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Research article

Approximate solution of nonlinear fuzzy Fredholm integral equations using bivariate Bernstein polynomials with error estimation

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Abstract: This paper is concerned with obtaining approximate solutions of fuzzy Fredholm integral equations using Picard iteration method and bivariate Bernstein polynomials. We first present the way to approximate the value of the multiple integral of any fuzzy-valued function based on the two dimensional Bernstein polynomials. Then, it is used to construct the numerical iterative method for finding the approximate solutions of two dimensional fuzzy integral equations. Also, the error analysis and numerical stability of the method are established for such fuzzy integral equations considered here in terms of supplementary Lipschitz condition. Finally, some numerical examples are considered to demonstrate the accuracy and the convergence of the method.

Keywords: fuzzy Fredholm integral equations; Picard iteration method; bivariate Bernstein polynomials; numerical approximation; error estimation
Mathematics Subject Classification: 03E72, 46S40

1. Introduction

Bernstein polynomials have been used recently to solve some linear as well as nonlinear fuzzy integral equations. These polynomials are positive and their sum is unity. The study of fuzzy integral equations was initially considered by Kaleva [32] to convert the initial value problem for first-order fuzzy differential equations into the equivalent fuzzy Volterra integral equation. Some authors investigated the existence of a unique solution for fuzzy integral equations using the Banach fixed point theorem (see [9, 10, 28, 36, 37]). Numerical methods for approximating the solution of fuzzy integral equations have been studied by many authors based on quadrature rules and Pcard's approximations (see [11, 12, 15, 16, 23, 38, 43, 48, 49, 53]). Several numerical iterative approaches have been proposed based on the Picard approximations and other techniques such as the Lagrange interpolation [26], divided and finite differences [35], hybrid block-pulse functions and Taylor series [7], triangular basis functions [8] and block-pulse basis functions [50, 53]. Some other existing numerical and analytical methods in literature are: Adomian decomposition [1], Nyström methods [2], Bernstein polynomials [22], fuzzy Haar wavelets [50] and block pulse functions [39], homotopy analysis [34] and homotopy perturbation [5]. Yang and Gong [47] introduced the concept of ill-posedness for the fuzzy Fredholm integral equation of the first kind using the Zadehs decomposition theorem of fuzzy number and obtained the approximate solution for this class of integral equations using regularization method based on classic Tikhonov regularization scheme. Early, Ziari et al. [55] presented an iterative method based on fuzzy Bernstein polynomials for solving nonlinear fuzzy Volterra integral equations. Recently, Ziari et al. [56] proposed an iterative method to solve fuzzy integral equations using generalized quadrature rule. The study of fuzzy integral equations in two dimensions based on iterative technique was started by Sadatrasoul and Ezzati (see [41]). Indeed, they developed the iterative method in [23] for two dimensional fuzzy integral equations. Also, Ezzati and Ziari in [24] proposed a non-iterative numerical method for solving fuzzy Fredholm integral equations in two dimensions based on Bernstein polynomials. Sadatrasoul and Ezzati in [42] presented an iterative method to find approximate solution of linear and nonlinear two dimensional Hammerstein fuzzy integral equations by considering optimal quadrature formula for fuzzy-number-valued functions of Lipschitz type. In [16], Bica and Popescu constructed the fuzzy trapezoidal cubature rule for fuzzy functions of Lipschitz type to approximate the solution of nonlinear fuzzy Fredholm integral equations in two variables. Recently, Akhavan et al. [3] developed an iterative approach and combining trapezoidal and midpoint formulas for such equations. In [33], the existence and uniqueness results for fuzzy Fredholm-Volterra integral have been investigated and the authors proposed an iterative numerical approach for these type of equations based on bivariate triangular functions. In this paper, we approximate the integral of two dimensional fuzzy function by the bivariate fuzzy Bernstein polynomials and investigate the error analysis of the numerical method. Also, an iterative procedure is constructed based on two dimensional fuzzy Bernstein polynomials for solving two dimensional nonlinear fuzzy Fredholm integral equations,

$$F(x,y) = f(x,y) \oplus \int_c^d \int_a^b H(x,y,s,t) \odot G(F(s,t))dsdt, \quad (s,t) \in [a,b] \times [c,d]. \tag{1.1}$$

The main advantage of proposed method is the reduction of computational cost with respect to existing methods in the literature such as iterative methods based on fuzzy trapezoidal cubature rule in [16] and [41]. For instance, the number of function evaluations of the proposed method at the knots is n^2 for $m = n$ and the same factor for methods in [16] and [41] equals to $4n^2$ for $m = n$, thus, the proposed approach is reasonable from aspect of reduction of computational cost in comparison with the existing methods. We provide some numerical experiments to confirm the numerical results with theoretical results. Finally, some concluding remarks will be presented.

2. Preliminaries

Definition 2.1. (See [20].) A fuzzy number is a function $w : R \to [0, 1]$ having the properties:

(1) w is normal , that is $\exists x_0 \in R$ such that $w(x_0) = 1$,
(2) w is fuzzy convex set

$$(i.e.\ w(\lambda x + (1-\lambda)y) \geq \min\{w(x), w(y)\} \quad \forall x, y \in R, \lambda \in [0,1]),$$

(3) w is upper semicontinuous on R,
(4) the $\overline{\{ x \in R : w(x) > 0\}}$ is compact set.

The set of all fuzzy numbers is denoted by E.

Definition 2.2. (See [27].) An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{w}(r), \overline{w}(r)), 0 \le r \le 1$, which satisfy the following requirements:

(1) $\underline{w}(r)$ is a bounded left continuous nondecreasing function over [0,1],
(2) $\overline{w}(r)$ is a bounded left continuous nonincreasing function over [0,1],
(3) $\underline{w}(r) \le \overline{w}(r)$, $0 \le r \le 1$.

For addition and scalar multiplication of fuzzy numbers in fuzzy set space E we have:

(1) $w_1 \oplus w_2 = (\underline{w_1}(r) + \underline{w_2}(r), \overline{w_1}(r) + \overline{w_2}(r))$,

(2) $(\lambda \odot w_1) = \begin{cases} (\lambda \underline{w_1}(r), \lambda \overline{w_1}(r)) & \lambda \ge 0, \\ \\ (\lambda \overline{w_1}(r), \lambda \underline{w_1}(r)) & \lambda < 0. \end{cases}$

Definition 2.3. (See [31].) For given fuzzy numbers $w_1 = (\underline{w_1}(r), \overline{w_1}(r))$ and $w_2 = (\underline{w_2}(r), \overline{w_2}(r))$, the quantity

$$D(w_1, w_2) = \sup_{r \in [0,1]} \max\{|\underline{w_1}(r) - \underline{w_2}(r)| , |\overline{w_1}(r) - \overline{w_2}(r)| \}$$

is the Hausdorff distance between fuzzy numbers w_1 and w_2.

Lemma 2.1. (See [46].) If $w_1, w_2, w_3, w_4 \in E$ and $k \in R$, then we have:

(1) $D(w_1 \oplus w_3, w_2 \oplus w_3) = D(w_1, w_2)$,
(2) $D(k \odot w_1, k \odot w_2) = |k| \, D(w_1, w_2)$,
(3) $D(w_1 \oplus w_2, w_3 \oplus w_4) \le D(w_1, w_3) + D(w_2, w_4)$,
(4) $D(w_1 \oplus w_2, \tilde{0}) \le D(w_1, \tilde{0}) + D(w_2, \tilde{0})$.

In [46], it is proved that (E, D) is a complete metric space.

Lemma 2.2. (see [4].) For any $k_1, k_2 \in \mathbb{R}$ with $k_1 \cdot k_2 \ge 0$ and any $w \in E$ we have $D\left(k_1 \odot w, k_2 \odot w\right) = |k_1 - k_2| \, D\left(w, \widetilde{0}\right)$.

Remark 2.1. The property (4) in Lemma (2.1) provide the definition of a function $\|.\| : E \to R^+$ by $\|w_1\| = D(w_1, \tilde{0})$, which has the properties of the usual norms. In [5] the properties of this function are exhibited as follows:
(i) $\|w_1\| \ge 0, \ \forall \ w_1 \in E$, and $\|w_1\| = 0$ iff $w_1 = \tilde{0}$,
(ii) $\|\lambda.w_1\| = |\lambda|\|w_1\|$ and $\|w_1 \oplus w_2\| \le \|w_1\| + \|w_2\|, \ \forall \ w_1, w_2 \in E, \ \forall \ \lambda \in R$,
(iii) $|\|w_1\| - \|w_2\|| \le D(w_1, w_2)$ and $D(w_1, w_2) \le \|w_1\| + \|w_2\| \ \forall \ w_1, w_2 \in E$.

Definition 2.4. (See [25].) A fuzzy-real-number-valued function $f : [a, b] \to E$ is said to be continuous in $t_0 \in [a, b]$, if for given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - x_0| < \delta$. implies that

$D(f(x), f(x_0)) < \varepsilon$, for all $x \in [a, b]$ and We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $t_0 \in [a, b]$, and we will use the notation $C([a, b], E)$ for the space of all continuous fuzzy functions.

Definition 2.5. (See [29].) Let $f : [a, b] \to E$ be given. The function f is fuzzy-Riemann integrable to $I(f) \in E$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v]\,; \xi\}$ of $[a, b]$ with the norms $\Delta(p) < \delta$, we have

$$D\left(\sum_{P}(v - u) \odot f(\xi), I(f)\right) < \varepsilon,$$

where $\sum$ denotes the fuzzy summation. In this case it is denoted by $I(f) = (FR) \int_a^b f(x)dx$.

Lemma 2.3. (See [29].) If $f,\ g : [a, b] \subseteq R \to E$ are fuzzy continuous functions, then the function $F : [a, b] \to R_+$ by $F(x) = D(f(x), g(x))$ is continuous on $[a, b]$ and

$$D\left((FR) \int_a^b f(x)dx, (FR) \int_a^b g(x)dx\right) \leq \int_a^b D(f(x), g(x))dx.$$

Theorem 2.1. (See [30].) If $f, g : [a, b] \to E$ are (FR) integrable fuzzy functions, and α, β are real numbers, then

$$(FR) \int_a^b (\alpha \odot f(x) \oplus \beta \odot g(x))dx = \alpha \odot (FR) \int_a^b f(x)dx \oplus \beta \odot (FR) \int_a^b g(x)dx.$$

Definition 2.6. (See [11].) For $L \geq 0$, a function $f : [a, b] \to E$ is L-Lipschitz if

$$D\left(f\left(x\right), f\left(y\right)\right) \leq L\left|x - y\right|$$

for any $x, y \in [a, b]$. A function $F : E \to E$ is Lipschitz if there exists $L' \geq 0$ such that $D\left(F\left(u\right), F\left(v\right)\right) \leq L' \cdot D\left(u, v\right)$, for any $u, v \in E$.
According to [11], any Lipschitz function is continuous.

For $f \in C_F[a, b]$, let us consider the Bernstein-type fuzzy polynomials

$$B_n^{(F)}(f)(x) = \sum_{k=0}^{n} f\left(a + \frac{k(b - a)}{n}\right) \odot p_{n,k}(x),\ n \in N,\ x \in [a, b],$$

where $P_{n,k}(x)$ $(k = 0, ..., n)$ are Bernstein polynomials of degree n defined on $[a, b]$ as:

$$P_{n,k}(x) = \binom{n}{k}\frac{(x - a)^k(b - x)^{n-x}}{(b - a)^n}, \quad k = 0, ..., n.$$

It is obvious that $P_{n,k}(x) \geq 0, \quad \forall\ x \in [a, b]$ and $P_{n,0}(x), P_{n,1}(x), ..., P_{n,n}(x)$ are linearly independent algebraic polynomials of degree n and $\sum_{k=0}^{n} p_{n,k}(x) = 1$.

The following definitions are related to fuzzy-number-valued functions in two variables.

Definition 2.9.(See [3].) Let $f:[a,b]\times[c,d]\to E$ be given. The function f is fuzzy-Riemann double integrable to $I(f)\in E$ if for any $\varepsilon>0$, there exists $\delta>0$ such that for any partition $\Delta=(\Delta_x,\Delta_y)$ of the region $[a,b]\times[c,d]$, with $\Delta_x: a=x_0<x_1<\ldots x_{m-1}<x_m=b$ and $\Delta_y: c=y_0<y_1<\ldots<y_{n-1}<y_n=d$ and any set of intermediate points $\left\{(\xi_i,\eta_j): i=1,2,...,m\ , j=1,2,...,n\right\}$, with norm $\upsilon(\Delta)=\max\{|x_i-x_{i-1}|+\left|y_j-y_{j-1}\right|: i=1,\ldots,m,\ j=1,\ldots,n\}<\delta$, we have

$$D\left(\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i-x_{i-1})\left(y_j-y_{j-1}\right)\odot f\left(\xi_i,\eta_j\right),I(f)\right)<\varepsilon, \tag{2.1}$$

In this case, the fuzzy number $I(f)$ will be denoted by

$$I(f)=(FR)\int_c^d(FR)\left(\int_a^b f(s,t)ds\right)dt. \tag{2.2}$$

Definition 2.10. (See [40].) A function $f:[a,b]\times[c,d]\to E$ is called:
(i) continuous in $(x_0,y_0)\in[a,b]\times[c,d]$ if for any $\varepsilon>0$ there exists $\delta>0$ such that for any $(x,y)\in[a,b]\times[c,d]$ with $|x-x_0|<\delta$, $|y-y_0|<\delta$ we have

$$D\left(f(x,y),f(x_0,y_0)\right)<\varepsilon.$$

The function f is continuous on $[a,b]\times[c,d]$ if it is continuous in each $(x,y)\in[a,b]\times[c,d]$.
(ii) bounded if there exists $M\geq 0$ such that

$$D\left(f(x,y),\widetilde{0}\right)\leq M,\quad \forall\,(x,y)\in[a,b]\times[c,d].$$

The set of all continuous functions $f:[a,b]\times[c,d]\to E$ is denoted by $C\left([a,b]\times[c,d],E\right)$.

Lemma 2.4. (See [40].) If $f\in C\left([a,b]\times[c,d],E\right)$ then

$$(FR)\int_c^d\left((FR)\int_a^b f(x,y)\,dx\right)dy$$

exists and

$$\underline{\left((FR)\int_c^d\left((FR)\int_a^b f(x,y)\,dx\right)dy\right)}(r)=\int_c^d\left(\int_a^b \underline{f(x,y,r)}dx\right)dy$$

$$\overline{\left((FR)\int_c^d\left((FR)\int_a^b f(x,y)\,dx\right)dy\right)}(r)=\int_c^d\left(\int_a^b \overline{f(x,y,r)}dx\right)dy.$$

Lemma 2.5. If $f,g:[a,b]\times[c,d]\to E$ are continuous fuzzy functions then the function $\varphi:[a,b]\times[c,d]\to\mathbb{R}_+$ defined by $\varphi(s,t)=D\left(f(x,y),g(x,y)\right)$ is continuous on $[a,b]\times[c,d]$ and

$$D((FR)\int_c^d\left((FR)\int_a^b f(x,y)\,dx\right)dy,(FR)\int_c^d\left((FR)\int_a^b g(x,y)\,dx\right)dy)\leq$$

$$\leq \int_c^d \left(\int_a^b D\left(f\left(x,y\right), g\left(x,y\right)\right) dx \right) dy.$$

Proof. For simplicity we consider only $m = n$, where the m and n are the number of points in the partition of $[a,b]$ and $[c,d]$, respectively, the case of $m \neq n$ follows similarly. Firstly, we prove that the function $\varphi\left(s,t\right) = D(f(s,t), g(s,t))$ be continuous in every point $(s_0,t_0) \in [a,b] \times [c,d]$, for this purpose, we let $\{(s_n,t_n)\}_{n\geq 1}$, $(s_n,t_n) \in [a,b] \times [c,d]$, such that $\lim_{n\to\infty}(s_n,t_n) = (s_0,t_0)$. In this case, we have:

$$D\left(f(s_n,t_n), g(s_n,t_n)\right) \leq D\left(f(s_n,t_n), f(s_0,t_0)\right) + D\left(f(s_0,t_0), g(s_0,t_0)\right) + D\left(g(s_0,t_0), g(s_n,t_n)\right),$$

and on the other hand, we have:

$$D\left(f(s_0,t_0), g(s_0,t_0)\right) \leq D\left(f(s_0,t_0), f(s_n,t_n)\right) + D\left(f(s_n,t_n), g(s_n,t_n)\right) + D\left(g(s_n,t_n), g(s_0,t_0)\right).$$

Taking to the limit when $n \to \infty$, and according to the continuity of f and g we obtain:

$$\lim_{n\to\infty} D\left(f(s_n,t_n), g(s_n,t_n)\right) = D\left(f(s_0,t_0), g(s_0,t_0)\right),$$

that is φ is continuous at each $(s_0,t_0) \in [a,b] \times [c,d]$. Now, let $P_n = \{([s_{n-1}, s_n]; \xi_n)\}$, $n \in N$ and $Q_n = \{([t_{n-1}, t_n]; \eta_n)\}$, $n \in N$ be two sequences of partitions of $[a,b]$ and $[c,d]$ with $\Delta(P_n) \to 0$, when $n \to \infty$ and $\Delta(Q_n) \to 0$, when $n \to \infty$, respectively.
So, we have:

$$D\left((FR)\int_c^d (FR)\left(\int_a^b f\left(s,t\right) ds\right) dt, (FR)\int_c^d (FR)\left(\int_a^b g\left(s,t\right) ds\right) dt\right) \leq$$

$$D\left((FR)\int_c^d (FR)\left(\int_a^b f\left(s,t\right) ds\right) dt, \sum_{i=1}^n \sum_{j=1}^n (x_i - x_{i-1})\left(y_j - y_{j-1}\right) \odot f\left(\xi_i, \eta_j\right)\right) +$$

$$D\left(\sum_{i=1}^n \sum_{j=1}^n (x_i - x_{i-1})\left(y_j - y_{j-1}\right) \odot f\left(\xi_i, \eta_j\right), \sum_{i=1}^n \sum_{j=1}^n (x_i - x_{i-1})\left(y_j - y_{j-1}\right) \odot g\left(\xi_i, \eta_j\right)\right) +$$

$$D\left(\sum_{i=1}^n \sum_{j=1}^n (x_i - x_{i-1})\left(y_j - y_{j-1}\right) \odot g\left(\xi_i, \eta_j\right), (FR)\int_c^d (FR)\left(\int_a^b g\left(s,t\right) ds\right) dt\right) \leq$$

$$D\left((FR)\int_c^d (FR)\left(\int_a^b f\left(s,t\right) ds\right) dt\right), , \sum_{i=1}^n \sum_{j=1}^n (x_i - x_{i-1})\left(y_j - y_{j-1}\right) \odot f\left(\xi_i, \eta_j\right)\Big) +$$

$$\sum_{i=1}^n \sum_{j=1}^n D\left(f\left(\xi_i, \eta_j\right), g\left(\xi_i, \eta_j\right)\right)(x_i - x_{i-1})\left(y_j - y_{j-1}\right) + D\Bigg(\sum_{i=1}^n \sum_{j=1}^n (x_i - x_{i-1})\left(y_j - y_{j-1}\right) \odot g\left(\xi_i, \eta_j\right),$$

$$(FR)\int_c^d (FR)\left(\int_a^b g\left(s,t\right) ds\right) dt\Bigg)$$

Taking to the limit when $n \to \infty$, we get:

$$D\left((FR)\int_c^d (FR)\left(\int_a^b f\left(s,t\right) ds\right) dt, (FR)\int_c^d (FR)\left(\int_a^b g\left(s,t\right) ds\right) dt\right) \leq$$

$$\int_c^d \int_a^b D\left(f\left(s,t\right), g\left(s,t\right)\right) dsdt.$$

Thus, the proof is complete. □

Moreover, it can be proved that any continuous function $f : [a, b] \times [c, d] \rightarrow E$ is bounded.

Consider a fuzzy-number-valued function $f : [a, b] \times [c, d] \rightarrow E$ having the following Lipschitz property: there exist $L_1, L_2 \geq 0$ such that

$$D\left(f\left(x_1, y_1\right), f\left(x_2, y_2\right)\right) \leq L_1 \left|x_1 - x_2\right| + L_2 \left|y_1 - y_2\right| \tag{2.3}$$

for all $x_1, x_2 \in [a, b]$, $y_1, y_2 \in [c, d]$.

Similar to the one dimensional case, for $f \in C_F\left([a, b] \times [c, d]\right)$ we express the fuzzy two-dimensional Bernstein operators as follows:

Definition 2.10. (See [4].) Two dimensional fuzzy Bernstein operator on the region $[a, b] \times [c, d]$ is defined as follows:

$$B_{m,n}^{(F)}(f)(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(x_i, y_j\right) \odot P_{m,i}(x) P_{n,j}(y),\ \forall\ (x, y) \in [a, b] \times [c, d],\ \forall\ (m, n) \in N^2, \tag{2.4}$$

where $x_i = a + \frac{i(b-a)}{m}$, $(i = 0, ..., m)$, $y_j = c + \frac{j(d-c)}{n}$, $(j = 0, ...n)$ and $P_{m,i}(x)$, $(i = 0, ..., m)$, $P_{n,j}(x)$, $(j = 0, ..., n)$ are Bernstein polynomials of degrees m and n respectively, moreover the polynomials $P_{m,i}(x)$ and $P_{n,j}$ $\forall i, j$, are defined on intervals $[a, b]$ and $[c, d]$, respectively.
Let $\varphi_{i,j}(x, y) = P_{m,i}(x).P_{n,j}(y)$, it is obvious that $\phi_{i,j}(x, y) \geq 0$, $\forall$ $(x, y) \in [a, b] \times [c, d]$, $\forall i, j$, and $\varphi_{0,0}(x, y), \varphi_{0,1}(x, y), ..., \varphi_{m,n}(x, y)$ are linearly independent, and

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \varphi_{i,j}(s, t) = 1. \tag{2.5}$$

3. Approximation of integral of fuzzy function

The fuzzy function $f \in C_F\left([a, b] \times [c, d]\right)$ can be approximated using fuzzy Two dimensional Bernstein-type polynomials as

$$f(s, t) \approx \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(s_i, t_j\right) \odot P_{m,i}(s) P_{n,j}(t),$$

where $s_i = a + \frac{i(b-a)}{m}$, $(i = 0, ..., m)$, $t_j = c + \frac{j(d-c)}{n}$, $(j = 0, ...n)$. Hence, the approximate value of the integral of fuzzy function can be obtained as follows:

$$\begin{aligned}(FR)\int_c^d (FR)\int_a^b f(s, t) dsdt &\approx (FR)\int_c^d (FR)\int_a^b \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(s_i, t_j\right) \odot P_{m,i}(s) P_{n,j}(t) dsdt \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(s_i, t_j\right) \odot \int_c^d \int_a^b P_{m,i}(s) P_{n,j}(t) dsdt.\end{aligned}$$

In [21], the derivative of Bernstein polynomials on interval $[a, b]$ for fixed m is expressed as follows:

$$P'_{m,i}(s) = \frac{m}{b-a}\left(P_{m-1,i-1}(s) + P_{m-1,i}(s)\right).$$

By integrating from the above equality, we obtain:

$$\int_a^b P_{m-1,i-1}(s)ds = \int_a^b P_{m-1,i}(s)ds,$$

namely, for fixed m all Bernstein basis functions have the same definite integral over the interval [a, b]. Since $\sum_{i=0}^{m} p_{m,i}(s) = 1$ we have:

$$\int_a^b P_{m,i}(s)ds = \frac{b-a}{m+1} \quad i = 0, 1, ..., m.$$

Similarity for two dimensional Bernstein polynomials, we obtain:

$$\int_c^d \int_a^b P_{m,i}(s)P_{n,j}(t)dsdt = \frac{(b-a)(d-c)}{(m+1)(n+1)}, \quad \forall\ i, j. \tag{3.1}$$

So, we have the following approximation for double integral of two dimensional fuzzy function:

$$(FR)\int_c^d (FR)\int_a^b f(s,t)dsdt \approx \frac{(b-a)(d-c)}{(m+1)(n+1)} \odot \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(s_i, t_j\right), \tag{3.2}$$

where $s_i = a + \frac{i(b-a)}{m},\ (i = 0, ..., m),\ t_j = c + \frac{j(d-c)}{n},\ (j = 0, ...n)$.

In the following theorem, we obtain the error estimate of the above approximation.
Theorem 3.1. Let $f \in C_F([a, b] \times [c, d])$ be a Lipschitzian function. Then we have:

$$\begin{aligned}\Sigma =& D\left((FR)\int_c^d (FR)\int_a^b f(s,t)dsdt, \frac{(b-a)(d-c)}{(m+1)(n+1)} \odot \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(s_i, t_j\right)\right)\\ &\leq \left(\frac{L_1(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2(b-a)(d-c)^2}{2\sqrt{n}}\right).\end{aligned}$$

Proof. According to (2.5), we have:

$$\begin{aligned}\Sigma &= D\left((FR)\int_c^d (FR)\int_a^b f(s,t)dsdt, \frac{(b-a)(d-c)}{(m+1)(n+1)} \odot \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(s_i, t_j\right)\right)\\ &= D\Bigg((FR)\int_c^d (FR)\int_a^b \sum_{i=0}^{m}\sum_{j=0}^{n} \varphi_{i,j}(s,t)f(s,t)dsdt\\ &\qquad, (FR)\int_c^d (FR)\int_a^b \sum_{i=0}^{m}\sum_{j=0}^{n} \varphi_{i,j}(s,t)f\left(s_i, t_j\right)dsdt\Bigg).\end{aligned}$$

where $\varphi_{i,j}(s,t) = P_{m,i}(s).P_{n,j}(t)$. Then, applying the parts of 2 and 3 of Lemma 2.1, we obtain:

$$\Sigma \leq \int_c^d \int_a^b \sum_{i=0}^{m} \sum_{j=0}^{n} \varphi_{i,j}(s,t) D\left(f(s,t), f\left(s_i, t_j\right)\right) dsdt.$$

As regards, f satisfies in Lipschitz condition namely, inequality (2.3) holds for f, we obtain:

$$\Sigma \leq \int_c^d \int_a^b \sum_{i=0}^{m} \sum_{j=0}^{n} \varphi_{i,j}(s,t) \left(L_1 \left|s - s_i\right| + L_2 \left|t - t_j\right|\right) dsdt.$$

It is well known following inequality which is expressed in [29]:

$$\sum_{i=0}^{m} |s - s_i| P_{m,i}(s) \leq \frac{b-a}{2\sqrt{m}}, \quad s \in [a,b], \quad m \in N. \tag{3.3}$$

According to $\varphi_{i,j}(s,t) = P_{m,i}(s).P_{n,j}(t)$ and considering the $\sum_{i=0}^{m} P_{m,i}(s) = 1$, $\sum_{j=0}^{n} P_{n,j}(t) = 1$, and inequality (3.3), we obtain:

$$\begin{aligned} \Sigma =& D\left((FR)\int_c^d (FR)\int_a^b f(s,t)dsdt, \frac{(b-a)(d-c)}{(m+1)(n+1)} \odot \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(s_i, t_j\right)\right) \\ \leq& \left(\frac{L_1(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2(b-a)(d-c)^2}{2\sqrt{n}}\right). \end{aligned}$$

4. Two dimensional fuzzy integral equations

Here, we consider the two dimensional nonlinear fuzzy Fredholm integral Eq. (1.1), where H(x,y,s, t) is a crisp kernel function over $([a,b] \times [c,d])^2$, f, F are continuous fuzzy-number-valued functions and $G : E \to E$ is a continuous fuzzy function. We assume that H is continuous and therefore it is uniformly continuous with respect to (s,t) and there exists $M_H > 0$, such that $M_H = \max_{a\leq x,s\leq b,\ c\leq y,t\leq d} |H(x,y,s,t)|$.

Let $\Omega = \{f : [a,b] \times [c,d] \to E; f \text{ is continuous}\}$ be the space of the two-dimensional fuzzy continuous functions with the metric $D^*(f,g) = \sup_{a\leq s\leq b,\ c\leq t\leq d} D(f(s,t), g(s,t))$, for $f, g \in \Omega$. In the following theorem, sufficient conditions for the existence of an unique solution of Eq (1.1) are given.

Theorem 4.1. (See [41].) Let $H(x,y,s,t)$ be continuous and positive for $a \leq x, s \leq b,\ c \leq y, t \leq d$ and $f : [a,b] \times [c,d] \to E$ be continuous on $[a,b] \times [c,d]$. Moreover assume that there exists $L > 0$, such that

$$D(G(F_1(s,t)), G(F_2(s,t))) \leq L.D(F_1(s,t), F_2(s,t)), \quad \forall (s,t) \in [a,b] \times [c,d],\ F_1,\ F_2 \in \Omega.$$

If $B = M_H L(b-a)(d-c) < 1$, then the iterative procedure for $k \in \mathbf{N}$

$$\begin{aligned} &F_0(x,y) = f(x,y), \\ &F_k(x,y) = f(x,y) \oplus (FR)\int_c^d (FR)\int_a^b H(x,y,s,t) \odot G(F_{k-1}(s,t))dsdt, \forall (x,y) \in [a,b] \times [c,d]. \end{aligned} \tag{4.1}$$

converges to the solution of F of (1.1). In addition, the following error bound holds:

$$D^*(F, F_k) \leq \frac{B^k}{1-B} D^*(F_1, F_0), \quad \forall k \in \mathbf{N}. \tag{4.2}$$

where Ω is the set of fuzzy continuous functions on $I = [a, b] \times [c, d]$.

Remark 4.1. If $F_0 = f$ the error estimate (4.2), becomes:

$$D^*(F, F_k) \leq \frac{B^{k+1}}{L(1-B)} \left(L\|f\|_F + M_0\right), \quad \forall k \in \mathbf{N}, \tag{4.3}$$

where $M_0 = \sup\limits_{(x,y)\in I} \|G(\tilde{0})\|_F$.
Proof. It is observed that

$$\begin{aligned}
D^*(F_0, F_1) &= \sup_{(x,y)\in I} D\Big(f(x,y), f(x,y) \oplus (FR)\int_c^d (FR)\int_a^b H(x,y,s,t) \odot G(F_0(s,t))dsdt\Big) \\
&\leq \sup_{(x,y)\in I} \int_c^d \int_a^b D\Big(\tilde{0}, H(x,y,s,t) \odot G(f(s,t))\Big)dsdt \\
&\leq \int_c^d \int_a^b \sup_{(x,y)\in I} \sup_{(s,t)\in I} |H(x,y,s,t)| D\Big(\tilde{0}, G(f(s,t))\Big)dsdt \\
&\leq M_H \int_c^d \int_a^b \sup_{(s,t)\in I} \Big[D\Big(\tilde{0}, G(\tilde{0})\Big) + D\Big(G(\tilde{0}), G(f(s,t))\Big)\Big]dsdt \\
&\leq M_H(b-a)(d-c)\Big[\sup_{(s,t)\in I} \|G(\tilde{0})\|_F + LD^*(\tilde{0}, f)\Big] \leq \frac{B}{L}(M_0 + L\|f\|_F),
\end{aligned}$$

and hence, it obtains the inequality (4.3). □

Now, we introduce the numerical method to find the approximate solution of the two dimensional nonlinear fuzzy Fredholm integral equation (1.1). In this way, we consider the following uniform partitions of the region $[a, b] \times [c, d]$:

$$D_x = a = s_1 < s_2 < ... < s_{m-1} < s_m = b \ , \ D_y = c = t_1 < t_2 < ... < t_{n-1} < t_n = d \tag{4.4}$$

with $s_i = a + \frac{i(b-a)}{m}$, $t_j = c + \frac{j(d-c)}{n}$, $0 \leq i \leq m$, $0 \leq j \leq n$. Then according to (3.2), the following iterative procedure gives the approximate solution of Eq (1.1) in the point $(x, y) \in [a, b] \times [a, b]$ using two dimensional fuzzy Bernstein-type polynomials:

$$\begin{aligned}
&z_0(x,y) = f(x,y), \\
&z_k(x,y) = f(x,y) \oplus \frac{(b-a)(d-c)}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} H\left(x,y,s_i,t_j\right) \odot G\left(z_{k-1}\left(s_i,t_j\right)\right).
\end{aligned} \tag{4.5}$$

5. Error analysis

In this section, we investigate the convergence of the iterative proposed method to the solution of Eq (1.1) under the following conditions:

(i) $f : [a,b] \times [c,d] \to E$ is fuzzy continuous;
(ii) $H : ([a,b] \times [c,d])^2 \to R^+$ is continuous;
(iii) There exist $\alpha, \beta \geq 0$ such that

$$D\left(f\left(s',t'\right), f\left(s'',t''\right)\right) \leq \alpha\left|s'-s''\right| + \beta\left|t'-t''\right|,$$

for any $s', s'' \in [a,b], t', t'' \in [c,d]$;
(iv) There exists $L > 0$ such that

$$D(G(F_1(s,t)), G(F_2(s,t))) \leq L.D(F_1(s,t), F_2(s,t)), \quad \forall (s,t) \in [a,b] \times [c,d],$$

where $F_1, F_2 : [a,b] \times [c,d] \to E$;
(v) $B = M_H L(b-a)(d-c) < 1$, where L is as given in the above item and $M_H \geq 0$ is such that $M_H = \max\limits_{a\leq x,s\leq b,\ c\leq y,t\leq d} H(x,y,s,t)$;
(vi) There exist $\mu, \lambda \geq 0$ such that

$$\left|H\left(x,y,s',t'\right) - H\left(x,y,s'',t''\right)\right| \leq \mu\left|s'-s''\right| + \lambda\left|t'-t''\right|,$$

for any $x, s', s'' \in [a,b]$, and $y, t', t'' \in [c,d]$;
(vii) There exist $\gamma, \eta \geq 0$ such that

$$\left|H\left(s',t',s,t\right) - H\left(s'',t'',s,t\right)\right| \leq \gamma\left|s'-s''\right| + \eta\left|t'-t''\right|,$$

for any $s, s', s'' \in [a,b], t, t', t'' \in [c,d]$;

Firstly, we prove that functions $z_k(s,t)$ for all k are Lipschitzian.

Lemma 5.1. Consider the iterative procedure (4.5). Under the conditions (i)–(vii) then functions z_k for all $k \geq 0$ are Lipschitzian.
Proof. Using the properties of the distance between fuzzy numbers stated in Lemma 2.1 we obtain

$$\begin{aligned}
D\left(z_k\left(s',t'\right), z_k\left(s'',t''\right)\right) &\leq D\left(f\left(s',t'\right), f\left(s'',t''\right)\right) + \\
&+ D\Big(\frac{(b-a)(d-c)}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} H\left(s',t',s_i,t_j\right) \odot G\left(z_{k-1}\left(s_i,t_j\right)\right), \\
&\frac{(b-a)(d-c)}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} H\left(s'',t'',s_i,t_j\right) \odot G\left(z_{k-1}\left(s_i,t_j\right)\right)\Big) \\
\leq D\left(f\left(s',t'\right), f\left(s'',t''\right)\right) + \\
\frac{(b-a)(d-c)}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} &|H(s',t',s_i,t_j) - H(s'',t'',s_i,t_j)| D(G(z_{k-1}(s_i,t_j)), \tilde{0}).
\end{aligned}$$

Consequently, using conditions (iii) and (vii), we obtain:

$$\begin{aligned}D\left(z_k\left(s',t'\right),z_k\left(s'',t''\right)\right) &\leq \alpha|s'-s''|+\beta|t'-t''|+\\ &+(b-a)(d-c)\Big(\gamma\,|s'-s''|+\eta\,|t'-t''|\Big)D^*(G(z_{k-1}),\tilde{0})\end{aligned} \tag{5.1}$$

By applying the properties of the norm function $||.||_F$ in Remark 1, we obtain

$$\begin{aligned}&D\Big(G\left(z_k(s,t)\right),\tilde{0}\Big)\leq D\Big(G\left(z_k(s,t)\right),G\big(\tilde{0}\big)\Big)+D\Big(G\big(\tilde{0}\big),\tilde{0}\Big)\\ &\leq LD\Big(z_k(s,t),\tilde{0}\Big)+\sup_{0\leq s,t\leq 1}||G(\tilde{0})||_F=LD\Big(z_k(s,t),\tilde{0}\Big)+M_0.\end{aligned} \tag{5.2}$$

By using again the properties of the norm function $||.||_F$ in Remark 1, we have

$$\begin{aligned}&D\Big(z_k(s,t),\tilde{0}\Big)=||z_k(s,t)||_F\leq D\Big(f(s,t),\tilde{0}\Big)+\\ &\frac{(b-a)(d-c)}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}D(H(s,t,s_i,t_j)\odot G(z_{k-1}(s_i,t_j)),\tilde{0}).\end{aligned}$$

By applying the inequality (5.2) into the above inequality, we have:

$$\begin{aligned}&||z_k(s,t||_F\leq||f(s,t)||_F+\\ &\frac{(b-a)(d-c)}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}|H(s,t,s_i,t_j)|\Big(LD(z_{k-1}(s_i,t_j),\tilde{0})+M_0\Big)\\ &\leq||f(s,t)||_F+M_H(b-a)(d-c)\Big(LD^*(z_{k-1},\tilde{0})+M_0\Big).\end{aligned}$$

Taking supremum from the above inequality for $(s,t)\in[a,b]\times[c,d]$, it follows that

$$||z_k||_F\leq||f||_F+M_H(b-a)(d-c)\left(L||z_{k-1}||_F+M_0\right)=||f||_F+B||z_{k-1}||_F+\frac{B}{L}M_0.$$

By successive substitutions on the above inequality, we obtain

$$||z_k||_F\leq\frac{1-B^{k+1}}{1-B}\cdot||f||_F+\frac{1-B^k}{1-B}\cdot\frac{B}{L}M_0. \tag{5.3}$$

Since, $\frac{1-B^k}{1-B}\leq\frac{1}{1-B}$, for all $k\in\mathbf{N}$, we obtain:

$$||z_k||_F\leq\frac{L||f||_F+BM_0}{L(1-B)}. \tag{5.4}$$

Taking supremum from the inequality (5.2), it follows that

$$D^*(G(z_{k-1}),\tilde{0})=||G(z_{k-1})||_F\leq L||z_k||_F+M_0$$

Now, by taking into account the inequality (5.4), from the above inequality we get:

$$D^*(G(z_{k-1}),\tilde{0})=||G(z_{k-1})||_F\leq\frac{L||f||_F+BM_0}{1-B}+M_0=\frac{L||f||_F+M_0}{1-B} \tag{5.5}$$

Finally, by considering the above inequality, from (5.1) we obtain:

$$D\left(z_k\left(s',t'\right),z_k\left(s'',t''\right)\right)\leq$$
$$\left(\alpha+\gamma(b-a)(d-c)\frac{L\|f\|_F+M_0}{1-B}\right)|s'-s''|+\left(\beta+\eta(b-a)(d-c)\frac{L\|f\|_F+M_0}{1-B}\right)|t'-t''|$$

By supposing $L_1=\alpha+\gamma(b-a)(d-c)\frac{L\|f\|_F+M_0}{1-B}$ and $L_2=\beta+\eta(b-a)(d-c)\frac{L\|f\|_F+M_0}{1-B}$, we have:

$$D(z_k(s',t'),z_k(s'',t''))\leq L_1|s'-s''|+L_2|t'-t''|.$$

Thus, the functions $z_k(s,t)$ for all k are Lipschitzian. □

Now, we prove an interesting result about the satisfying functions $H(x,y,s,t)\odot G(z_k(s,t))$ in Lipschitz condition which is used in the proof of the main result.

Lemma 5.2. Consider the iterative procedure (4.5). Under the conditions (i)–(vii) and supposing functions z_k are Lipschitzian then the functions $\varphi_k(s,t)=H(x,y,s,t)\odot G(z_k(s,t))$ are Lipschitzian.
Proof. Using fuzzy Distance, we have:

$$D(\varphi_k(s',t'),\varphi_k(s'',t''))=D\left(H(x,y,s',t')\odot G(z_k(s',t')),H(x,y,s'',t'')\odot G(z_k(s'',t''))\right)\leq$$
$$\leq D\left(H(x,y,s',t')\odot G(z_k(s',t')),H(x,y,s',t')\odot G(z_k(s'',t''))\right)+$$
$$+D\left(H(x,y,s',t')\odot G(z_k(s'',t'')),H(x,y,s'',t'')\odot G(z_k(s'',t''))\right)$$

Using part (2) of Lemma 2.1, Lemma 2.2 and condition (iii), we obtain:

$$D(\varphi_k(s',t'),\varphi_k(s'',t''))\leq|H(x,y,s',t')|LD\left(z_k(s',t'),z_k(s'',t'')\right)+$$
$$+|H(x,y,s',t')-H(x,y,s'',t'')|D\left(G\left(z_k(s'',t'')\right),\tilde{0}\right)$$

According to Lemma 5.1 and using condition (vi), we get:

$$D(\varphi_k(s',t'),\varphi_k(s'',t''))\leq M_HL(L_1|s'-s''|+L_2|t'-t''|)+(\mu|s'-s''|+\lambda|t'-t'')D\left(G\left(z_k(s'',t'')\right),\tilde{0}\right)\quad(5.6)$$

By applying the inequality (5.5) from the proof of Lemma 5.1, we obtain

$$D\left(G\left(z_k(s'',t'')\right),\tilde{0}\right)\leq D^*(G(z_k),\tilde{0})=\|G(z_k)\|_F\leq\frac{L\|f\|_F+M_0}{1-B}$$

Now, by taking into account the above inequality, from (5.6) we get:

$$D(\varphi_k(s',t'),\varphi_k(s'',t''))\leq$$
$$\left(M_HLL_1+\frac{\mu(L\|f\|_F+M_0)}{1-B}\right)|s'-s''|+\left(M_HLL_2+\frac{\lambda(L\|f\|_F+M_0)}{1-B}\right)|t'-t''|$$

By assuming $L_1'=\left(M_HLL_1+\frac{\mu(L\|f\|_F+M_0)}{1-B}\right)$ and $L_2'=\left(M_HLL_2+\frac{\lambda(L\|f\|_F+M_0)}{1-B}\right)$. we have:

$$D(\varphi_k(s',t'),\varphi_k(s'',t''))\leq L_1'|s'-s''|+L_2'|t'-t''|.$$

Hence, the functions $\varphi_k(s,t)$ for all k are Lipschitzian. $\square$

Theorem 5.1. Under the conditions (i)–(vii) the iterative procedure Eq (4.5) converges to the unique solution of Eq (1.1), F, and its error estimate is as follows:

$$D^*(F, z_k) \leq \frac{B^{k+1}}{L(1-B)}(L\|f\|_F + M_0) + \frac{1}{1-B}\left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}}\right),$$

where $M_0 = \sup_{0\leq x,y\leq 1} \|F(\tilde{0})\|_F$.

Proof. Let $\varphi_k(s,t) = H(x,y,s,t) \odot G(F_k(s,t))$ and $\psi_k(s,t) = H(x,y,s,t) \odot G(z_k(s,t))$ for all $k \in \mathbb{N}^*$. Since

$$F_1(x,y) = f(x,y) \oplus (FR)\int_c^d (FR)\int_a^b H(x,y,s,t) \odot G(F_0(s,t)dsdt,$$

we have

$$\begin{aligned}
D(F_1(x,y), z_1(x,y)) &= D(f(x,y), f(x,y)) + \\
&+ D\left((FR)\int_c^d (FR)\int_a^b \varphi_0(s,t)dsdt, \frac{(b-a)(d-c)}{(m+1)(n+1)}\sum_{i=0}^m\sum_{j=0}^n \psi_0(s_i,t_j)\right) \\
&\leq D\left((FR)\int_c^d (FR)\int_a^b \varphi_0(s,t)dsdt, (FR)\int_c^d (FR)\int_a^b \psi_0(s,t)dsdt\right) + \\
&+ D\left((FR)\int_c^d (FR)\int_a^b \psi_0(s,t)dsdt, \frac{(b-a)(d-c)}{(m+1)(n+1)}\sum_{i=0}^m\sum_{j=0}^n \psi_0(s_i,t_j)\right).
\end{aligned}$$

Regarding to part (2) of Lemma 2.1, Theorem 3.1 and condition (iii), we have:

$$\begin{aligned}
D(F_1(x,y), z_1(x,y)) \leq &\int_c^d \int_a^d |H(x,y,s,t)|LD\,(F_0(s,t)), z_0(s,t))\,dsdt + \\
&+ D\left((FR)\int_c^d (FR)\int_a^b \psi_0(s,t)dsdt, \frac{(b-a)(d-c)}{(m+1)(n+1)}\sum_{i=0}^m\sum_{j=0}^n \psi_0(s_i,t_j)\right).
\end{aligned}$$

Applying Lemma 5.1 and Theorem 3.1 for second part of above expression, we obtain:

$$\begin{aligned}
D(F_1(x,y), z_1(x,y)) \leq &\int_c^d \int_a^b |H(x,y,s,t)|LD\,(F_0(s,t)), z_0(s,t))\,dsdt \\
&+ \left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}}\right).
\end{aligned}$$

Taking supremum from the above inequality for $(x,y) \in [a,b] \times [c,d]$, and according to $D\,(F_0(s,t)), z_0(s,t)) = 0$, we deduce:

$$D^*(F_1, z_1) \leq \left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2)}{2\sqrt{n}}\right).$$

Now, since

$$F_2(x,y) = f(x,y) \oplus (FR)\int_c^d (FR)\int_a^b H(x,y,s,t) \odot G(F_1(s,t))dsdt,$$

we conclude that

$$\begin{aligned}
D(F_2(x,y), z_2(x,y)) &= D(f(x,y), f(x,y)) + \\
&+ D\left((FR)\int_c^d (FR)\int_a^b \varphi_1(s,t)dsdt, \frac{(b-a)(d-c)}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\psi_1(s_i,t_j)\right) \\
&\le D\left((FR)\int_c^d (FR)\int_a^b \varphi_1(s,t), (FR)\int_c^d (FR)\int_a^b \psi_1(s,t)dsdt\right) + \\
&+ D\left((FR)\int_0^1 (FR)\int_0^1 \psi_1(s,t)dsdt, \frac{(b-a)(d-c)}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\psi_1(s_i,t_j)\right).
\end{aligned}$$

Using again part (2) of Lemma 2.1, Theorem 3.1, condition (iii) and Lemma 5.1, we obtain:

$$D(F_2(x,y), z_2(x,y)) \le BD^*(F_1, z_1) + \left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}}\right).$$

Then, we have:

$$D(F_2(x,y), z_2(x,y)) \le (1+B)\left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}}\right).$$

By induction, for $k \ge 3$, we obtain:

$$D(F_k(x,y), z_k(x,y)) \le \left(1 + B + \ldots + B^{k-1}\right)\left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}}\right).$$

Therefore, we have:

$$D^*(F_k, z_k) \le \left(\frac{1-B^k}{1-B}\right)\left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}}\right).$$

Since $B < 1$, we conclude that

$$D^*(F_k, z_k) \le \frac{1}{1-B}\left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}}\right). \tag{5.7}$$

Using Eqs (4.3) and (5.7), we obtain:

$$\begin{aligned}
D^*(F, z_k) \le D^*(F, F_k) + D^*(F_k, z_k) \le &\frac{B^{k+1}}{L(1-B)}(L\|f\|_F + M_0) \\
&+ \frac{1}{1-B}\left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}}\right). \qquad \square
\end{aligned}$$

Remark 5.1. Since $B < 1$, it is easy to show that

$$\lim_{k\to\infty m\to\infty n\to\infty} D^*(F, z_k) = 0,$$

this result shows that the proposed method is convergent.

6. Numerical stability analysis

For the iterative numerical method, it is more suitable investigating the stability of the obtained numerical solution with respect to the choice of the first iteration. So, in order to study the numerical stability of the iterative method (4.5) with respect to small changes in the starting approximation, we consider an another initial iteration term $\hat{F}_0 \in \Omega$ such that there exists $\epsilon > 0$ for which $D(F_0(x,y), \hat{F}_0(x,y) < \epsilon$, for all $(x,y) \in [a,b] \times [c,d]$. So, we have the new sequence of successive approximations as:

$$\hat{F}_k(x,y) = \hat{f}(x,y) \oplus (FR) \int_c^d (FR) \int_a^b H(x,y,s,t) \odot G(\hat{F}_{k-1}(s,t)) dsdt, \ (x,y) \in [a,b] \times [c,d]. \quad (6.1)$$

Using the same iterative method, the terms of produced sequence are:

$$\begin{aligned} &\hat{z}_0(x,y) = \hat{f}(x,y), \\ &\hat{z}_k(x,y) = \hat{f}(x,y) \oplus \frac{(b-a)(d-c)}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} H\left(x,y,s_i,t_j\right) \odot G\left(\hat{z}_{k-1}\left(s_i,t_j\right)\right). \end{aligned} \quad (6.2)$$

Definition 6.1. We say that the method of successive approximations applied to solve Eq. (1.1) is numerically stable with respect to the choice of the first iteration iff there exist positive numbers p, q and constants $k_1, k_2, k_3 > 0$ which are independent by step-sizes $h_1 = \frac{b-a}{m}$ and $h_2 = \frac{d-c}{n}$ respectively, such that

$$D(z_k(x,y), \hat{z}_k(x,y)) < K_1 \epsilon + K_2 h_1^p + K_3 h_2^q. \quad (6.3)$$

Theorem 6.1. Under the conditions of Theorem 5.1, the iterative method (6.2) is numerically stable with respect to the choice of the first iteration.

Proof. Clearly, we observe that:

$$D(z_k(x,y), \hat{z}_k(x,y)) \leq D(z_k(x,y), F_k(x,y)) + D(F_k(x,y), \hat{F}_k(x,y)) + D(\hat{F}_k(x,y), \hat{z}_k(x,y)). \quad (6.4)$$

Using Eq (5.7), we obtain:

$$D(z_k(x,y), F_k(x,y)) \leq \frac{1}{1-B} \left(\frac{L_1'(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2'(b-a)(d-c)^2}{2\sqrt{n}} \right). \quad (6.5)$$

Also, by similar reasoning we have:

$$D(\hat{F}_k(x,y), z_k(x,y)) \leq \frac{1}{1-B} \left(\frac{L_1''(b-a)^2(d-c)}{2\sqrt{m}} + \frac{L_2''(b-a)(d-c)^2}{2\sqrt{n}} \right). \quad (6.6)$$

In order to obtain the upper bound of $D(F_k(x,y), \hat{F}_k(x,y))$, we observe that

$$D(F_0(x,y), \hat{F}_0(x,y)) < \epsilon, \quad \forall (x,y) \in [a,b] \times [c,d],$$

and thus

$$D(F_1(x,y), \hat{F}_1(x,y)) = D(f(x,y), \hat{f}(x,y)) + D\Big((FR) \int_c^d (FR) \int_a^b H(x,y,s,t) \odot G(F_0(s,t)) dsdt,$$

$$
\begin{aligned}
&, (FR)\int_c^d (FR)\int_a^b H(x,y,s,t)\odot(G(\hat{F}_0(s,t))dsdt\Big)\\
&\le \epsilon + \int_c^d\int_a^b |H(x,y,s,t)| D\left(G(f(s,t)), G(\hat{f}(s,t)\right) dsdt\\
&\le \epsilon + M_H \int_c^d\int_a^b D\left(G(f(s,t)), G(\hat{f}(s,t))\right) dsdt\\
&\le \epsilon + LM_H(b-a)(d-c)D^*\left(f,\hat{f}\right).
\end{aligned}
$$

Therefor, we obtain:

$$D(F_1(x,y),\hat{F}_1(x,y)) < \epsilon + B\epsilon.$$

By induction for $k \geq 2$, we get:

$$
\begin{aligned}
D(F_k(x,y),\hat{F}_k(x,y)) &\le \epsilon + B\cdot\epsilon + B^2\cdot\epsilon + .. + B^{k-1}\cdot\epsilon + B^k\cdot\epsilon\\
&\le \epsilon\cdot\frac{1-B^{k+1}}{1-B}, \quad \forall x,y\in[a,b]\times[c,d],\ \ k\in\mathbf{N}.
\end{aligned}
$$

According to $B < 1$, this inequality becomes:

$$D\left(F_k(x,y),\hat{F}_k(x,y)\right) < \frac{\epsilon}{1-B}, \quad \forall x,y\in[a,b]\times[c,d],\ \ k\in\mathbf{N}.$$

Then, from eq-stab1, we get:

$$
\begin{aligned}
&D(z_k(x,y),\hat{z}_k(x,y)) \le\\
\le&\frac{\epsilon}{1-B} + \left(\frac{(b-a)^2(d-c)(L_1'+L_1'')}{2(1-B)}\right)\left(\frac{1}{\sqrt{m}}\right) + \left(\frac{(b-a)(d-c)^2(L_2'+L_2'')}{2(1-B)}\right)\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
\tag{6.7}
$$

By comparing inequalities (6.3) and (6.7), we deduce that

$$K_1 = \frac{1}{1-ML},\ \ K_2 = \frac{(b-a)^2(d-c)(L_1'+L_1'')}{2(1-B)},\ \ K_3 = \frac{(b-a)(d-c)^2(L_2'+L_2'')}{2(1-B)},\ \ p=q=\frac{1}{2}.$$

So, the proof is complete. □

7. Numerical experiments

In this section, we present numerical results obtained by applying the proposed algorithm on two examples in order to demonstrate the validity and accuracy of the method. The approximate solution is calculated for different values of k, m and n.

Example 1. Consider

$$F(x,y) = f(x,y)\oplus(FR)\int_0^1(FR)\int_0^1 H(x,y,s,t)\odot(F(s,t))^2dsdt, \ \ (x,y)\in[0,1]^2 \tag{7.1}$$

where

$$H(x,y,s,t) = \frac{xyst}{2}, \qquad s,t,x,y\in[0,1]$$

$$\underline{f}(x,y,r) = rxy - \frac{1}{32}r^2xy, \qquad s,t,r \in [0,1]$$

$$\overline{f}(x,y,r) = (2-r)xy - \frac{1}{32}(2-r)^2xy, \qquad s,t,r \in [0,1].$$

The above fuzzy equation has the exact solution

$$\Big(\underline{F}(x,y,r), \overline{F}(x,y,r)\Big) = \Big(rxy, (2-r)xy\Big).$$

The results of the proposed iterative algorithm in given points are shown in terms of errors in Tables 1 and 2.

Table 1. The accuracy on the level sets for Example 1 in $(x,y) = (0.5, 0.5)$.

	k=10, m=10, n=10		k=15, m=30, n=30	
r-level	$\lvert\underline{F} - \underline{y}_k\rvert$	$\lvert\overline{F} - \overline{y}_k\rvert$	$\lvert\underline{F} - \underline{y}_k\rvert$	$\lvert\overline{F} - \overline{y}_k\rvert$
0	0.0000E+0	7.7426E-3	0.0000E+0	2.4452E-3
0.25	1.0452E-4	5.7966E-3	3.3656E-5	1.8366E-3
0.50	4.2630E-4	4.1667E-3	1.3695E-4	1.3242E-3
0.75	9.7850E-4	2.8324E-3	3.1356E-4	9.0278E-4
1	1.7754E-3	1.7754E-3	5.6742E-4	5.6742E-4

Table 2. The accuracy on the level sets for Example 1 in $(x,y) = (0.8, 0.8)$.

	k=10, m=10, n=10		k=15, m=30, n=30	
r-level	$\lvert\underline{F} - \underline{y}_k\rvert$	$\lvert\overline{F} - \overline{y}_k\rvert$	$\lvert\underline{F} - \underline{y}_k\rvert$	$\lvert\overline{F} - \overline{y}_k\rvert$
0	0.0000E+0	8.0781E-3	0.0000E+0	4.7666E-3
0.25	2.3896E-4	6.9781E-3	8.5789E-5	3.8241E-3
0.50	8.6241E-4	5.4650E-3	3.4468E-4	2.9150E-3
0.75	1.7319E-3	3.6661E-3	7.7281E-4	2.0815E-3
1	2.7111E-3	2.7111E-3	1.3583E-3	1.3583E-3

This method well performs for linear two dimensional fuzzy integral equation.

Example 2. Consider

$$F(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t) \odot F(s,t)dsdt, \quad (x,y) \in [0,1]^2 \tag{7.2}$$

where

$$H(x,y,s,t) = \frac{xs+yt}{10}, \qquad s,t,x,y \in [0,1]$$

$$\underline{f}(x,y,r) = rxy - \frac{1}{60}rx - \frac{1}{60}ry, \qquad s,t,r \in [0,1]$$

$$\overline{f}(x,y,r) = 2xy - rxy - \frac{1}{3}x - \frac{1}{3}y + \frac{1}{60}rx + \frac{1}{60}ry, \qquad s,t,r \in [0,1].$$

The above fuzzy equation has the exact solution

$$\Big(\underline{F}(x,y,r),\overline{F}(x,y,r)\Big) = \Big(rxy,(2-r)xy\Big).$$

The results of the proposed iterative algorithm in given points are shown in terms of errors in Tables 3 and 4.

Table 3. The accuracy on the level sets for Example 2 in $(x, y) = (0.5, 0.5)$.

	k=10, m=10, n=10		k=10, m=30, n=30	
r-level	$\lvert\underline{F}-\underline{y}_k\rvert$	$\lvert\overline{F}-\overline{y}_k\rvert$	$\lvert\underline{F}-\underline{y}_k\rvert$	$\lvert\overline{F}-\overline{y}_k\rvert$
0	0.0000E+0	1.7731E-3	0.0000E+0	5.9032E-4
0.25	2.2163E-4	1.5514E-3	7.3790E-5	5.1653E-4
0.50	4.4326E-4	1.3298E-3	1.4758E-4	4.4274E-4
0.75	6.6489E-4	1.1082E-3	2.2137E-4	3.6895E-4
1	8.8652E-4	8.8652E-4	2.9516E-4	2.9516E-4

Table 4. The accuracy on the level sets for Example 2 in $(x, y) = (0.8, 0.8)$.

	k=10, m=10, n=10		k=10, m=30, n=30	
r-level	$\lvert\underline{F}-\underline{y}_k\rvert$	$\lvert\overline{F}-\overline{y}_k\rvert$	$\lvert\underline{F}-\underline{y}_k\rvert$	$\lvert\overline{F}-\overline{y}_k\rvert$
0	0.0000E+0	2.6347E-3	0.0000E+0	7.5628E-4
0.25	3.2933E-4	2.3053E-3	9.4535E-5	6.6174E-4
0.50	6.5867E-4	1.9760E-3	1.8907E-4	5.6721E-4
0.75	9.8800E-4	1.6467E-3	2.8361E-4	4.7268E-4
1	1.3173E-3	1.3173E-3	3.7814E-4	3.7814E-4

8. Conclusions

In this paper, we have developed an numerical iterative method for the solution of two dimensional fuzzy Fredholm integral equations of the second kind. The method presented in this work was applied Picard iterations and numerical approximation to evaluate the iterated integral based on bivariate Bernstein polynomials. Moreover, we have considered the conditions to prove the sequence of approximations of the solution at given point converge to the exact solution. The concept of numerical stability is defined based on the choice of the first iteration and then the numerical stability of the proposed iterative algorithm is proven. Finally, some numerical examples are included to examine the accuracy and the convergence of the proposed method.

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. S. Abbasbandy, T. Allahviranloo, The Adomian decomposition method applied to the fuzzy system of Fredholm integral equations of the second kind, *Int. J. Uncertain. Fuzz.*, **14** (2006), 101–110. https://doi.org/10.1142/S0218488506003868
2. S. Abbasbandy, E. Babolian, M. Alavi, Numerical method for solving linear Fredholm fuzzy integral equations of the second kind, *Chaos Soliton. Fract.*, **31** (2007), 138–146. https://doi.org/10.1016/j.chaos.2005.09.036
3. K. Akhavan Zakeri, S. Ziari, M. A. Fariborzi Araghi, I. Perfilieva, Efficient numerical solution to a bivariate nonlinear fuzzy fredholm integral equation, *IEEE T. Fuzzy Syst.*, 2019. https://doi.org/10.1109/TFUZZ.2019.2957100
4. G. A. Anastassiou, Fuzzy Mathematics: Approximation Theory. Springer, Berlin, 2010. https://doi.org/10.1007/978-3-642-11220-1
5. H. Attari, Y. Yazdani, A computational method for for fuzzy Volterra-Fredholm integral equations, *Fuzzy Inf. Eng.*, **2** (2011), 147–156. https://doi.org/10.1007/s12543-011-0073-x
6. E. Babolian, H. Sadeghi Goghary, S. Abbasbandy, Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method, *Appl. Math. Comput.*, **161** (2005), 733–744. https://doi.org/10.1016/j.amc.2003.12.071
7. M. Baghmisheh, R. Ezzati, Numerical solution of nonlinear fuzzy Fredholm integral equations of the second kind using hybrid of block-pulse functions and Taylor series, *Adv. Differ. Equ-Ny*, **51** (2015).
8. M. Baghmisheh, R. Ezzati, Error estimation and numerical solution of nonlinear fuzzy Fredholm integral equations of the second kind using triangular functions, *J. Intell. Fuzzy Syst.*, **30** (2016), 639–649. https://doi.org/10.3233/IFS-151783
9. K. Balachandran, P. Prakash, Existence of solutions of nonlinear fuzzy Volterra-Fredholm integral equations, *Indian J. Pure Ap. Mat.*, **33** (2002), 329–343.
10. K. Balachandran, K. Kanagarajan, Existence of solutions of general nonlinear fuzzy Volterra-Fredholm integral equations, *J. Appl. Math. Stochastic Anal.*, **3** (2005), 333–343. https://doi.org/10.1155/JAMSA.2005.333
11. B. Bede, S. G. Gal, Quadrature rules for integrals of fuzzy-number-valued functions, *Fuzzy Set. Syst.*, **145** (2004), 359–380. https://doi.org/10.1016/S0165-0114(03)00182-9
12. A. M. Bica, Error estimation in the approximation of the solution of nonlinear fuzzy Fredholm integral equations, *Inf. Sci.*, **178** (2008), 1279–1292. https://doi.org/10.1016/j.ins.2007.10.021
13. A. M. Bica, C. Popescu, Approximating the solution of nonlinear Hammerstein fuzzy integral equations, *Fuzzy Set. Syst.*, **245** (2014), 1–17. https://doi.org/10.1016/j.fss.2013.08.005
14. A. M. Bica, Algebraic structures for fuzzy numbers from categorial point of view, *Soft Comput.*, **11** (2007), 1099–1105. https://doi.org/10.1007/s00500-007-0167-x
15. A. M. Bica, S. Ziari, Iterative numerical method for solving fuzzy Volterra linear integral equations in two dimensions, *Soft Comput.* **21** (2017), 1097–1108. https://doi.org/10.1007/s00500-016-2085-2

16. A. M. Bica, C. Popescu, Fuzzy trapezoidal cubature rule and application to two-dimensional fuzzy Fredholm integral equations, *Soft Comput.*, **21** (2017), 1229–1243.

17. A. M. Bica, S. Ziari, Open fuzzy cubature rule with application to nonlinear fuzzy Volterra integral equations in two dimensions, *Fuzzy Set. Syst.*, **358** (2019), 108–131. https://doi.org/10.1016/j.fss.2018.04.010

18. Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations, *Chaos Soliton. Fract.*, **38** (2008), 112–119. https://doi.org/10.1016/j.chaos.2006.10.043

19. W. Congxin, W. Cong, The supremum and infimum of the set of fuzzy numbers and its applications, *J. Math. Anal. Appl.*, **210** (1997), 499–511. https://doi.org/10.1006/jmaa.1997.5406

20. D. Dubois, H. Prade, Fuzzy numbers: An overview. In: Analysis of Fuzzy Information, CRC Press, BocaRaton, **1** (1987), 3–39.

21. E. H. Dohaa, A. H. Bhrawyb, M. A. Saker, Integrals of Bernstein polynomials: An application for the solution of high even-order differential equations, *Appl. Math. Lett.*, **24** (2011), 559–565. https://doi.org/10.1016/j.aml.2010.11.013

22. R. Ezzati, S. Ziari, Numerical solution and error estimation of fuzzy Fredholm integral equation using fuzzy Bernstein polynomials, *Aust. J. Basic Appl. Sci.*, **5** (2011), 2072–2082.

23. R. Ezzati, S. Ziari, Numerical solution of nonlinear fuzzy Fredholm integral equations using iterative method, *Appl. Math. Comput.*, **225** (2013), 33–42. https://doi.org/10.1016/j.amc.2013.09.020

24. R. Ezzati, S. Ziari, Numerical solution of two-dimensional fuzzy Fredholm integral equations of the second kind using fuzzy bivariate Bernstein polynomials, *Int. J. Fuzzy Syst.*, **15** (2013), 84–89. https://doi.org/10.1017/S1466046612000579

25. J. X. Fang, Q. Y. Xue, Some properties of the space fuzzy-valued continuous functions on a compact set, *Fuzzy Set. Syst.*, **160** (2009), 1620–1631. https://doi.org/10.1016/j.fss.2008.07.014

26. M. A. Fariborzi Araghi, N. Parandin, Numerical solution of fuzzy Fredholm integral equations by the Lagrange interpolation based on the extension principle, *Soft Comput.*, **15** (2011), 2449–2456. https://doi.org/10.1007/s00500-011-0706-3

27. M. Friedman, M. Ma, A. Kandel, Numerical solutions of fuzzy differential and integral equations, *Fuzzy Set. Syst.*, **106** (1999), 35–48. https://doi.org/10.1016/S0165-0114(98)00355-8

28. M. Friedman, M. Ma, A. Kandel, Solutions to fuzzy integral equations with arbitrary kernels, *Int. J. Approximating Reasoning*, **20** (1999), 249–262. https://doi.org/10.1016/S0888-613X(99)00005-5

29. S. G. Gal, Approximation theory in fuzzy setting, In: Anastassiou, GA (ed.) Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton, 2000, 617–666.

30. R. Goetschel, W. Voxman, Elementary fuzzy calculus, *Fuzzy Set. Syst.*, (1986), 31–43. https://doi.org/10.1016/0165-0114(86)90026-6

31. Z. Gong, C. Wu, Bounded variation absolute continuity and absolute integrability for fuzzy-number-valued functions, *Fuzzy Set. Syst.*, **129** (2002), 83–94. https://doi.org/10.1016/S0165-0114(01)00132-4

32. O. Kaleva, Fuzzy differential equations, *Fuzzy Set. Syst.*, **24** (1987), 301–317. https://doi.org/10.1016/0165-0114(87)90029-7

33. S. Karamseraji, R. Ezzati, S. Ziari, Fuzzy bivariate triangular functions with application to nonlinear fuzzy Fredholm–Volterra integral equations in two dimensions, *Soft Comput.*, **24** (2020), 9091–9103. https://doi.org/10.1007/s00500-019-04439-9

34. A. Molabahrami, A. Shidfar, A. Ghyasi, An analytical method for solving linear Fredholm fuzzy integral equations of the second kind, *Comput. Math. Appl.*, **61** (2011), 2754–2761. https://doi.org/10.1016/j.camwa.2011.03.034

35. N. Parandin, M. A. Fariborzi Araghi, The numerical solution of linear fuzzy Fredholm integral equations of the second kind by using finite and divided differences methods, *Soft Comput.*, **15** (2010), 729–741. https://doi.org/10.1007/s00500-010-0606-y

36. J. Y. Park, H. K. Han, Existence and uniqueness theorem for a solution of fuzzy Volterra integral equations, *Fuzzy Set. Syst.*, **105** (1999), 481–488. https://doi.org/10.1016/S0165-0114(97)00238-8

37. J. Y. Park, J. U. Jeong, On the existence and uniqueness of solutions of fuzzy Volttera-Fredholm integral equations, *Fuzzy Set. Syst.*, **115** (2000), 425–431. https://doi.org/10.1016/S0165-0114(98)00341-8

38. J. Y. Park, S. Y. Lee, J. U. Jeong, The approximate solution of fuzzy functional integral equations, *Fuzzy Set. Syst.*, **110** (2000), 79–90. https://doi.org/10.1016/S0165-0114(98)00008-6

39. A. Rivaz, F. Yousefi, H. Salehinejad, Using block pulse functions for solving two-dimensional fuzzy Fredholm integral equations of the second kind, *Int. J. Appl. Math.*, **25** (2012), 571–582. https://doi.org/10.1590/S0103-51502012000300013

40. S. M. Sadatrasoul, R. Ezzati, Quadrature rules and iterative method for numerical solution of two-dimensional fuzzy integral equations, *Abstract Appl. Anal.*, 2014, https://doi.org/10.1155/2014/413570

41. S. M. Sadatrasoul, R. Ezzati, Iterative method for numerical solution of two-dimensional nonlinear fuzzy integral equations, *Fuzzy Set. Syst.*, **280** (2015), 91–106. https://doi.org/10.1016/j.fss.2014.12.008

42. S. M. Sadatrasoul, R. Ezzati, Numerical solution of two-dimensional nonlinear Hammerstein fuzzy integral equations based on optimal fuzzy quadrature formula, *J. Comput. Appl. Math.*, **292** (2016), 430–446. https://doi.org/10.1016/j.cam.2015.07.023

43. V. Samadpour Khalifeh Mahaleh, R. Ezzati, Numerical solution of linear fuzzy Fredholm integral equations of second kind using iterative method and midpoint quadrature formula, *J. Int. Fuzzy Syst.*, **33** (2017), 1293 –1302. https://doi.org/10.3233/JIFS-162044

44. P. V. Subrahmanyam, S. K. Sudarsanam, A note on fuzzy Volterra integral equations, *Fuzzy Set. Syst.*, **81** (1996), 237–240. https://doi.org/10.1016/0165-0114(95)00180-8

45. C. Wu, S. Song, H. Wang, On the basic solutions to the generalized fuzzy integral equation, *Fuzzy Set. Syst.*, (1998), 255–260.

46. C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions (I), *Fuzzy Set. Syst.*, **102** (2001), 523–532. https://doi.org/10.1016/S0165-0114(99)00057-3

47. H. Yang, Z. Gong, Ill-posedness for fuzzy Fredholm integral equations of the first kind and regularization methods, *Fuzzy Set. Syst.*, **358** (2019), 132–149. https://doi.org/10.1016/j.fss.2018.05.010
48. S. Ziari, R. Ezzati, S. Abbasbandy, Numerical solution of linear fuzzy Fredholm integral equations of the second kind using fuzzy Haar wavelet. In: Advances in Computational Intelligence. Communications in Computer and Information Science, **299** (2012), 79–89.
49. S. Ziari, A. M. Bica, New error estimate in the iterative numerical method for nonlinear fuzzy Hammerstein-Fredholm integral equations, *Fuzzy Set. Syst.*, **295** (2016), 136–152. https://doi.org/10.1016/j.fss.2015.09.021
50. S. Ziari, R. Ezzati, S. Abbasbandy, Fuzzy block-pulse functions and its application to solve linear fuzzy Fredholm integral equations of the second kind, In: Advances in Computational Intelligence, Communications in Computer and Information Science, (2016) 821–832.
51. S. Ziari, Iterative method for solving two-dimensional nonlinear fuzzy integral equations using fuzzy bivariate block-pulse functions with error estimation, *Iran J. Fuzzy Syst.*, **15** (2018), 55–76.
52. S. Ziari, Towards the accuracy of iterative numerical methods for fuzzy Hammerstein-Fredholm integral equations, *Fuzzy Set. Syst.*, **375** (2019), 161–178. https://doi.org/10.1016/j.fss.2018.09.006
53. S. Ziari, I. Perfilieva, S. Abbasbandy, Block-pulse functions in the method of successive approximations for nonlinear fuzzy Fredholm integral equations, *Differ. Equ. Dyn. Syst.*, (2019), https://doi.org/10.1007/s12591-019-00482-y.
54. S. Ziari, A. M. Bica, An iterative numerical method to solve nonlinear fuzzy Volterra-Hammerstein integral equations, *J. Intel. Fuzzy Syst.*, **37** (2019), 6717–6729. https://doi.org/10.3233/JIFS-190149
55. S. Ziari, A. M. Bica, R. Ezzati, Iterative fuzzy Bernstein polynomials method for nonlinear fuzzy Volterra integral equations, *Comp. Appl. Math.*, **39** (2020). https://doi.org/10.1007/s40314-020-01361-x.
56. S. Ziari, T. Allahviranloo, W. Pedrycz, An improved numerical iterative method for solving nonlinear fuzzy Fredholm integral equations via Picards method and generalized quadrature rule, *Comp. Appl. Math.*, **40** (2021). https://doi.org/10.1007/s40314-021-01616-1.