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Research article

A note on the preconditioned tensor splitting iterative method for solving strong $\mathcal{M}$-tensor systems

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Abstract: In this note, we present a new preconditioner for solving the multi-linear systems, which arise from many practical problems and are different from the traditional linear systems. Based on the analysis of the spectral radius, we give new comparison results between some preconditioned tensor splitting iterative methods. Numerical examples are given to demonstrate the efficiency of the proposed preconditioned method.

Keywords: multi-linear systems; Tensor splitting; preconditioned method; spectral radius; comparison theorem
Mathematics Subject Classification: 15A69, 65F10

1. Introduction

In this note, we consider the following multi-linear system

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}, \tag{1.1}$$

where $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is an order m dimension n tensor, $\mathbf{x}$ and $\mathbf{b}$ are n dimensional vectors, and the tensor-vector product $\mathcal{A}\mathbf{x}^{m-1}$ is defined as [1]

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \quad i = 1, 2, \cdots, n, \tag{1.2}$$

where x_i denotes the i-th component of $\mathbf{x}$. The multi-linear system (1.1) arises from a number of scientific computing and engineering applications [1, 2, 6], such as data analysis [10], the sparsest solutions to tensor complementarity problems [11], and so on.

One of the applications of the multi-linear system (1.1) is the numerical solution of the partial differential equation with Dirichlet's boundary condition

$$\begin{cases} u(x)^{m-2} \cdot \triangle u(x) = -f(x) \quad in \quad \Omega, \\ u(x) = g(x) \quad on \quad \partial\Omega, \end{cases} \quad (m = 3, 4, \cdots) \tag{1.3}$$

where $\triangle = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ and $\Omega = [0, 1]^d$. When $f(\cdot)$ is a constant function, this PDE is a nonlinear Klein-Gordon equation (see [9, 13, 14]). Just as authors studied in [13, 14], $u \mapsto u^{\theta} \cdot \triangle u$ can also be discretized into an mth-order nonsingular $\mathcal{M}$-tensor

$$\mathcal{L}_h^{(d)} = \sum_{k=0}^{d-1} \underbrace{\mathcal{I} \otimes \cdots \otimes \mathcal{I}}_{k} \otimes \mathcal{L}_h \otimes \underbrace{\mathcal{I} \otimes \cdots \otimes \mathcal{I}}_{d-k-1},$$

which satisfies

$$(\mathcal{L}_h \mathbf{u}^{m-1})_i = u_i^{m-2} \cdot (L_h \mathbf{u})_i$$

for $i = 1, 2, \cdots, n$, where $\mathcal{L}_h$ is an mth-order tensor $\mathcal{M}$-tensor with

$$\begin{cases} (\mathcal{L}_h)_{1,1,\cdots,1} = (\mathcal{L}_h)_{n,n,\cdots,n} = \frac{1}{h^2}, \\ (\mathcal{L}_h)_{i,i,\cdots,i} = \frac{2}{h^2}, \\ (\mathcal{L}_h)_{i,i-1,i,\cdots,i} = (\mathcal{L}_h)_{i,i,i-1,\cdots,i} = \cdots = (\mathcal{L}_h)_{i,i,i,\cdots,i-1} = -\frac{1}{h^2(m-1)}, i = 2, 3, \cdots, n-1, \\ (\mathcal{L}_h)_{i,i+1,i,\cdots,i} = (\mathcal{L}_h)_{i,i,i+1,\cdots,i} = \cdots = (\mathcal{L}_h)_{i,i,i,\cdots,i+1} = -\frac{1}{h^2(m-1)}, i = 2, 3, \cdots, n-1, \end{cases}$$

The PDE in (1.3) is discretized into an $\mathcal{M}$-equation $\mathcal{L}_h^{(d)} \mathbf{u}^{m-1} = \mathbf{f}$. This class of multi-linear equations can be regarded as a higher-order generalization of the one discussed in [9, 15, 16].

To solve the multi-linear system (1.1), Ng, Qi and Zhou [12] proposed an algorithm for $\mathbf{b} = \mathbf{0}$. When $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, Ding and Wei [9] generalized the Jacobi method, the Gauss-Seidel method and the Newton algorithm. Liu et al. [3] discussed the tensor splitting $\mathcal{A} = \mathcal{E} - \mathcal{F}$, and then proposed a general tensor splitting iterative method for solving the multi-linear system (1.1) as follows:

$$x_k = [M(\mathcal{E})^{-1} \mathcal{F} \mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})^{-1} \mathbf{b}]^{[\frac{1}{m-1}]}, k = 1, 2, \cdots, \tag{1.4}$$

where $\mathbf{x}_0$ is a given initial vector and the tensor $\mathcal{T} = M(\mathcal{E})^{-1} \mathcal{F}$ is called the iterative tensor of the splitting method (see [3, 4]). They discussed the convergence rate for the tensor splitting iterative method and showed that the spectral radius $\rho(M(\mathcal{E})^{-1} \mathcal{F})$ can be seen as an approximate convergence rate of the iteration (1.4).

For matrix splitting iterative methods, it is well known that the preconditioning technique is very important, which can be used to improve the rate of convergence of the iterative method when a suitable preconditioner is chosen [4, 5, 7]. In [3], Liu et al. explored preconditioning techniques for tensor splitting methods and discussed the preconditioned Gauss-Seidel type and SOR type iterative methods, and proved that the Guass-Seidel type method demonstrates faster convergence than the Jacobi method, that is to say, the spectral radius of the iterative matrix of the Guass-Seidel method is not larger than the one of the Jacobi method. Recently, Cui et al. [5] proposed a new preconditioner for solving $\mathcal{M}$-tensor systems and gave some comparison theorems of the preconditioned Gauss-Seidel type method. The preconditioned iterative method is to transform the original system into the preconditioned form

$$P\mathcal{A}\mathbf{x}^{m-1} = P\mathbf{b}, \tag{1.5}$$

where the matrix P is a nonsingular preconditioner. Let $P\mathcal{A} = \mathcal{E}_P - \mathcal{F}_P$ be a splitting of $P\mathcal{A}$. Then the corresponding preconditioned tensor splitting iterative method is given as follows:

$$x_k = [M(\mathcal{E}_P)^{-1}\mathcal{F}_P\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E}_P)^{-1}P\mathbf{b}]^{[\frac{1}{m-1}]}, k = 1, 2, \cdots . \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2 we introduce some definitions and some related lemmas which will be used in the sequel. In Section 3, we first present a counter-example for existing research results and then propose a new special preconditioner. Meanwhile, we give the comparison theorems for the preconditioned Gauss-Seidel type iterative methods. The final section is the concluding remark.

2. Preliminaries

Let $\mathbf{0}$, O and $\mathcal{O}$ denote a zero vector, a zero matrix and a zero tensor, respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be two tensors with the same sizes. The order $\mathcal{A} \geq \mathcal{B}(> \mathcal{B})$ means that each entry of $\mathcal{A}$ is no less than (larger than) corresponding one of $\mathcal{B}$.

For a positive integer n, let $\langle n \rangle = \{1, 2, \cdots, n\}$. A tensor $\mathcal{A}$ consists of $n_1 \times \cdots \times n_m$ entries in the real field $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_m}), a_{i_1 i_2 \cdots i_m} \in \mathbb{R}, i_j \in \langle n_j \rangle, j = 1, \cdots, m.$$

If $n_1 = \cdots = n_m = n$, $\mathcal{A}$ is called an order m dimension n tensor. We denote the set of all order m dimension n tensors by $\mathbb{R}^{[m,n]}$. When $m = 1$, $\mathbb{R}^{[1,n]}$ is simplified as $\mathbb{R}^n$, which is the set of all n-dimension real vectors. When $m = 2$, $\mathbb{R}^{[2,n]}$ denotes the set of all $n \times n$ real matrices. Similarly, the above notions can be generalized to the complex number field $\mathbb{C}$. Let $\mathbb{R}_+$ be the nonnegative real field. If each entry of $\mathcal{A}$ is nonnegative, we call $\mathcal{A}$ a nonnegative tensor, and the set of all the order m dimension n nonnegative tensors is denoted by $\mathbb{R}_+^{[m,n]}$.

Let $A \in \mathbb{R}^{[2,n]}$ and $\mathcal{B} \in \mathbb{R}^{[k,n]}$. The matrix-tensor product $\mathcal{C} = A\mathcal{B} \in \mathbb{R}^{[k,n]}$ is defined by

$$c_{ji_2\cdots i_k} = \sum_{j_2=1}^{n} a_{jj_2} b_{j_2 i_2 \cdots i_k}. \tag{2.1}$$

The formular (2.1) can be written as follows (see [3]):

$$\mathcal{C}_{(1)} = (A\mathcal{B})_{(1)} = A\mathcal{B}_{(1)}, \tag{2.2}$$

where $\mathcal{C}_{(1)}$ and $\mathcal{B}_{(1)}$ are the matrices obtained from $\mathcal{C}$ and $\mathcal{B}$ flattened along the first index (see [3]), For example, if $\mathcal{B} = (b_{ijk}) \in \mathbb{C}^{[3,n]}$, then

$$\mathcal{B}_{(1)} = \begin{pmatrix} b_{111} & \cdots & b_{1n1} & b_{112} & \cdots & b_{1n2} & \cdots & b_{11n} & \cdots & b_{1nn} \\ b_{211} & \cdots & b_{2n1} & b_{212} & \cdots & b_{2n2} & \cdots & b_{21n} & \cdots & b_{2nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n11} & \cdots & b_{nn1} & b_{n12} & \cdots & b_{nn2} & \cdots & b_{n1n} & \cdots & b_{nnn} \end{pmatrix}.$$

Next we recall some definitions and lemmas for the completeness our presentation.

Definition 2.1. *([1]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C} \setminus 0)$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$ if they satisfy the equation*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{2.3}$$

where $\mathbf{x}^{[m-1]} = (x_1^{m-1}, \cdots, x_n^{m-1})^T$. We call $(\lambda, \mathbf{x})$ an H-eigenpair if both λ and $\mathbf{x}$ are real.

Let $\rho(\mathcal{A}) = max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$ be the spectral radius of $\mathcal{A}$, where $\sigma(\mathcal{A})$ is the set of all eigenvalue of $\mathcal{A}$. We use $\mathcal{I}_k = (\delta_{i_1 \cdots i_k})$ to denote a unit tensor with its entries given by:

$$\delta_{i_1 \cdots i_k} = \begin{cases} 1, i_1 = \cdots = i_k, \\ 0, else. \end{cases}$$

Definition 2.2. *([6]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is called a $\mathcal{Z}$-tensor if its off-diagonal entries are non-positive. $\mathcal{A}$ is called an $\mathcal{M}$-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $\eta \geq \rho(\mathcal{B})$ such that*

$$\mathcal{A} = \eta \mathcal{I}_m - \mathcal{B}.$$

If $\eta > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong $\mathcal{M}$-tensor.

Definition 2.3. *([2]) Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. Then the majorization matrix $M(\mathcal{A})$ of $\mathcal{A}$ is the $n \times n$ matrix with the entries*

$$M(\mathcal{A})_{ij} = a_{ij \cdots j}, i, j = 1, \cdots, n.$$

Lemma 2.1. *([4, 8]) For $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, the following inequalities hold:*
$\mu \mathbf{x}^{[m-1]} \leq (<) \mathcal{A}\mathbf{x}^{m-1}, \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}$*, implies* $\mu \leq (<) \rho(\mathcal{A})$*,*
and
$\mathcal{A}\mathbf{x}^{m-1} \leq \nu \mathbf{x}^{[m-1]}, \mathbf{x} > \mathbf{0}$*, implies* $\rho(\mathcal{A}) \leq \nu$*.*

Lemma 2.2. *([6]) Let $\mathcal{A}$ be a $\mathcal{Z}$-tensor, then $\mathcal{A}$ is a strong $\mathcal{M}$-tensor if and only if $\mathcal{A}$ is a semi-positive; that is, there exists $\mathbf{x} > 0$ with $\mathcal{A}\mathbf{x}^{m-1} > 0$.*

3. Comparison theorems

The preconditioner P_α was introduced in [4] as follows:

$$P_\alpha = I + S_\alpha,$$

where

$$S_\alpha = \begin{pmatrix} 0 & -\alpha_1 a_{12\cdots 2} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{23\cdots 3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1,n,\cdots,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.1}$$

and I is an identity matrix, $\boldsymbol{\alpha} = (\alpha_i)$ and α_i is a parameter, $i = 1, \cdots, n-1$.

In [5], Cui et al. considered the preconditioner with $P_{max} = I + S_{max}$, where S_{max} was given by

$$S_{max} = (s^m_{i,k_i}) = \begin{cases} -a_{ik_i\cdots k_i}, i = 1, \cdots, n-1, k_i > i, \\ 0, \qquad otherwise, \end{cases} \tag{3.2}$$

where

$$k_i = \min\left\{ j \Big| \max_j |a_{ij\cdots j}|, i < n, j > i \right\}.$$

Some results in [4, 5] were given below.

Lemma 3.1. *([4]) Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. If $\boldsymbol{\alpha} = (\alpha_i)$ and $\alpha_i \in [0, 1], i = 1, 2, \cdots, n-1$, $\mathcal{A}_\alpha = P_\alpha \mathcal{A} = \mathcal{E}_\alpha - \mathcal{F}_\alpha$ and $(\rho_\alpha, \mathbf{x}_\alpha)$ is Perron eigenpair of $\mathcal{T}_\alpha = M(\mathcal{E}_\alpha)^{-1}\mathcal{F}_\alpha$, then $\mathcal{A}\mathbf{x}_\alpha^{m-1} \geq 0$.*

Lemma 3.2. *([4]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\mathcal{A} = \mathcal{I}_m - \mathcal{L} - \mathcal{F}, \mathcal{F} \geq \mathcal{O}$, where $\mathcal{L} = L\mathcal{I}_m$, $-L$ is the strictly lower part of $M(\mathcal{A})$. If $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, then for all $\alpha_i \in [0, 1], i = 1, \cdots, n-1$, $(I + S_\alpha)\mathcal{A}$ is a strong $\mathcal{M}$-tensor.*

Lemma 3.3. *([5]) Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$. If $\mathcal{A}$ is a strong $\mathcal{M}$-tensor and $\mathcal{A} = \mathcal{I}_m - \mathcal{L} - \mathcal{F}$, where $\mathcal{L} = L\mathcal{I}_m$, $-L$ is the strictly lower triangle part of $M(\mathcal{A})$, then $\mathcal{A}_{max} = (I + S_{max})\mathcal{A}$ is a strong $\mathcal{M}$-tensor.*

Lemma 3.4. *([5], Lemma 3) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathcal{M}$-tensor. For $\mathcal{A}_{max} = \mathcal{E}_{max} - \mathcal{F}_{max}$, we have the following inequality holds if*

$$0 < \alpha_i a_{i,i+1,\cdots,i+1} a_{i+1,j,\cdots,j} \leq a_{ik_i\cdots k_i} a_{k_i j\cdots j} < 1, k_i > i, j \leq i.$$

$$M(\mathcal{E}_{max})^{-1} \geq M(\hat{\mathcal{E}}_\alpha)^{-1}. \tag{3.3}$$

Theorem 3.1. *([5], Theorem 4) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathcal{M}$-tensor. For*

$$\hat{\mathcal{A}}_\alpha = (I + S_\alpha)\mathcal{A} = \hat{\mathcal{E}}_\alpha - \hat{\mathcal{F}}_\alpha$$

and

$$\mathcal{A}_{max} = (I + S_{max})\mathcal{A} = \mathcal{E}_{max} - \mathcal{F}_{max},$$

under the conditions made in Lemma 3.4, there exists a positive vector $\mathbf{x}$ such that $0 \leq \hat{\mathcal{A}}\mathbf{x}^{m-1} \leq \mathcal{A}_{max}\mathbf{x}^{m-1}$, then we have the following inequality holds

$$\rho(\mathcal{T}_{max}) \leq \rho(\hat{\mathcal{T}}_\alpha), \tag{3.4}$$

where $\mathcal{T}_{max} = M(\mathcal{E}_{max})^{-1}\mathcal{F}_{max}$ and $\hat{\mathcal{T}}_\alpha = M(\hat{\mathcal{E}}_\alpha)^{-1}\hat{\mathcal{F}}_\alpha$.

We first consider a counter-example for Theorem 3.1.

Example 3.1. *We consider a tensor $\mathcal{A} \in \mathbb{R}^{[3,4]}$, where*

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 1 & -0.04 & -0.04 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 \end{pmatrix}, \mathcal{A}(:,:,2) = \begin{pmatrix} -0.04 & -0.04 & -0.04 & -0.04 \\ -0.04 & 1 & -0.04 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 \end{pmatrix},$$

$$\mathcal{A}(:,:,3) = \begin{pmatrix} -0.04 & -0.04 & -0.1 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 \\ -0.04 & -0.04 & 1 & -0.04 \\ -0.04 & -0.04 & -0.04 & -0.04 \end{pmatrix}, \mathcal{A}(:,:,4) = \begin{pmatrix} -0.04 & -0.04 & -0.04 & -0.04 \\ -0.04 & 1 & -0.04 & -0.1 \\ -0.04 & -0.04 & -0.04 & -0.1 \\ -0.04 & -0.04 & -0.04 & 1 \end{pmatrix}.$$

It is easy to show that $\mathcal{A}$ is a strong $\mathcal{M}$-tensor and

$$M(\hat{\mathcal{E}}_1)^{-1} = \begin{pmatrix} 1.0016 & 0 & 0 & 0 \\ 0.0417 & 1.0016 & 0 & 0 \\ 0.0461 & 0.0442 & 1.004 & 0 \\ 0.0436 & 0.0418 & 0.0402 & 1 \end{pmatrix},$$

$$M(\mathcal{E}_{max})^{-1} = \begin{pmatrix} 1.004 & 0 & 0 & 0 \\ 0.0444 & 1.004 & 0 & 0 \\ 0.0463 & 0.0444 & 1.004 & 0 \\ 0.0438 & 0.0419 & 0.0402 & 1 \end{pmatrix}.$$

From the above computation we can see that $M(\mathcal{E}_{max})^{-1} \geq M(\hat{\mathcal{E}}_1)^{-1} \geq 0$. That is, the tensor $\mathcal{A}$ satisfies the condition of the Lemma 3.4. But by computation, we have $\rho(\mathcal{T}_{max}) = 0.5896 > 0.5877 = \rho(\hat{\mathcal{T}}_\alpha)$. This contradicts the conclusion of the Theorem 3.1 in [5].

We next propose a new preconditioner $\tilde{P} = I + \tilde{S}$, where

$$\tilde{S} = \begin{pmatrix} 0 & -a_{12\cdots 2} & 0 & \cdots & -a_{1k_1\cdots k_1} & \cdots & \cdots & 0 \\ 0 & 0 & -a_{23\cdots 3} & \cdots & \cdots & -a_{2k_2\cdots k_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & -a_{n-1,n,\cdots,n} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

where

$$k_i = \min\left\{ j \middle| \max_j |a_{ij\cdots j}|, i < n, j \geq i+1 \right\}.$$

Without loss of generality, we assume that each diagonal entry of the tensor $\mathcal{A}$ is 1. Let

$$\tilde{\mathcal{A}} = \tilde{P}\mathcal{A} = \tilde{\mathcal{E}} - \tilde{\mathcal{F}} = \tilde{\mathcal{D}} - \tilde{\mathcal{L}} - \tilde{\mathcal{U}},$$

where $\tilde{\mathcal{D}} = \tilde{D}\mathcal{I}_m$, $\tilde{\mathcal{L}} = \tilde{L}\mathcal{I}_m$, and $\tilde{D}, -\tilde{L}$ are the diagonal part, the strictly lower triangular part of $M(\tilde{\mathcal{A}})$. If $a_{i,i+1,\cdots,i+1}a_{i+1,i,\cdots,i} + a_{ik_i\cdots k_i}a_{k_ii\cdots i} \neq 1$, then $M(\tilde{\mathcal{D}} - \tilde{\mathcal{L}})^{-1}$ exists, we get the Gauss-Seidel type iteration tensor $\tilde{\mathcal{T}}$ can be defined by $M(\tilde{\mathcal{D}} - \tilde{\mathcal{L}})^{-1}\tilde{\mathcal{U}}$.

Lemma 3.5. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathcal{M}$-tensor, then $\tilde{\mathcal{A}} = \tilde{P}\mathcal{A} = (I + \tilde{S})\mathcal{A}$ is a strong $\mathcal{M}$-tensor.*

Proof. We first show that $\tilde{\mathcal{A}}$ is a $\mathcal{Z}$-tensor. Since

$$\tilde{a}_{ii_2\cdots i_m} = \begin{cases} a_{ii_2\cdots i_m} - a_{i,i+1,\cdots,i+1}a_{i+1,i_2,\cdots,i_m} - a_{ik_i\cdots k_i}a_{k_ii_2\cdots i_m}, i = 1,\cdots,n-2, \\ a_{n-1,i_2,\cdots,i_m} - a_{n-1,i+1,\cdots,i+1}a_{i+1,i_2,\cdots,i_m}, i = n-1, \\ a_{ni_2\cdots i_m}, j = n, \end{cases} \tag{3.5}$$

for $(i, i_2, \cdots, i_m) \neq (i, i, \cdots, i)$, we have $\tilde{a}_{ii_2\cdots i_m} \leq 0$, that is, $\tilde{\mathcal{A}}$ is a $\mathcal{Z}$-tensor.

As $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, from Lemma 2.2, there exists a positive vector $\mathbf{x}$ such that $\mathcal{A}\mathbf{x}^{m-1} > 0$. Thus, $\tilde{\mathcal{A}}\mathbf{x}^{m-1} = (I + \tilde{S})\mathcal{A}\mathbf{x}^{m-1} > 0$. That is to say, there exists a positive vector $\mathbf{x}$ such that $\tilde{\mathcal{A}}\mathbf{x}^{m-1} > 0$, Then from Lemma 2.2 again, we know that $\tilde{\mathcal{A}}$ is a strong $\mathcal{M}$-tensor. □

Lemma 3.6. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathcal{M}$-tensor, let*

$$\hat{\mathcal{A}}_{\mathbf{1}} = (I + S_{\mathbf{1}})\mathcal{A} = \mathcal{E}_{\mathbf{1}} - \mathcal{F}_{\mathbf{1}},$$

$$\mathcal{A}_{max} = (I + S_{max})\mathcal{A} = \mathcal{E}_{max} - \mathcal{F}_{max},$$

and

$$\tilde{\mathcal{A}} = (I + \tilde{S})\mathcal{A} = \tilde{\mathcal{E}} - \tilde{\mathcal{F}},$$

then

$$M(\tilde{\mathcal{E}})^{-1} \geq M(\mathcal{E}_{max})^{-1} \geq M(\mathcal{E}_{\mathbf{1}})^{-1}.$$

Proof. Let $\mathcal{A}_{max} = (a^m_{ii_2\cdots i_m})$ and $\tilde{\mathcal{A}} = (\tilde{a}_{ii_2\cdots i_m})$, we have

$$a^m_{ii_2\cdots i_m} = \begin{cases} a_{ii_2\cdots i_m} - a_{ik_i\cdots k_i}a_{k_ii_2\cdots i_m}, i = 1, \cdots, n-1, \\ a_{ni_2\cdots i_m}, j = n, \end{cases}$$

and $\tilde{a}_{ii_2\cdots i_m}$ is defined by (3.5). Since $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, from Lemma 3.3 and Lemma 3.5, we know that $\mathcal{A}_{max}$ and $\tilde{\mathcal{A}}$ are strong $\mathcal{M}$-tensors. Thus we have $M(\mathcal{E}_{max})$ and $M(\tilde{\mathcal{E}})$ are nonsingular $\mathcal{M}$-matrices. Since

$$a_{ii_2\cdots i_m} - a_{i,i+1,\cdots,i+1}a_{i+1,i_2,\cdots,i_m} - a_{ik_i\cdots k_i}a_{k_ii_2\cdots i_m} \leq a_{ii_2\cdots i_m} - a_{ik_i\cdots k_i}a_{k_ii_2\cdots i_m},$$

for $i = 1, \cdots, n-2$ and $\hat{a}_{ii_2\cdots i_m} = \tilde{a}_{ii_2\cdots i_m}$, for $i = n-1, n$, then we get $M(\tilde{\mathcal{E}})^{-1} \geq M(\mathcal{E}_{max})^{-1}$. The later inequality can be obtained from Lemma 3.4. □

Theorem 3.2. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. For $\hat{\mathcal{A}}_{\mathbf{1}} = \mathcal{E}_{\mathbf{1}} - \mathcal{F}_{\mathbf{1}}$ and $\tilde{\mathcal{A}} = \tilde{\mathcal{E}} - \tilde{\mathcal{F}}$, let $\hat{\mathcal{T}}_{\mathbf{1}} = M(\mathcal{E}_{\mathbf{1}})^{-1}\mathcal{F}_{\mathbf{1}}$ and $\tilde{\mathcal{T}} = M(\tilde{\mathcal{E}})^{-1}\tilde{\mathcal{F}}$, then we have*

$$\rho(\tilde{\mathcal{T}}) \leq \rho(\hat{\mathcal{T}}_{\mathbf{1}}). \tag{3.6}$$

Proof. From Lemma 3.1, we know there exists a positive Perron vector $\mathbf{x}$ of $\hat{\mathcal{T}}_{\mathbf{1}}$ such that $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$. Thus we have $\hat{\mathcal{A}}_{\mathbf{1}}\mathbf{x}^{m-1} = (I + \hat{S}_{\mathbf{1}})\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$. Notice that

$$\tilde{\mathcal{A}}\mathbf{x}^{m-1} - \hat{\mathcal{A}}_{\mathbf{1}}\mathbf{x}^{m-1} = (\tilde{\mathcal{A}} - \hat{\mathcal{A}}_{\mathbf{1}})\mathbf{x}^{m-1} = (\tilde{S} - S_{\mathbf{1}})\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}.$$

This means that $\tilde{\mathcal{A}}\mathbf{x}^{m-1} \geq \hat{\mathcal{A}}_{\mathbf{1}}\mathbf{x}^{m-1} \geq \mathbf{0}$. From Lemma 3.6, we have $M(\tilde{\mathcal{E}})^{-1} \geq M(\mathcal{E}_{\mathbf{1}})^{-1} \geq O$. Hence, we have

$$\begin{aligned} &M(\tilde{\mathcal{E}})^{-1}\tilde{\mathcal{A}}\mathbf{x}^{m-1} - M(\mathcal{E}_{\mathbf{1}})^{-1}\hat{\mathcal{A}}_{\mathbf{1}}\mathbf{x}^{m-1} \\ &= M(\tilde{\mathcal{E}})^{-1}(\tilde{\mathcal{A}} - \hat{\mathcal{A}}_{\mathbf{1}})\mathbf{x}^{m-1} + (M(\tilde{\mathcal{E}})^{-1} - M(\mathcal{E}_{\mathbf{1}})^{-1})\hat{\mathcal{A}}_{\mathbf{1}}\mathbf{x}^{m-1} \geq \mathbf{0}. \end{aligned}$$

Since

$$M(\tilde{\mathcal{E}})^{-1}\tilde{\mathcal{A}} = \mathcal{I}_m - M(\tilde{\mathcal{E}})^{-1}\tilde{\mathcal{F}}, M(\mathcal{E}_{\mathbf{1}})^{-1}\hat{\mathcal{A}}_{\mathbf{1}} = \mathcal{I}_m - M(\mathcal{E}_{\mathbf{1}})^{-1}\mathcal{F}_{\mathbf{1}}.$$

Then, we obtain

$$M(\tilde{\mathcal{E}})^{-1}\tilde{\mathcal{F}}\mathbf{x}^{m-1} \leq M(\mathcal{E}_{\mathbf{1}})^{-1}\mathcal{F}_{\mathbf{1}}\mathbf{x}^{m-1} = \hat{\mathcal{T}}_{\mathbf{1}}\mathbf{x}^{m-1} = \rho(M(\mathcal{E}_{\mathbf{1}})^{-1}\mathcal{F}_{\mathbf{1}})\mathbf{x}^{[m-1]}.$$

By Lemma 2.1, we have $\rho(M(\tilde{\mathcal{E}})^{-1}\tilde{\mathcal{F}}) \leq \rho(M(\mathcal{E}_{\mathbf{1}})^{-1}\mathcal{F}_{\mathbf{1}})$, i.e. $\rho(\tilde{\mathcal{T}}) \leq \rho(\hat{\mathcal{T}}_{\mathbf{1}})$. □

Remark 3.1. *We consider the Example 3.1, it is easy to show that*

$$M(\tilde{\mathcal{E}})^{-1} = \begin{pmatrix} 1.0056 & 0 & 0 & 0 \\ 0.0461 & 1.0056 & 0 & 0 \\ 0.0465 & 0.0444 & 1.004 & 0 \\ 0.0439 & 0.0420 & 0.0402 & 1 \end{pmatrix}.$$

That is, $M(\tilde{\mathcal{E}})^{-1} \geq M(\mathcal{E}_{\mathbf{1}})^{-1}$, by computation, we have $\rho(\hat{\mathcal{T}}_{\mathbf{1}}) = 0.5877$, and its corresponding Perron vector

$$\mathbf{x} = (x_1, x_2, x_3, x_4)^T = (1, 0.9124, 0.9496, 0.9557)^T.$$

Hence, according to Theorem 3.2, we have $\rho(\tilde{\mathcal{T}}) = 0.5816 < 0.5877 = \rho(\hat{\mathcal{T}}_{\mathbf{1}})$.

Theorem 3.3. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor. For $\mathcal{A}_{max} = \mathcal{E}_{max} - \mathcal{F}_{max}$ and $\tilde{\mathcal{A}} = \tilde{\mathcal{E}} - \tilde{\mathcal{F}}$, let*

$$\mathcal{T}_{max} = M(\mathcal{E}_{max})^{-1}\mathcal{F}_{max}$$

and

$$\tilde{\mathcal{T}} = M(\tilde{\mathcal{E}})^{-1}\tilde{\mathcal{F}}.$$

If there exists a positive Perron vector $\mathbf{x}$ of $\mathcal{T}_{max}$ such as $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$. then we have

$$\rho(\tilde{\mathcal{T}}) \leq \rho(\mathcal{T}_{max}). \tag{3.7}$$

Proof. The proof is similar to those in Theorem 3.2, here we omit it. □

Example 3.2. *We consider the tensor $\mathcal{A} \in \mathbb{R}^{[3,3]}$ in [5], where*

$$\mathcal{A}(:,:,1) = \begin{pmatrix} 1 & -0.12 & -0.13 \\ -0.12 & -0.03 & -0.06 \\ -0.13 & -0.02 & -0.1 \end{pmatrix},$$

$$\mathcal{A}(:,:,2) = \begin{pmatrix} -0.04 & -0.02 & -0.03 \\ -0.01 & 1 & -0.02 \\ -0.03 & -0.04 & -0.02 \end{pmatrix},$$

$$\mathcal{A}(:,:,3) = \begin{pmatrix} -0.03 & -0.02 & -0.04 \\ -0.02 & -0.06 & -0.03 \\ -0.02 & -0.1 & 1 \end{pmatrix}.$$

We can show that $\mathcal{A}$ is a strong $\mathcal{M}$-tensor. Let $\alpha = (\alpha_1, \alpha_2) = (1, 1)$, we get:

$$M(\hat{\mathcal{E}}_{\mathbf{1}})^{-1} = \begin{pmatrix} 1.0024 & 0 & 0 \\ 0.0125 & 1.0012 & 0 \\ 0.0136 & 0.04 & 1 \end{pmatrix},$$

$$M(\mathcal{E}_{max})^{-1} = \begin{pmatrix} 1.0052 & 0 & 0 \\ 0.01247 & 1.0012 & 0 \\ 0.01357 & 0.04 & 1 \end{pmatrix},$$

$$M(\tilde{\mathcal{E}})^{-1} = \begin{pmatrix} 1.0077 & 0 & 0 \\ 0.0125 & 1.0012 & 0 \\ 0.0136 & 0.04 & 1 \end{pmatrix}.$$

From the above we can see that

$$M(\tilde{\mathcal{E}})^{-1} \geq M(\mathcal{E}_{max})^{-1} \geq O, M(\tilde{\mathcal{E}})^{-1} \geq M(\hat{\mathcal{E}}_{\mathbf{1}})^{-1} \geq O.$$

By computation, we have

$$\rho(\tilde{\mathcal{T}}) = 0.3346 < 0.3386 = \rho(\mathcal{T}_{max}) < 0.3451 = \rho(\hat{\mathcal{T}}_{\mathbf{1}}).$$

4. Conclusions

In this paper, we present a new preconditioner $I + \tilde{S}$ for solving multi-linear sysyems and give new comparison results between two different preconditioned tensor splitting iterative methods. Comparison theorems show that the spectral radius of the proposed preconditioner is less than those of the preconditioners in [5]. We present two numerical experiments to validate the effectiveness of the proposed preconditioner.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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