2), (2.5) and (2.6)(i.e. the operator T), $f(x)$ and $g(x)$ are corresponding eigenfunctions of λ_1 and λ_2 respectively, if $\lambda_1 \neq \lambda_2$, then the eigenfunctions $f(x)$ and $g(x)$ are orthogonal to each other, i.e.*

$$\int_a^b f(x)\bar{g}(x)w(x)dx = 0.$$

3. Continuous dependence of eigenvalues and eigenfunctions

In this section we establish the characterization of the eigenvalues as zeros of an entire function and prove the continuity of the eigenvalues and eigenfunctions for the regular fourth-order BVP (2.2), (2.5) and (2.6).

Let

$$\Omega = \left\{ \omega = (A, B, a, b, q_0, q_1, \frac{1}{q_2}, s_1, s_2, s_3, s_4, w) \right\}$$

such that (2.3), (2.4), (2.6) hold. For the special case of the separated BCs (2.7) we also use the notation

$$\Omega_s = \left\{ \omega = (\alpha, \beta, a, b, q_0, q_1, \frac{1}{q_2}, s_1, s_2, s_3, s_4, w) \right\},$$

and for the real coupled case (2.8) we let

$$\Omega_{rc} = \left\{ \omega = (K, a, b, q_0, q_1, \frac{1}{q_2}, s_1, s_2, s_3, s_4, w) \right\}$$

and for the complex coupled case (2.10) we let

$$\Omega_c = \left\{\omega = (\theta, K, a, b, q_0, q_1, \frac{1}{q_2}, s_1, s_2, s_3, s_4, w)\right\},$$

to highlight the dependence of the parameters of each case on the problem.

Observe that the values of $q_0, q_1, \frac{1}{q_2}, s_1, s_2, s_3, s_4, w$ outside the interval J, that is, in $J'\backslash J$, do not affect the spectrum of the problem determined by ω. To account for this and to facilitate comparisons between eigenvalues of problems defined on different intervals, we let

$$\check{\Omega} = \left\{\check{\omega} = (A, B, a, b, \check{q}_0, \check{q}_1, \frac{1}{\check{q}_2}, \check{s}_1, \check{s}_2, \check{s}_3, \check{s}_4, \check{w})\right\},$$

where

$$\check{q}_0 = \begin{cases} q_0, & x \in [a, b], \\ 0, & otherwise \end{cases}$$

and $\check{q}_1, \frac{1}{\check{q}_2}, \check{s}_1, \check{s}_2, \check{s}_3, \check{s}_4, \check{w}$ are defined similarly. Now, we introduce the Banach space

$$\mathcal{B} := M_{4\times 4}(\mathbb{C}) \oplus M_{4\times 4}(\mathbb{C}) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \underbrace{L(J') \oplus \cdots \oplus L(J')},$$

equipped with the norm

$$\| \omega \| = \| \check{\omega} \| = \| A \| + \| B \| + |a| + |b| + \int_a^b (|q_0| + |q_1| + |\frac{1}{q_2}| + |s_1| + |s_2| + |s_3| + |s_4| + |w|)\mathrm{d}x$$

for any $\omega = (A, B, a, b, q_0, q_1, \frac{1}{q_2}, s_1, s_2, s_3, s_4, w) \in \mathcal{B}$, where $\| A \|$ is any fixed matrix norm.

It is clear that $\check{\Omega}$ is a subset of $\mathcal{B}$ but Ω is not. We identify Ω with $\check{\Omega}$ as a subset of $\mathcal{B}$ to inherit the norm from $\mathcal{B}$ and the convergence in Ω which is determined by this norm.

Let $\Phi(x, \lambda)$ be the matrix solution of the initial value problem

$$\mathcal{Y}' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -s_2 & s_1 & -\frac{1}{q_2} & 0 \\ q_1 s_4 - q_2 s_1 s_2 - s_3 & -q_1 & -s_1 & -1 \\ q_1 s_4^2 + q_0 + q_2 s_2^2 - \lambda w & -(q_1 s_4 + q_2 s_1 s_2 - s_3) & s_2 & 0 \end{pmatrix} \mathcal{Y} \quad on\ (a', b'), \tag{3.1}$$

with $\mathcal{Y}(a) = I$, where I is the identity matrix, i.e. the first row of $\mathcal{Y}$ is a fundamental system of the equation (2.2). Then we have the following conclusions.

Lemma 3. *The complex number λ is an eigenvalue of the BVP (2.2), (2.5) and (2.6) if and only if*

$$\Delta(\lambda) := \mathbf{det}[A + B\Phi(b, \lambda)] = 0.$$

Proof. The proof is omitted since it is routine. □

Theorem 4. *Let* $\widetilde{\omega} = \left(\widetilde{A}, \widetilde{B}, \widetilde{a}, \widetilde{b}, \widetilde{q_0}, \widetilde{q_1}, \frac{1}{\widetilde{q_2}}, \widetilde{s_1}, \widetilde{s_2}, \widetilde{s_3}, \widetilde{s_4}, \widetilde{w}\right) \in \Omega$, *assume that* $\mu = \lambda(\widetilde{\omega})$ *is the eigenvalue of BVP (2.2), (2.5) and (2.6) determined by* $\widetilde{\omega}$. *Then, given any* $\varepsilon > 0$ *sufficiently small, there exists a* $\delta > 0$ *such that for any* $\omega = (A, B, a, b, q_0, q_1, \frac{1}{q_2}, s_1, s_2, s_3, s_4, w) \in \Omega$ *satisfying*

$$\| \omega - \widetilde{\omega} \| = \| A - \widetilde{A} \| + \| B - \widetilde{B} \| + |a - \widetilde{a}| + |b - \widetilde{b}| + \int_a^b (|q_0 - \widetilde{q_0}| + |q_1 - \widetilde{q_1}|$$
$$+ |\frac{1}{q_2} - \frac{1}{\widetilde{q_2}}| + |s_1 - \widetilde{s_1}| + |s_2 - \widetilde{s_2}| + |s_3 - \widetilde{s_3}| + |s_4 - \widetilde{s_4}| + |w - \widetilde{w}|) < \delta,$$

then

$$|\lambda(\omega) - \lambda(\widetilde{\omega})| < \varepsilon.$$

Proof. For $\omega \in \Omega$ and $\lambda \in \mathbb{C}$ let $\Phi(\cdot, a, q_0, q_1, \frac{1}{q_2}, s_1, s_2, s_3, s_4, w, \lambda)$ be the matrix solution of (3.1) and use this notation to highlight the dependence of the parameters here. According to Lemma 3, the complex number $\lambda(\omega)$ is an eigenvalue of the BVP (2.2), (2.5) and (2.6) if and only if $\Delta(\omega, \lambda) = \mathbf{det}[A + B\Phi(b, \lambda)] = 0$ holds, hence $\Delta(\widetilde{\omega}, \mu) = 0, \mu = \lambda(\widetilde{\omega})$. Furthermore, for any $\omega \in \Omega$, $\Delta(\omega, \lambda)$ is an entire function of λ and it is continuous in ω, see Theorems 2.7, 2.8 of [13]. It is obvious that $\Delta(\widetilde{\omega}, \lambda)$ is not a constant in λ since μ is an isolated eigenvalue. Hence there exists $\rho_0 > 0$ such that $\Delta(\widetilde{\omega}, \lambda) \neq 0$ for $\lambda \in S_{\rho_0} := \{\lambda \in \mathbb{C} : |\lambda - \mu| = \rho_0\}$. By the well known theorem on continuity of the roots of an equation as a function of parameters [4], the result follows. □

Remark 1. *The statement of Theorem 4 also holds true for* $\omega \in \Omega_s$, $\omega \in \Omega_{rc}$ *and* $\omega \in \Omega_c$, *respectively.*

Theorem 5. *Let (2.4) hold, let* $c \in [a, b]$ *and* $d_1, d_2, d_3, d_4 \in \mathbb{C}$. *Consider the initial value problem*

$$\begin{cases} -(y^{[3]})' + s_2 y^{[2]} - (q_2 s_2 s_1 - s_3 + q_1 s_4) y^{[1]} + (q_2 s_2^2 + q_1 s_4^2 + q_0) y = \lambda w y, \\ y^{[0]}(c) = d_1, \;\; y^{[1]}(c) = d_2, \;\; y^{[2]}(c) = d_3, \;\; y^{[3]}(c) = d_4. \end{cases} \tag{3.2}$$

Then the unique solution $y = y(\cdot, c, d_1, d_2, d_3, d_4)$ *is a continuous function of all its variables. More precisely, given* $\varepsilon > 0$ *and any compact subinterval* J *of* (a', b'), *there exists a* $\delta > 0$ *such that if*

$$|c - c_0| + |d_1 - d_{10}| + |d_2 - d_{20}| + |d_3 - d_{30}| + |d_4 - d_{40}| < \delta,$$

then

$$|y^{[0]}(x, c, d_1, d_2, d_3, d_4) - y^{[0]}(x, c_0, d_{10}, d_{20}, d_{30}, d_{40})| < \varepsilon,$$
$$|y^{[1]}(x, c, d_1, d_2, d_3, d_4) - y^{[1]}(x, c_0, d_{10}, d_{20}, d_{30}, d_{40})| < \varepsilon,$$
$$|y^{[2]}(x, c, d_1, d_2, d_3, d_4) - y^{[2]}(x, c_0, d_{10}, d_{20}, d_{30}, d_{40})| < \varepsilon,$$
$$|y^{[3]}(x, c, d_1, d_2, d_3, d_4) - y^{[3]}(x, c_0, d_{10}, d_{20}, d_{30}, d_{40})| < \varepsilon,$$

for all $x \in J$.

Proof. The proof is similar to [13], it extends readily to our case. □

Definition 3. *A normalized eigenfunction* u *of the BVP (2.2), (2.5) and (2.6) we mean an eigenfunction* u *that satisfies*

$$\int_a^b |u|^2 w = 1.$$

Theorem 6. *Let the notation and hypotheses of Theorem 4 hold.*

(1) Assume the eigenvalue $\lambda(\widetilde{\omega})$ is simple for some $\widetilde{\omega} \in \Omega$ and let $u = u(\cdot, \widetilde{\omega})$ denote a normalized eigenfunction of $\lambda(\widetilde{\omega})$, then there exists normalized eigenfunction $u = u(\cdot, \omega)$ of $\lambda(\omega)$ for $\omega \in \Omega$ such that when $\omega \to \widetilde{\omega}$ in Ω, we have

$$u^{[k]}(\cdot, \omega) \to u^{[k]}(\cdot, \widetilde{\omega}), \quad k = 0, 1, 2, 3, \tag{3.3}$$

all uniformly on any compact subinterval J of (a', b').

(2) Assume that $\lambda(\omega)$ is an eigenvalue of multiplicity l $(l = 2, 3, 4)$ for all ω in some neighborhood M of $\widetilde{\omega}$ in Ω. Then there exist l linearly independent normalized eigenfunctions $u_1 = u_1(\cdot, \omega), \cdots, u_l = u_l(\cdot, \omega)(l = 2, 3, 4)$ of $\lambda(\omega)$ such that when $\omega \to \widetilde{\omega}$ in Ω, we have

$$u_i^{[k]}(\cdot, \omega) \to u_i^{[k]}(\cdot, \widetilde{\omega}), \quad k = 0, 1, 2, 3, \;\; i = 1, 2, \cdots, l, \tag{3.4}$$

all uniformly on any compact subinterval J of (a', b').

Proof. The proof is similar to the papers in [26] and [29], only to note the quasi derivatives are different from those in [26] and [29], but they do not affect the conclusions, hence we omitted here. □

4. Differential expressions of eigenvalues

In this section, we shall show the eigenvalues determined in Theorem 4 are differentiable, and in particular, we give the derivative formulas of the eigenvalues for the parameters. To this end, we first introduce the definition of Frechet derivative, which is different from the classical derivative, to show the derivative formulas of some parameters.

Definition 4. *A map $\mathcal{T}$ from a Banach space X into another Banach space Y is differentiable at a point $x \in X$ if there exists a bounded linear operator $d\mathcal{T}_x : X \to Y$ such that for $h \in X$*

$$|\mathcal{T}(x + h) - \mathcal{T}(x) - d\mathcal{T}_x(h)| = o(h) \qquad as \qquad h \to 0.$$

Theorem 7. *Let $\lambda(\omega)$ be an eigenvalue of the BVP (2.2), (2.5) and (2.6) with $\omega \in \Omega$, and let $u = u(\cdot, \omega)$ be a normalized eigenfunction for $\lambda(\omega)$, then λ is differentiable with respect to the parameters in ω. Namely that λ is continuously differentiable with respect to each variable α, β for the separated BC (2.7); continuously differentiable with respect to each variable θ, K for the coupled BCs (2.8) and (2.10), and more precisely, the derivative formulas of λ are given as follows:*

1. Fix all parameters of ω except α and let $\lambda = \lambda(\alpha)$ and $u = u(\cdot, \alpha)$ denote the eigenvalue and the corresponding real normalized eigenfunction. Then λ is differentiable and

$$\lambda'(\alpha) = -2\left\{u^{[0]}(a)u^{[2]}(a) + u^{[1]}(a)u^{[3]}(a)\right\}.$$

2. Fix all parameters of ω except β and let $\lambda = \lambda(\beta)$ and $u = u(\cdot, \beta)$ denote the eigenvalue and the corresponding real normalized eigenfunction. Then λ is differentiable and

$$\lambda'(\beta) = 2\left\{u^{[0]}(b)u^{[2]}(b) + u^{[1]}(b)u^{[3]}(b)\right\}.$$

3. Fix all parameters of ω except θ and let $\lambda = \lambda(\theta)$ and $u = u(\cdot, \theta)$ denote the eigenvalue and the corresponding normalized eigenfunction. Then λ is differentiable at θ for any θ satisfying $-\pi < \theta < 0$ or $0 < \theta < \pi$ and

$$\lambda'(\theta) = -2\mathbf{Im}[u^{[0]}\bar{u}^{[3]} + u^{[2]}\bar{u}^{[1]}](b) = -2\mathbf{Im}[u^{[0]}\bar{u}^{[3]} + u^{[2]}\bar{u}^{[1]}](a).$$

4. Fix all parameters of ω except K and let $\lambda = \lambda(K)$ and $u = u(\cdot, K)$ denote the eigenvalue and the corresponding normalized eigenfunction. Assume K satisfies (2.9) and for all H is chosen so that $K + H$ satisfies (2.9). Then λ is differentiable within $SL_4(\mathbb{R})$ and its Frechet derivative is given by:

$$\begin{aligned} d\lambda_K(H) &= (\bar{u}^{[3]}, -\bar{u}^{[2]}, \bar{u}^{[1]}, -\bar{u}^{[0]})(b)HK^{-1}\begin{pmatrix} u^{[0]} \\ u^{[1]} \\ u^{[2]} \\ u^{[3]} \end{pmatrix}(b) \\ &= (\bar{u}^{[3]}, -\bar{u}^{[2]}, \bar{u}^{[1]}, -\bar{u}^{[0]})(a)H(K+H)^{-1}\begin{pmatrix} u^{[0]} \\ u^{[1]} \\ u^{[2]} \\ u^{[3]} \end{pmatrix}(a), \end{aligned}$$

where $H \in M_{4,4}(\mathbb{R})$ such that $K + H \in SL_4(\mathbb{R})$.

Proof. We first prove part 2, and part 1 can be proved similarly.

We fix all data except β, let $u = u(\cdot, \beta)$ and $v = u(\cdot, \beta + \varepsilon)$ denotes the real normalized eigenfunctions of $\mu = \lambda(\beta)$ and $\nu = \lambda(\beta + \varepsilon)$, respectively. From BC (2.7) we have $[u, v](a) = 0$, and thus

$$\begin{aligned} &(\lambda(\beta) - \lambda(\beta + \varepsilon)) \int_a^b uvw \\ &= [u, v](b) - [u, v](a) \\ &= [u^{[0]}v^{[3]} - u^{[3]}v^{[0]} - u^{[1]}v^{[2]} + u^{[2]}v^{[1]}](b) \\ &= \tan\beta u^{[1]}(b)v^{[3]}(b) - \tan(\beta + \varepsilon)u^{[3]}(b)v^{[1]}(b) - \tan(\beta + \varepsilon)u^{[1]}(b)v^{[3]}(b) + \tan\beta u^{[3]}(b)v^{[1]}(b) \\ &= [\tan\beta - \tan(\beta + \varepsilon)]\left[u^{[1]}v^{[3]} + u^{[3]}v^{[1]}\right](b). \end{aligned} \tag{4.1}$$

Dividing both sides of (4.1) by ε and taking the limit as $\varepsilon \to 0$, then by Theorem 6 we can get

$$\begin{aligned} -\lambda'(\beta) &= -\sec^2\beta u^{[1]}(b)u^{[3]}(b) - \sec^2\beta u^{[3]}(b)u^{[1]}(b) \\ &= -\tan^2\beta u^{[1]}(b)u^{[3]}(b) - u^{[1]}(b)u^{[3]}(b) - \left[\tan^2\beta u^{[3]}(b)u^{[1]}(b) + u^{[3]}(b)u^{[1]}(b)\right] \\ &= -2\left\{u^{[0]}(b)u^{[2]}(b) + u^{[1]}(b)u^{[3]}(b)\right\}. \end{aligned}$$

Next we prove part 3. For any increment ε, let the eigenvalues be $\lambda(\theta)$ and $\lambda(\theta + \varepsilon)$, and their corresponding normalized eigenfunctions are $u = u(\cdot, \theta)$, $v = u(\cdot, \theta + \varepsilon)$, respectively. Let

$U = \left(u^{[0]}, u^{[1]}, u^{[2]}, u^{[3]}\right)^T$ and $V = \left(v^{[0]}, v^{[1]}, v^{[2]}, v^{[3]}\right)^T$, then we have

$$\begin{aligned}
&[\lambda(\theta) - \lambda(\theta+\varepsilon)] \int_a^b u\bar{v}w \\
&= [u^{[0]}\overline{v}^{[3]} - u^{[3]}\overline{v}^{[0]} - u^{[1]}\overline{v}^{[2]} + u^{[2]}\overline{v}^{[1]}]_a^b \\
&= V^*(b)EU(b) - V^*(a)EU(a) \\
&= e^{i\theta}V^*(b)EKU(a) - e^{i(\theta+\varepsilon)}V^*(b)(K^*)^{-1}EU(a) \\
&= e^{i\theta}V^*(b)EKU(a) - e^{i(\theta+\varepsilon)}V^*(b)EKU(a) \\
&= e^{i\theta}(1-e^{i\varepsilon})V^*(b)EKU(a).
\end{aligned}$$

Dividing both sides of the above equation by ε and taking the limit as $\varepsilon \to 0$, then by Theorem 6 we obtain

$$\begin{aligned}
-\lambda'(\theta) &= -ie^{i\theta}U^*(b)EKU(a) \\
&= -iU^*(b)EU(b) \\
&= -i\left(\bar{u}^{[3]}, -\bar{u}^{[2]}, \bar{u}^{[1]}, -\bar{u}^{[0]}\right)(b)\begin{pmatrix} u^{[0]} \\ u^{[1]} \\ u^{[2]} \\ u^{[3]} \end{pmatrix}(b) \\
&= -i[u^{[0]}\overline{u}^{[3]} - u^{[3]}\overline{u}^{[0]} - u^{[1]}\overline{u}^{[2]} + u^{[2]}\overline{u}^{[1]}](b) \\
&= 2\mathbf{Im}\left[u^{[0]}\overline{u}^{[3]} + u^{[2]}\overline{u}^{[1]}\right](b).
\end{aligned}$$

Similarly, we can get

$$-\lambda'(\theta) = 2\mathbf{Im}\left[u^{[0]}\overline{u}^{[3]} + u^{[2]}\overline{u}^{[1]}\right](a).$$

Finally we prove part 4. For any increment H, let the eigenvalues be $\lambda(K)$ and $\lambda(K+H)$, and their corresponding normalized eigenfunctions are $u = u(\cdot, K),\ v = u(\cdot, K+H)$, then ones have

$$\begin{aligned}
&[\lambda(K) - \lambda(K+H)] \int_a^b u\bar{v}w \\
=&[u^{[0]}\overline{v}^{[3]} - u^{[3]}\overline{v}^{[0]} - u^{[1]}\overline{v}^{[2]} + u^{[2]}\overline{v}^{[1]}]_a^b \\
=&V^*(b)EU(b) - V^*(a)EU(a) \\
=&e^{i\theta}V^*(b)EKU(a) - e^{i\theta}V^*(b)E(K+H)U(a) \\
=&-e^{i\theta}V^*(b)EHU(a).
\end{aligned}$$

Let $H \to 0$, then by the definition of Frechet derivative, we arrive at

$$\begin{aligned}
d\lambda_K(H) &= e^{i\theta}U^*(b)EHU(a) \\
&= U^*(b)EHK^{-1}U(b) \\
&= (\bar{u}^{[3]}, -\bar{u}^{[2]}, \bar{u}^{[1]}, -\bar{u}^{[0]})(b)HK^{-1}\begin{pmatrix} u^{[0]} \\ u^{[1]} \\ u^{[2]} \\ u^{[3]} \end{pmatrix}(b).
\end{aligned}$$

Similarly, we can get

$$\begin{aligned} d\lambda_K(H) &= e^{i\theta}e^{-i\theta}U^*(a)EH(K+H)^*U(a) \\ &= U^*(a)EH(K+H)^{-1}U(a) \\ &= (\bar{u}^{[3]}, -\bar{u}^{[2]}, \bar{u}^{[1]}, -\bar{u}^{[0]})(a)H(K+H)^{-1}\begin{pmatrix} u^{[0]} \\ u^{[1]} \\ u^{[2]} \\ u^{[3]} \end{pmatrix}(a). \end{aligned}$$

□

Because there are too many coefficient functions in the original equation (2.1), it is difficult to solve the differential expressions of the coefficient functions. For conveniences, here we set $s_1 = s_2 = 0$, at this time the equation (2.1) can be simplified to

$$\left\{\left[q_2(y^{(2)})\right]^{(1)} - q_1(y^{(1)} + s_4 y) + s_3 y\right\}^{(1)} - s_3 y^{(1)} + q_1 s_4(y^{(1)} + s_4 y) + q_0 y = \lambda w y. \tag{4.2}$$

The following theorem will be given under this vanishing condition.

Theorem 8. *Let $s_1 = s_2 = 0$, $\lambda(\omega)$ be an eigenvalue for the BVP (2.2), (2.5) and (2.6) with $\omega \in \Omega$, and $u = u(\cdot, \omega)$ be a normalized eigenfunction for $\lambda(\omega)$. Then λ is differentiable with respect to the coefficient functions in ω and more precisely, the derivative formulas of λ are given as follows:*

1. Fix all parameters of ω except q_2. Then

$$d\lambda_{\frac{1}{q_2}}(h) = -\int_a^b |u^{[2]}|^2 h, \quad h \in L(J, \mathbb{R}).$$

2. Fix all parameters of ω except q_1. Then

$$d\lambda_{q_1}(h) = \int_a^b |u^{[1]}|^2 h + \int_a^b s_4^2 |u^{[0]}|^2 h - 2\int_a^b s_4 \mathbf{Re}(u^{[0]}\bar{u}^{[1]})h, \quad h \in L(J, \mathbb{R}).$$

Particularly, if $u = u(\cdot, \omega)$ is the real normalized eigenfunction for $\lambda(\omega)$, then

$$d\lambda_{q_1}(h) = \int_a^b (u^{[1]} - s_4 u^{[0]})^2 h, \quad h \in L(J, \mathbb{R}).$$

3. Fix all parameters of ω except q_0. Then

$$d\lambda_{q_0}(h) = \int_a^b |u^{[0]}|^2 h, \quad h \in L(J, \mathbb{R}).$$

4. Fix all parameters of ω except s_3. Then

$$d\lambda_{s_3}(h) = 2\int_a^b \mathbf{Re}(\bar{u}^{[1]} u^{[0]})h, \quad h \in L(J, \mathbb{R}).$$

5. Fix all parameters of ω except s_4. Then

$$d\lambda_{s_4}(h) = 2\int_a^b q_1\left[s_4|u^{[0]}|^2 - \mathbf{Re}(\bar{u}^{[1]}u^{[0]})\right]h, \quad h \in L(J,\mathbb{R}).$$

6. Fix all parameters of ω except w. Then

$$d\lambda_w(h) = -\lambda\int_a^b |u^{[0]}|^2 h, \quad h \in L(J,\mathbb{R}).$$

Proof. To prove part 1. Let us fix all data except q_2 and let $u = u(\cdot, \frac{1}{q_2})$ and $v = u(\cdot, \frac{1}{q_2} + h)$, then direct computation yields that

$$\begin{aligned}
&\left[\lambda(\frac{1}{q_2}+h)-\lambda(\frac{1}{q_2})\right](u,v)\\
=&\left[\lambda(\frac{1}{q_2}+h)-\lambda(\frac{1}{q_2})\right]\int_a^b u\bar{v}w\\
=&\lambda(\frac{1}{q_2}+h)\int_a^b u\bar{v}w-\lambda(\frac{1}{q_2})\int_a^b u\bar{v}w\\
=&\int_a^b\left[-(\bar{v}^{[3]})'-(q_1s_4-s_3)\bar{v}^{[1]}+(q_1s_4^2+q_0)\bar{v}^{[0]}\right]u^{[0]}\\
&-\int_a^b\left[-(u^{[3]})'-(q_1s_4-s_3)u^{[1]}+(q_1s_4^2+q_0)u^{[0]}\right]\bar{v}^{[0]}\\
=&\left[-u^{[0]}\bar{v}^{[3]}+u^{[3]}\bar{v}^{[0]}\right]_a^b+\int_a^b\bar{v}^{[3]}(u^{[0]})'-\int_a^b u^{[3]}(\bar{v}^{[0]})'-\int_a^b(q_1s_4-s_3)\bar{v}^{[1]}u^{[0]}\\
&+\int_a^b(q_1s_4^2+q_0)\bar{v}^{[0]}u^{[0]}+\int_a^b(q_1s_4-s_3)u^{[1]}\bar{v}^{[0]}-\int_a^b(q_1s_4^2+q_0)\bar{v}^{[0]}u^{[0]}\\
=&\left[-u^{[0]}\bar{v}^{[3]}+u^{[3]}\bar{v}^{[0]}\right]_a^b-\int_a^b\left[-(\bar{v}^{[2]})'+q_1(\bar{v}^{[1]}+s_4\bar{v})-s_3\bar{v}\right]u^{[1]}\\
&+\int_a^b\left[-(u^{[2]})'+q_1(u^{[1]}+s_4u)-s_3u\right]\bar{v}^{[1]}-\int_a^b(q_1s_4-s_3)\bar{v}^{[1]}u^{[0]}\\
&+\int_a^b(q_1s_4^2+q_0)\bar{v}^{[0]}u^{[0]}+\int_a^b(q_1s_4-s_3)u^{[1]}\bar{v}^{[0]}-\int_a^b(q_1s_4^2+q_0)\bar{v}^{[0]}u^{[0]}\\
=&\left[-u^{[0]}\bar{v}^{[3]}+u^{[3]}\bar{v}^{[0]}+u^{[1]}\bar{v}^{[2]}-u^{[2]}\bar{v}^{[1]}\right]_a^b-\int_a^b\bar{v}^{[2]}(u^{[1]})'+\int_a^b u^{[2]}(\bar{v}^{[1]})'\\
&+\int_a^b q_1\bar{v}^{[1]}u^{[1]}-\int_a^b(q_1s_4-s_3)u^{[1]}\bar{v}^{[0]}-\int_a^b q_1\bar{v}^{[1]}u^{[1]}+\int_a^b(q_1s_4-s_3)\bar{v}^{[1]}u^{[0]}\\
&-\int_a^b(q_1s_4-s_3)\bar{v}^{[1]}u^{[0]}+\int_a^b(q_1s_4-s_3)u^{[1]}\bar{v}^{[0]}.
\end{aligned}$$

For all self-adjoint BCs, one obtains

$$\left[-u^{[0]}\bar{v}^{[3]}+u^{[3]}\bar{v}^{[0]}+u^{[1]}\bar{v}^{[2]}-u^{[2]}\bar{v}^{[1]}\right]_a^b=0,$$

combining the above equations, one has

$$\left[\lambda(\frac{1}{q_2}+h)-\lambda(\frac{1}{q_2})\right](u,v)=-\int_a^b \bar{v}^{[2]}(u^{[1]})'+\int_a^b u^{[2]}(\bar{v}^{[1]})'.$$

As $s_1 = s_2 = 0$, $y^{[2]} = q_2 y^{(2)}$, hence

$$\begin{aligned}&\left[\lambda(\frac{1}{q_2}+h)-\lambda(\frac{1}{q_2})\right](u,v)\\ &=-\int_a^b \bar{v}^{[2]}(u^{[1]})'+\int_a^b u^{[2]}(\bar{v}^{[1]})'\\ &=-\int_a^b \bar{v}^{[2]}(-\frac{1}{q_2})u^{[2]}+\int_a^b u^{[2]}\left[-(\frac{1}{q_2}+h)\right]\bar{v}^{[2]}\\ &=-\int_a^b u^{[2]}\bar{v}^{[2]}h,\end{aligned}$$

let $h\to 0$, then the desired result can be obtained by Theorem 6.

Now we prove part 2. Fix all data except q_1 and let $u = u(\cdot, q_1)$ and $v = u(\cdot, q_1 + h)$, then direct computation yields that

$$\begin{aligned}&\left[\lambda(q_1+h)-\lambda(q_1)\right](u,v)\\ =&\left[\lambda(q_1+h)-\lambda(q_1)\right]\int_a^b u\bar{v}w\\ =&\lambda(q_1+h)\int_a^b u\bar{v}w-\lambda(q_1)\int_a^b u\bar{v}w\\ =&\int_a^b\left\{-(\bar{v}^{[3]})'-\left[(q_1+h)s_4-s_3\right]\bar{v}^{[1]}+\left[(q_1+h)s_4^2+q_0\right]\bar{v}^{[0]}\right\}u^{[0]}\\ &-\int_a^b\left\{-(u^{[3]})'-(q_1s_4-s_3)u^{[1]}+(q_1s_4^2+q_0)u^{[0]}\right\}\bar{v}^{[0]}\\ =&\left[-u^{[0]}\bar{v}^{[3]}+u^{[3]}\bar{v}^{[0]}+u^{[1]}\bar{v}^{[2]}-u^{[2]}\bar{v}^{[1]}\right]_a^b-\int_a^b \bar{v}^{[2]}(u^{[1]})'+\int_a^b u^{[2]}(\bar{v}^{[1]})'\\ &-\int_a^b\left[(q_1+h)s_4-s_3\right]\bar{v}^{[1]}u^{[0]}+\int_a^b\left[(q_1+h)s_4^2+q_0\right]\bar{v}^{[0]}u^{[0]}+\int_a^b(q_1s_4-s_3)u^{[1]}\bar{v}^{[0]}\\ &-\int_a^b(q_1s_4^2+q_0)\bar{v}^{[0]}u^{[0]}+\int_a^b(q_1+h)\bar{v}^{[1]}u^{[1]}-\int_a^b\left[(q_1+h)s_4-s_3\right]u^{[1]}\bar{v}^{[0]}\\ &-\int_a^b q_1\bar{v}^{[1]}u^{[1]}+\int_a^b(q_1s_4-s_3)\bar{v}^{[1]}u^{[0]}\\ =&\int_a^b \bar{v}^{[1]}u^{[1]}h+\int_a^b s_4^2\bar{v}^{[0]}u^{[0]}h-\int_a^b s_4\bar{v}^{[1]}u^{[0]}h-\int_a^b s_4\bar{v}^{[0]}u^{[1]}h.\end{aligned}$$

Let $h\to 0$ in $L(J,\mathbb{R})$, then by Theorem 6, we arrive at

$$d\lambda_{q_1}(h)=\int_a^b |u^{[1]}|^2h+\int_a^b s_4^2|u^{[0]}|^2h-2\int_a^b s_4\mathbf{Re}(u^{[0]}\bar{u}^{[1]})h,\quad h\in L(J,\mathbb{R}).$$

While $u = u(\cdot, \omega)$ is the real normalized eigenfunction, then the above equation will take the form

$$d\lambda_{q_1}(h) = \int_a^b (u^{[1]} - s_4 u^{[0]})^2 h, \quad h \in L(J, \mathbb{R}).$$

Next we prove part 3. Fix all data except q_0 and let $u = u(\cdot, q_0)$ and $v = u(\cdot, q_0 + h)$, then direct computation yields that

$$\begin{aligned}
&\left[\lambda(q_0 + h) - \lambda(q_0)\right](u, v) \\
=&\lambda(q_0 + h) \int_a^b u\bar{v}w - \lambda(q_0) \int_a^b u\bar{v}w \\
=&\int_a^b \left\{-(\bar{v}^{[3]})' - \left[q_1 s_4 - s_3\right] \bar{v}^{[1]} + \left[q_1 s_4^2 + (q_0 + h)\right] \bar{v}^{[0]}\right\} u^{[0]} \\
&- \int_a^b \left\{-(u^{[3]})' - (q_1 s_4 - s_3)u^{[1]} + (q_1 s_4^2 + q_0)u^{[0]}\right\} \bar{v}^{[0]} \\
=&\int_a^b \left[(q_1 + h)s_4^2 + q_0\right] \bar{v}^{[0]}u^{[0]} - \int_a^b (q_1 s_4^2 + q_0)\bar{v}^{[0]}u^{[0]} \\
=&\int_a^b \bar{v}^{[0]}u^{[0]}h,
\end{aligned}$$

let $h \to 0$, then by Theorem 6, we arrive at

$$d\lambda_{q_0}(h) = \int_a^b |u^{[0]}|^2 h, \quad h \in L(J, \mathbb{R}).$$

At last let us prove part 5. We fix all data except s_4 and let $u = u(\cdot, s_4)$ and $v = u(\cdot, s_4 + h)$, then direct computation yields that

$$\begin{aligned}
&\left[\lambda(s_4 + h) - \lambda(s_4)\right](u, v) \\
=&\lambda(s_4 + h) \int_a^b u\bar{v}w - \lambda(s_4) \int_a^b u\bar{v}w \\
=&\int_a^b \left\{-(\bar{v}^{[3]})' - \left[q_1(s_4 + h) - s_3\right] \bar{v}^{[1]} + \left[q_1(s_4 + h)^2 + q_0\right] \bar{v}^{[0]}\right\} u^{[0]} \\
&- \int_a^b \left\{-(u^{[3]})' - (q_1 s_4 - s_3)u^{[1]} + (q_1 s_4^2 + q_0)u^{[0]}\right\} \bar{v}^{[0]} \\
=&\left[-u^{[0]}\bar{v}^{[3]} + u^{[3]}\bar{v}^{[0]}\right]_a^b + \int_a^b \bar{v}^{[3]}(u^{[0]})' - \int_a^b u^{[3]}(\bar{v}^{[0]})' - \int_a^b \left[q_1(s_4 + h) - s_3\right] \bar{v}^{[1]}u^{[0]} \\
&+ \int_a^b \left[q_1(s_4 + h)^2 + q_0\right] \bar{v}^{[0]}u^{[0]} + \int_a^b (q_1 s_4 - s_3)u^{[1]}\bar{v}^{[0]} - \int_a^b (q_1 s_4^2 + q_0)\bar{v}^{[0]}u^{[0]} \\
=&\left[-u^{[0]}\bar{v}^{[3]} + u^{[3]}\bar{v}^{[0]}\right]_a^b - \int_a^b \left[-(\bar{v}^{[2]})' + q_1(\bar{v}^{[1]} + (s_4 + h)\bar{v}) - s_3\bar{v}\right] u^{[1]} \\
&+ \int_a^b \left[-(u^{[2]})' + q_1(u^{[1]} + s_4 u) - s_3 u\right] \bar{v}^{[1]} - \int_a^b \left[q_1(s_4 + h) - s_3\right] \bar{v}^{[1]}u^{[0]}
\end{aligned}$$

$$+\int_a^b \left[q_1(s_4+h)^2+q_0\right]\bar{v}^{[0]}u^{[0]}+\int_a^b (q_1s_4-s_3)u^{[1]}\bar{v}^{[0]}-\int_a^b (q_1s_4^2+q_0)\bar{v}^{[0]}u^{[0]}$$

$$=-\int_a^b (q_1(s_4+h)-s_3)u^{[1]}\bar{v}^{[0]}+\int_a^b (q_1s_4-s_3)\bar{v}^{[1]}u^{[0]}$$

$$-\int_a^b \left[q_1(s_4+h)-s_3\right]\bar{v}^{[1]}u^{[0]}+\int_a^b \left[q_1(s_4+h)^2+q_0\right]\bar{v}^{[0]}u^{[0]}$$

$$+\int_a^b (q_1s_4-s_3)u^{[1]}\bar{v}^{[0]}-\int_a^b (q_1s_4^2+q_0)\bar{v}^{[0]}u^{[0]}$$

$$=\int_a^b q_1(2hs_4+h^2)\bar{v}^{[0]}u^{[0]}-\int_a^b q_1h\bar{v}^{[1]}u^{[0]}-\int_a^b q_1h\bar{v}^{[0]}u^{[1]},$$

let $h\to 0$, still by Theorem 6, ones have

$$d\lambda_{s_4}(h)=2\int_a^b q_1\left[s_4|u^{[0]}|^2-\mathbf{Re}(\bar{u}^{[1]}u^{[0]})\right]h,\quad h\in L(J,\mathbb{R}).$$

Using the similar method, we can obtain part 4 and part 6. □

Lemma 4. *[16] Assume the function $f\in L_{loc}(a,b')$, then*

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f=f(t)\ a.e.\ b\in(a,b'). \tag{4.3}$$

Theorem 9. *Let the BVP be given as (2.2), (2.5) and (2.6) on [a,b]. Fix the BCs and the endpoint a. Let $\lambda=\lambda(b)$ for $b\in(a,b')$, then*

- *1. $\lambda(b)$ is a continuous function of b for $b\in(a,b')$.*
- *2. If $\lambda(b)$ is simple for some $b\in(a,b')$, then $\lambda(b)$ is simple for every $b\in(a,b')$.*
- *3. There exists a normalized eigenfunction $u(\cdot,b)$ of $\lambda(b)$ for $b\in(a,b')$ such that $u^{[j]}(\cdot,b)$ (j=0,1,2,3) are uniformly convergent in b on any compact subinterval of (a,b'), i.e.,*

$$u^{[j]}(\cdot,b+h)\to u^{[j]}(\cdot,b),\quad j=0,1,2,3\ \ as\ \ h\to 0,$$

and this convergence is uniform on any compact subinterval of (a,b').

Proof. The proof is given for classical second-order Sturm–Liouville case in [16], it can extend readily to our case, only to note that here is the fourth-order case and the quasi-derivatives are different. □

Theorem 10. *Let (2.4) hold, consider the BVP (2.2), (2.7) with $0\le\alpha<\pi$ and $\beta=\pi$. Fix all components of ω except b and let $\lambda=\lambda(b)$ and $u=u(\cdot,b)$ be the eigenvalue and the corresponding real normalized eigenfunction. Then λ is differentiable and*

$$\lambda'(b)=2u^{[1]}(b,b)u^{[3]}(b,b)+q_1(b)(u^{[1]}(b,b))^2\ \ a.e.\ b\in(a,b'). \tag{4.4}$$

Proof. For small h, we choose $\mu=\lambda(b)$, $\nu=\lambda(b+h)$ and $u=u(\cdot,b)$, $v=u(\cdot,b+h)$, from Lemma 1 and the BCs (2.7), noting that $[u,v](a)=0,\ u^{[0]}(b,b)=0,\ u^{[2]}(b,b)=0$, we have

$$\begin{aligned}&[\lambda(b)-\lambda(b+h)]\int_a^b u(r,b)u(r,b+h)w(r)dr=[u,v]_a^b\\&=[u^{[0]}v^{[3]}-u^{[3]}v^{[0]}-u^{[1]}v^{[2]}+u^{[2]}v^{[1]}](b)\\&=-u^{[3]}(b,b)u^{[0]}(b,b+h)-u^{[1]}(b,b)u^{[2]}(b,b+h).\end{aligned} \tag{4.5}$$

On the one hand

$$
\begin{aligned}
u^{[0]}(b,b+h) &= u^{[0]}(b,b+h) - u^{[0]}(b+h,b+h) \\
&= -\int_b^{b+h} u'(r,b+h)dr \\
&= \int_b^{b+h} u^{[1]}(r,b+h)dr \\
&= \int_b^{b+h} u^{[1]}(r,b)dr + \int_b^{b+h} [u^{[1]}(r,b+h) - u^{[1]}(r,b)]dr,
\end{aligned}
$$

and by Lemma 4

$$
\lim_{h\longrightarrow 0} \frac{u^{[0]}(b,b+h)}{h} = u^{[1]}(b,b). \tag{4.6}
$$

On the other hand

$$
\begin{aligned}
&u^{[2]}(b,b+h) \\
=&u^{[2]}(b,b+h) - u^{[2]}(b+h,b+h) \\
=&-\int_b^{b+h} (u^{[2]})'(r,b+h)dr \\
=&-\int_b^{b+h} \left[-u^{[3]}(r,b+h) - s_1(r)u^{[2]}(r,b+h) - q_1(r)u^{[1]}(r,b+h)\right] dr \\
&-\int_b^{b+h} -\left[(q_2 s_1 s_2 - q_1 s_4 + s_3)(r)u^{[0]}(r,b+h)\right] dr \\
=&\int_b^{b+h} u^{[3]}(r,b)dr + \int_b^{b+h} [u^{[3]}(r,b+h) - u^{[3]}(r,b)]dr \\
&+\int_b^{b+h} s_1(r)u^{[2]}(r,b)dr + \int_b^{b+h} s_1(r)[u^{[2]}(r,b+h) - u^{[2]}(r,b)]dr \\
&+\int_b^{b+h} q_1(r)u^{[1]}(r,b)dr + \int_b^{b+h} q_1(r)[u^{[1]}(r,b+h) - u^{[1]}(r,b)]dr \\
&+\int_b^{b+h} (q_2 s_1 s_2 - q_1 s_4 + s_3)(r)u^{[0]}(r,b)dr \\
&-\int_b^{b+h} (q_2 s_1 s_2 - q_1 s_4 + s_3)(r)[u^{[0]}(r,b) - u^{[0]}(r,b+h)]dr,
\end{aligned}
$$

and by Lemma 4

$$
\lim_{h\longrightarrow 0} \frac{u^{[2]}(b,b+h)}{h} = u^{[3]}(b,b) + q_1(b)u^{[1]}(b,b). \tag{4.7}
$$

Observe that

$$
\int_a^b u(r,b)u(r,b+h)w(r)dr \to \int_a^b u^2(r,b)w(r)dr = 1, \text{ as } h \to 0, \tag{4.8}
$$

then putting (4.6)–(4.8) into (4.5) and noting that $h \to 0$, we arrive at (4.4). □

Theorem 11. *Let (2.4) hold, consider the BVP (2.2), (2.7) with $0 \leq \alpha < \pi$ and $\beta = \frac{\pi}{2}$. Fix all components of ω except b and let $\lambda = \lambda(b)$ and $u = u(\cdot, b)$ be the eigenvalue and the corresponding real normalized eigenfunction. Then λ is differentiable and*

$$\lambda'(b) = (q_2 s_2^2 + q_1 s_4^2 + q_0 - \lambda w)(b)(u^{[0]}(b,b))^2 - \frac{1}{q_2(b)}(u^{[2]}(b,b))^2 \quad a.e.\ b \in (a, b'). \tag{4.9}$$

Proof. For small h, we choose $\mu = \lambda(b)$, $\nu = \lambda(b+h)$ and $u = u(\cdot, b)$, $v = u(\cdot, b+h)$, from Lemma 1 and the BCs (2.7), noting that $[u, v](a) = 0$, $u^{[1]}(b, b) = 0$, $u^{[3]}(b, b) = 0$, we have

$$\begin{aligned}
&[\lambda(b) - \lambda(b+h)] \int_a^b u(r,b)u(r,b+h)w(r)dr = [u,v]_a^b \\
&= [u^{[0]}v^{[3]} - u^{[3]}v^{[0]} - u^{[1]}v^{[2]} + u^{[2]}v^{[1]}](b) \\
&= u^{[0]}(b,b)u^{[3]}(b,b+h) + u^{[2]}(b,b)u^{[1]}(b,b+h).
\end{aligned} \tag{4.10}$$

Since

$$\begin{aligned}
&u^{[1]}(b,b+h) \\
=&u^{[1]}(b,b+h) - u^{[1]}(b+h,b+h) \\
=&-\int_b^{b+h} (u^{[1]})'(r,b+h)dr \\
=&-\int_b^{b+h} -u^{(2)}(r,b+h)dr \\
=&-\int_b^{b+h} -\left[\frac{1}{q_2(r)}u^{[2]}(r,b+h) + s_1(r)u^{[1]}(r,b+h) + s_2(r)u^{[0]}(r,b+h)\right]dr \\
=&\int_b^{b+h} \frac{1}{q_2(r)}u^{[2]}(r,b)dr + \int_b^{b+h} \frac{1}{q_2(r)}[u^{[2]}(r,b+h) - u^{[2]}(r,b)]dr \\
&+\int_b^{b+h} s_2(r)u^{[0]}(r,b)dr - \int_b^{b+h} s_2(r)[u^{[0]}(r,b) - u^{[0]}(r,b+h)]dr,
\end{aligned}$$

and by Lemma 4 we have

$$\lim_{h \longrightarrow 0} \frac{u^{[1]}(b,b+h)}{h} = \frac{1}{q_2(b)}u^{[2]}(b,b) + s_2(b)u^{[0]}(b,b). \tag{4.11}$$

Similarly

$$
\begin{aligned}
&u^{[3]}(b, b+h) \\
=&u^{[3]}(b, b+h) - u^{[3]}(b+h, b+h) \\
=&-\int_b^{b+h} (u^{[3]})'(r, b+h)dr \\
=&-\int_b^{b+h} \left[s_2(r)u^{[2]}(r, b+h) - (q_2 s_1 s_2 - s_3 + q_1 s_4)(r)u^{[1]}(r, b+h)\right] dr \\
&-\int_b^{b+h} \left[(q_2 s_2^2 + q_1 s_4^2 + q_0 - \lambda w)(r)u^{[0]}(r, b+h)\right] dr \\
=&-\int_b^{b+h} s_2(r)u^{[2]}(r, b)dr + \int_b^{b+h} s_2(r)[u^{[2]}(r, b) - u^{[2]}(r, b+h)]dr \\
&-\int_b^{b+h} (q_2 s_2^2 + q_1 s_4^2 + q_0 - \lambda w)(r)u^{[0]}(r, b)dr \\
&-\int_b^{b+h} (q_2 s_2^2 + q_1 s_4^2 + q_0 - \lambda w)(r)[u^{[0]}(r, b+h) - u^{[0]}(r, b)]dr,
\end{aligned}
$$

and by Lemma 4 we have

$$
\lim_{h \longrightarrow 0} \frac{u^{[3]}(b, b+h)}{h} = -s_2(b)u^{[2]}(b, b) - (q_2 s_2^2 + q_1 s_4^2 + q_0 - \lambda w)(b)u^{[0]}(b, b). \tag{4.12}
$$

Observe that

$$
\int_a^b u(r, b)u(r, b+h)w(r)dr \to \int_a^b u^2(r, b)w(r)dr = 1, \text{ as } h \to 0, \tag{4.13}
$$

then putting (4.11)–(4.13) into (4.10) and noting that $h \to 0$, we arrive at (4.9). □

Theorem 12. *Let (2.4) hold, consider the BVP (2.2), (2.7) with $0 \le \alpha < \pi$ and $0 < \beta \le \pi$.*

(1) Fix all components of ω except a and let $\lambda = \lambda(a)$ and $u = u(\cdot, a)$ be the eigenvalue and the corresponding real normalized eigenfunction. Then λ is differentiable and

$$
\begin{aligned}
\lambda'(a) =& -(q_2 s_2^2 + q_1 s_4^2 + q_0 - \lambda w)(a)(u^{[0]}(a, a))^2 - q_1(a)(u^{[1]}(a, a))^2 + \frac{1}{q_2(a)}(u^{[2]}(a, a))^2 \\
&- 2u^{[1]}(a, a)u^{[3]}(a, a) \quad a.e.\ a \in (a', b).
\end{aligned} \tag{4.14}
$$

(2) Fix all components of ω except b and let $\lambda = \lambda(b)$ and $u = u(\cdot, b)$ be the eigenvalue and the corresponding real normalized eigenfunction. Then λ is differentiable and

$$
\begin{aligned}
\lambda'(b) =& (q_2 s_2^2 + q_1 s_4^2 + q_0 - \lambda w)(b)(u^{[0]}(b, b))^2 + q_1(b)(u^{[1]}(b, b))^2 - \frac{1}{q_2(b)}(u^{[2]}(b, b))^2 \\
&+ 2u^{[1]}(b, b)u^{[3]}(b, b) \quad a.e.\ b \in (a, b').
\end{aligned} \tag{4.15}
$$

Proof. Here we prove (2), and (1) can be proved similarly. For small h, we choose $\mu = \lambda(b)$, $\nu = \lambda(b+h)$ and $u = u(\cdot, b)$, $v = u(\cdot, b+h)$, respectively. From Lemma 1 and the BCs (2.7), noting that $[u, v](a) = 0$, we have

$$
\begin{aligned}
&[\lambda(b)-\lambda(b+h)]\int_a^b u(r,b)u(r,b+h)w(r)dr\\
=&[u,v]_a^b=[u^{[0]}v^{[3]}-u^{[3]}v^{[0]}-u^{[1]}v^{[2]}+u^{[2]}v^{[1]}](b)\\
=&u^{[0]}(b,b)u^{[3]}(b,b+h)-u^{[3]}(b,b)u^{[0]}(b,b+h)\\
&-u^{[1]}(b,b)u^{[2]}(b,b+h)+u^{[2]}(b,b)u^{[1]}(b,b+h).
\end{aligned}
\tag{4.16}
$$

Now dividing both sides of (4.16) by h, and taking the limit as $h\to 0$. By putting (4.6), (4.7), (4.11) and (4.12) into (4.16), and using the continuity of λ at b, the uniform convergence of $u(\cdot,b+h)$ to $u(\cdot,b)$, then by Theorem 9 we can obtain (4.15). □

Theorem 13. *Let (2.4) hold, consider the BVP (2.2), (2.10) with $-\pi<\theta\le\pi$.*

(1) Fix all components of ω except a and let $\lambda=\lambda(a)$ and $u=u(\cdot,a)$. Then λ is differentiable and

$$
\begin{aligned}
\lambda'(a)=&-(q_2s_2^2+q_1s_4^2+q_0-\lambda w)(a)|u^{[0]}(a,a)|^2-q_1(a)|u^{[1]}(a,a)|^2+\frac{1}{q_2(a)}|u^{[2]}(a,a)|^2\\
&-2\mathbf{Re}[u^{[1]}(a,a)\bar{u}^{[3]}(a,a)]\quad a.e.\ a\in(a',b).
\end{aligned}
\tag{4.17}
$$

(2) Fix all components of ω except b and let $\lambda=\lambda(b)$ and $u=u(\cdot,b)$. Then λ is differentiable and

$$
\begin{aligned}
\lambda'(b)=&(q_2s_2^2+q_1s_4^2+q_0-\lambda w)(b)|u^{[0]}(b,b)|^2+q_1(b)|u^{[1]}(b,b)|^2-\frac{1}{q_2(b)}|u^{[2]}(b,b)|^2\\
&+2\mathbf{Re}[u^{[1]}(b,b)\bar{u}^{[3]}(b,b)]\quad a.e.\ b\in(a,b').
\end{aligned}
\tag{4.18}
$$

Proof. Here we prove (2), and (1) can be proved similarly.

For small h, we choose $\mu=\lambda(b)$, $\nu=\lambda(b+h)$ and $u=u(\cdot,b)$, $v=u(\cdot,b+h)$, from Lemma 1 we have

$$
\begin{aligned}
&[\lambda(b)-\lambda(b+h)]\int_a^b u\bar{v}w\\
&=[u^{[0]}\bar{v}^{[3]}-u^{[3]}\bar{v}^{[0]}-u^{[1]}\bar{v}^{[2]}+u^{[2]}\bar{v}^{[1]}]_a^b\\
&=V^*(b)EU(b)-V^*(a)EU(a)\\
&=V^*(b)EU(b)-V^*(b+h)e^{i\theta}(K^{-1})^*Ee^{-i\theta}K^{-1}U(b)\\
&=V^*(b)EU(b)-V^*(b+h)EU(b)\\
&=\left(\bar{v}^{[3]},-\bar{v}^{[2]},\bar{v}^{[1]},-\bar{v}^{[0]}\right)(b)\begin{pmatrix}u^{[0]}\\u^{[1]}\\u^{[2]}\\u^{[3]}\end{pmatrix}(b)-\left(\bar{v}^{[3]},-\bar{v}^{[2]},\bar{v}^{[1]},-\bar{v}^{[0]}\right)(b+h)\begin{pmatrix}u^{[0]}\\u^{[1]}\\u^{[2]}\\u^{[3]}\end{pmatrix}(b)\\
&=\left[\left(\bar{v}^{[3]},-\bar{v}^{[2]},\bar{v}^{[1]},-\bar{v}^{[0]}\right)(b)-\left(\bar{v}^{[3]},-\bar{v}^{[2]},\bar{v}^{[1]},-\bar{v}^{[0]}\right)(b+h)\right]\begin{pmatrix}u^{[0]}\\u^{[1]}\\u^{[2]}\\u^{[3]}\end{pmatrix}(b).
\end{aligned}
\tag{4.19}
$$

Now dividing both sides of (4.19) by h, and taking the limit as $h\to 0$. By putting (4.6), (4.7), (4.11) and (4.12) into (4.19), and using the continuity of λ at b, and the uniform convergence of $u(\cdot,b+h)$ to $u(\cdot,b)$, then by Theorem 9 we can obtain (4.18). □

Remark 2. *Theorems 10, 11, 12 list the derivative formulaes of eigenvalues with respect to the endpoints for the separated self-adjoint BCs, respectively. Theorem 13 is for complex coupled self-adjoint BCs. For conveniences, in Theorems 10, 11, 12 we choose the real normalized eigenfunctions, while in Theorem 13 we choose the complex normalized eigenfunctions. The conclusions in Theorem 13 are also applicable when the BCs are real coupled self-adjoint case.*

5. Concluding Remarks

In the present paper we consider the fourth-order boundary value problems with distributional potentials. Beside the basic eigenvalue properties of the considered problem, the dependence of eigenvalues of the following fourth-order differential equation

$$\begin{aligned}&\left\{\left[q_2(y^{(2)}-s_1y^{(1)}-s_2y)\right]^{(1)}+q_2s_1(y^{(2)}-s_1y^{(1)})-q_1(y^{(1)}+s_4y)+s_3y\right\}^{(1)}\\&+q_2s_2y^{(2)}-s_3y^{(1)}+q_1s_4(y^{(1)}+s_4y)+q_0y=\lambda wy,\end{aligned}\tag{5.1}$$

defined on the interval $J\subset J'=(a',b')$ are investigated. This is the general fourth-order differential equation with distributional potentials. For equation (5.1), in [29] the authors investigated the deficiency index theory of the operator generated by (5.1). As $s_j\equiv 0, 1\le j\le 4$, and $q_1=0$, equation (5.1) will reduce to

$$(q_2y^{(2)})^{(2)}+q_0y=\lambda wy,\tag{5.2}$$

it is the classical fourth-order Sturm–Liouville equation and the corresponding results of eigenvalue dependence are given in [26]. Compared to [26], the results in this paper are more general. Due to the appearance of the distributional potential functions, the equation becomes more complex, so we adopt new quasi-derivatives to express it. Besides the self-adjointness and some eigenvalue properties of the fourth-order BVPs with distributional potentials, the derivative formulas of several distributional potential functions are given.

To our best knowledge, for higher order boundary value problems with distributional potentials, the corresponding results have not been studied yet. Here, for conveniences, we get the derivative formulas of the coefficient functions under the vanishing conditions $s_1=s_2=0$. In fact, if $s_1\neq 0, s_2\neq 0$ the corresponding derivative formulas still exist. However, due to the complexity we will not consider here and left them to the readers who are interested in.

The eigenvalue problems and eigenvalue dependence problems of differential operators play the important role in mathematics and other fields of sciences. Such problems can be viewed as the theoretical basis of the ordinary differential equations, and give the effective way for numerical computation of eigenvalues of a differential operator. The results here are more general than the previously known results.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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