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Research article

On the exponential Diophantine equation $(a(a-l)m^2+1)^x+(alm^2-1)^y=(am)^z$

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Abstract: Suppose that a, l, m are positive integers with $a \equiv 1 \pmod 2$ and $a^2m^2 \equiv -2 \pmod p$, where p is a prime factor of l. In this paper, we prove that the title exponential Diophantine equation has only the positive integer solution $(x, y, z) = (1, 1, 2)$. As an another result, we show that if $a = l$, then the title equation has positive integer solutions $(x, y, z) = (n, 1, 2)$, $n \in \mathbb{N}$. The proof is based on elementary methods, Bilu-Hanrot-Voutier Theorem on primitive divisors of Lehmer numbers, and some results on generalized Ramanujan-Nagell equations.

Keywords: exponential Diophantine equations; integer solution; Fibonacci number
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1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ denote the sets of integers, positive integers, respectively. Let a, b, c be fixed relatively prime positive integers greater than one. There have been many papers investigating the positive integer solutions of the exponential Diophantine equation

$$a^x + b^y = c^z. \tag{1.1}$$

A lot of interesting results have been obtained. In recent years, T. Miyazaki, N. Terai and other scientific research workers have also made certain progress. For detailed historical research background, see [15, 21].

In 1956, Sierpinski [20] considered the case of (a, b, c)=$(3, 4, 5)$, and showed that $(x, y, z) = (2, 2, 2)$ is the only positive integer solution. Jesmanowicz [9] conjectured that if a, b, c are Pythagorean numbers, then the Eq (1.1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$. As an analogue of Jesmanowicz's conjecture, Terai [22] conjectured that the exponential Diophantine equation

$$a^x + b^y = c^z, a^p + b^q = c^r, p, q, r \in \mathbb{N}, r \geq 2 \tag{1.2}$$

has only the positive integer solution $(x, y, z) = (p, q, r)$ except for a handful of triples (a, b, c). Exceptional cases are listed explicitly in [24] and [16]. This conjecture has been proved to be true in many special cases, see [5, 8, 14, 17, 23], but is still unsolved in general.

Now take the Diophantine equation

$$(am^2 + 1)^x + (bm^2 - 1)^y = (cm)^z \tag{1.3}$$

where a, b, c and m are given positive integers such that $a + b = c^2$. Some authors studied Eq (1.3) for some special values, see [1, 10, 18, 25, 26]. Terai and Hibino [25] studied the equation

$$(3pm^2 - 1)^x + (p(p - 3)m^2 + 1)^y = (pm)^z, \tag{1.4}$$

where m be a positive integer and p be a prime. As $m \equiv 1 \pmod 4$, $m \equiv 1, 2 \pmod 3$, $p \equiv 1 \pmod 4$, and $p < 3784$, they proved that the only solution of Eq (1.4) is $(x, y, z) = (1, 1, 2)$. The proof of this result is based on elementary methods and Baker's method. In 2019, Deng, Wu and Yuan [7] studied the equation

$$(3am^2 - 1)^x + (a(a - 3)m^2 + 1)^y = (am)^z, \tag{1.5}$$

where a and m be positive integers such that $am \equiv 1, 2 \pmod 3$, $a \equiv 1 \pmod 2$, and $a \geq 5$. They showed that the exponential Diophantine Eq (1.5) has only the positive integer solution $(x, y, z)=(1, 1, 2)$ by using some results on generalized Ramanujan-Nagell equations and some elementary methods. In 2020, Kizildere and Soydan [11] studied the equation

$$(5pm^2 - 1)^x + (p(p - 5)m^2 + 1)^y = (pm)^z, \tag{1.6}$$

where p be a prime with $p \geq 5$, $p \equiv 3 \pmod 4$ and m be a positive integer. They proved that the Diophantine Eq (1.6) has only the positive integer solution $(x, y, z) = (1, 1, 2)$ where $pm \equiv 1, 4 \pmod 5$ by the methods of [25]. In the same year, Kizildere, Le and Soydan [12] studied the equation

$$(a(a - l)m^2 + 1)^x + (alm^2 - 1)^y = (am)^z \tag{1.7}$$

where l, m, a be positive integers such that $a \equiv 1 \pmod 2$, $am \equiv 1, 2 \pmod 3$, $l \equiv 0 \pmod 3$. They showed that the exponential Diophantine Eq (1.7) has only the positive integer solution $(x, y, z)=(1, 1, 2)$ with $\min\{a(a-l)m^2+1, alm^2-1\} > 30$ by using the Bilu-Hanrot-Voutier Theorem on primitive divisors of Lehmer numbers.

In this paper, we consider the Diophantine equation

$$(a(a - l)m^2 + 1)^x + (alm^2 - 1)^y = (am)^z \tag{1.8}$$

where $a \equiv 1 \pmod 2$, $a > l \geq 3$ and there exists a prime factor p of l satisfying $a^2m^2 \equiv -2 \pmod p$. The following results are the main theorems of this paper.

Theorem 1.1. *Suppose that a, l and m are positive integers with $a \equiv 1 \pmod 2$, $a > l \geq 3$ and there exists a prime factor p of l satisfying $a^2m^2 \equiv -2 \pmod p$. If $l \equiv 1 \pmod 2$ or $l \equiv 0 \pmod 2$ and $\min\{a(a - l)m^2 + 1, alm^2 - 1\} > 30$, then Diophantine Eq (1.8) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

Theorem 1.2. *Suppose that a and m are positive integers with $a = l$. Then Diophantine Eq* (1.8) *has positive integer solutions* $(x, y, z) = (n, 1, 2)$, $n \in \mathbb{N}$.

As a corollary to Theorem 1.1, taking $l = 3 \pmod 6$, we get a general result as follows :

Corollary 1.3. *Suppose that a and m are positive integers with* $a \geq 5$, $a \equiv 1 \pmod 2$, *and* $3 \nmid am$. *Then Diophantine equation*

$$(a(a-l)m^2 + 1)^x + (alm^2 - 1)^y = (am)^z \tag{1.9}$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

As another corollary to Theorem 1.1, taking $l \equiv 0 \pmod 3$, we give a general result as follows:

Corollary 1.4. [12] *Suppose that a, m and l are positive integers with* $a \geq 5$, $a \equiv 1 \pmod 2$, $3 \nmid am$ *and* $l \equiv 0 \pmod 3$. *Then Diophantine equation*

$$(a(a-l)m^2 + 1)^x + (alm^2 - 1)^y = (am)^z \tag{1.10}$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$ *with* $\min\{a(a-l)m^2 + 1, alm^2 - 1\} > 30$.

By Corollary 1.3, we know that Eq (1.8) has only the positive integer solution (x, y, z)=$(1, 1, 2)$ when $a = 11$ and $3 \nmid m$. By using a result on linear forms in p-adic logarithms due to Bugeaud [2], we give a general result as follows:

Corollary 1.5. *If* $m > 2360$ *or* $3 \nmid m$, *then the exponential Diophantine equation*

$$(88m^2 + 1)^x + (33m^2 - 1)^y = (11m)^z \tag{1.11}$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

It's easy to see, letting a a prime and $l = 3$ in Theorem 1.1 gives us the result of [25]. Picking $l = 3$ in Theorem 1.1 yields the result of Deng, Wu and Yuan [7] and taking $l \equiv 0 \pmod 3$ in Theorem 1.1 yields the result of Kizildere, Le and Soydan [12], respectively.

This paper is organized as follows. First of all, in Section 2, we show some preliminary lemmas which are needed in the proofs of our main results. Then in Section 3, we give the proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. Finally, Section 5 is devoted to the proof of Corollary 1.5.

2. Preliminaries

In this section, we present several auxiliary lemmas that are needed in the proof of Theorem 1.1 and Theorem 1.2. Firstly, we will introduce here some definitions.

For any positive integer n, let F_n and L_n be the n-th Fibonacci number and Lucas number, respectively.

Next, we recall some useful lemmas.

Lemma 2.1. [13] *For a fixed solution* (X, Y, z) *of the equation*

$$D_1X^2 + D_2Y^2 = k^z, \gcd(X, Y) = 1, z \geq 1, X, Y, z \in \mathbb{Z}, \tag{2.1}$$

there exists a unique positive integer L such that

$$L = D_1\alpha X + D_2\beta Y, \ \ 1 \leq L < k,$$

where α, β *are integers with* $\beta X - \alpha Y = 1$ *and* $\{D_1, D_2, k\} \in \mathbb{Z}$*,*$\min\{D_1, D_2, k\} > 1$*,*$\gcd(D_1, D_2) = 1$*,*$2 \nmid k$*.*

The positive integer L defined as in Lemma 2.1 is called the characteristic number of the solution (X, Y, z) and is denoted by $\langle X, Y, z\rangle$.

Lemma 2.2. [13] *If* $\langle X, Y, z\rangle = L$ *, then* $D_1X \equiv -LY \pmod{k}$.

For a fixed positive integer L_1, if Eq (2.1) has a solution (X_1, Y_1, z_1) with $(X_1, Y_1, z_1) = L_1$, then the set of all solutions (X, Y, z) of Eq (2.1) with $(X, Y, z) \equiv \pm L_1 \pmod{k}$ is called a solution class of (2.1) and is denoted by $S(L_1)$.

Lemma 2.3. [13] *For any fixed solution class* $S(L_1)$ *of Eq* (2.1)*, there exists a unique solution* $(X_1, Y_1, z_1) \in S(L_1)$ *such that* $X_1 \geq 1$*,* $Y_1 \geq 1$*, and* $z_1 \leq z$*, where z runs through all solution* $(X, Y, z) \in S(L_1)$*.The solution* (X_1, Y_1, z_1) *is called the least solution of* $S(L_1)$*. Every solution* $(X, Y, z) \in S(L_1)$ *can be expressed as*

$$z = z_1t, 2 \nmid t, t \in \mathbb{N},$$
$$X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1(X_1\sqrt{D_1} + \lambda_2Y_1\sqrt{-D_2})^t, \lambda_1, \lambda_1 \in \{1, -1\}.$$

Lemma 2.4. [3] *Let* (X_1, Y_1, z_1) *be the least solution of* $S(L_1)$*. If Eq* (2.1) *has solution* $(X, Y, z) \in S(L_1)$ *satisfying* $X \geq 1$ *and* $Y = 1$*, then* $Y_1 = 1$*. Furthermore, if* $(X, z) \neq (X_1, z_1)$*, then one of the following conditions is satisfied:*

(1) $D_1X_1^2 = \frac{1}{4}(k^{z_1} \pm 1), D_2 = \frac{1}{4}(3k^{z_1} \mp 1), (X, z) = (X_1|D_1X_1^2 - 3D_2|, 3z_1)$.

(2) $D_1X_1^2 = \frac{1}{4}F_{3r+3\varepsilon}, D_2 = \frac{1}{4}L_{3r}, k^{z_1} = F_{3r+\varepsilon}, (X, z) = (X_1|D_1^2X_1^4 - 10D_1D_2X_1^2 + 5D_2^2|, 5z_1)$*, where r is a positive integer,* $\varepsilon \in \{1, -1\}$*.*

Lemma 2.5. [6] *For any positive integer n, let* F_n *be the n-th Fibonacci number. The equation*

$$F_n = u^2, n, u \in \mathbb{N} \tag{2.2}$$

has only the solutions $(n, u) = (1, 1), (2, 1)$*, and* $(12, 12)$*.*

Lemma 2.6. [19] *Suppose that x, y, m, n are positive integers greater than one. Then the equation*

$$x^m - y^n = 1 \tag{2.3}$$

has only the positive integer solution $(x, y, m, n) = (3, 2, 2, 3)$*.*

Finally, we need a result on primitive divisors of Lehmer numbers due to Bilu, Hanrot, Voutier [4].

Let α, β be algebraic integers. If $(\alpha + \beta)^2$, $\alpha\beta$ are nonzero coprime integers, and $\frac{\alpha}{\beta}$ is not a root of unity, then (α,β) is called a Lehmer pair. Let $E = (\alpha + \beta)^2$ and $G = \alpha\beta$. Then we have

$$\alpha = \frac{1}{2}(\sqrt{E} + \lambda\sqrt{F}), \beta = \frac{1}{2}(\sqrt{E} - \lambda\sqrt{F}), \lambda \in \{-1, 1\}, \tag{2.4}$$

where $F = E - 4G$. Further, one defines the corresponding sequence of Lehmer numbers by

$$L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Obviously, $L_n(\alpha,\beta), (n \geq 1)$ are nonzero integers.

A prime q is called a primitive divisor of the Lehmer number $L_n(\alpha,\beta), (n \geq 1)$ if $q|L_n(\alpha,\beta)$ and $q \nmid FL_1(\alpha,\beta)...L_{n-1}(\alpha,\beta)$.

Lemma 2.7. [4] *If $n > 30$, then $L_n(\alpha,\beta)$ has primitive divisors.*

Lemma 2.8. *Let $m \geq 2100$ be a positive integer and (x, y, z) be a positive integer solution of Eq (1.11). Suppose that $N = \max\{x, y\}$, then $z > 1.87N$.*

Proof. Let (x, y, z) be a positive integer solution of Eq (1.11), then we derive that

$$(11m)^z = (88m^2 + 1)^x + (33m^2 - 1)^y > (88m^2 + 1)^x \tag{2.5}$$

and

$$(11m)^z = (88m^2 + 1)^x + (33m^2 - 1)^y > (33m^2 - 1)^y. \tag{2.6}$$

By (2.5) and (2.6), we get that

$$z > f(m) \cdot x := \frac{\log(88m^2 + 1)}{\log(11m)} \cdot x \tag{2.7}$$

and

$$z > g(m) \cdot y := \frac{\log(33m^2 - 1)}{\log(11m)} \cdot y. \tag{2.8}$$

Notice that $f(m)$ and $g(m)$ are increasing and $m \geq 2100$, we get that

$$z > f(m) \cdot x \geq f(2100) \cdot x > 1.96 \cdot x \tag{2.9}$$

and

$$z > g(m) \cdot y \geq g(2100) \cdot y > 1.87 \cdot y. \tag{2.10}$$

If $N = \max\{x, y\} = x$, then

$$z > 1.96 \cdot x > 1.87N.$$

If $N = \max\{x, y\} = y$, then

$$z > 1.87 \cdot y = 1.87N.$$

Therefore, we conclude that if $m \geq 2100$ and $N = \max\{x, y\}$, then

$$z > 1.87N.$$

This completes the proof of Lemma 2.8. □

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1.

Lemma 3.1. *Let* (x, y, z) *be a positive integer solution of Eq* (1.8). *Suppose there exists a prime factor* p *of* l *satisfying* $a^2m^2 \equiv -2 \pmod{p}$. *Then both* x *and* y *are odd.*

Proof. Since $a > l \geq 3$, we see from Eq (1.8) that

$$0 \equiv (am)^z \equiv (a(a-l)m^2 + 1)^x + (alm^2 - 1)^y \equiv 1 + (-1)^y \pmod{a},$$

and y is odd.

Since $a^2m^2 \equiv -2 \pmod{p}$, by Eq (1.8) and y is odd , we have

$$0 \not\equiv (am)^z \equiv (a(a-l)m^2 + 1)^x + (alm^2 - 1)^y \equiv (-1)^x + (-1)^y \pmod{p}. \tag{3.1}$$

Hence, we obtain from (3.1) that x is odd. □

Lemma 3.1 is proved.

3.1. The case $m \equiv 0 \pmod{2}$

Lemma 3.2. *Suppose that* $m \equiv 0 \pmod{2}$. *Then Eq* (1.8) *has only the positive integer solution* $(x, y, z) = (1, 1, 2)$.

Proof. If $z \leq 2$, then $(x, y, z) = (1, 1, 2)$ from Eq (1.8). Hence, we may assume that $z \geq 3$. Taking Eq (1.8) modulo m^3 implies that

$$a(a-l)x + aly \equiv 0 \pmod{m}.$$

That is,

$$a^2x \equiv 0 \pmod{2},$$

which is impossible, since both x and a are odd. Therefore, we conclude that if m is even, then Eq (1.8) has only the positive integer solution $(x, y, z) = (1, 1, 2)$. □

Lemma 3.2 is proved.

3.2. The case $m \equiv 1 \pmod{2}$

Lemma 3.3. *Suppose that* m *is odd and there exists a prime factor* p *of* l *satisfying* $a^2m^2 \equiv -2 \pmod{p}$. *Then Eq* (1.8) *has only the positive integer solution* $(x, y, z) = (1, 1, 2)$.

Proof. We assume that Eq (1.8) has another positive integer solution $(x, y, z) \neq (1, 1, 2)$. By Lemma 3.1, we get that both x and y are odd positive integers. Thus, Eq (1.8) yields the following equation:

$$(a(a-l)m^2 + 1)X^2 + (alm^2 - 1)Y^2 = (am)^z, \tag{3.2}$$

where

$$X = (a(a-l)m^2 + 1)^{\frac{x-1}{2}}, Y = (alm^2 - 1)^{\frac{y-1}{2}}, z \geq 2.$$

Hence, $(X, Y, z) = ((a(a-l)m^2+1)^{\frac{x-1}{2}}, (alm^2-1)^{\frac{y-1}{2}}, z) \neq (1, 1, 2)$ is a positive integer solution of Eq (3.2) with $\gcd(X, Y) = 1$.

Let $L = \langle (a(a-l)m^2+1)^{\frac{x-1}{2}}, (alm^2-1)^{\frac{y-1}{2}}, z) \rangle$. By Lemma 2.2 and Eq (3.2), we know that L satisfies

$$1 \equiv (a(a-l)m^2+1)^{\frac{x+1}{2}} \equiv -L(alm^2-1)^{\frac{y-1}{2}} \equiv (-1)^{\frac{y+1}{2}} L \pmod{am}. \tag{3.3}$$

On the other hand, let $L_1 = \langle 1, 1, 2 \rangle$, since $(X_1, Y_1, z_1) = (1, 1, 2)$ is a positive integer solution of Eq (3.2). Then by Lemma 2.2 we have

$$1 \equiv a(a-l)m^2+1 \equiv -L_1 \pmod{am}.$$

It implies that $L \equiv \pm L_1 \pmod{am}$, and it is obvious that $(X_1, Y_1, z_1) = (1, 1, 2)$ is the least solution of $S(L_1)$, since $z \geq 2$. Hence, $(X, Y, z) \in S(L_1)$. Therefore, using Lemma 2.3, we get that

$$z = z_1 t = 2t, 2 \nmid t, t \in \mathbb{N},$$

$$\begin{aligned}(a(a-l)m^2+1)^{\frac{x-1}{2}} \sqrt{a(a-l)m^2+1} + (alm^2-1)^{\frac{y-1}{2}} \sqrt{1-alm^2} \\ = \lambda_1(\sqrt{a(a-l)m^2+1} + \lambda_2 \sqrt{1-alm^2}))^t, \lambda_1, \lambda_2 \in \{-1, 1\}.\end{aligned} \tag{3.4}$$

By Eq (3.4), we get

$$(alm^2-1)^{\frac{y-1}{2}} = \lambda_1\lambda_2 \sum_{i=0}^{\frac{t-1}{2}} \binom{t}{2i+1} (a(a-l)m^2+1)^{\frac{t-1}{2}-i}(1-alm^2)^i. \tag{3.5}$$

and

$$(a(a-l)m^2+1)^{\frac{x-1}{2}} = \lambda_1 \sum_{i=0}^{\frac{t-1}{2}} \binom{t}{2i} (a(a-l)m^2+1)^{\frac{t-1}{2}-i}(1-alm^2)^i. \tag{3.6}$$

□

Now, we consider the following two cases.

Case 1. $l \equiv 1 \pmod 2$. Since $2|(alm^2-1)$ and $2 \nmid (a(a-l)m^2+1)$, we see from Eq (3.5) that $y = 1$ and $(alm^2-1)^{\frac{y-1}{2}} = 1$. It implies that (X, Y, z) is a solution of $S(L_1)$ satisfying $Y = 1$ and $(X, z) \neq (X_1, z_1) = (1, 2)$.

Therefore, by Lemma 2.4, we get either

$$(a(a-l)m^2+1) = (a(a-l)m^2+1)X_1^2 = \frac{1}{4}((am)^2 \pm 1) \tag{3.7}$$

or

$$(am)^2 = (am)^{z_1} = F_{3r+\varepsilon}, \varepsilon \in \{-1, 1\}. \tag{3.8}$$

By Eq (3.7), we can obtain

$$3a^2m^2 + 3 = 4alm^2, \tag{3.9}$$

and

$$am^2(4l-3a) = 5 \tag{3.10}$$

If Eq (3.9) holds, taking modulo 4, we get $6 \equiv 0 \pmod 4$, which is impossible.

If Eq (3.10) holds, we get $am^2 \mid 5$, that is, $a = 5$, $m = 1$ and $l = 4$, which is impossible.

By Lemma 2.5, we know that Eq (3.8) has no positive integer solution, since $am \equiv 1 \pmod 2$.

Hence, Eq (1.8) has only the positive integer solution $(x, y, z) = (1, 1, 2)$ with $l \equiv 1 \pmod 2$.

Case 2. $l \equiv 0 \pmod 2$ and $\min\{a(a-l)m^2 + 1, alm^2 - 1\} > 30$.

By Eqs (3.5) and (3.6), we get either $a(a-l)m^2 + 1|t$ or $alm^2 - 1|t$. This implies that

$$t \geq \min\{a(a-l)m^2 + 1, alm^2 - 1\}. \tag{3.11}$$

Further, since $\min\{a(a-l)m^2 + 1, alm^2 - 1\} > 30$, we see from (3.11) that

$$t > 30. \tag{3.12}$$

Let

$$\alpha = \sqrt{alm^2 - 1} + \sqrt{a(l-a)m^2 - 1}, \beta = \sqrt{alm^2 - 1} - \sqrt{a(l-a)m^2 - 1} \tag{3.13}$$

Then $(\alpha + \beta)^2 = 4(alm^2 - 1), \alpha\beta = (am)^2$ are nonzero coprime integers, and $\frac{\alpha}{\beta}$ is not a root of unity. Hence, (α, β) is a Lehmer pair. By Eqs (3.5) and (3.13), we have

$$(alm^2 - 1)^{\frac{y-1}{2}} = \left|\frac{\alpha^t - \beta^t}{\alpha - \beta}\right| = \left|L_t(\alpha, \beta)\right|. \tag{3.14}$$

We see from (3.14) that the Lehmer number $L_t(\alpha, \beta)$ has no primitive divisors. But, by Lemma 2.7, we find from (3.14) that this is false.

Thus, under the assumption, Eq (1.8) has only the positive integer solution $(x, y, z) = (1, 1, 2)$ with $l \equiv 0 \pmod 2$.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Proof. Since $l = a$, Eq (1.8) can be written as

$$(am)^z - (a^2m^2 - 1)^y = 1. \tag{4.1}$$

By Lemma 2.6, we obtain that Eq (4.1) has only positive integer solutions $(x, y, z) = (n, 1, 2), n \in \mathbb{N}$. □

This completes the proof of Theorem 1.2.

5. Proof of Corollary 1.5

In order to obtain an upper bound for m, we shall quote a result on linear forms in p-adic logarithms due to Bugeaud [2]. Here we consider the case where $y_1 = y_2 = 1$ in the notation of [2], page 375.

Let p be an odd prime. Let a_1 and a_2 be nonzero integers prime to p. Let g be the least positive integer such that

$$v(a_1^g - 1) \geq 1, v(a_2^g - 1) \geq 1,$$

where we denote the p-adic valuation by $v_p(\cdot)$. Assume that there exists a real number E such that

$$\frac{1}{p-1} < E \leq v_p(a_1^g - 1).$$

We consider the integer

$$\Lambda = a_1^{b_1} - a_2^{b_2}$$

where b_1 and b_2 are positive integers. We let A_1 and A_2 be real numbers greater than one with

$$\log A_i \geq \max\{\log |a_i|, E \log p\}, i = 1, 2,$$

and we put

$$B = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

Proposition 5.1. [2] *With the above notation, if a_1 and a_2 are multiplicatively independent, then we have the upper estimates*

$$v_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4}(\max\{\log B + \log(E \log p) + 0.4, 6E \log p, 5\})^2 \log A_1 \log A_2. \tag{5.1}$$

Proof. Now let (x, y, z) be a solution of Eq (1.11). By Corollary 1.3 and Lemma 3.2, we know that Eq (1.11) has only the positive integer solution $(x, y, z) = (1, 1, 2)$ when $3 \nmid m$ or $2|m$. Thus, the proof of Corollary 1.5 suffices to prove the case $m > 2360$, $3|m$, $2 \nmid m$. Recall that y is odd. Here, we apply Proposition 5.1. For this, we set $p = 11$, $a_1 = 88m^2 + 1$, $a_2 = 1 - 33m^2$, $b_1 = x$, $b_2 = y$, and

$$\Lambda = (88m^2 + 1)^x - (1 - 33m^2)^y.$$

Then we may take $g = 1$, $E = 1$, $A_1 = 88m^2 + 1$, $A_2 = 33m^2 - 1$. Hence by (5.1), we have

$$z \leq \frac{36.1}{(\log 11)^4}(\max\{\log B + \log(\log 11) + 0.4, 6 \log 11, 5\})^2 \log(88m^2 + 1) \log(33m^2 - 1), \tag{5.2}$$

where

$$B = \frac{x}{\log(33m^2 - 1)} + \frac{y}{\log(88m^2 + 1)}.$$

Suppose that $z \geq 4$. Taking Eq (1.11) modulo m^4 we have

$$88x + 33y \equiv 0 \pmod{m^2}.$$

In particular, we put $N = \max\{x, y\}$, then $N \geq \frac{m^2}{121}$. Since $z > N$ and $B \leq \frac{N}{\log 5m}$ we have that

$$N < z \leq \frac{36.1}{(\log 11)^4}(\max\{\log(\frac{N}{\log 5m})+\log(\log 11)+0.4, 6 \log 11, 5\})^2 \log(88m^2+1) \log(33m^2-1). \tag{5.3}$$

Let

$$H(N, m) = \max\{\log(\frac{N}{\log 5m}) + \log(\log 11) + 0.4, 6 \log 11, 5\}.$$

Note that $6\log 11 > 5$, then

$$H(N,m) = \max\{\log(\frac{N}{\log 5m}) + \log(\log 11) + 0.4, 6\log 11\}.$$

If

$$H(N,m) = \log(\frac{N}{\log 5m}) + \log(\log 11) + 0.4 \geq 6\log 11.$$

Then the inequality $\log N > 6\log 11 - \log(\log 11) - 0.4$ implies that $N > 495231$. On the other hand, from (5.3) we have that

$$N < 1.092 \cdot (\log N + 1.275)^2 \log(88 \cdot 121N + 1)\log(33 \cdot 121N - 1), \tag{5.4}$$

which implies that $N < 66033$, a contradiction. Hence $H(N,m) = 6\log 11$ and therefore from (5.3) we have the inequality

$$m^2 \leq 27348.57 \cdot \log(88m^2 + 1)\log(33m^2 - 1). \tag{5.5}$$

This implies that $m \leq 3342$.

If $2100 \leq m \leq 3342$, by Lemma 2.8 we have

$$1.87m^2 < 27348.57 \cdot \log(88m^2 + 1)\log(33m^2 - 1).$$

Then $m \leq 2359$, which contradicts the fact that $m > 2360$.

We conclude $z \leq 3$. If $z = 3$, let $a = 88m^2 + 1, b = 33m^2 - 1$ we obtain that

$$a^{2x} + 2a^x b^y + b^{2y} = a^3 + 3a^2 b + 3ab^2 + b^3. \tag{5.6}$$

If $y = 1$, we get from (5.6) that $a|b^2(b-1)$. It follows that $a = 88m^2 + 1|b - 1 = 33m^2 - 2$ since $\gcd(a,b) = 1$, which is impossible. If $x = 1$, we get from (5.6) that $b|a^2(a-1)$. It follows that $b = 33m^2 - 1|a - 1 = 88m^2$, so $33m^2 - 1|11m^2 - 3$, which is impossible. Thus we have $x > 1$ and $y > 1$. This will lead to

$$a^3 + 3a^2b + 3ab^2 + b^3 = a^{2x} + 2a^x b^y + b^{2y} \geq a^4 + 2a^2b^2 + b^4 > a^3 + 3a^2b + 3ab^2 + b^3,$$

a contradiction.

Hence $z \leq 2$. In this case, one can easily show that $(x,y,z) = (1,1,2)$.

□

This completes the proof of Corollary 1.5.

Remark. It is easy to see that the Diophantine equation $(a(a-l)m^2 + 1)^x + (alm^2 - 1)^y = (am)^z$ has solution $(x,y,z) = (1,1,2)$ if we remove all the conditions in Theorem 1.1. How can prove it? On the other hand, it is worth noting that from Theorem 1.1, the condition $\min\{a(a-l)m^2 + 1, alm^2 - 1\} \leq 30$ still has the value of further research.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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