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Research article

Global existence of classical solutions for the 2D chemotaxis-fluid system with logistic source

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Abstract: In this paper, we consider the incompressible chemotaxis-Navier-Stokes equations with logistic source in spatial dimension two. We first show a blow-up criterion and then establish the global existence of classical solutions to the system for the Cauchy problem under some rough conditions on the initial data.

Keywords: chemotaxis; Navier-Stokes; blow-up; classical solutions; Cauchy problem
Mathematics Subject Classification: 35K55, 35Q92, 35Q35, 92C17

1. Introduction

The coupled chemotaxis-Navier-Stokes model with logistic source terms is as following: $Q_T = (0, T] \times \mathbb{R}^2$

$$\begin{cases} n_t + u \cdot \nabla n = \mathcal{D}_n \Delta n - \nabla \cdot (n\chi(c)\nabla c) + \kappa n - \mu n^2, & (x,t) \in Q_T, \\ c_t + u \cdot \nabla c = \mathcal{D}_c \Delta c - g(c)n, & (x,t) \in Q_T, \\ u_t + (u \cdot \nabla)u - \nabla P = \mathcal{D}_u \Delta u + n\nabla\Phi, & (x,t) \in Q_T, \\ \nabla \cdot u = 0, & (x,t) \in Q_T. \end{cases} \tag{1.1}$$

Here the unknows are the concentration of bacteria $n = n(x,t) : Q_T \to \mathbb{R}^+$; the oxygen concentration $c = c(x,t) : Q_T \to \mathbb{R}^+$; the given vector field $u = u(x,t) : Q_T \to \mathbb{R}^2$; and the pressure of the fluid $P = P(x,t) : Q_T \to \mathbb{R}$. We denote the corresponding diffusion coefficients for the cells, the oxygen and the fluid by $\mathcal{D}_n, \mathcal{D}_c$ and $\mathcal{D}_u$.

Apart from that, $\chi(c)$ represents the chemotactic sensitivity, and $g(c)$ is the consumption rate of oxygen. In [19], it is shown that the functions $g(c)$ and $\chi(c)$ are constants at large c and rapidly approach zero below some critical c^*. Moreover, the experimentalists in [19] used multiples of the Heaviside step function to model $\chi(\cdot)$ and $g(\cdot)$. The scalar valued function Φ is also given.

The model (1.1) was proposed in [12], which in order to describe the movement of bacteria in response to the presence of a chemical signal substance (oxygen or another nutrient) in their fluid environment. κ is the effective growth rate of the population and μ controlling death by overcrowding. For more the logistic term related, see [10, 11, 24, 35].

In recent years, the study of the dynamics of solutions to (1.1) has attracted many researchers. Finite time blow-up phenomena is one of the most important dynamical problems in (1.1). One can refer to [3,30]. It is worth pointing that the logistic term $\kappa n-\mu n^2$ of (1.1) can avoid blow-up phenomena in [9].

To make the system (1.1) be well-posed, we add some initial conditions as follows:

$$(n, c, u)|_{t=0} = (n_0(x), c_0(x), u_0(x)), \quad x \in \mathbb{R}^2. \tag{1.2}$$

Besides, we also have the corresponding chemotaxis-only subsystem of (1.1) on letting $u \equiv 0$, that is, the system

$$\begin{cases} n_t = \mathcal{D}_n \Delta n - \nabla \cdot (n\chi(c)\nabla c) + \kappa n - \mu n^2, \\ c_t = \mathcal{D}_c \Delta c - c + n, \end{cases} \tag{1.3}$$

is a variant of the classical Keller-Segel model. Up to now, one notices that the homogeneity of model (1.3) may be undermined by the cross-diffusive term $-\nabla \cdot (n\chi(c)\nabla c)$ and even enforce blow-up of solutions [1]. Although we are facing great challenges, there are still numerous analytic results which mainly fixed attention on the local and global solvability of corresponding initial(-boundary)-value problems in either bounded or unbounded domains Ω, under diverse technical conditions on $\chi(\cdot)$ and $g(\cdot)$.

As for initial boundary value problem (1.3) in higher space dimensions, Winkler [22] proved the global existence of a small solution to (1.3) and that it asymptotically behaves like the solution of a decoupled system of linear parabolic equations. On the other hand, a result concerning that blow-up behavior occurring for some radially symmetric positive initial data in higher dimension was recently obtained in [23]. And one can see a relevant reference in [25].

If system (1.3) is coupled with Navier-Stokes equations, we have the following model

$$\begin{cases} n_t + u \cdot \nabla n = \mathcal{D}_n \Delta n - \nabla \cdot (n\chi(c)\nabla c) + \kappa n - \mu n^2, \\ c_t + u \cdot \nabla c = \mathcal{D}_c \Delta c - c + n, \\ u_t + u \cdot \nabla u = \mathcal{D}_u \Delta u - \nabla P - n\nabla\phi + f(x, t). \end{cases} \tag{1.4}$$

As far as we know, results on smooth global solvability could be established for the two-dimensional version of (1.4) whenever μ is positive ([18]), while a similar statement could be derived when $\mu \geq 23$ in three dimension at least for a Stokes simplification of (1.4) in which the nonlinear convective term $u \cdot \nabla u$ is neglected [17]. In a recent paper [31], Winkler reveals that whenever $\omega > 0$, requiring that

$$\frac{\kappa}{\min\{\mu, \mu^{\frac{3}{2}+\omega}\}} < \eta$$

with some $\eta = \eta(\omega) > 0$, and that f satisfies a suitable assumption on ultimate smallness, is sufficient to ensure that each of these generalized solutions becomes eventually smooth and classical.

For system (1.1), when $\kappa = \mu = 0$, [6] demonstrates that there exists a global weak solutions for the 2D chemotaxis-Stokes equations, *i.e.*, the nonlinear convective term $u \cdot \nabla u$ is removed in the fluid equation of (1.1), by making use of quasi-energy functionals associated with (1.1), under the following conditions on $\chi(\cdot)$ and $g(\cdot)$:

$$\chi(c) > 0, \quad \chi^{'}(c) \geq 0, \quad g(0) = 0, \quad g^{'}(c) > 0, \quad \frac{d^2}{dc^2}\Big(\frac{g(c)}{\chi(c)}\Big) < 0, \tag{1.5}$$

which were relaxed in [8, 13],

$$\begin{aligned} &\chi(c) \geq 0, \quad \chi^{'}(c) \geq 0, \quad g(c) \geq 0, \quad g^{'}(c) \geq 0, \\ &\frac{d^2}{dc^2}\Big(\frac{g(c)}{\chi(c)}\Big) \leq 0, \quad \frac{\chi^{'}(c)g(c) + \chi(c)g^{'}(c)}{\chi(c)} > 0, \end{aligned} \tag{1.6}$$

for the fully chemotaxis-Navier-Stokes system (1.1) without any smallness conditions on the initial data.

Furthermore, for the smooth initial data, Chae et al. establish the global existence of smooth solutions [3] by assuming that $\chi(c)$ and $g(c)$ satisfy

$$\chi(c) \geq 0, \quad \chi^{'}(c) \geq 0, \quad g(c) \geq 0, \quad g^{'}(c) \geq 0, \quad g(0) = 0, \tag{1.7}$$

and that there exists a constant η such that

$$\sup_{c\geq 0} |\chi(c) - \eta g(c)| < \epsilon \quad \text{for a sufficiently small} \quad \epsilon > 0, \tag{1.8}$$

which is taken away in [4]. For the full 3D chemotaxis-Navier-Stokes system, the global classical solution near constant steady states and the global weak solutions under the special situation that $\chi(\cdot)$ precisely coincides with a fixed multiple of $g(\cdot)$ are constructed in [6] and [3], respectively.

In the case of $\kappa = \mu = 0$, Winkler establishes the existence of global classical solutions in 2D bounded convex domains Ω in [26]. Then he [27] asserts that a solution of the two-dimensional chemotaxis-Navier-Stokes system stabilizes to the spatially uniform equilibrium $(\bar{n}, 0, 0)$ with $\bar{n_0} = \frac{1}{|\Omega|}\int_\Omega n_0(x)dx$ in the sense of $L^\infty(\Omega)$. If $\kappa \geq 0$, $\mu > 0$, in a bounded smooth domain $\Omega \subset \mathbb{R}^3$, J. Lankeit [12] shows that after some waiting time weak solutions constructed become smooth and finally converge to the semi-trivial steady state $(\frac{\kappa}{\mu}, 0, 0)$. More related paper on the bounded domain case, for the 2D case, one may refer to [16, 20, 21] and for the 3D case, one also refer to [2, 16, 21, 28, 29]. Additionally, under some strong structural assumptions on χ and g , the global existence of weak solutions is proved as well as their eventual smoothness and stabilization to the 3D version of the chemotaxis-Navier-Stokes system is established in [28] and [29].

Recently, Duan et al. [7] established the global existence of weak solutions and classical solutions for both the Cauchy problem and the initial-boundary value problem supplemented with some initial data. In addition, Wu et al. [32] improved the results by taking more careful calculations than those of [7].

The main purpose of this paper is to demonstrate the global existence of classical solutions to (1.1) for the Cauchy problem. We first introduce the corresponding blowup criterion.

Now, it is position to state our main theorems in this article.

Theorem 1.1. *Let $m \geq 3$. Assume that $\chi(\cdot)$, $g(\cdot) \in C^m(\mathbb{R})$ with $g(0) = 0$, and that $\|\nabla^l \Phi\|_{L^\infty(\mathbb{R}^2)} < \infty$ for $1 \leq |l| \leq m$.*

(A1) *Then there exists $T^* > 0$, the maximal time of existence, such that, if the initial data $(n_0, c_0, u_0) \in H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2; \mathbb{R}^2)$, then there exists a unique classical solution (n, c, u) to system (1.1)-(1.2) satisfying for any $T < T^*$*

$$(n, c, u) \in L^\infty(0, T; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2; \mathbb{R}^2))$$

and

$$(\nabla n, \nabla c, \nabla u) \in L^2(0, T; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2; \mathbb{R}^2)).$$

(A2) *Moreover, if the maximal time of existence $T^* < \infty$, then*

$$\int_0^{T^*} \|\nabla c(\tau)\|^2_{L^\infty(\mathbb{R}^2)} d\tau = \infty \tag{1.9}$$

The proof of A1 in Theorem 1.1 is standard, one can refer to [3, 33]. We will focus on proving part A2.

Theorem 1.2. *Let $m \geq 3$. Under the assumptions of Theorem 1.1, then system (1.1)-(1.2) admits a unique global-in-time classical solution (n, c, u) satisfying for any $T > 0$*

$$(n, c, u) \in L^\infty(0, T; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2; \mathbb{R}^2))$$

and

$$(\nabla n, \nabla c, \nabla u) \in L^2(0, T; H^{m-1}(\mathbb{R}^2) \times H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2; \mathbb{R}^2)).$$

Before we are going to prove the main results, we give an important proposition, which can be found in [14].

Let

$$X_0 = \{n_0 \in L^1 \cap L^2,\ n_0 > 0, \quad \nabla \sqrt{c_0} \in L^2,\ c_0 \in L^1 \cap L^\infty,\ c_0 > 0, \quad u_0 \in L^2\}.$$

Proposition 1.1. *Let the triple $(n_0, c_0, u_0) \in X_0$, $\nabla \Phi \in L^\infty(\mathbb{R}^2)$ and $\chi(\cdot)$, $g(\cdot) \in C^m(\mathbb{R})$ with $m \geq 3$ and $g(0) = 0$. Then, system (1.1) has a unique global solution (n, c, u) such that*

$$\begin{aligned}
&n \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2));\\
&c \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\mathbb{R}^2));\\
&u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2)) \cap L^2_{loc}(\mathbb{R}^+; H^1(\mathbb{R}^2)).
\end{aligned}$$

The rest of this article is organized as follows. Let's briefly establish a blow-up criterion in Section 2, discuss the Cauchy problem in Section 3.

2. Proof of Theorem 1.1

Now, we show the proof of the blow-up criterion for the fluid chemotaxis equations.

Proof. Above all, we take the L^2 estimate of n into account. Multiplying n on both sides of $(1.1)_1$ and integrating in spaces, by the fact that χ is continuous and c is uniformly bounded, we deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|n\|_{L^2}^2+\mathcal{D}_n\|\nabla n\|_{L^2}^2+\mu\|n\|_{L^3}^3 &\le C\|\chi(c)n\nabla c\|_{L^2}\|\nabla n\|_{L^2}+\kappa\|n\|_{L^2}^2\\ &\le C\|\nabla c\|_{L^\infty}^2\|n\|_{L^2}^2+\frac{1}{4}\|\nabla n\|_{L^2}^2+\kappa\|n\|_{L^2}^2.\end{aligned}$$

Next, testing $-\Delta c$ to both sides of $(1.1)_2$ and making use of the integration by parts, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla c\|_{L^2}^2+\mathcal{D}_c\|\Delta c\|_{L^2}^2\\ \le\ &\|\nabla(u\cdot\nabla c)-(u\cdot\nabla)\nabla c\|_{L^2}\|\nabla c\|_{L^2}+C\|g(c)n\|_{L^2}\|\Delta c\|_{L^2}\\ \le\ &C\|\nabla u\|_{L^\infty}\|\nabla c\|_{L^2}^2+C\|n\|_{L^2}^2+\frac{1}{4}\|\Delta c\|_{L^2}^2.\end{aligned}$$

Similary, taking the L^2 inner product $-\Delta u$ to both sides of $(1.1)_3$ and using the integration by parts, we also deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\mathcal{D}_u\|\Delta u\|_{L^2}^2 &\le \|\nabla((u\cdot\nabla)u)-(u\cdot\nabla)\nabla u\|_{L^2}\|\nabla u\|_{L^2}+C\|n\|_{L^2}\|\Delta u\|_{L^2}\\ &\le C\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2+C\|n\|_{L^2}^2+C\|\Delta u\|_{L^2}^2.\end{aligned}$$

From all the estimates, it means that

$$\begin{aligned}&\frac{d}{dt}(\|n\|_{L^2}^2+\|\nabla c\|_{L^2}^2+\|\nabla u\|_{L^2}^2)+(\|\nabla n\|_{L^2}^2+\|\Delta c\|_{L^2}^2+\|\Delta u\|_{L^2}^2)\\ \le\ &C(1+\|\nabla u\|_{L^\infty}+\|\nabla c\|_{L^\infty}^2)(\|n\|_{L^2}^2+\|\nabla c\|_{L^2}^2+\|\nabla u\|_{L^2}^2).\end{aligned}$$

An application of Gronwall's inequality yields that

$$\begin{aligned}&\sup(\|n\|_{L^2}^2+\|\nabla c\|_{L^2}^2+\|\nabla u\|_{L^2}^2)+\int_0^T(\|\nabla n\|_{L^2}^2+\|\Delta c\|_{L^2}^2+\|\Delta u\|_{L^2}^2)dt\\ \le\ &C(\|n_0\|_{L^2}^2+\|\nabla c_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2)\exp\Big(\int_0^T 1+\|\nabla u\|_{L^\infty}+\|\nabla c\|_{L^\infty}^2\Big).\end{aligned}$$

Observe that $\|n\|_{L^\infty(0,T;L^2)}$ and $\|\nabla n\|_{L^2(0,T;L^2)}$ are uniformly bounded if $\int_0^T\|\nabla u\|_{L^\infty}+\|\nabla c\|_{L^\infty}^2dt$ is bounded. Meanwhile, we notice that $n\in L_x^qL_t^\infty$ and $\nabla n^{\frac{q}{2}}\in L_x^2L_t^2$ for all $2<q<\infty$, and, owing to the following deduction

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|n\|_{L^q}^q+\mathcal{D}_n\|\nabla n^{\frac{q}{2}}\|_{L^2}^2+\mu\int_{R^2}n^{q+1}dx &\le C\int_{R^2}|n\chi(c)\nabla c\nabla n^{q-1}|dx+\kappa\int_{R^2}n^qdx\\ &\le C\|\nabla c\|_{L^\infty}^2\|n\|_{L^q}^q+\frac{1}{2}\|\nabla n^{\frac{q}{2}}\|_{L^2}^2+\kappa\|n\|_{L^q}^q.\end{aligned}$$

Collecting the above inequality, it ensures that $\|n(t)\|_{L^q} \le C$, with C is independent of q. Then, letting $q \to \infty$, $n \in L^\infty_x L^\infty_t$ is obtained.

Additionally, we consider the estimate in the space $(n, c, u) \in H^1 \times H^2 \times H^2$. It follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla n\|^2_{L^2} + \mathcal{D}_n\|\Delta n\|^2_{L^2} + 2\mu\|n^{\frac{1}{2}}\nabla n\|^2_{L^2} \\
\le\ & C\|\nabla(u\cdot\nabla n) - (u\cdot\nabla)\nabla n\|_{L^2}\|\nabla n\|_{L^2} + C\|\chi(c)\nabla n\nabla c\|_{L^2}\|\nabla^2 n\|_{L^2} + C\|\chi(c)n\Delta c\|_{L^2}\|\nabla^2 n\|_{L^2} \\
& + C\|\chi'(c)n\nabla c\nabla c\|_{L^2}\|\nabla^2 n\|_{L^2} + \kappa\|\nabla n\|^2_{L^2} \\
\le\ & C\|\nabla u\|_{L^\infty}\|\nabla n\|^2_{L^2} + C\|\nabla n\|_{L^2}\|\nabla c\|_{L^\infty}\|\nabla^2 n\|_{L^2} + C\|n\|_{L^\infty}\|\Delta c\|_{L^2}\|\nabla^2 n\|_{L^2} \\
& + C\|n\|_{L^\infty}\|\nabla c\|_{L^\infty}\|\nabla c\|_{L^2}\|\nabla^2 n\|_{L^2} + \kappa\|\nabla n\|^2_{L^2} \\
\le\ & C\|\nabla u\|_{L^\infty}\|\nabla n\|^2_{L^2} + C\|\nabla n\|^2_{L^2}\|\nabla c\|^2_{L^\infty} + \frac{1}{8}\|\nabla^2 n\|^2_{L^2} + C\|n\|^2_{L^\infty}\|\Delta c\|^2_{L^2} + \frac{1}{8}\|\nabla^2 n\|^2_{L^2} \\
& + C\|n\|^2_{L^\infty}\|\nabla c\|^2_{L^\infty}\|\nabla c\|^2_{L^2} + \frac{1}{8}\|\nabla^2 n\|^2_{L^2} + \kappa\|\nabla n\|^2_{L^2}.
\end{aligned}$$

In conjunction with Young's inequality and Gronwall's inequality, we conclude

$$\begin{aligned}
&\sup\|\nabla n\|^2_{L^2} + \int_0^T \|\nabla^2 n\|^2_{L^2}dt \\
\le\ & \Big(\|\nabla n_0\|^2_{L^2} + C\|n\|^2_{L^\infty(0,T;L^\infty)}\Big(\int_0^T \|\Delta c\|^2_{L^2}dt + \|\nabla c\|^2_{L^\infty(0,T;L^2)}\int_0^T \|\nabla c\|^2_{L^\infty}dt\Big)\Big) \\
& \times\exp\Big(\int_0^T 1 + \|\nabla u\|_{L^\infty} + \|\nabla c\|^2_{L^\infty}dt\Big).
\end{aligned}$$

Afterwards, $n \in H^1_x L^\infty_t \cap H^2_x L^2_t$. For the H^2 estimate of c, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta c\|^2_{L^2} + \mathcal{D}_c\|\nabla\Delta c\|^2_{L^2} \le\ & C\|\nabla u\|_{L^\infty}\|\Delta c\|^2_{L^2} + C\|\Delta u\|_{L^2}\|c\|_{L^\infty}\|\nabla\Delta c\|_{L^2} \\
& + C\|g'(c)n\nabla c\|_{L^2}\|\nabla\Delta c\|_{L^2} + C\|g(c)n\nabla c\|_{L^2}\|\nabla\Delta c\|_{L^2} \\
\le\ & C\|\nabla u\|_{L^\infty}\|\Delta c\|^2_{L^2} + C\|\Delta u\|_{L^2}\|c\|_{L^\infty}\|\nabla\Delta c\|_{L^2} \\
& + C\|\nabla c\|_{L^2}\|n\|_{L^\infty}\|\nabla\Delta c\|_{L^2} + C\|\nabla n\|_{L^2}\|\nabla\Delta c\|_{L^2} \\
\le\ & C\|\nabla u\|_{L^\infty}\|\Delta c\|^2_{L^2} + C\|\Delta u\|^2_{L^2}\|c\|^2_{L^\infty} + \frac{1}{8}\|\nabla\Delta c\|^2_{L^2} \\
& + C\|\nabla c\|^2_{L^2}\|n\|^2_{L^\infty} + \frac{1}{8}\|\nabla\Delta c\|^2_{L^2} + C\|\nabla n\|^2_{L^2} + \frac{1}{8}\|\nabla\Delta c\|^2_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2_{L^2} + \mathcal{D}_u\|\nabla\Delta u\|^2_{L^2} \le\ & C\|\nabla u\|_{\infty}\|\nabla u\|^2_{L^2} + C\|\Delta n\|_{L^2}\|\Delta u\|_{L^2} \\
\le\ & C\|\nabla u\|_{\infty}\|\nabla u\|^2_{L^2} + C\|\Delta n\|^2_{L^2} + \frac{1}{2}\|\Delta u\|^2_{L^2}.
\end{aligned}$$

Then we get $c \in H^2_x L^\infty_t \cap H^3_x L^2_t$ and $u \in H^2_x L^\infty_t \cap H^3_x L^2_t$ by Gronwall's inequality. In what follows, we show the estimate in the space $(n, c, u) \in H^2 \times H^3 \times H^3$. In a similar way, it indicates

$$\frac{1}{2}\frac{d}{dt}\|n\|^2_{H^2} + \mathcal{D}_n\|\nabla n\|^2_{H^2} \le C\|u\|_{L^\infty}\|n\|_{H^2}\|\nabla n\|_{H^2} + C\|\nabla u\|_{L^\infty}\|n\|_{H^1}\|\nabla n\|_{H^2}$$

$$
\begin{aligned}
& C\|\chi(c)n\nabla c\|_{H^2}\|\nabla n\|_{H^2}+\kappa\|n\|_{H^2}^2+C\|n\|_{L^\infty}\|n\|_{H^2}^2 \\
\leq\ & C\|u\|_{L^\infty}^2\|n\|_{H^2}^2+C\|\nabla u\|_{L^\infty}^2\|n\|_{H^1}^2+C\|\chi(c)n\nabla c\|_{H^2}^2 \\
& +\kappa\|n\|_{H^2}^2+C\|n\|_{L^\infty}\|n\|_{H^2}^2+\frac{1}{2}\|\nabla n\|_{H^2}^2.
\end{aligned}
$$

We control

$$\|\chi(c)n\nabla c\|_{H^2}\leq C\|n\|_{H^2}\|\chi(c)\nabla c\|_{H^2},$$

and

$$\|\nabla^2(\chi(c)\nabla c)\|_{L^2}\leq C\|\nabla^3 c\|_{L^2}+C\|\nabla^2 c\|_{L^2}\|\nabla c\|_{L^\infty}+C\|\nabla c\|_{L^6}^3.$$

We also obtained $c\in H_x^2L_t^\infty\cap H_x^3L_t^2$. Consequently, in view of Young's inequality and Gronwall's inequality, it follows that

$$\sup\|n\|_{H^2}^2+\int_0^T\|\nabla n\|_{H^2}^2dt\leq\|n_0\|_{H^2}^2\exp\Big(C+\int_0^T 1+\|\nabla u\|_{L^\infty}+\|\nabla c\|_{L^\infty}^2dt\Big).$$

In what follows, for the estimate of c, we further have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|c\|_{H^3}^2+\mathcal{D}_c\|\nabla c\|_{H^3}^2 \leq\ & C\|\nabla u\|_{L^\infty}\|c\|_{H^3}\|\nabla c\|_{H^3}+C\|u\|_{H^3}\|\nabla c\|_{L^\infty}\|\nabla c\|_{H^3} \\
& +C\|g(c)n\|_{H^2}^2+\frac{1}{4}\|\nabla c\|_{H^3}^2 \\
\leq\ & C\|\nabla u\|_{L^\infty}^2\|c\|_{H^3}^2+C\|u\|_{H^3}^2\|\nabla c\|_{L^\infty}^2 \\
& +C(\|c\|_{H^2}^2\|n\|_{H^2}^2+\|c\|_{H^1}^2\|\nabla c\|_{L^\infty}^2\|n\|_{H^2}^2)+\frac{1}{2}\|\nabla c\|_{H^3}^2.
\end{aligned}
$$

For the estimate of u, we thereby obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{H^3}^2+\mathcal{D}_u\|\nabla u\|_{H^3}^2 \leq\ & C\|\nabla u\|_{L^\infty}\|u\|_{H^3}\|\nabla u\|_{H^3}+C\|n\|_{H^2}\|\nabla u\|_{H^3} \\
\leq\ & C\|\nabla u\|_{L^\infty}^2\|u\|_{H^3}^2+C\|n\|_{H^2}^2+\frac{1}{2}\|\nabla u\|_{H^3}^2.
\end{aligned}
$$

Applying Gronwall's inequality, it implies $(c,u)\in(H_x^3L_t^\infty\cap H_x^4L_t^2)\times(H_x^3L_t^\infty\cap H_x^4L_t^2)$. Let us demonstrate $H^{m-1}\times H^m\times H^m$ estimates. We've proved that the case $m=2,3$ and 4, and therefore we consider the case $m\geq 5$. Operating $\partial^\alpha(|\alpha|\leq m-1)$ and multiplying $\partial^\alpha n$ on both sides of $(1.1)_1$ and integrating in spaces, we thereby obtain

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|n\|_{H^{m-1}}^2+\mathcal{D}_n\|\nabla n\|_{H^{m-1}}^2 \\
\leq\ & C\|\nabla u\|_{L^\infty}\|n\|_{H^{m-1}}\|\nabla n\|_{H^{m-1}}+C\|u\|_{H^{m-1}}\|\nabla n\|_{L^\infty}\|\nabla n\|_{H^{m-1}} \\
& +C\|\chi(c)n\nabla c\|_{H^{m-1}}\|\nabla n\|_{H^{m-1}}+\kappa\|n\|_{H^{m-1}}^2+C\|n\|_{L^\infty}\|n\|_{H^{m-1}}^2 \\
\leq\ & C\|\nabla u\|_{L^\infty}^2\|n\|_{H^{m-1}}^2+C\|u\|_{H^{m-1}}^2\|\nabla n\|_{L^\infty}^2+C\|\chi(c)n\nabla c\|_{H^{m-1}}^2 \\
& +\kappa\|n\|_{H^{m-1}}^2+C\|n\|_{L^\infty}\|n\|_{H^{m-1}}^2+\frac{1}{2}\|\nabla n\|_{H^{m-1}}^2.
\end{aligned}
$$

The estimate for the case $m = 4$ was already obtained, thus $\|\nabla c\|_{L^\infty(0,T;L^\infty)}$ is bounded. Subsequently, we derive

$$\|\nabla\chi(c)\|_{H^{m-2}} \leq C(1 + \|\nabla c\|_{L^\infty})\|\nabla\chi^{'}(c)\|_{H^{m-3}}.$$

the classical product lemma on each step of iteration shows

$$\|\nabla\chi(c)\|_{H^{m-2}} \leq C(1 + \|\nabla c\|_{L^\infty})^{m-1}.$$

Therefore we also obtain

$$\|\chi(c)n\nabla c\|_{H^{m-1}} \leq C(1 + \|c\|_{H^m} + \|\nabla c\|^m_{L^\infty})\|n\|_{H^{m-1}}$$

applying the product lemma. Add up the inequality above and we get

$$\begin{aligned}\frac{d}{dt}\|n\|^2_{H^{m-1}} + \|\nabla n\|^2_{H^{m-1}} \quad \leq \quad & C\|\nabla u\|^2_{L^\infty}\|n\|^2_{H^{m-1}} + C\|u\|^2_{H^{m-1}}\|\nabla n\|^2_{L^\infty} \\ & + C(1 + \|c\|_{H^m} + \|\nabla c\|^m_{L^\infty})^2\|n\|^2_{H^{m-1}} + \kappa\|n\|^2_{H^{m-1}} \\ & + C\|n\|_{L^\infty}\|n\|^2_{H^{m-1}}.\end{aligned}$$

Similarly, for the H^m estimate of c, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|c\|^2_{H^m} + \mathcal{D}_c\|\nabla c\|^2_{H^m} \quad \leq \quad & C\|\nabla u\|_{L^\infty}\|c\|_{H^m}\|\nabla c\|_{H^m} + C\|u\|_{H^m}\|\nabla c\|_{L^\infty}\|\nabla c\|_{H^m} \\ & + C(1 + \|\nabla c\|^{2m-2}_{L^\infty})\|n\|^2_{H^{m-1}} + \frac{1}{4}\|\nabla c\|^2_{H^m} \\ \leq \quad & C\|\nabla u\|^2_{L^\infty}\|c\|^2_{H^m} + C\|u\|^2_{H^m}\|\nabla c\|^2_{L^\infty} \\ & + C(1 + \|\nabla c\|^{2m-2}_{L^\infty})\|n\|^2_{H^{m-1}} + \frac{1}{2}\|\nabla c\|^2_{H^m}.\end{aligned}$$

For the estimate of u, we also deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{H^m} + \mathcal{D}_u\|\nabla u\|^2_{H^m} \quad \leq \quad & C\|\nabla u\|_{L^\infty}\|u\|_{H^m}\|\nabla u\|_{H^m} + C\|n\|_{H^{m-1}}\|\nabla u\|_{H^m} \\ \leq \quad & C\|\nabla u\|^2_{L^\infty}\|u\|^2_{H^m} + C\|n\|^2_{H^{m-1}} + \frac{1}{2}\|\nabla u\|^2_{H^m}.\end{aligned}$$

Finally, by summing up all the above estimates and utilizing Gronwall's inequality, the fact $(n, c, u) \in (H^{m-1}_x L^\infty_t \cap H^m_x L^2_t) \times (H^m_x L^\infty_t \cap H^{m+1}_x L^2_t) \times (H^m_x L^\infty_t \cap H^{m+1}_x L^2_t)$ is obtained.

As a matter of fact that $\|\nabla c\|_{L^\infty}$ is solely responsible for $n \in L^2_x L^\infty_t$ and $\nabla n \in L^2_x L^2_t$ by the above process. That is

$$\begin{aligned}\frac{d}{dt}\|n\|^2_{L^2} + \|\nabla n\|^2_{L^2} + \mu\int_{\mathbb{R}^2} n^3 dx \quad \leq \quad & C\int_{\mathbb{R}^2} |n\chi(c)\nabla c\nabla n| dx + \kappa\|n\|^2_{L^2} \\ \leq \quad & C\|\nabla c\|^2_{L^\infty}\|n\|^2_{L^2} + \frac{1}{2}\|\nabla n\|^2_{L^2} + \kappa\|n\|^2_{L^2}.\end{aligned}$$

This yields $u \in L^2_x L^\infty_t$ and $\nabla u \in L^2_x L^2_t$ by

$$\frac{d}{dt}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} \leq C\|n\|_{L^2}\|u\|_{L^2} \leq C\|n\|^2_{L^2} + C\|u\|^2_{L^2}.$$

Furthermore, we obtain $n \in L^q_x L^\infty_t$ and $\nabla n^{\frac{q}{2}} \in L^2_x L^2_t$ for all $2 < q < \infty$

$$\begin{aligned}\frac{d}{dt}\|n\|^q_{L^q} + \|\nabla n^{\frac{q}{2}}\|^2_{L^2} + \mu \int_{\mathbb{R}^2} n^{q+1} dx &\leq C_q \int_{\mathbb{R}^2} |n\chi(c)\nabla c \nabla n^{q-1}| dx + \kappa \int_{\mathbb{R}^d} n^q dx \\ &\leq C_q \|\nabla c\|^2_{L^\infty} \|n\|^q_{L^q} + \frac{1}{2}\|\nabla n^{\frac{q}{2}}\|^2_{L^2} + \kappa \|n\|^q_{L^q}.\end{aligned}$$

Besides, we note that $\nabla c \in L^2_x L^\infty_t$ and $\nabla^2 c \in L^2_x L^2_t$. Actually,

$$\begin{aligned}\frac{d}{dt}\|\nabla c\|^2_{L^2} + \|\nabla^2 c\|^2_{L^2} &\leq C\|\nabla c\|_{L^\infty}\|u\|_{L^2}\|\nabla^2 c\|_{L^2} + C\|n\|_{L^2}\|\nabla^2 c\|_{L^2} \\ &\leq C\|\nabla c\|^2_{L^\infty}\|u\|^2_{L^2} + C\|n\|^2_{L^2} + \frac{1}{2}\|\nabla^2 c\|^2_{L^2}.\end{aligned}$$

Finally, we denote vorticity as $\omega := \nabla \times u$; where $\omega = \partial_1 u_2 - \partial_2 u_1$ in two dimensions. Then, let us set $\nabla^\perp n = (-\partial_2 n, \partial_1 n)$. We investigate the vorticity equation

$$\omega_t - \Delta\omega + u\nabla\omega = \nabla^\perp n \nabla\Phi,$$

We notice that $\omega \in L^2_x L^\infty_t$ and $\nabla\omega \in L^2_x L^2_t$, on account of

$$\frac{d}{dt}\|\omega\|^2_{L^2} + \|\nabla\omega\|^2_{L^2} \leq C\|\nabla n\|_{L^2}\|\omega\|_{L^2} \leq C\|\nabla n\|^2_{L^2} + C\|\omega\|^2_{L^2}.$$

In addition, we note that $\nabla\omega \in L^2_x L^\infty_t$ and $\nabla^2\omega \in L^2_x L^2_t$. In fact, testing $-\Delta\omega$, this implies that

$$\begin{aligned}\frac{d}{dt}\|\nabla\omega\|^2_{L^2} + \|\nabla^2\omega\|^2_{L^2} &\leq \|u\|_{L^4}\|\nabla\omega\|_{L^4}\|\Delta\omega\|_{L^2} + \|\nabla n\|_{L^2}\|\Delta\omega\|_{L^2} \\ &\leq C\|u\|^{\frac{1}{2}}_{L^2}\|\nabla u\|^{\frac{1}{2}}_{L^2}\|\nabla\omega\|^{\frac{1}{2}}_{L^2}\|\nabla^2\omega\|^{\frac{3}{2}}_{L^2} + \|\nabla n\|_{L^2}\|\Delta\omega\|_{L^2} \\ &\leq C\|u\|^2_{L^2}\|\nabla u\|^2_{L^2}\|\nabla\omega\|^2_{L^2} + C\|\nabla n\|^2_{L^2} + \frac{1}{2}\|\nabla^2\omega\|^2_{L^2}.\end{aligned}$$

Hence, via embedding, we get

$$\int_0^T \|\nabla u\|_{L^\infty} dt \leq \int_0^T \|\nabla u\|_{H^2} dt \leq C\int_0^T \|\omega\|_{H^2} dt < \infty.$$

This implies the desired result. □

3. Proof of Theorem 1.2

In this section, we will show that the local classical solutions can be extended at any time $T > 0$.

Proof of Theorem 1.2. The process is similar to the proof of Theorem 1.2 in [7]. Using contradictory methods, supposing that the maximal time T^* is finite, we will prove

$$\int_0^{T^*} \|\nabla c(\tau)\|^2_{L^\infty(\mathbb{R}^2)} d\tau < \infty,$$

which contradicts the extensibility criterion (1.9). Actually, according to the fact

$$\|\nabla c(\tau)\|^2_{L^\infty(\mathbb{R}^2)} \leq \widetilde{C}_{GN}\|\nabla c\|_{L^2(\mathbb{R}^2)}\|\nabla^3 c\|_{L^2(\mathbb{R}^2)} \leq \frac{\widetilde{C}_{GN}}{2}(\|\nabla c\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^3 c\|^2_{L^2(\mathbb{R}^2)}),$$

where $\widetilde{C}_{GN}$ is a positive constant resulted from the Gagliardo-Nirenberg inequality.

Now, we just need to verify that

$$\int_0^{T^*}\int_{\mathbb{R}^2} |\nabla c(x,\tau)|^2 dx d\tau < \infty, \tag{3.1}$$

and

$$\int_0^{T^*}\int_{\mathbb{R}^2} |\nabla^3 c(x,\tau)|^2 dx d\tau < \infty. \tag{3.2}$$

According to Proposition 1.1, we obtain (3.1).

As for (3.2), applying Δ to the $(1.1)_2$, multiplying Δc with the resulted equation, and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\Delta c|^2 dx + \mathcal{D}_c \int_{\mathbb{R}^2} |\nabla\Delta c|^2 dx \\ = & \int_{\mathbb{R}^2} (\nabla(u\cdot\nabla c) + g'(c)n\nabla c + g(c)\nabla n)\cdot\nabla\Delta c dx \\ := & I_1 + I_2 + I_3. \end{aligned}$$

For I_1, we get

$$\begin{aligned} I_1 \leq & \frac{1}{\mathcal{D}_c}\int_{\mathbb{R}^2}|\nabla u|^2|\nabla c|^2 dx + \frac{1}{\mathcal{D}_c}\int_{\mathbb{R}^2}|u|^2|D^2 c|^2 dx + \frac{\mathcal{D}_c}{4}\int_{\mathbb{R}^2}|\nabla\Delta c|^2 dx \\ \leq & \frac{1}{\mathcal{D}_c}\|\nabla u\|^2_{L^3(\mathbb{R}^2)}\|\nabla c\|^2_{L^6(\mathbb{R}^2)} + \frac{1}{\mathcal{D}_c}\|u\|^2_{L^\infty(\mathbb{R}^2)}\|D^2 c\|^2_{L^2(\mathbb{R}^2)} + \frac{\mathcal{D}_c}{4}\int_{\mathbb{R}^2}|\nabla\Delta c|^2 dx. \end{aligned}$$

Similarly, let $M = \|c_0\|_{L^\infty}$, we conclude

$$\begin{aligned} I_2 \leq & \frac{1}{\mathcal{D}_c}\sup_{0\leq s\leq M} g'^2(s)\int_{\mathbb{R}^2}|n|^2|\nabla c|^2 dx + \frac{\mathcal{D}_c}{8}\int_{\mathbb{R}^2}|\nabla\Delta c|^2 dx \\ \leq & \frac{1}{\mathcal{D}_c}\sup_{0\leq s\leq M} g'^2(s)\|n\|^2_{L^3(\mathbb{R}^2)}\|\nabla c\|^2_{L^6(\mathbb{R}^2)} + \frac{\mathcal{D}_c}{8}\int_{\mathbb{R}^2}|\nabla\Delta c|^2 dx, \end{aligned}$$

and

$$\begin{aligned} I_3 \leq & \frac{1}{\mathcal{D}_c}\sup_{0\leq s\leq M} g^2(s)\int_{\mathbb{R}^2}|\nabla n|^2 dx + \frac{\mathcal{D}_c}{8}\int_{\mathbb{R}^2}|\nabla\Delta c|^2 dx \\ \leq & \frac{1}{\mathcal{D}_c}\sup_{0\leq s\leq M} g^2(s)\int_{\mathbb{R}^2}|\nabla n|^2 dx + \frac{\mathcal{D}_c}{8}\int_{\mathbb{R}^2}|\nabla\Delta c|^2 dx. \end{aligned}$$

Collecting $I_1 - I_3$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\Delta c|^2 dx + \mathcal{D}_c\int_{\mathbb{R}^2}|\nabla\Delta c|^2 dx$$

$$\leq \quad C_2(\|u\|^2_{H^2(\mathbb{R}^2)} + \|n\|^2_{H^1(\mathbb{R}^2)})\|\nabla c\|^2_{H^1(\mathbb{R}^2)} + C_2 \int_{\mathbb{R}^2} |\nabla n|^2 dx,$$

with C_2 is a positive constant depending on the Sobolev's embedding and the initial data. The Gronwall's inequality implies that

$$\begin{aligned} & \|\nabla c(t)\|^2_{H^1(\mathbb{R}^2)} + \mathcal{D}_c \int_0^{T^*} \|\nabla^2 c(\tau)\|^2_{H^1(\mathbb{R}^2)} d\tau \\ \leq \; & \|c_0\|^2_{H^2(\mathbb{R}^2)} e^{C_2 \int_0^{T^*} (\|n(\tau)\|^2_{H^1(\mathbb{R}^2)} + \|u(\tau)\|^2_{H^2(\mathbb{R}^2)}) d\tau} + C_2 \int_0^{T^*} \|n(\tau)\|^2_{H^1(\mathbb{R}^2)} d\tau, \end{aligned}$$

for all $t \in (0, T^*)$. Therefore we have verified (3.2) provided that

$$\int_0^{T^*} (\|n\|^2_{H^1(\mathbb{R}^2)} + \|u\|^2_{H^1(\mathbb{R}^2)})(\tau) d\tau < \infty.$$

According to Proposition 1.1, we verify the above inequality. For convenience of readers, we give a sketch of the proof. To begin with, taking the L^2-inner product with n for equation $(1.1)_1$, we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} n^2 dx + \mathcal{D}_n \int_{\mathbb{R}^2} |\nabla n|^2 dx + \mu \int_{\mathbb{R}^2} n^3 dx \\ = \; & \int_{\mathbb{R}^2} n\chi(c) \nabla c \cdot \nabla n dx + \kappa \int_{\mathbb{R}^2} n^2 dx \\ \leq \; & \frac{\mathcal{D}_n}{4} \int_{\mathbb{R}^2} |\nabla n|^2 dx + \frac{\sup_{0\leq s\leq M} \chi^2(s)}{\mathcal{D}_n} \int_{\mathbb{R}^2} n^2 |\nabla c|^2 dx + \kappa \int_{\mathbb{R}^2} n^2 dx. \end{aligned}$$

Next, we control $\int_{\mathbb{R}^2} n^2 |\nabla c|^2 dx$ of the above inequality

$$\begin{aligned} \int_{\mathbb{R}^2} n^2 |\nabla c|^2 dx \; & \leq \; \|n\|^2_{L^4(\mathbb{R}^2)} \|\nabla c\|^2_{L^4(\mathbb{R}^2)} \\ & \leq \; C_{GN} \|n\|_{L^2(\mathbb{R}^2)} \|\nabla n\|^2_{L^2(\mathbb{R}^2)} \|c\|_{L^\infty(\mathbb{R}^2)} \|\Delta c\|_{L^2(\mathbb{R}^2)} \\ & \leq \; \frac{\mathcal{D}_n}{4} \int_{\mathbb{R}^2} |\nabla n|^2 dx + C_3 \int_{\mathbb{R}^2} |\Delta c|^2 dx \int_{\mathbb{R}^2} n^2 dx, \end{aligned}$$

with $C_3 = \frac{\sup_{0\leq s\leq M} \chi^4(s) C_{GN}^2 M^2}{\mathcal{D}_n^3}$, by using Sobolev's embedding, Young's inequality and the boundedness of c. Therefore, we have

$$\frac{d}{dt} \int_{\mathbb{R}^2} n^2 dx + \mathcal{D}_n \int_{\mathbb{R}^2} |\nabla n|^2 dx \leq (2C_3 \int_{\mathbb{R}^2} |\Delta c|^2 dx + \kappa) \int_{\mathbb{R}^2} n^2 dx.$$

In view of Gronwall's inequality, we deduce

$$\int_{\mathbb{R}^2} n^2(x,t) dx + \mathcal{D}_n \int_0^{T^*} \int_{\mathbb{R}^2} |\nabla n(x,\tau)|^2 dx \leq C_4,$$

for all $t \in (0, T^*)$ and some positive constant C_4 depending on the initial data and the maximal time T^*, where we also used the inequality

$$\int_0^{T^*} \int_{\mathbb{R}^2} |\Delta c(x,\tau)|^2 dx d\tau < \infty,$$

by recalling Proposition 1.1. A direct integration on $[0, T^*]$ implies that

$$\int_0^{T^*} \|n(\tau)\|_{H^1(\mathbb{R}^2)}^2 d\tau < \infty. \tag{3.3}$$

Similarly, we also have

$$\int_0^{T^*} \|u(\tau)\|_{H^1(\mathbb{R}^2)}^2 d\tau < \infty. \tag{3.4}$$

Let's first investigate the integrability of the second derivative of u. For convenience, let $\omega := \nabla^{\perp} u$ be the vorticity of u and then the vorticity equation as follow:

$$\omega_t + (u \cdot \nabla)\omega = \mathcal{D}_u \Delta\omega + \nabla^{\perp}(n\nabla\Phi).$$

Next, a direct energy method follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\omega|^2 dx + \mathcal{D}_u \int_{\mathbb{R}^2} |\nabla\omega|^2 dx &= \int_{\mathbb{R}^2} \nabla^{\perp}\omega \cdot (n\nabla\Phi) dx \\ &\leq \frac{\mathcal{D}_u}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2 dx + \frac{1}{2\mathcal{D}_u}\int_{\mathbb{R}^2} n^2 |\nabla\Phi|^2 dx.\end{aligned}$$

This leads to the vorticity estimate

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\omega|^2 dx + \mathcal{D}_u \int_{\mathbb{R}^2} |\nabla\omega|^2 dx \leq C_5 \int_{\mathbb{R}^2} n^2 dx,$$

where C_5 is a positive constant depending on $\nabla\Phi$. In conjunction with Gronwall's inequality and (3.3), we infer

$$\int_{\mathbb{R}^2} |\omega|^2(x,t) dx + \mathcal{D}_u \int_0^{T^*}\int_{\mathbb{R}^2} |\nabla\omega(x,\tau)|^2 dx d\tau \leq C_6,$$

for all $t \in (0, T^*)$ with $C_6 := \int_{\mathbb{R}^2} |\omega|^2(x,0)dx + C_4 C_5 T^*$. Application of the above inequality and the Biot-Savart law, we derive

$$\int_0^{T^*} \|\nabla^2 u(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau < \infty. \tag{3.5}$$

Sum up (3.3)-(3.5), it implies the desired result. This completes the Proof of Theorem 1.2. □

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Conflict of interest

All authors declare that there is no interests in this paper.

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A. Appendix

In this appendix, we will give a sketch proof of local existence.

We first introduce the dynamic partition of the unity to define Besov spaces. One may check [15] for more details. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be supported in $C = \{\xi \in \mathbb{R}^d, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\sum_{q \in Z} \varphi(2^{-q}\xi) = 1, \quad \text{for} \quad \xi \neq 0.$$

Defining $\chi(\xi) = 1 - \sum_{q \in N} \varphi(2^{-q}\xi)$. For $f \in \mathcal{S}'$, we set Littlewood-Paley operators as follows

$$\Delta_{-1} f = \chi(D) f;\ \forall q \in \mathbb{N},\ \Delta_q f = \varphi(2^{-q}D) f \quad \text{and} \quad \forall q \in \mathbb{Z},\ \dot{\Delta}_q f = \varphi(2^{-q}D) f.$$

The following low-frequency cut-off will be also used:

$$S_q f = \sum_{-1 \le q' \le q-1} \Delta_{q'} f \quad \text{and} \quad \dot{S}_q f = \sum_{q' \le q-1} \dot{\Delta}_{q'} f.$$

Now, let us recall the definition of the Besov space. For $s \in \mathbb{R}$, $1 \le p, r \le \infty$, we define the homogenous Besov space $\dot{B}^s_{p,r}$ as the set of tempered distributions of $f \in \mathcal{S}'/\mathcal{P}$ such that

$$\|f\|_{\dot{B}^s_{p,r}} := \Big(\sum_{q \in Z} 2^{qsr} \|\dot{\Delta}_q f\|_p^r \Big)^{\frac{1}{r}} < \infty,$$

where $\mathcal{P}$ is the polynomial space. The inhomogeneous space $B^s_{p,r}$ is the set of tempered distribution f such that

$$\|f\|_{B^s_{p,r}} := \Big(\sum_{q\geq -1} 2^{qsr}\|\Delta_q f\|_p^r\Big)^{\frac{1}{r}} < \infty.$$

It is worthwhile to remark that $B^s_{2,2}$ and $B^s_{\infty,\infty}$ coincide with the usual Sobolev spaces H^s and the usual Hölder space C^s for $s \in \mathbb{R} \setminus \mathbb{Z}$, respectively.

In our study, we require the space-time Besov spaces as following manner: for $T > 0$ and $n \geq 1$, we denote by $L^\rho_T B^s_{p,r}$ the set of all tempered distribution f satisfying

$$\|f\|_{L^\rho_T B^s_{p,r}} \triangleq \Big\|(\sum_{q\in\mathbb{Z}} 2^{qsr}\|\dot{\Delta}_q f\|^r_{L^p(\mathbb{R}^d)})^{\frac{1}{r}}\Big\|_{L^\rho_T} < \infty.$$

Lemma A.1. *[15] Let $1 \leq p \leq q \leq \infty$. Assume that $f \in L^p$, then there exists a constant C independent of f, j such that*

$$\text{supp}\hat{f} \subset \{|\xi| \leq C2^j\} \Longrightarrow \|\partial^\alpha f\|_q \leq C2^{j|\alpha|+dj(\frac{1}{p}-\frac{1}{q})}\|f\|_p,$$

$$\text{supp}\hat{f} \subset \Big\{\frac{1}{C}2^j \leq |\xi| \leq C2^j\Big\} \Longrightarrow \|f\|_p \leq C2^{-j|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial^\beta f\|_p.$$

Lemma A.2. *[15] There exists a constant $C > 0$ such that for $S > 0$, we have*

$$\|uv\|_{H^s} \leq C\|u\|_\infty\|v\|_{H^s} + C\|u\|_{H^s}\|v\|_\infty.$$

Lemma A.3. *[5] Let u be a solution of the transport equation*

$$\begin{cases} u_t + v\cdot\nabla u = 0 \\ u(x,0) = u_0 \end{cases}$$

and define $R_q := v\cdot\nabla\Delta_q u - \Delta_q(v\cdot\nabla u)$, $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$ such that

$$s > -d\ \min(\frac{1}{p_1}, \frac{1}{p'}) \quad \Big(\text{or } s > -1 - d\ \min(\frac{1}{p_1}, \frac{1}{p'}) \quad \text{if div } v = 0\Big).$$

There exists a sequence $c_q \in \ell^r(\mathbb{Z})$ such that $\|c_q\|_{\ell^r} = 1$ and a constant C depending only on d, r, s, p, and p_1, which satisfy

$$\forall q \in \mathbb{Z}, 2^{qs}\|R_q\|_p \leq Cc_q Z'(t)\|u\|_{B^s_{p,r}},$$

with

$$Z'(t) := \begin{cases} \|\nabla v\|_{B^{\frac{d}{p_1}}_{p_1,\infty}\cap L^\infty}, & \text{if} \quad s < 1 + \dfrac{d}{p_1}, \\ \|\nabla v\|_{B^{s-1}_{p_1,r}}, & \text{if either} \quad s > 1 + \dfrac{d}{p_1} \quad \text{or} \quad s = 1 + \dfrac{d}{p_1} \quad \text{for} \quad r = 1. \end{cases}$$

Theorem A.4. *Let $s \geq 2$. Assume that $\chi(\cdot)$, $g(\cdot) \in C^s(\mathbb{R})$ with $g(0) = 0$, and that $\|\nabla^l \Phi\|_{L^\infty(\mathbb{R}^2)} < \infty$ for $1 \leq |l| \leq s$.*

(A1) *Then there exists $T^* > 0$, the maximal time of existence, such that, if the initial data $(n_0, c_0, u_0) \in H^{s-1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2)$, then there exists a unique classical solution (n, c, u) to system (1.1)-(1.2) satisfying for any $T < T^*$*

$$(n, c, u) \in L^\infty(0, T; H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2; \mathbb{R}^2))$$

and

$$(\nabla n, \nabla c, \nabla u) \in L^2(0, T; H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2; \mathbb{R}^2)).$$

Proof. We construct the following regularized system:

$$\begin{cases} n_t^k + u^k \cdot \nabla n^k = \Delta n^k - \nabla \cdot (n^k \chi(c^k) \nabla c^k) + n^k - (n^k)^2, k \in \mathcal{N}, \\ c_t^k + u^k \cdot \nabla c^k = \Delta c^k - g(c^k) n^k, \\ u_t^k + (u^k \cdot \nabla) u^k - \nabla P^k = \Delta u^k + n^k \nabla \Phi, \\ \nabla \cdot u^k = 0, \\ (n^k, c^k, u^k)|_{t=0} = (S_k n^0, S_k c^0, S_k u^0). \end{cases} \tag{A.1}$$

Step 1. Uniform estimates.

Taking the operation Δ_q with $q \geq -1$ on the first equation of (A.1), we have

$$\Delta_q n_t^k + \Delta_q (u^k \cdot \nabla n^k) = \Delta \Delta_q n^k - \nabla \cdot \Delta_q (n^k \chi(c^k) \nabla c^k) + \Delta_q n^k - \Delta_q (n^k)^2. \tag{A.2}$$

Multiplying (A.2) by $\Delta_q n^k$ and integrating by parts gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta_q n^k\|_2^2 + \|\nabla \Delta_q n^k\|_2^2 \quad = \quad & -\int_{\mathbb{R}^2} \Delta_q (u^k \cdot \nabla n^k) \Delta_q n^k dx - \int_{\mathbb{R}^2} \nabla \cdot \Delta_q (n^k \chi(c^k) \nabla c^k) \Delta_q n^k dx \\ & + \int_{\mathbb{R}^2} \Delta_q n^k \Delta_q n^k dx - \int_{\mathbb{R}^2} \Delta_q (n^k)^2 \Delta_q n^k dx \\ \leq \quad & \|\Delta_q (u^k \cdot \nabla n^k)\|_2 \|\Delta_q n^k\|_2 + \|\Delta_q (n^k \nabla c^k)\|_2 \|\nabla \Delta_q n^k\|_2 \\ & + \|\Delta_q n^k\|_2^2 + \|\Delta_q (n^k)^2\|_2 \|\Delta_q n^k\|_2. \end{aligned}$$

Multiplying 2^{2qs} on both sides of the above inequality, then taking the ℓ^1 norm, using Hölder's inequality and Young's inequality together with Lemma 4.2, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|n^k\|_{H^s}^2 + \|n^k\|_{H^{s+1}}^2 \quad \leq \quad & \|u^k \cdot \nabla n^k\|_{H^s} \|n^k\|_{H^s} + \|n^k \nabla c^k\|_{H^s} \|n^k\|_{H^{s+1}} \\ & + \|n^k\|_{H^s}^2 + \|(n^k)^2\|_{H^s} \|n^k\|_{H^s} \\ \leq \quad & C\|u^k\|_{H^s} \|n^k\|_{H^{s+1}} \|n^k\|_{H^s} + C\|n^k\|_{H^s} \|c^k\|_{H^{s+1}} \|n^k\|_{H^{s+1}} \\ & + \|n^k\|_{H^s}^2 + C\|n^k\|_{H^s}^2 \|n^k\|_{H^s} \\ \leq \quad & C\|u^k\|_{H^s}^2 \|n^k\|_{H^s}^2 + \frac{1}{8}\|n^k\|_{H^{s+1}}^2 + C\|n^k\|_{H^s}^2 \|c^k\|_{H^{s+1}}^2 + \frac{1}{8}\|n^k\|_{H^{s+1}}^2 \end{aligned}$$

$$+\|n^k\|_{H^s}^2 + C(\|n^k\|_{H^s}^4 + \|n^k\|_{H^s}^2).$$

from which we have

$$\begin{aligned} \frac{d}{dt}\|n^k\|_{H^s}^2 + \|n^k\|_{H^{s+1}}^2 &\leq C(\|u^k\|_{H^s}^2\|n^k\|_{H^s}^2 + \|n^k\|_{H^s}^2\|c^k\|_{H^{s+1}}^2 \\ &\quad +\|n^k\|_{H^s}^2 + \|n^k\|_{H^s}^4). \end{aligned} \tag{A.3}$$

In a similarly way to (A.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|c^k\|_{H^{s+1}}^2 + \|c^k\|_{H^{s+2}}^2 \leq C\|u^k\|_{H^{s+1}}^2\|c^k\|_{H^{s+1}}^2 + \frac{1}{8}\|c^k\|_{H^{s+2}}^2 + C\|c^k\|_{H^{s+1}}^2 + \frac{1}{8}\|n^k\|_{H^{s+1}}^2.$$

it follows that

$$\frac{d}{dt}\|c^k\|_{H^{s+1}}^2 + \|c^k\|_{H^{s+2}}^2 \leq C(\|u^k\|_{H^{s+1}}^2\|c^k\|_{H^{s+1}}^2 + \|c^k\|_{H^{s+1}}^2) + \frac{1}{8}\|n^k\|_{H^{s+1}}^2. \tag{A.4}$$

Applying Δ_q with $q \geq -1$ to the third equation of (A.1) yields

$$\Delta_q u_t^k + (u^k \cdot \nabla)\Delta_q u^k - \nabla\Delta_q P^k = (u^k \cdot \nabla)\Delta_q u^k - \Delta_q((u^k \cdot \nabla)u^k) + \Delta_q(n^k\nabla\Phi).$$

Multiplying the above equality with $\Delta_q u^k$ yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta_q u^k\|_2^2 + \|\nabla\Delta_q u^k\|_2^2 &= \int_{\mathbb{R}^d}((u^k \cdot \nabla)\Delta_q u^k - \Delta_q((u^k \cdot \nabla)u^k))\Delta_q u^k dx + \int_{\mathbb{R}^d}\Delta_q(n^k\nabla\Phi)\Delta_q u^k dx \\ &\leq \|\,(u^k \cdot \nabla)\Delta_q u^k - \Delta_q((u^k \cdot \nabla)u^k)\|_2\|\Delta_q u^k\|_2 + \|\Delta_q(n^k\nabla\Phi)\|_2\|\Delta_q u^k\|_2. \end{aligned}$$

Multiplying $2^{2q(s+1)}$ on both side of the above inequality and taking the l^1 norm, we obtain

$$\begin{aligned} \frac{d}{dt}\|u^k\|_{H^{s+1}}^2 + \|u^k\|_{H^{s+2}}^2 &\leq \|\nabla u^k\|_\infty\|u^k\|_{H^{s+1}}^2 + \frac{1}{8}\|n^k\|_{H^{s+1}}^2 + \|u^k\|_{H^{s+1}}^2 \\ &\leq C(\|u^k\|_{H^{s+1}}^4 + \|u^k\|_{H^{s+1}}^2) + \frac{1}{8}\|n^k\|_{H^{s+1}}^2. \end{aligned} \tag{A.5}$$

Summing up (A.3)-(A.5), we obtain

$$\begin{aligned} &\frac{d}{dt}(\|n^k\|_{H^s}^2 + \|c^k\|_{H^{s+1}}^2 + \|u^k\|_{H^{s+1}}^2) + \|n^k\|_{H^{s+1}}^2 + \|c^k\|_{H^{s+2}}^2 + \|u^k\|_{H^{s+2}}^2 \\ \leq\ & C(\|n^k\|_{H^s}^2 + \|c^k\|_{H^{s+1}}^2 + \|u^k\|_{H^{s+1}}^2)(1 + \|n^k\|_{H^s}^2 + \|c^k\|_{H^{s+1}}^2 + \|u^k\|_{H^{s+1}}^2) \\ \leq\ & (1 + \|n^k\|_{H^s}^2 + \|c^k\|_{H^{s+1}}^2 + \|u^k\|_{H^{s+1}}^2)^2. \end{aligned}$$

We conclude from the Gronwall inequality that

$$1 + \|n^k\|_{H^s}^2 + \|c^k\|_{H^{s+1}}^2 + \|u^k\|_{H^{s+1}}^2 \leq \frac{1 + \|n_0^k\|_{H^s}^2 + \|c_0^k\|_{H^{s+1}}^2 + \|u_0^k\|_{H^{s+1}}^2}{1 - C(1 + \|n_0^k\|_{H^s}^2 + \|c_0^k\|_{H^{s+1}}^2 + \|u_0^k\|_{H^{s+1}}^2)t}.$$

We can choose

$$T = \frac{1}{2C(1 + \|n_0^k\|_{H^s}^2 + \|c_0^k\|_{H^{s+1}}^2 + \|u_0^k\|_{H^{s+1}}^2)} > 0.$$

such that

$$
\begin{aligned}
\sup_{t\in[0,T]} \quad & (\|n^k(t)\|_{H^s}^2 + \|c^k(t)\|_{H^{s+1}}^2 + \|u^k(t)\|_{H^{s+1}}^2) \\
& + \int_0^t (\|n^k\|_{H^{s+1}}^2 + \|c^k\|_{H^{s+2}}^2 + \|u^k\|_{H^{s+2}}^2)(\tau)d\tau \\
\leq \quad & 2(1 + \|n_0^k\|_{H^s}^2 + \|c_0^k\|_{H^{s+1}}^2 + \|u_0^k\|_{H^{s+1}}^2).
\end{aligned} \tag{A.6}
$$

Step 2. Compactness.
From (A.6), we obtain

$$
\begin{aligned}
& n^k \in L^\infty([0,T], H^s) \cap L^2([0,T], H^{s+1}), \\
& c^k \in L^\infty([0,T], H^{s+1}) \cap L^2([0,T], H^{s+2}), \\
& u^k \in L^\infty([0,T], H^{s+1}) \cap L^2([0,T], H^{s+2}).
\end{aligned}
$$

In order to show the convergence, we also need uniform boundedness for $\partial_t n^k$, $\partial_t c^k$ and $\partial_t u^k$. From the first equation of (A.1), we know

$$
\begin{aligned}
\|\partial_t n^k\|_{L_t^\infty H^{-1}} \quad \leq \quad & \|\Delta n^k\|_{L_t^\infty H^{-1}} + \|u^k \cdot \nabla n^k\|_{L_t^\infty H^{-1}} + \|\nabla \cdot (n^k \nabla c^k)\|_{L_t^\infty H^{-1}} \\
& + \|n^k\|_{L_t^\infty H^{-1}} + \|(n^k)^2\|_{L_t^\infty H^{-1}} \\
\leq \quad & \|n^k\|_{L_t^\infty H^s} + \|u^k\|_{L_t^\infty H^{s+1}} \|n^k\|_{L_t^\infty H^s} + \|n^k\|_{L_t^\infty H^s} \|c^k\|_{L_t^\infty H^{s+1}} \\
& + \|n^k\|_{L_t^\infty H^s} + \|n^k\|_{L_t^\infty H^s}^2 \\
\leq \quad & C.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|\partial_t c^k\|_{L_t^\infty H^{-1}} \quad \leq \quad & \|\Delta c^k\|_{L_t^\infty H^{-1}} + \|u^k \cdot \nabla c^k\|_{L_t^\infty H^{-1}} + \|c^k n^k\|_{L_t^\infty H^{-1}} \\
\leq \quad & \|c^k\|_{L_t^\infty H^{s+1}} + \|u^k\|_{L_t^\infty H^{s+1}} \|c^k\|_{L_t^\infty H^{s+1}} + \|c^k\|_{L_t^\infty H^{s+1}} \|n^k\|_{L_t^\infty H^s} \\
\leq \quad & C.
\end{aligned}
$$

and

$$
\begin{aligned}
\|\partial_t u^k\|_{L_t^\infty H^{-1}} \quad \leq \quad & \|\Delta u^k\|_{L_t^\infty H^{-1}} + \|(u^k \cdot \nabla) u^k\|_{L_t^\infty H^{-1}} + \|n^k \nabla \Phi\|_{L_t^\infty H^{-1}} \\
\leq \quad & \|u^k\|_{L_t^\infty H^{s+1}} + \|u^k\|_{L_t^\infty H^{s+1}}^2 + \|n^k\|_{L_t^\infty H^s} \\
\leq \quad & C.
\end{aligned}
$$

Since L^2 is locally compactly embedded in H^{-1}, we apply the Aubin-Lions Lemma to conclude that, up to an extraction of subsequence, the approximate solution sequence (n^k, c^k, u^k) strongly converges in $L^\infty([0,T]; H^{-1})$ to some function (n, c, u) such that

$$
\begin{aligned}
& n^k \in L^\infty([0,T]; H^s) \cap L^2([0,T], H^{s+1}), \\
& c^k \in L^\infty([0,T]; H^{s+1}) \cap L^2([0,T], H^{s+2}), \\
& u^k \in L^\infty([0,T]; H^{s+1}) \cap L^2([0,T], H^{s+2}).
\end{aligned}
$$

Using the above estimates, it is easy to pass the limit in the approximate system (A.1) and (n, c, u) solve (1.1) in the sense of distribution. By a classical deduction [34], we get $n \in C([0,T]; H^s)$, $c \in C([0,T]; H^{s+1})$ and $u \in C([0,T]; H^{s+1})$.

By virtue of Heat equation theory, we can prove the time differentiation. For example, we consider the following equations:

$$u_t - \Delta u = f(x,t), \quad \text{then} \quad u_t = \Delta u - f(x,t).$$

Suppose $u \in L_t^\infty H^s$ and $f \in L_t^\infty H^{s-2}$, we obtain $u_t \in L_t^\infty H^{s-2}$. For the arbitrariness of s, we have $u_t \in L_t^\infty H^s, \forall\ s > 0$. In a similar way, we get

$$\partial_t(u_t) - \Delta u_t = f_t$$

and $u_{tt} \in L_t^\infty H^{s-2}$. Thus we show the time differentiation.

Step 3. Uniqueness.

Let us consider the two solutions (n_1, c_1, u_1) and (n_2, c_2, u_2) associated with the same initial data and satisfy (1.1). We use the notation $\delta n = n_1 - n_2$, $\delta c = c_1 - c_2$ and $\delta u = u_1 - u_2$. Then we have

$$\begin{cases} \partial_t \delta n + \delta u \cdot \nabla n_1 + u_2 \cdot \nabla \delta n = \Delta \delta n - \nabla \cdot (\delta n \chi(c_1) \nabla c_1) \\ \quad - \nabla \cdot (n_2(\chi(c_1) - \chi(c_2)) \nabla c_1) - n_2 \chi(c_2) \nabla(\delta c) + \delta n - n_1 \delta n - n_2 \delta n, \\ \partial_t \delta c + \delta u \cdot \nabla c_1 + u_2 \cdot \nabla \delta c = \Delta \delta c - n_1(g(c_1) - g(c_2)) - g(c_2)\delta n, \\ \partial_t \delta u + (\delta u \cdot \nabla) u_1 + (u_2 \cdot \nabla)\delta u - \nabla(P_1 - P_2) = \Delta \delta u + \delta n \nabla \Phi. \end{cases} \tag{A.7}$$

Taking the L^2-inner product of the first equation with δn, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\delta n(t)\|_2^2 + \|\nabla \delta n(t)\|_2^2 \\ = \;& -\int_{\mathbb{R}^2} (\delta u \cdot \nabla n_1)\delta n dx - \int_{\mathbb{R}^2} \nabla \cdot (\delta n \chi(c_1)\nabla c_1)\delta n dx - \int_{\mathbb{R}^2} \nabla \cdot (n_2(\chi(c_1) - \chi(c_2))\nabla c_1)\delta n dx \\ & - \int_{\mathbb{R}^2} n_2 \chi(c_2) \nabla \delta c \delta n dx + \int_{\mathbb{R}^2} \delta n \delta n dx - \int_{\mathbb{R}^2} n_1 \delta n \delta n dx - \int_{\mathbb{R}^2} n_2 \delta n \delta n dx \\ \leq \;& C(\|\delta u\|_2^2 + \|\delta n\|_2^2 \|n_1\|_{H^s}^2) + C\|\delta n\|_2^2 \|c_1\|_{H^{s+1}}^2 + C\|\delta c\|_2^2 \|n_2\|_{H^s}^2 \|c_1\|_{H^{s+1}}^2 + \frac{1}{8}\|\nabla \delta n\|_2^2 \\ & + C\|n_2\|_{H^s}^2 \|\nabla \delta c\|_2^2 + C\|\delta n\|_2^2 + \|n_1\|_{H^s}\|\delta n\|_2^2 + \|n_2\|_{H^s}\|\delta n\|_2^2. \end{aligned}$$

from which we get

$$\begin{aligned} & \frac{d}{dt}\|\delta n(t)\|_2^2 + \|\nabla \delta n(t)\|_2^2 \\ \leq \;& C(\|\delta u\|_2^2 + \|\delta n\|_2^2 \|n_1\|_{H^s}^2 + \|\delta n\|_2^2 \|c_1\|_{H^{s+1}}^2 + \|n_2\|_{H^s}^2 \|\nabla \delta c\|_2^2 \\ & + \|\delta n\|_2^2 + \|n_1\|_{H^s}\|\delta n\|_2^2 + \|n_2\|_{H^s}\|\delta n\|_2^2 + \|\delta c\|_2^2 \|n_2\|_{H^s}^2 \|c_1\|_{H^{s+1}}^2). \end{aligned} \tag{A.8}$$

Next, taking the L^2-inner product of the second equation of Eqs.(A.7) with δc, we know

$$\frac{1}{2}\frac{d}{dt}\|\delta c(t)\|_2^2 + \|\nabla \delta c(t)\|_2^2 \;=\; -\int_{\mathbb{R}^2} (\delta u \cdot \nabla c_1)\delta c dx - \int_{\mathbb{R}^2} (g(c_1) - g(c_2)) n_1 \delta c dx - \int_{\mathbb{R}^2} \delta n g(c_2) \delta c dx$$

$$\leq \quad C(\|\delta u\|_2^2 + \|\delta c\|_2^2\|c_1\|_{H^{s+1}}^2 + \|\delta c\|_2^2\|n_1\|_{H^s} + \|\delta n\|_2^2 + \|\delta c\|_2^2).$$

Hence we get

$$\begin{aligned}\frac{d}{dt}\|\delta c(t)\|_2^2 + \|\nabla\delta c(t)\|_2^2 \quad &\leq \quad C(\|\delta u\|_2^2 + \|\delta c\|_2^2\|c_1\|_{H^{s+1}}^2 + \|\delta c\|_2^2\|n_1\|_{H^s} \\ &\quad +\|\delta n\|_2^2 + \|\delta c\|_2^2).\end{aligned} \tag{A.9}$$

Taking ∂_i on both sides of the second equation of Eqs (A.7) yields

$$\partial_t\partial_i\delta c + u_2\cdot\nabla\partial_i\nabla c - \Delta\partial_i\delta c = -\partial_i(\delta u\cdot\nabla c_1) - \partial_i u_2\cdot\nabla\delta c - \partial_i(n_1(g(c_1)-g(c_2))) - \partial_i(g(c_2)\delta n).$$

Multiplying the above equation with $\partial_i\delta c$ and integrating with respect to space variable, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\delta c(t)\|_2^2 + \|\Delta\delta c(t)\|_2^2 \quad &= \quad -\sum_i\int_{\mathbb{R}^d}\partial_i(\delta u\cdot\nabla c_1)\partial_i\delta c dx - \sum_i\int_{\mathbb{R}^d}\partial_i u_2\cdot\nabla\delta c\partial_i\delta c dx \\ &\quad -\sum_i\int_{\mathbb{R}^d}\partial_i(n_1(g(c_1)-g(c_2)))\partial_i\delta c dx - \sum_i\int_{\mathbb{R}^d}\partial_i(g(c_2)\delta n)\partial_i\delta c dx \\ &\leq \quad \int_{\mathbb{R}^d}(\delta u\cdot\nabla c_1)\Delta\delta c dx - \int_{\mathbb{R}^d}(\nabla\delta c\cdot\nabla)u_2\cdot\nabla\delta c dx \\ &\quad +C\int_{\mathbb{R}^d}n_1\delta c\Delta\delta c dx + \int_{\mathbb{R}^d}g(c_2)\delta n\Delta\delta c dx \\ &\leq \quad C\|\delta u\|_2^2\|c_1\|_{H^{s+1}}^2 + \frac{1}{8}\|\Delta\delta c\|_2^2 + C\|\nabla\delta c\|_2^2\|u_2\|_{H^{s+1}} + C\|\delta c\|_2^2\|n_1\|_{H^s}^2 \\ &\quad +\frac{1}{8}\|\Delta\delta c\|_2^2 + C\|\delta n\|_2^2 + \frac{1}{8}\|\Delta\delta c\|_2^2.\end{aligned}$$

Then we get

$$\begin{aligned}\frac{d}{dt}\|\nabla\delta c(t)\|_2^2 + \|\Delta\delta c(t)\|_2^2 \quad &\leq \quad C(\|\delta u\|_2^2\|c_1\|_{H^{s+1}}^2 + \|\nabla\delta c\|_2^2\|u_2\|_{H^{s+1}} \\ &\quad +\|\delta c\|_2^2\|n_1\|_{H^s}^2 + \|\delta n\|_2^2).\end{aligned} \tag{A.10}$$

Performing the L^2-inner product of the third equation of system (A.7) with δu, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta u(t)\|_2^2 + \|\nabla\delta u(t)\|_2^2 \quad &= \quad -\int_{\mathbb{R}^d}((\delta u\cdot\nabla)u_1)\cdot\delta u dx + \int_{\mathbb{R}^d}\delta n\nabla\Phi\cdot\delta u dx \\ &\leq \quad C\|\delta u\|_2^2\|u_1\|_{H^{s+1}} + C(\|\delta n\|_2^2 + \|\delta u\|_2^2).\end{aligned}$$

Such that

$$\frac{d}{dt}\|\delta u(t)\|_2^2 \quad \leq \quad C(\|\delta u\|_2^2\|u_1\|_{H^{s+1}} + \|\delta n\|_2^2 + \|\delta u\|_2^2). \tag{A.11}$$

From ((A.8)-(A.11), we have

$$\frac{d}{dt}(\|\delta n(t)\|_2^2 + \|\delta c(t)\|_2^2 + \|\nabla\delta c(t)\|_2^2 + \|\delta u(t)\|_2^2) + \|\nabla\delta n\|_2^2 + \|\nabla\delta c\|_2^2 + \|\Delta\delta c\|_2^2 + \|\nabla\delta u(t)\|_2^2$$

$$\begin{aligned}\le\ & C(\|\delta u\|_2^2+\|\delta n\|_2^2\|n_1\|_{H^s}^2+\|\delta n\|_2^2\|c_1\|_{H^{s+1}}^2+\|n_2\|_{H^s}^2\|\nabla\delta c\|_2^2+\|\delta n\|_2^2+\|n_1\|_{H^s}\|\delta n\|_2^2\\ & +\|n_2\|_{H^s}\|\delta n\|_2^2+\|\delta c\|_2^2\|n_2\|_{H^s}^2\|c_1\|_{H^{s+1}}^2+\|\delta u\|_2^2+\|\delta c\|_2^2\|c_1\|_{H^{s+1}}^2+\|\delta c\|_2^2\|n_1\|_{H^s}+\|\delta n\|_2^2+\|\delta c\|_2^2\\ & +\|\delta u\|_2^2\|c_1\|_{H^{s+1}}^2+\|\nabla\delta c\|_2^2\|u_2\|_{H^{s+1}}+\|\delta c\|_2^2\|n_1\|_{H^s}^2+\|\delta n\|_2^2+\|\delta u\|_2^2\|u_1\|_{H^{s+1}}).\end{aligned}$$

Consequently, we have

$$\begin{aligned}& \frac{d}{dt}(\|\delta n(t)\|_2^2+\|\delta c(t)\|_2^2+\|\nabla\delta c(t)\|_2^2+\|\delta u(t)\|_2^2)\\ \le\ & CF(t)(\|\delta n\|_2^2+\|\delta c\|_2^2+\|\nabla\delta c\|_2^2+\|\delta u\|_2^2)\end{aligned}$$

where

$$\begin{aligned}F(t) \ =\ & 1+\|n_1\|_{H^s}+\|n_2\|_{H^s}+\|u_1\|_{H^{s+1}}+\|u_2\|_{H^{s+1}}+\|n_1\|_{H^s}^2+\|n_2\|_{H^s}^2\\ & +\|c_1\|_{H^{s+1}}^2+\|c_2\|_{H^{s+1}}^2+\|n_2\|_{H^s}^2\|c_1\|_{H^{s+1}}^2.\end{aligned}$$

By the above estimates, we know that $F(t)$ is integrable. Therefore, we finally obtain the uniqueness by using the Gronwall inequality. □

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