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*Research article*

# Analysis of a derivative with two variable orders

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**Abstract:** In this paper, we investigate a derivative with the two variable orders. The first one shows the variable order fractal dimension and the second one presents the fractional order. We consider these derivatives with the power law kernel, exponential decay kernel and Mittag-Leffler kernel. We give the theory of this derivative in details. We also present the numerical approximation. The results we obtained in this work are very useful for researchers to improve many things for fractal fractional derivative with two variable orders.

**Keywords:** fractal fractional derivative; two variables order; theory and applications
**Mathematics Subject Classification:** 26A33, 33F05

## 1. Introduction

Very recently due to the complexities of nature and the inaccuracy of existing mathematical model to replicate physical observed problem, a new mathematical concept was introduced and called fractal fractional differential and integral operators. These operators appear to be superior to fractal and fractional operators as when the fractal dimension turns to 1, we recover the fractional differential operators and also when the fractional order turns to 1, one recovers the fractal differential operators. Additionally when the fractional order and fractal dimension are 1 then we recover the classical differential operators. Nevertheless, to further empower the concept of differentiation, Atangana and Anum suggested an extension of the fractal dimension into variable order fractal dimension; this idea was also applied in some problem with great success. While the idea introduced by Atangana and Anum was precious, to modeling real world problem, it become important to recall that fractional differential operator with variable orders are leader in terms of modeling anomalous diffusion and many other important anomalous problems that cannot be described by fractional differential and integral operators with constant orders [1]. Due to the wider applicability of the concept of variable order, one will ask the question to know if the concept of fractal-fractional differential operator could not be extended totally to the concept of variable orders. This is to say the fractional order and the fractal dimension are converted into variable order. This is extended version of to the extension make by Atangana and Anum [1]. As when the fractional variable order is constant, we obtain the operators suggested by Atangana and Anum. Nevertheless, if the fractal dimension is constant that a new operator is obtained. In this paper, we will present generalization of the fractal-fractional differential operators with three different kernels to the concept of variable order fractal-fractional differential operators. We have to admit that such operators may not have a corresponding integral [2–10]. For more details see [11–19].

## 2. Main results

We give the following definitions for real variables.

**Definition 2.1.** *Let $u$ be a differentiable function. Let $\gamma(z)$ and $\theta(z)$ be two continuous functions such that $0 < \gamma(z) < 1$. We define a fractal fractional derivative with fractional of variable order $\gamma(z)$ and variable fractal order $\theta(z)$ as:*

$$ {}_0^{C}D_z^{\gamma(z),\theta(z)}u(z) = \frac{1}{\Gamma(1-\gamma(z))}\int_0^z \frac{du(p)}{dp^{\theta(p)}}(z-p)^{-\gamma(p)}dp. \tag{2.1}$$

$$ {}_0^{R}D_z^{\gamma(z),\theta(z)}u(z) = \frac{1}{\Gamma(1-\gamma(z))}\frac{d}{dz^{\theta(z)}}\int_0^z u(p)(z-p)^{-\gamma(p)}dp. \tag{2.2}$$

**Definition 2.2.** *Let $u$ be a differentiable function. Let $\gamma(z)$ and $\theta(z)$ be two continuous functions such that $0 < \gamma(z) < 1$. We define a fractal fractional derivative with fractional of variable order $\gamma(z)$ and variable fractal order $\theta(z)$ with exponential decay as:*

$$ {}_0^{CE}D_z^{\gamma(z),\theta(z)}u(z) = \frac{M(\gamma(z))}{1-\gamma(z)}\int_0^z \frac{du(p)}{dp^{\theta(p)}}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)\right)dp. \tag{2.3}$$

$$ {}_0^{RE}D_z^{\gamma(z),\theta(z)}u(z) = \frac{M(\gamma(z))}{1-\gamma(z)}\frac{d}{dz^{\theta(z)}}\int_0^z u(p)\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)\right)dp. \tag{2.4}$$

**Definition 2.3.** *Let $u$ be a differentiable function. Let $\gamma(z)$ and $\theta(z)$ be two continuous functions such that $0 < \gamma(z) < 1$. We define a fractal fractional derivative with fractional of variable order $\gamma(z)$ and variable fractal order $\theta(z)$ with the generalized Mittag-Leffler kernel as:*

$$ {}_0^{CM}D_z^{\gamma(z),\theta(z)}u(z) = \frac{AB(\gamma(z))}{1-\gamma(z)}\int_0^z \frac{du(p)}{dp^{\theta(p)}}E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp. \tag{2.5}$$

$$ {}_0^{RM}D_z^{\gamma(z),\theta(z)}u(z) = \frac{AB(\gamma(z))}{1-\gamma(z)}\frac{d}{dz^{\theta(z)}}\int_0^z u(u)E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp. \tag{2.6}$$

We present some properties of the new differential operators.

**Corollary 1.** *Let $\gamma(z)$ and $\theta(z)$ be two continuous and bounded functions. Let $u(z)$ be continuous such that*

$$\begin{aligned}
\left|{}_{0}^{C}D_{z}^{\gamma(z),\theta(z)}u(z)\right| &= \left|\frac{1}{\Gamma(1-\gamma(z))}\int_{0}^{z}\frac{du(p)}{dp^{\theta(z)}}(z-p)^{-\gamma(p)}dp\right| \\
&\leq \left|\frac{1}{\Gamma(1-\gamma(z))}\right|\int_{0}^{z}\left|\frac{du(p)}{dp^{\theta(p)}}(z-p)^{-\gamma(p)}\right|dp \\
&\leq \left|\frac{1}{\Gamma(1-\gamma(z))}\right|\int_{0}^{z}\left|\frac{du(p)}{dp}(z-p)^{-\gamma(p)}\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right|dp \\
&< \sup_{z\in[0,Z]}\left|\frac{1}{\Gamma(1-\gamma(z))}\right|\int_{0}^{z}\sup_{l\in[0,p]}\left|\frac{du(l)}{dl}\right| \\
&\quad\times\sup_{l\in[0,p]}\left|\frac{l^{-\theta(l)}}{\theta'(l)\ln(l)+\frac{\beta(l)}{l}}\right|(z-p)^{-\gamma(p)}dp \\
&< M_1M_2M_3\int_{0}^{t}(t-\tau)^{-\alpha(\tau)}d\tau
\end{aligned}$$

*where*

$$M_1 = \sup_{z\in[0,Z]}\left|\frac{1}{\Gamma(1-\gamma(z))}\right| \tag{2.7}$$

$$M_2 = \sup_{z\in[0,Z]}\left|\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{z}}\right| \tag{2.8}$$

$$M_3 = \sup_{z\in[0,Z]}\left|\frac{du(z)}{dz}\right| \tag{2.9}$$

*Nevertheless, since $\gamma(z)$ is continuously bounded then there exists $\xi \in [0, Z]$. $\forall z \in [0, Z]$ $\gamma(\xi) > M$ in this case*

$$\int_{0}^{z}(z-p)^{-\gamma(p)}dp < \int_{0}^{z}(z-p)^{-M}dp < \frac{z^{1-M}}{1-M}. \tag{2.10}$$

*Thus, we obtain*

$$\left|{}_{0}^{C}D_{z}^{\gamma(z),\theta(z)}u(z)\right| < M_1M_2M_3\frac{z^{1-M}}{1-M}.$$

Also

$$\left|{}_{0}^{R}D_{z}^{\gamma(z),\theta(z)}u(z)\right| = \left|\frac{1}{\Gamma(1-\gamma(z))}\frac{d}{dz^{\theta(z)}}\int_{0}^{z}u(p)(z-p)^{-\gamma(p)}dp\right| \tag{2.11}$$

Using the fact that the integral is differentiable then the above can be reformulated as:

$$\begin{aligned}
\left|{}_0^R D_z^{\gamma(z),\theta(z)} u(z)\right| &= \left|\frac{1}{\Gamma(1-\gamma(z))}\frac{d}{dz^{\theta(z)}}\int_0^z u(p)(z-p)^{-\gamma(p)}dp\right| \\
&\leq \left|\frac{1}{\Gamma(1-\gamma(z))}\frac{d}{dz}\int_0^z u(p)(z-p)^{\gamma(p)}\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{z}}dp\right| \\
&\leq \left|\frac{1}{\Gamma(1-\gamma(z))}\right|\left|\frac{d}{dz}\int_0^z u(p)(z-p)^{-\gamma(p)}dp\right|\left|\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{z}}\right| \\
&< \sup_{z\in[0,Z]}\left|\frac{1}{\Gamma(1-\gamma(z))}\right|\left|\sup_{z\in[0,Z]}\right|\left|\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{z}}\right| \\
&\quad\times\left|\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{-\alpha(\tau)}d\tau\right|
\end{aligned}$$

Noting that the function $\gamma(z)$ is continuous and bounded then we can find $\xi \in [0, Z]$ such that

$$\left|\frac{d}{dz}\int_0^z u(p)(z-p)^{-\alpha(p)}dp\right| < \left|\frac{d}{dz}\int_0^z u(p)(z-p)^{-M}dp\right|$$

Nevertheless, we have that

$$\begin{aligned}
\left|\frac{d}{dz}\int_0^z u(p)(z-p)^{-M}dp\right| &< \left|u(z)+\int_0^z \frac{d}{dp}u(p)(z-p)^{-M}dp\right| \\
&< |u(z)|+\left|\int_0^z \frac{d}{dp}u(p)(z-p)^{-M}dp\right| \\
&< \sup_{z\in[0,Z]}|u(z)|+\left|\int_0^z \frac{d}{dp}u(p)(z-p)^{-M}dp\right| \\
&< \|u\|_\infty+\int_0^z\left|\frac{d}{dp}u(p)\right|(z-p)^{-M}dp \\
&< \|u\|_\infty+\int_0^z \sup_{z\in[0,Z]}\left|\frac{d}{dp}u(p)\right|(z-p)^{-M}dp \\
&< \|u\|_\infty+\left\|\frac{du}{dz}\right\|_\infty\int_0^z (z-p)^{-M}dp \\
&< \|u\|_\infty+\left\|\frac{d}{dz}\right\|_\infty\frac{z^{1-M}}{1-M}
\end{aligned}$$

Thus putting all together, we have

$$\left|{}_0^R D_z^{\gamma(z),\theta(z)} u(z)\right| < M_1 M_2\left(\|u\|_\infty+\left\|\frac{du}{dz}\right\|_\infty\frac{z^{1-M}}{1-M}\right)$$

We consider the case when the kernel is the exponential decay law.

$$
\begin{aligned}
\left|{}_0^{CE}D_z^{\gamma(z),\theta(z)}u(z)\right| &= \left|\frac{M(\gamma(z))}{1-\gamma(z)}\int_0^z \frac{du(p)}{dp^{\theta(p)}}\exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)\right)dp\right| \\
&\leq \left|\frac{M(\gamma(z))}{1-\gamma(z)}\right|\left|\int_0^z \frac{du(p)}{dp}\exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\right| \\
&< \sup_{z\in[0,Z]}\left|\frac{M(\gamma(z))}{1-\gamma(z)}\right|\int_0^z\left|\frac{du(p)}{dp}\right|\exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)\right) \\
&\quad \times\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
&< M_1\int_0^z \sup_{z\in[0,Z]}\left|\frac{du(p)}{dp}\right|\exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
&< M_1\int_0^z\left\|\frac{du(p)}{dp}\right\|_\infty \exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
&< M_1\left\|\frac{du(z)}{dz}\right\|_\infty\int_0^z \exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(t-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp
\end{aligned}
$$

Since $\gamma(z)$ is continuously bounded there exists $\xi \in [0, Z]$ such that

$$
\begin{aligned}
&\left|\int_0^z \exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\right| \\
<\ &\left|\int_0^z \exp\left(-\frac{\gamma}{1-\gamma}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\right|
\end{aligned}
$$

Now

$$
\int_0^z \exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp
$$

Since $\gamma(p)$ is continuous and bounded there exists $\xi \in [0, Z]$ such that $\gamma(\xi) = M$ and

$$
\begin{aligned}
&\left|\int_0^z \exp\left(-\frac{M}{1-M}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\right| \\
>\ &\left|\int_0^z \exp\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\right|
\end{aligned}
$$

Nevertheless

$$
\begin{aligned}
&\left|\int_0^z \exp\Big(-\frac{M}{1-M}(z-p)\Big)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\right| \\
<&\int_0^z \left|\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right|\exp\Big(-\frac{M}{1-M}(t-p)\Big)dp \\
<&\int_0^z \sup_{p\in[0,Z]}\left|\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right|\exp\Big(-\frac{M}{1-M}(z-p)\Big)dp \\
<&\sup_{p\in[0,Z]}\left|\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{z}}\right|\int_0^z \exp\Big(-\frac{M}{1-M}(z-p)\Big)dp \\
<&\sup_{z\in[0,Z]}\left|\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{z}}\right|\int_0^z dp \\
<&\sup_{z\in[0,Z]}\left|\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{z}}\right|z
\end{aligned}
$$

So replacing the above in the formulas, we obtain

$$
\left|{}_0^{CE}D_z^{\gamma(z),\theta(z)}u(z)\right| \;<\; M_1\left\|\frac{du(z)}{dz}\right\|_\infty M_2 z
$$

Now we consider the case when the kernel is the generalized Mittag-Leffler function.

$$
\begin{aligned}
\left|{}_0^{CM}D_z^{\gamma(z),\theta(z)}u(z)\right| &= \left|\frac{AB(\gamma(z))}{1-\gamma(z)}\int_0^z \frac{du(p)}{dp^{\theta(p)}}E_{\gamma(p)}\Big(-\frac{\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\Big)dp\right| \\
&\le \left|\frac{AB(\gamma(z))}{1-\gamma(z)}\right|\int_0^t \left|\frac{du(p)}{dp^{\theta(p)}}\right|E_{\gamma(p)}\Big(-\frac{\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\Big)dp \\
&< \sup_{z\in[0,Z]}\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\right|\int_0^z \left|\frac{du(p)}{dp}\right|\left|\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right| \\
&\quad \times E_{\gamma(p)}\Big(-\frac{\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\Big)dp \\
&< \sup_{z\in[0,Z]}\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\right|\int_0^z \sup_{z\in[0,Z]}\left|\frac{du(p)}{dp}\right|\sup_{z\in[0,Z]}\left|\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right| \\
&\quad \times E_{\gamma(p)}\Big(-\frac{\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\Big)dp \\
&< \sup_{z\in[0,Z]}\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\right|\sup_{z\in[0,Z]}\left|\frac{du(p)}{dp}\right|\sup_{z\in[0,Z]}\left|\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right|
\end{aligned}
$$

$$\times \int_0^z E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp$$

Since the function $\gamma(z)$ is continuous and bounded there exists $\xi \in [0, Z]$ such that $\forall p \in [0, Z]$,

$$E_{\gamma(\xi)}\left(-\frac{\gamma(\xi)}{1-\gamma(\xi)}(z-\xi)^{\gamma(\xi)}\right) \geq E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right). \tag{2.12}$$

Therefore, we have the following inequality

$$\begin{aligned}\left|{}_0^{CM}D_z^{\gamma(z),\theta(z)}u(z)\right| &< \sup_{z\in[0,Z]}\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\right| \left\|\frac{du(p)}{dp}\right\|_\infty \sup_{z\in[0,Z]}\left|\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{z}}\right| \\ &\quad \times \int_0^z E_{\gamma(\xi)}\left(-\frac{\gamma(\xi)}{1-\gamma(\xi)}(z-p)^{\gamma(\xi)}\right)dp\end{aligned}$$

Let $\theta_1 = \gamma(\xi)$ then we have

$$\int_0^z E_{\theta_1}\left(-\frac{\theta_1}{1-\theta_1}(z-p)^{\theta_1}\right)dp = \int_0^z E_{\gamma(\xi)}\left(-\frac{\gamma(\xi)}{1-\gamma(\xi)}(z-p)^{\gamma(\xi)}\right)dp. \tag{2.13}$$

By putting $\lambda = \frac{\theta_1}{1-\theta_1}$, we obtain

$$\begin{aligned}\int_0^z E_{\theta_1}\left(-\lambda(z-p)^{\theta_1}\right)dp &= \int_0^t \sum_{k=0}^{\infty}\frac{\left(-\lambda(z-p)^{\theta_1}\right)^k}{\Gamma(\theta_1 k+1)}dp \\ &\leq \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{\Gamma(\theta_1 k+1)}\int_0^t (z-p)^{\theta_1 k}\,dp \\ &\leq \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{\Gamma(\theta_1 k+1)}\int_0^1 (z-xh)^{\theta_1 k}\,tdh \\ &\leq \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{\Gamma(\theta_1 k+1)}\int_0^1 t^{\theta_1 k+1}(1-h)^{\theta_1 k}\,dh \\ &\leq z\sum_{k=0}^{\infty}\frac{(-\lambda t^{\theta_1})^k}{\Gamma(\theta_1 k+1)}\int_0^1 h^{1-1}(1-h)^{\theta_1 k+1-1}\,dh \\ &\leq z\sum_{k=0}^{\infty}\frac{(-\lambda t^{\theta_1})^k}{\Gamma(\theta_1 k+1)}B(1,\theta_1 k+1) \\ &\leq z\sum_{k=0}^{\infty}\frac{(-\lambda t^{\theta_1})^k}{\Gamma(\theta_1 k+1)}\frac{\Gamma(1)\Gamma(\theta_1 k+1)}{\Gamma(\theta_1 k+2)} \\ &\leq zE_{\theta_1,2}\left(-\lambda t^{\theta_1}\right).\end{aligned}$$

Replacing in the inequality, we get

$$\left|{}_0^{CM}D_z^{\gamma(z),\theta(z)}u(z)\right| < \sup_{z\in[0,Z]}\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\right| \left\|\frac{du(p)}{dp}\right\|_\infty \sup_{z\in[0,Z]}\left|\frac{z^{-\theta(z)}}{\theta'(z)\ln(z)+\frac{\theta(z)}{x}}\right| \times xE_{\gamma(\xi),2}\left(-\frac{\gamma(\xi)}{1-\gamma(\xi)}z^{\gamma(\xi)}\right)$$

We consider two continuous functions $f$ and $g$. Let $\gamma(z)$ and $\theta(z)$ be two continuous bounded functions. We aim to evaluate

$$\begin{aligned}
&\left|{}_0^{CM}D_z^{\gamma(z),\theta(z)}u(z) -{}_0^{CM} D_z^{\gamma(z),\theta(z)}g(z)\right| \\
= &\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\int_0^z \frac{du(p)}{dp^{\theta(p)}}E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp\right. \\
&\left.-\frac{AB(\gamma(z))}{1-\gamma(z)}\int_0^z \frac{dg(p)}{dp^{\theta(p)}}E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp\right| \\
\leq &\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\int_0^z \left(\frac{du(p)}{dp}-\frac{dg(p)}{dp}\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right. \\
&\left.\times E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp\right| \\
< &\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\right|\int_0^z \left(\frac{d}{dp}(u(p)-g(p)\right)\left|\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right| \\
&\times E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp \\
< &\sup_{z\in[0,Z]}\left|\frac{AB(\gamma(z))}{1-\gamma(z)}\right|\int_0^z \sup_{p\in[0,Z]}\left|\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\right| \\
&\times\left(\frac{d}{dp}(u(p)-g(p))\right)E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp \\
< &M_1M_2\int_0^z \left|\frac{d}{dp}(u(p)-g(p))\right|E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp \\
< &M_1M_2\int_0^z \sup_{p\in[0,Z]}\left|\frac{d}{dp}(u(p)-g(p))\right| \\
&\times E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp \\
< &M_1M_2\left\|\frac{d}{dz}(u(z)-g(z))\right\|_\infty\int_0^z E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\right)dp.
\end{aligned}$$

As presented earlier

$$\int_0^z E_{\gamma(p)}\left(-\frac{\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\right)dp = xE_{\gamma(\xi),2}\left(-\frac{\gamma(\xi)}{1-\gamma(\xi)}t^{\gamma(\xi)}\right). \tag{2.14}$$

Thus, we have

$$\begin{aligned}\left|{}_{0}^{CM}D_{z}^{\gamma(t),\theta(t)}u(z)-{}_{0}^{CM}D_{z}^{\gamma(t),\theta(t)}g(z)\right| &< M_{1}M_{2}t\left\|\frac{d}{dt}(u(z)-g(z))\right\|_{\infty}\\ &\quad\times E_{\gamma(\xi),2}\left(-\frac{\gamma(\xi)}{1-\gamma(\xi)}t^{\gamma(\xi)}\right).\end{aligned}$$

**Theorem 2.4.** *Let u and g be continuous functions. Let $\gamma(z)$ and $\theta(z)$ be two continuous functions such that* $0<\gamma(z),\theta(z)<1$. *If*

$$\left\|\frac{d}{dz}(u-g)\right\|_{\infty}<\theta\left\|u-g\right\|_{\infty} \tag{2.15}$$

*then*

$$\begin{aligned}\left|{}_{0}^{CM}D_{z}^{\gamma(z),\theta(z)}u(z)-{}_{0}^{CM}D_{z}^{\gamma(z),\theta(z)}g(z)\right| &< M_{1}M_{2}\theta\left\|u-g\right\|_{\infty}xE_{\gamma(\xi),2}\left(-\frac{\gamma(\xi)}{1-\gamma(\xi)}z^{\gamma(\xi)}\right)\\ &< K\left\|u-g\right\|_{\infty}.\end{aligned}$$

The proof is directly obtained from the above derivation.

**Corollary 2.** *Let u and g be two continuous bounded functions. Let* $0<\gamma(z),\theta(z)<1$. *Then, we have*

$${}_{0}^{C}D_{z}^{\gamma(z),\theta(z)}\left(u(z)g(z)\right)=\theta_{1}\left({}_{0}^{C}D_{z}^{\gamma(z)}u(z)\right)+\theta_{2}\left({}_{0}^{C}D_{z}^{\gamma(z)}g(z)\right).$$

*Proof.* Since $u$ and $g$ are continuous there exist $M_u$ and $M_g$ such that $\|u\|_{\infty}=M_u$ and $\|g\|_{\infty}=M_g$. Thus, we obtain

$$\begin{aligned}&{}_{0}^{C}D_{z}^{\gamma(z),\theta(z)}\left(u(z)g(z)\right)\\ =\;&\frac{1}{\Gamma(1-\gamma(t))}\int_{0}^{z}\frac{d(u(p)g(p))}{dp^{\theta(p)}}(z-p)^{-\gamma(p)}dp\\ \leq\;&\frac{1}{\Gamma(1-\gamma(z))}\int_{0}^{z}\frac{d(u(p)g(p))}{dp}(z-p)^{-\gamma(p)}\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\\ \leq\;&\frac{1}{\Gamma(1-\gamma(z))}\int_{0}^{z}\left(g(p)\frac{du(p)}{dp}+u(p)\frac{dg(p)}{dp}\right)(z-p)^{-\gamma(p)}\\ &\times\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\\ \leq\;&\|g\|_{\infty}\frac{1}{\Gamma(1-\gamma(z))}\int_{0}^{z}\frac{du(p)}{dp}(z-p)^{-\gamma(p)}\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\\ &+\|u\|_{\infty}\frac{1}{\Gamma(1-\gamma(z))}\int_{0}^{z}\frac{dg(p)}{dp}(z-p)^{-\gamma(p)}\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp\end{aligned}$$

$$\begin{aligned}
&\leq \|g\|_\infty \frac{1}{\Gamma(1-\gamma(z))}\int_0^z \frac{du(p)}{dp}(z-p)^{-\gamma(p)}dp \\
&\quad + \|u\|_\infty \frac{1}{\Gamma(1-\gamma(z))}\int_0^z \frac{dg(p)}{dp}(z-p)^{-\gamma(p)}dp \\
&< M_g\left({}_0^C D_z^{\gamma(t)}u(z)\right) + M_u\left({}_0^C D_z^{\gamma(z)}g(z)\right).
\end{aligned}$$

This completes the proof. □

**Corollary 3.** *Let u and g be two continuous bounded functions. Let* $0 < \gamma(z), \theta(z) < 1$. *Then, we have*

$${}_0^{CE}D_z^{\gamma(z),\theta(z)}\left(u(z)g(z)\right) = \theta_1\left({}_0^{CE}D_z^{\gamma(z)}u(z)\right) + \theta_2\left({}_0^{CE}D_z^{\gamma(t)}g(z)\right).$$

*Proof.* Since $u$ and $g$ are continuous there exist $M_u$ and $M_g$ such that $\|u\|_\infty = M_u$ and $\|g\|_\infty = M_g$. Thus, we obtain

$$\begin{aligned}
&{}_0^{CE}D_z^{\gamma(z),\theta(z)}\left(u(z)g(z)\right) \\
&= \frac{M(\gamma)}{1-\gamma(z)}\int_0^z \frac{d(u(p)g(p))}{dp^{\theta(p)}}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)\right)dp \\
&\leq \frac{M(\gamma)}{1-\gamma(z)}\int_0^z \frac{d(u(p)g(p))}{dp}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(t-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
&\leq \frac{M(\gamma)}{1-\gamma(z)}\int_0^z \left(g(p)\frac{du(p)}{dp} + u(p)\frac{dg(p)}{dp}\right)\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)\right) \\
&\quad \times \frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
&\leq \|g\|_\infty \frac{M(\gamma)}{1-\gamma(z)}\int_0^z \frac{du(p)}{dp}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
&\quad + \|u\|_\infty \frac{M(\gamma)}{1-\gamma(z)}\int_0^z \frac{dg(p)}{dp}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
&\leq \|g\|_\infty \frac{M(\gamma)}{1-\gamma(z)}\int_0^z \frac{du(p)}{dp}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)\right)dp \\
&\quad + \|u\|_\infty \frac{M(\gamma)}{1-\gamma(z)}\int_0^z \frac{dg(p)}{dp}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)\right)dp \\
&< M_g\left({}_0^{CE}D_z^{\gamma(z)}u(z)\right) + M_u\left({}_0^{CE}D_z^{\gamma(z)}g(z)\right).
\end{aligned}$$

This completes the proof. □

**Corollary 4.** *Let u and g be two continuous bounded functions. Let* $0 < \gamma(z), \theta(z) < 1$. *Then, we have*

$${}_0^{CM}D_z^{\gamma(z),\theta(z)}\left(u(z)g(z)\right) = \theta_1\left({}_0^{CM}D_z^{\gamma(z)}u(z)\right) + \theta_2\left({}_0^{CM}D_z^{\gamma(z)}g(z)\right).$$

*Proof.* Since $u$ and $g$ are continuous there exist $M_u$ and $M_g$ such that $\|u\|_\infty = M_f$ and $\|g\|_\infty = M_g$. Thus, we obtain

$$
\begin{aligned}
& {}_{0}^{CM}D_z^{\gamma(z),\theta(z)}\,(u(z)g(z)) \\
= \; & \frac{AB(\gamma)}{1-\gamma(z)}\int_0^z \frac{d(u(p)g(p))}{dp^{\theta(p)}} E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\right)dp \\
\leq \; & \frac{AB(\gamma)}{1-\gamma(z)}\int_0^z \frac{d(u(p)g(p))}{dp} E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
\leq \; & \frac{AB(\gamma)}{1-\gamma(z)}\int_0^z \left(g(p)\frac{du(p)}{dp}+u(p)\frac{dg(p)}{dp}\right) E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right) \\
& \times \frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
\leq \; & \|g\|_\infty \frac{AB(\gamma)}{1-\gamma(t)}\int_0^z \frac{du(p)}{dp} E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
& + \|u\|_\infty \frac{M(\gamma)}{1-\gamma(z)}\int_0^z \frac{dg(p)}{dp} E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}dp \\
\leq \; & \|g\|_\infty \frac{AB(\gamma)}{1-\gamma(z)}\int_0^z \frac{du(p)}{dp} E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\right)dp \\
& + \|u\|_\infty \frac{AB(\gamma)}{1-\gamma(z)}\int_0^t \frac{dg(p)}{dp} E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(z-p)^{\gamma(p)}\right)dp \\
< \; & M_g\left({}_{0}^{CM}D_z^{\gamma(t)}u(z)\right) + M_u\left({}_{0}^{CM}D_z^{\gamma(z)}g(z)\right).
\end{aligned}
$$

This completes the proof. □

Let us consider the following general Cauchy problem where the derivative is given as:

$$
\begin{cases}
{}_{0}^{CM}D_z^{\gamma(z),\theta(z)}u(z) = f(z,u(z)), & 0 < z \\
y(0) = y_0
\end{cases}
$$

We aim to prove that the above equation has a unique solution under certain conditions. To achieve the conditions of existence it is important to note that there exists $\xi \in [0, Z]$ such that $\forall \xi \in [0, Z]$

$$
{}_{0}^{CM}D_z^{\gamma(\xi),\theta(\xi)}u(z) \geq {}_{0}^{CM}\, D_z^{\gamma(z),\theta(z)}u(z). \tag{2.16}
$$

For this point, there exist two positive numbers $\gamma_1$ and $\theta_1$ such that $\gamma(\xi) = \gamma_1$ and $\theta(\xi) = \theta_1$. Thus, we obtain

$$
{}_{0}^{CM}D_z^{\gamma(\xi),\theta(\xi)}u(z) \leq {}_{0}^{CM}\, D_z^{\gamma_1,\theta_1}u(z). \tag{2.17}
$$

that is to say

$$
f(z,u(z)) \leq {}_{0}^{CM}\, D_z^{\gamma_1,\theta_1}u(z). \tag{2.18}
$$

Applying the corresponding integral on both sides, we get

$$\frac{\theta_1}{\Gamma(\gamma_1)}\int_0^z p^{\theta_1-1}(z-p)^{\gamma_1-1}f(y,u(p))dp \geq u(z)-u(0). \tag{2.19}$$

So that

$$u(z)-u(0) \leq \frac{\theta_1}{\Gamma(\gamma_1)}\int_0^z p^{\theta_1-1}(z-p)^{\gamma_1-1}f(y,u(p))dp. \tag{2.20}$$

In this case, we can set a compact closed set of real numbers. But also we consider the following interval for time $[\Omega,\gamma]$. So $\sqcap_{\Omega,\gamma} = \overline{C_\Omega(z_0)} \times \overline{\wedge_\gamma(y(0))}$. We have defined $\overline{C_\Omega(z_0)} = [x_0-\Omega, x_0+\Omega]$ and $\overline{\wedge_\gamma(y(0))} = [y(0)-\gamma, y(0)+\gamma]$.

The real compact cylinder $\sqcap_{\Omega,\gamma}$ will be the interval within which the function $f(z,u(z))$ is defined. We assume that the function $f(z,u(z))$ is bounded within the compact cylinder $\sqcap_{\Omega,\gamma}$. Then we can find a positive number $\xi$ such that $\xi = \sup_{\sqcap_{\Omega,\gamma}}|f|$. To make the proof simple, we assume that the function $f(z,u(z))$ Lipschitz with respect to the second part $u(z)$. That is to say, there exists a real positive number $K$ such that $\forall u_1,u_2 \in \overline{\wedge_\gamma(u(0))}$,

$$\|f(z,u_1(z))-f(z,u_2(z))\|_\infty < K\|u_1-u_2\|_\infty. \tag{2.21}$$

Of course we have

$$\|u\|_\infty = \sup|u(z)|. \tag{2.22}$$

Between the functional space of continuous function, we defined a Picard's operator $\varphi_{\gamma,\theta} = \sqcap_{\Omega,\gamma} \rightarrow \sqcap_{\Omega,\gamma}$.

$$\varphi_{\gamma,\theta}\psi \leq u_0 + \int_0^z \frac{\theta_1 p^{\theta_1-1}}{\Gamma(\gamma_1)}(z-p)^{\gamma_1-1}f(y,\psi(p))dy. \tag{2.23}$$

We must prove that the defined map is complete.

$$\begin{aligned}
\|\varphi_{\gamma,\theta}\psi - u_0\|_\infty &\leq \left\|\frac{\theta_1}{\Gamma(\gamma_1)}\int_0^z p^{\theta_1-1}(z-p)^{\gamma_1-1}f(y,\psi(p))dy\right\|_\infty \\
&\leq \frac{\theta_1}{\Gamma(\gamma_1)}\int_0^z p^{\theta_1-1}(t-p)^{\gamma_1-1}\|f(y,\psi(p))\|_\infty dy \\
&< \frac{\theta_1}{\Gamma(\gamma_1)}\int_0^t p^{\theta_1-1}(t-p)^{\gamma_1-1}\sup_{y\in[0,t]}|f(y,\psi(p))|dy \\
&< \frac{\xi\theta_1}{\Gamma(\gamma_1)}\int_0^z p^{\theta_1-1}(z-p)^{\gamma_1-1}dy \\
&< \frac{\xi\theta_1}{\Gamma(\gamma_1)}\int_0^1 (zh)^{\theta_1-1}(z-xh)^{\gamma_1-1}xdh \\
&< \frac{\xi z^{\theta_1+\gamma_1-1}\theta_1}{\Gamma(\gamma_1)}\int_0^1 h^{\theta_1-1}(1-h)^{\gamma_1-1}dh \\
&< \frac{\xi z^{\theta_1+\gamma_1-1}\theta_1}{\Gamma(\gamma_1)}B(\theta_1,\gamma_1)
\end{aligned}$$

Also given two functions $\psi_1$ and $\psi_2$ we want to show that

$$||\varphi_{\gamma_1,\theta_1}\psi_1 - \varphi_{\gamma_1,\theta_1}\psi_2||_\infty \leq L||\psi_1 - \psi_2||_\infty \tag{2.24}$$

where $L < 1$. So let $x$ such that

$$||\varphi_{\gamma_1,\theta_1}\psi_1 - \varphi_{\gamma_1,\theta_1}\psi_2||_\infty = ||\left(\varphi_{\gamma_1,\theta_1}\psi_1 - \varphi_{\gamma,\theta}\psi_2\right)(z)||_\infty \tag{2.25}$$

Therefore from the definition, we have

$$\begin{aligned}
||\varphi_{\gamma_1,\theta_1}\psi_1 - \varphi_{\gamma_1,\theta_1}\psi_2||_\infty &= \frac{\theta_1}{\Gamma(\gamma_1)}\left\|\int_0^z p^{\theta_1-1}(z-p)^{\gamma_1-1}\left(f(y,\psi_1(p))\right.\right. \\
&\quad \left. -f(y,\psi_2(p))\right) dy||_\infty \\
&\leq \frac{\theta_1}{\Gamma(\gamma_1)}\int_0^z p^{\theta_1-1}(t-p)^{\gamma_1-1}||f(y,\psi_1(p)) \\
&\quad -f(y,\psi_2(p))||_\infty dy \\
&\leq \frac{\theta_1}{\Gamma(\gamma_1)}\int_0^z p^{\theta_1-1}(z-p)^{\gamma_1-1}L||\psi_1-\psi_2||_\infty dy \\
&\leq L||\psi_1-\psi_2||_\infty \frac{\theta_1}{\Gamma(\gamma_1)}\int_0^z p^{\theta_1-1}(z-p)^{\gamma_1-1}dy \\
&\leq z^{\theta_1+\gamma_1-1}L||\psi_1-\psi_2||_\infty \frac{\theta_1}{\Gamma(\gamma_1)}B(\gamma_1,\theta_1)
\end{aligned}$$

Therefore, we obtain

$$||\varphi_{\gamma_1,\theta_1}\psi_1 - \varphi_{\gamma_1,\theta_1}\psi_2||_\infty \quad < \quad L_1||\psi_1 - \psi_2||_\infty$$

where

$$L = z^{\theta_1+\gamma_1-1}L\frac{\theta_1}{\Gamma(\gamma_1)}B(\gamma_1,\theta_1).$$

Then, we have

$$L_1 < 1 \implies L < \frac{\Gamma(\gamma_1)}{\theta_1 B(\gamma_1,\theta_1)t^{\theta_1+\gamma_1-1}}.$$

In conclusion, the contraction is reached if and only if

$$\left\{\frac{\xi z^{\theta_1+\gamma_1-1}\theta_1}{\Gamma(\gamma_1)}B(\theta_1,\gamma_1) < \gamma, L < \frac{\Gamma(\gamma_1)}{\theta_1 B(\gamma_1,\theta_1)t^{\theta_1+\gamma_1-1}}\right\}$$

That is to say

$$\Omega < \min\left\{\left(\frac{\gamma_1\Gamma(\gamma_1)}{\theta_1\xi B(\gamma_1,\theta_1)}\right)^{\frac{1}{\theta_1+\gamma_1-1}}, \left(\frac{\Gamma(\gamma_1)}{B(\gamma_1,\theta_1)L\theta_1}\right)^{\frac{1}{\theta_1+\gamma_1-1}}\right\}$$

Within the above inequality the defined operator be said Lipschitz.

## 3. Numerical approximation

In this section, we derive the numerical approximation of the defined operators. We consider first the case with exponential decay kernel.

$$ {}_{0}^{CE}D_{t}^{\gamma(t),\theta(t)}u(z) = \frac{M(\gamma(t))}{1-\gamma(t)}\int_{0}^{t}\frac{du(p)}{dp^{\theta(p)}}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(t-p)\right)dp. $$

We consider the above when $t = t_{n+1}$, then the right hand side will be

$$ \begin{aligned} & \frac{M(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\int_{0}^{t_{n+1}}\frac{du(p)}{dp}\frac{p^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}\exp\left(\frac{-\gamma(p)}{1-\gamma(p)}(t_{n+1}-p)\right)dp \\ = & \frac{M(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{f^{j+1}-f^{j}}{\Delta t}\frac{t_j^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} \\ & \times\exp\left(\frac{-\gamma(t_j)}{1-\gamma(t_j)}(t_{n+1}-p)\right)dp \end{aligned} $$

For simplicity we let

$$ \gamma(t_j) = \frac{t_j^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}}, \quad p(t_j) = \frac{\gamma(t_j)}{1-\gamma(t_j)} $$

Such that

$$ \begin{aligned} & \frac{M(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{f^{j+1}-f^{j}}{\Delta t}\gamma(t_j)\exp\left(-p(t_j)(t_{n+1}-p)\right)dp \\ = & \frac{M(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^{j}}{\Delta t}\gamma(t_j)\int_{t_j}^{t_{j+1}}\exp\left(-p(t_j)(t_{n+1}-p)\right)dp \\ = & \frac{M(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^{j}}{\Delta t}\gamma(t_j)\left[\frac{1}{p(t_j)}\exp\left(-p(t_j)(t_{n+1}-p)\right)\right]_{t_j}^{t_{j+1}} \\ = & \frac{M(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^{j}}{\Delta t}\gamma(t_j)\frac{1}{p(t_j)}\left[\exp\left(-p(t_j)(t_{n+1}-t_{j+1}\right)\right. \\ & \left.-\exp\left(-p(t_j)(t_{n+1}-t_j)\right)\right] \end{aligned} $$

Therefore replacing all by their values, the variable order with exponential decay can be approximated as

$$ {}_{0}^{CE}D_{t}^{\gamma(t),\theta(t)}u(z)|_{t=t_{n+1}} \simeq \frac{M(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^{j}}{\Delta t}\frac{t_j^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} $$

$$\times\frac{1}{p(t_j)}\left[\exp\left(-p(t_j)(t_{n+1}-t_{j+1}\right)-\exp\left(-p(t_j)(t_{n+1}-t_j)\right)\right]$$

We continue with the power law case

$${}_0^C D_t^{\gamma(t),\theta(t)}u(z)=\frac{1}{\Gamma(1-\gamma(t))}\int_0^t\frac{du(p)}{dp^{\theta(p)}}(t-p)^{-\gamma(p)}dp$$

at $t=t_{n+1}$

$$\begin{aligned}
{}_0^C D_t^{\gamma(t),\theta(t)}u(z)|_{t=t_{n+1}} &= \frac{1}{\Gamma(1-\gamma(t_{n+1}))}\int_0^{t_{n+1}}\frac{du(p)}{dp}\frac{t^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}}(t_{n+1}-p)^{-\gamma(p)}dp\\
&\simeq \frac{1}{\Gamma(1-\gamma(t_{n+1}))}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}}\\
&\quad\times(t_{n+1}-p)^{-\gamma(t_j)}dp\\
&\simeq \frac{1}{\Gamma(1-\gamma(t_{n+1}))}\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}}\\
&\quad\times\int_{t_j}^{t_{j+1}}(t_{n+1}-p)^{-\gamma(t_j)}dp\\
&\simeq \frac{1}{\Gamma(1-\gamma(t_{n+1}))}\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}}\\
&\quad\times\left[\frac{(t_{n+1}-t_j)^{1-\gamma(t_j)}}{1-\gamma(t_j)}-\frac{(t_{n+1}-t_{j+1})^{1-\gamma(t_j)}}{1-\gamma(t_j)}\right]
\end{aligned}$$

In the light of what is presented above, the variable order derivative can be approximated as:

$$\begin{aligned}
{}_0^C D_t^{\gamma(t),\theta(t)}u(z) &\simeq \frac{1}{\Gamma(1-\gamma(t_{n+1}))}\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}}\\
&\quad\times\left[\frac{(t_{n+1}-t_j)^{1-\gamma(t_j)}}{1-\gamma(t_j)}-\frac{(t_{n+1}-t_{j+1})^{1-\gamma(t_j)}}{1-\gamma(t_j)}\right]
\end{aligned}$$

We continue with the Mittag-Leffler kernel case:

$${}_0^{CM} D_t^{\gamma(t),\theta(t)}u(z)=\frac{AB(\gamma(t))}{1-\gamma(t)}\int_0^t\frac{du(p)}{dp^{\theta(p)}}E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(t-p)^{\gamma(p)}\right)dp.$$

at $t=t_{n+1}$

$$
\begin{aligned}
{}_0^{CM}D_t^{\gamma(t),\theta(t)}u(z)|_{t=t_{n+1}} &= \frac{AB(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\int_0^{t_{n+1}}\frac{du(p)}{dp}\frac{t^{-\theta(p)}}{\theta'(p)\ln(p)+\frac{\theta(p)}{p}} \\
&\times E_{\gamma(p)}\left(\frac{-\gamma(p)}{1-\gamma(p)}(t_{n+1}-p)^{\gamma(p)}\right)dp \\
&\simeq \frac{AB(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} \\
&\times E_{\gamma(t_j)}\left(\frac{-\gamma(t_j)}{1-\gamma(t_j)}(t_{n+1}-p)^{\gamma(t_j)}\right)dp \\
&\simeq \frac{AB(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} \\
&\times\int_{t_j}^{t_{j+1}}E_{\gamma(t_j)}\left(\frac{-\gamma(t_j)}{1-\gamma(t_j)}(t_{n+1}-p)^{\gamma(t_j)}\right)dp \\
&\simeq \frac{AB(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} \\
&\times\int_{t_j}^{t_{j+1}}\sum_{k=0}^{\infty}\frac{(-\frac{\gamma(t_j)}{1-\gamma(t_j)})^k(t_{n+1}-p)^{\gamma(t_j)k}}{\Gamma(\gamma(t_j)k+1)}dp \\
&\simeq \frac{AB(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} \\
&\times\sum_{k=0}^{\infty}\frac{(-\frac{\gamma(t_j)}{1-\gamma(t_j)})^k}{\Gamma(\gamma(t_j)k+1)}\int_{t_j}^{t_{j+1}}(t_{n+1}-p)^{\gamma(t_j)k}dp \\
&\simeq \frac{AB(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} \\
&\times\sum_{k=0}^{\infty}\frac{(-\frac{\gamma(t_j)}{1-\gamma(t_j)})^k}{\Gamma(\gamma(t_j)k+1)}\left[\frac{(t_{n+1}-t_{j+1})^{\gamma(t_j)k+1}}{\gamma(t_j)k+1}-\frac{(t_{n+1}-t_j)^{\gamma(t_j)k+1}}{\gamma(t_j)k+1}\right] \\
&\simeq \frac{AB(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} \\
&\times\Bigg[(t_{n+1}-t_{j+1})E_{\gamma(t_j),2}\left(-\frac{\gamma(t_j)}{1-\gamma(t_j)}\left((t_{n+1}-t_{j+1})^{\gamma(t_j)}\right)\right) \\
&-(t_{n+1}-t_j)E_{\gamma(t_j),2}\left(-\frac{\gamma(t_j)}{1-\gamma(t_j)}\left((t_{n+1}-t_j)^{\gamma(t_j)}\right)\right)\Bigg]
\end{aligned}
$$

In the light of what is presented above, the variable order derivative with the Mittag-Leffler kernel can be approximated as:

$$
\begin{aligned}
{}_0^{CM}D_t^{\gamma(t),\theta(t)}u(z) \simeq\; & \frac{AB(\gamma(t_{n+1}))}{1-\gamma(t_{n+1})}\sum_{j=0}^{n}\frac{f^{j+1}-f^{j}}{\Delta t}\frac{t^{-\theta(t_j)}}{\frac{\theta(t_{j+1})-\theta(t_j)}{\Delta t}\ln(t_j)+\frac{\theta(t_j)}{t_j}} \\
& \times\Bigg[(t_{n+1}-t_{j+1})E_{\gamma(t_j),2}\left(-\frac{\gamma(t_j)}{1-\gamma(t_j)}\left((t_{n+1}-t_{j+1})^{\gamma(t_j)}\right)\right) \\
& -(t_{n+1}-t_j)E_{\gamma(t_j),2}\left(-\frac{\gamma(t_j)}{1-\gamma(t_j)}\left((t_{n+1}-t_j)^{\gamma(t_j)}\right)\right)\Bigg]
\end{aligned}
$$

## 4. Numerical simulations

We consider the following problem:

$$
m{}_0^CD_t^{\alpha(t),\theta(t)}V(t)+c\frac{dV}{dt}+kV=0
$$

We demonstrate the numerical simulation of the problem by Figure 1. Then, we consider

$$
m\frac{d^2V}{dt^2}+c{}_0^CD_t^{\alpha(t),\theta(t)}V(t)+kV=0
$$

We demonstrate the numerical simulations of this problem by Figure 2.

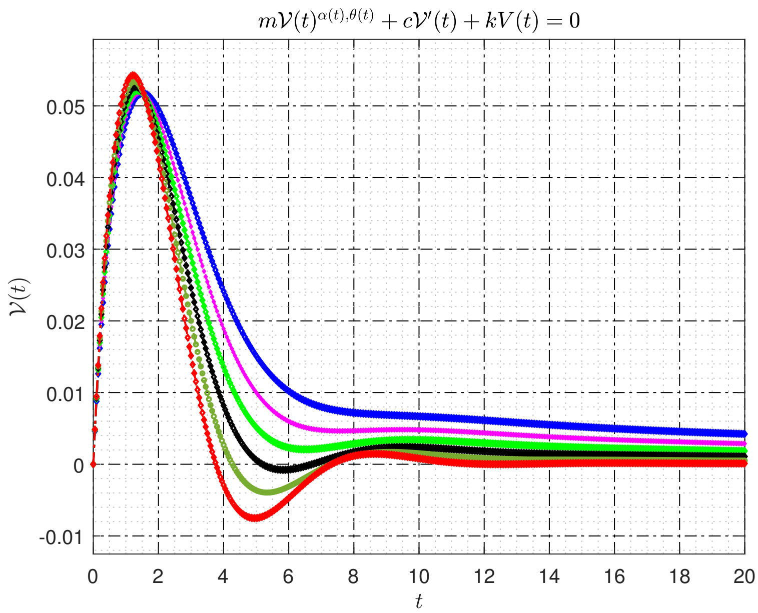


**Figure 1.** Simulation for $m = c = k = 1$ and $\theta = 1.0$.

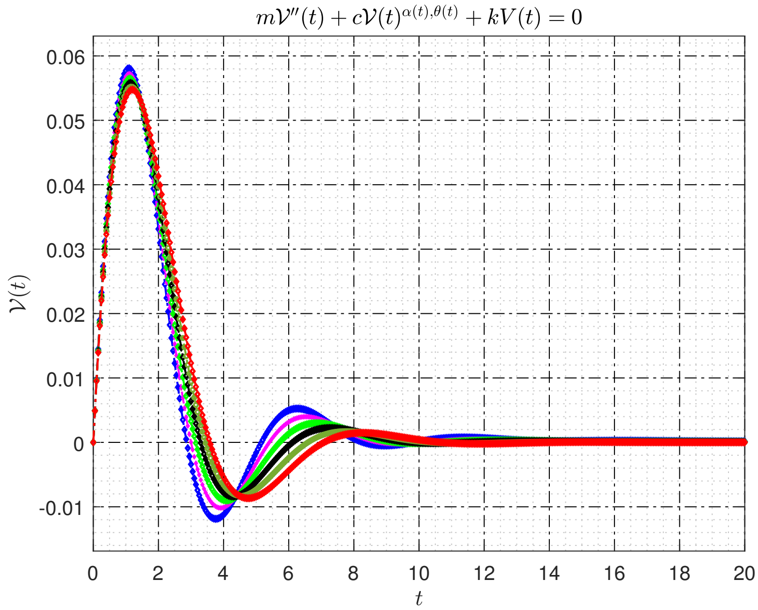


**Figure 2.** Simulation for $m = c = k = 1$ and $\theta = 1.0$.

In the Figures 1 and 2, $V(t)$ shows the approximate solution of the equations.

## 5. Conclusions

Recently, differential operators with variable orders have being acknowledged as more convenient mathematical operators. Therefore, we presented an updated version of the so-called fractal-fractional derivative, where the constant fractal dimension is replaced by variable order fractal dimension and constant fractional order is replaced by the variable order function. We demonstrated some important properties of this new derivative and showed the numerical approximation of it. When the two variable orders are constants, we get well the so-called fractal-fractional differential operators.

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## Conflict of interest

The authors declare no conflict of interests.

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