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Research article

Existence of ground state solutions for the modified Chern-Simons-Schrödinger equations with general Choquard type nonlinearity

Yingying Xiao[1,2], **Chuanxi Zhu**[1,*] **and Li Xie**[2]

[1] Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, China
[2] Nanchang JiaoTong Institute, Nanchang, Jiangxi, 330031, China

* **Correspondence:** Email: chuanxizhu@126.com.

Abstract: In this paper, we are concerned with the following modified Schrödinger equation

$$\begin{aligned}&-\Delta u+V(|x|)u-\kappa u\Delta(u^2)+q\frac{h^2(|x|)}{|x|^2}(1+\kappa u^2)u\\&\quad+q\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}(2+\kappa u^2(s))u^2(s)\mathrm{d}s\right)u=(I_\alpha*F(u))f(u),\qquad x\in\mathbb{R}^2,\end{aligned}$$

where $\kappa, q>0$, I_α is a Riesz potential, $\alpha\in(0,2)$ and $V\in C(\mathbb{R}^2,\mathbb{R})$, $F(t)=\int_0^t f(s)\mathrm{d}s$. Under appropriate assumptions on f and $V(x)$, by using the variational methods, we establish the existence of ground state solutions of the above equation.

Keywords: Chern-Simons-Schrödinger equations; ground state solutions; variational methods; monotone trick; general Choquard type
Mathematics Subject Classification: 35J60, 35J20

1. Introduction

In this paper, we establish the existence of ground state solutions to the following modified Chern-Simons-Schrödinger equation

$$\begin{aligned}&-\Delta u+V(|x|)u-\kappa u\Delta(u^2)+q\frac{h^2(|x|)}{|x|^2}(1+\kappa u^2)u\\&\quad+q\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}(2+\kappa u^2(s))u^2(s)\mathrm{d}s\right)u=(I_\alpha*F(u))f(u),\qquad x\in\mathbb{R}^2,\end{aligned}\tag{1.1}$$

where κ, $q > 0$, $\alpha \in (0, 2)$ and $h(l) = \int_0^l \varrho u^2(\varrho)d\varrho$ $(l \geq 0)$, u is a radially symmetric function, I_α is a Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{2-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\Pi 2^\alpha |x|^{2-\alpha}} := \frac{A_\alpha}{|x|^{2-\alpha}},$$

and Γ is the Gamma function, $F(t) = \int_0^t f(s)ds$, the potential V is supposed to satisfies:

$(\mathcal{V}_1)$ $V \in C(\mathbb{R}^2, \mathbb{R})$;

$(\mathcal{V}_2)$ $V(|x|) = V(x)$ and there exists $\beta \geq \gamma > 0$, such that $\beta \geq V(|x|) \geq \gamma$ for all $x \in \mathbb{R}^2$.

Moreover, we assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ verifies:

(f_1) $f \in C(\mathbb{R}^2, \mathbb{R})$, $f(t) = o(t)$;

(f_2) There exist constant $p \in (2 + \alpha, +\infty)$ and $C > 0$ such that

$$|f(t)| \leq C(1 + |t|^{p-1}), \quad \forall t \in \mathbb{R};$$

(f_3) There exists a constant $\vartheta > 8$ such that

$$0 < \vartheta F(t) \leq t f(t), \quad \forall t \in \mathbb{R}.$$

As we all know, Eq (1.1) originates from seeking the standing waves of the following nonlinear Chern-Simons-Schrödinger system

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi + f(\phi) = 0, \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}(\bar{\phi} D_2 \phi), \\ \partial_0 A_2 - \partial_2 A_0 = -\mathrm{Im}(\bar{\phi} D_1 \phi), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2, \end{cases} \tag{1.2}$$

where i denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex scalar field, $A_j : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, $D_j = \partial_j + iA_j$ is the covariant derivative for $j = 0, 1, 2$. The system was first proposed by Jackiw and Pi, consisting of Schrödinger equation augmented by the gauge field. The two-dimensional Chern-Simons-Schrödinger system is a non-relativistic quantum model describing the dynamics of a large number of particles in the plane, in which these particles interact directly through the spontaneous magnetic field. Moreover, the important applications of the system are also reflected in the study of the high temperature superconductors and fractional quantum Hall effect and Aharovnov-Bohm scattering. For more physical backgrounds of (1.2), we refer readers to [16, 17] and the references therein.

As far as we know, Byeon et al.'s [1] was the first article investigate the standing wave solutions of this system by the variational method. They considered the standing waves of system (1.2) with power type nonlinearity, that is, $f(u) = \lambda|u|^{p-1}u$, and obtained the existence and nonexistence results for (1.2) of type

$$\begin{aligned} \phi(t, x) &= u(|x|)e^{iwt}, & A_0(t, x) &= k(|x|), \\ A_1(t, x) &= \frac{x_2}{|x|^2}h(|x|), & A_2(t, x) &= -\frac{x_1}{|x|^2}h(|x|), \end{aligned} \tag{1.3}$$

where $w > 0$ is a given frequency, $\lambda > 0$ and $p > 1$, u, k, h are real valued functions depending only on $|x|$. The ansatz (1.3) satisfies the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$. After then, many researchers began to pay attention to this field. see e.g. [2, 3, 5, 7, 11–13, 15, 20, 21, 29] and the references therien. However, through the study of large number of literatures, it is found that there are few papers studying the modified Chern-Simons-Schrödinger equation, except for [8, 23, 24]. To best of our knowledge, there is no article to pay attention to general Choquard type nonlinearity for modified Chern-Simons-Schrödinger equation. Motivated by the previously mentioned paper [26], we shall study the existence of ground state solutions for Eq (1.1) using a change of variable and variational argument.

The problem (1.1) is the Euler-Lagrange equation of the energy functional

$$\begin{aligned}\mathcal{I}(u) =&\frac{1}{2}\int_{\mathbb{R}^2}\Big((1+2\kappa u^2)|\nabla u|^2 + V(|x|)u^2\Big) + \frac{q}{2}\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\Big(\int_0^{|x|} su^2(s)\mathrm{d}s\Big)^2\\ &+\frac{q}{4}\kappa\int_{\mathbb{R}^2}\frac{u^4(x)}{|x|^2}\Big(\int_0^{|x|} su^2(s)\mathrm{d}s\Big)^2 - \frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(u))F(u).\end{aligned}$$

From the variational point of view, the main difficulty of this problem is the energy functional $\mathcal{I}$ can not be well defined for $u \in H_r^1(\mathbb{R}^2)$. To solve this problem, we intend to adopt the Liu and Wang's [18] approach, considering the change of variable $g : \mathbb{R} \to \mathbb{R}$ given by

$$g'(t) = \frac{1}{\sqrt{1+2g^2(t)}} \quad \text{on} \quad [0, +\infty),$$

$g(0) = 0$ and $g(-t) = -g(t)$ on $(-\infty, 0]$. By the change of $u = g(v)$ of variable, Eq (1.1) is transformed into a semilinear problem

$$\begin{aligned}&-\Delta v + V(|x|)g(v)g'(v) + q\frac{\hat{h}^2[g(v(|x|))]}{|x|^2}(1+\kappa g^2(v))g(v)g'(v)\\ &+q\left(\int_{|x|}^{+\infty}\frac{\hat{h}[g(v(s))]}{s}(2+\kappa g^2(v(s)))g^2(v(s))\mathrm{d}s\right)g(v)g'(v)\\ &\quad = (I_\alpha * F(g(v)))f(g(v))g'(v),\end{aligned} \tag{1.4}$$

where

$$\hat{h}^2[g(v(|x|))] := \Big(\int_0^{|x|} sg^2(v(s))\mathrm{d}s\Big)^2.$$

Furthermore, the functional $\mathcal{I}(u)$ can be reduced to

$$\begin{aligned}\mathcal{J}(v) =&\frac{1}{2}\int_{\mathbb{R}^2}|\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^2}V(|x|)g^2(v) + \frac{q}{2}\mathcal{C}(g(v))\\ &+\frac{q}{4}\kappa\mathcal{D}(g(v)) - \frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(g(v)))F(g(v)),\end{aligned} \tag{1.5}$$

where

$$\mathcal{C}(g(v)) := \int_{\mathbb{R}^2}\frac{g^2(v(|x|))}{|x|^2}\Big(\int_0^{|x|} sg^2(v(s))\mathrm{d}s\Big)^2,$$

$$\mathcal{D}(g(v)) := \int_{\mathbb{R}^2}\frac{g^4(v(|x|))}{|x|^2}\Big(\int_0^{|x|} sg^2(v(s))\mathrm{d}s\Big)^2.$$

Obviously, the energy functional $\mathcal{J}(u)$ is well defined in $H_r^1(\mathbb{R}^2)$. It is easy to see that v is a critical point of $\mathcal{J}$,

$$\langle \mathcal{J}'(v), \psi\rangle = \int_{\mathbb{R}^2} \nabla v \nabla \psi + \int_{\mathbb{R}^2} V(|x|)g(v)g'(v)\psi + q\int_{\mathbb{R}^2} \Big\{\frac{\hat{h}^2[g(v(|x|))]}{|x|^2}(1+\kappa g^2(v)) \tag{1.6}$$

$$+ \int_{|x|}^{+\infty} \frac{\hat{h}[g(v(s))]}{s}(2+\kappa g^2(v(s)))g^2(v(s))\mathrm{d}s\Big\} g(v)g'(v)\psi$$
$$- \int_{\mathbb{R}^2} (I_\alpha * F(g(v)))f(g(v))g'(v)\psi,$$

for any $\psi \in H_r^1(\mathbb{R}^2)$, then v is a weak solution of (1.4), that is $u = g(v)$ solves (1.1). In particular, for $\tau = 2$ or $\tau = 4$, using the integrate by parts, one has

$$\int_{\mathbb{R}^2} \frac{\hat{h}^2[g(v(|x|))]}{|x|^2} g^\tau(v) = \int_{\mathbb{R}^2} \left(\int_{|x|}^{+\infty} \frac{g^\tau(v(s))\hat{h}[g(v(s))]}{s}\mathrm{d}s\right) g^2(v). \tag{1.7}$$

Note that by the Cauchy inequality, there exists a constant $C_0 > 0$ such that

$$\hat{h}^2[g(v(|x|))] := \Big(\int_{B_{|x|}} \frac{g^2(v(y))}{2\pi}\mathrm{d}y\Big)^2 \le C_0|x|^2\|g(v)\|_4^4.$$

Then for $v \in H_r^1(\mathbb{R}^2)$, we have

$$\mathcal{C}(g(v)) \le C_0\|g(v)\|_2^2\|g(v)\|_4^4, \tag{1.8}$$

$$\mathcal{D}(g(v)) \le C_0\|g(v)\|_4^8. \tag{1.9}$$

Now, we give our result in the following.

Theorem 1.1. *Under assumptions* $(\mathcal{V}_1)$, $(\mathcal{V}_2)$ *and* (f_1)–(f_3), *problem* (1.1) *has a ground state solution.*

Notations. To facilitate expression, hereafter, we recall the following basic notes:

- $H^1(\mathbb{R}^2)$: $= \{u \in L^2(\mathbb{R}^2), \nabla u \in L^2(\mathbb{R}^2)\}$ with the norm $\|v\| = \left(\int_{\mathbb{R}^2}(v^2 + |\nabla v|^2)\right)^{1/2}$;
- $H_r^1(\mathbb{R}^2)$: $= \{v \in H^1(\mathbb{R}^2)\colon v(x) = v(|x|)\}$;
- $L^s(\mathbb{R}^2)$ denotes the Lebesgue space with the norm $\|v\|_s = \left(\int_{\mathbb{R}^2} |v|^s\right)^{1/s}$, where $1 \le s < +\infty$;
- The embedding $H_r^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ is continuous for $2 \le s < +\infty$;
- The embedding $H_r^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ is compact for $2 < s < +\infty$;
- $H_r^1(\mathbb{R}^2) \hookrightarrow L^{\frac{4\upsilon}{2+\alpha}}(\mathbb{R}^2)$ if and only if $\frac{2+\alpha}{2} \le \upsilon < +\infty$;
- $\int_{\mathbb{R}^2} \spadesuit$ denotes $\int_{\mathbb{R}^2} \spadesuit \mathrm{d}x$;
- The weak convergence in $H_r^1(\mathbb{R}^2)$ is denoted by $\rightharpoonup$, and the strong convergence by $\to$;
- We use C, C_0 denote various positive constants.

The remainder of the paper is organized as follows. In section 2, we present some preliminary results. Section 3 devote to some required results and complete the proof details of Theorem 1.1.

2. Preliminaries

In this section, we give some useful lemmas and proposition, which play an important role in the proof of our result. Next, let us recall some properties of the variable g, which are proved in [6,18,27].

Lemma 2.1. ([6,18,27]) *The function $g(t)$ and its derivative satisfy the following properties:*
(g_1) $|g(t)| \leq |t|$ *for all* $t \in \mathbb{R}$*;*
(g_2) $|g(t)| \leq 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*
(g_3) $g(t)/2 \leq tg'(t) \leq g(t)$ *for all* $t \geq 0$*;*
(g_4) $g^2(t)/2 \leq tg(t)g'(t) \leq g^2(t)$ *for all* $t \in \mathbb{R}$*;*
(g_5) $|g(t)g'(t)| \leq 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$*;*
(g_6) *There exists a constant* $C > 0$ *such that*

$$|g(t)| \geq \begin{cases} C|t|, & if\ |t| \leq 1, \\ C|t|^{1/2}, & if\ |t| \geq 1. \end{cases}$$

Next, the following inequality holds if and only if the functions in $H_r^1(\mathbb{R}^2)$.

Proposition 2.2. ([1]) *For* $v \in H_r^1(\mathbb{R}^2)$*, there holds*

$$\int_{\mathbb{R}^2} |v|^4 \leq 2\Big(\int_{\mathbb{R}^2} |\nabla v|^2\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^2} \frac{v^2}{|x|^2}\Big(\int_0^{|x|} sv^2(s)ds\Big)^2\Big)^{\frac{1}{2}}.$$

In order to achieve our main result, we would like to recall the well-known Hardy-Littlewood-Sobolev inequality in [19].

Lemma 2.3. ([19]) *Let* $\mu, \nu > 1$ *and* $0 < \alpha < N(N = 1, 2...)$ *be such that*

$$\frac{1}{\mu} + \frac{1}{\nu} = 1 + \frac{\alpha}{N}.$$

Where $\zeta \in L^\mu(\mathbb{R}^N)$ *and* $\eta \in L^\nu(\mathbb{R}^N)$*, there exists a constant C, independent of* ζ*,* η*, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\zeta(x)\eta(y)}{|x-y|^{N-\alpha}} \leq C(\mu, \nu, N, \alpha)||\zeta||_\mu ||\eta||_\nu.$$

Finally, for functional $C(v)$, $\mathcal{D}(v)$, we give the following compactness lemma:

Lemma 2.4. ([8]) *Suppose that a sequence* $\{v_n\}$ *converges weakly to a function v in* $H_r^1(\mathbb{R}^2)$ *as* $n \to +\infty$*. Then for each* $\psi \in H_r^1(\mathbb{R}^2)$*,* $C(v_n)$*,* $C'(v_n)\psi$ *and* $C'(v_n)v_n$*,* $\mathcal{D}(v_n)$ *and* $\mathcal{D}'(v_n)\psi$*,* $\mathcal{D}'(v_n)v_n$ *converges up to a subsequence to* $C(v)$*,* $C'(v)\psi$ *and* $C'(v)v$*,* $\mathcal{D}(v)$ *and* $\mathcal{D}'(v)\psi$*,* $\mathcal{D}'(v)v$*, respectively, as* $n \to +\infty$*.*

3. Proof of Theorem 1.1

In this section, we would like to complete the proof of Theorem 1.1.

Theorem 3.1. ([14]) *Set* $(E, ||\cdot||)$ *be a Banach space,* $\mathbb{I} \subset \mathbb{R}^+$ *be a real interval. Consider a family* Ψ_η *of* C^1*-functional on E*

$$\Psi_\lambda(v) = \mathcal{A}(v) - \lambda\mathcal{B}(v), \qquad for\ all\ \lambda \in \mathbb{I},$$

where $B(v)$ is non-negative and when $\| v \| \to +\infty$, either $\mathcal{A}(v) \to +\infty$ or $\mathcal{B}(v) \to +\infty$. Assume that there exist two points v_1, v_2 holds

$$\max\left\{\Psi_\lambda(v_1), \Psi_\lambda(v_2)\right\} < \inf_{\gamma\in\bar{\Gamma}_\lambda} \max_{t\in[0,1]} \Psi_\lambda(\gamma(t)) = c_\lambda, \qquad for\ all\ \lambda \in \mathbb{I},$$

where

$$\bar{\Gamma}_\lambda = \left\{\gamma \in C([0,1], E) : \gamma(0) = v_1, \gamma(1) = v_2\right\}.$$

Then for a.e. $\lambda \in \mathbb{I}$, there exist a sequence $\{v_n\} \subset E$ such that

(1) *$\{v_n\}$ is bounded in E;*

(2) $\lim\limits_{n\to+\infty} \Psi_\lambda(v_n) = c_\lambda$*;*

(3) $\lim\limits_{n\to+\infty} \Psi'_\lambda(v_n) = 0$ *in the dual space E^{-1} of E.*

Furthermore, the map $\lambda \mapsto c_\lambda$ is non-increasing and left continuous.

Let $\mathbb{I} = [\frac{1}{2}, 1]$, we define the following energy functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla v|^2 + V(|x|)g^2(v)\right) + \frac{q}{2}C(g(v)) + \frac{q}{4}\kappa\mathcal{D}(g(v)) - \frac{\lambda}{2}\int_{\mathbb{R}^2}(I_\alpha * F(g(v)))F(g(v)),$$

where $\lambda \in \mathbb{I}$. Moreover, let

$$\mathcal{A}(v) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla v|^2 + V(|x|)g^2(v)\right) + \frac{q}{2}C(g(v)) + \frac{q}{4}\kappa\mathcal{D}(g(v)),$$

and

$$\mathcal{B}(v) = \frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(g(v)))F(g(v)).$$

Setting $\| v \| \to +\infty$, then $\mathcal{A}(v) \to +\infty$. Furthermore, $\mathcal{B}(v) \geq 0$.

Next, we prove that the functional $\mathcal{J}$ exhibits the mountain pass geometry.

Lemma 3.2. *Under assumptions $(\mathcal{V}_1)$ and $(\mathcal{V}_2)$, then there holds:*

(i) There exists $v \in H_r^1(\mathbb{R}^2)\backslash\{0\}$ such that $\mathcal{J}_\lambda(v) < 0$ for all $\lambda \in \mathbb{I}$;

(ii) $c_\lambda = \inf\limits_{\gamma\in\bar{\Gamma}_\lambda} \max\limits_{t\in[0,1]} \mathcal{J}_\lambda(\gamma(t)) > \max\left\{\mathcal{J}_\lambda(0), \mathcal{J}_\lambda(v)\right\}$ for all $\lambda \in \mathbb{I}$, where

$$\bar{\Gamma}_\lambda = \left\{\gamma \in C([0,1], H_r^1(\mathbb{R}^2)) : \gamma(0) = 0, \gamma(1) = v\right\}.$$

Proof. (i) Let $v \in H_r^1(\mathbb{R}^2)\backslash\{0\}$ be fixed. For any $\lambda \in \mathbb{I} = [\frac{1}{2}, 1]$, we have

$$\begin{aligned}&\mathcal{J}_\lambda(v) \leq \mathcal{J}_{\frac{1}{2}}(v)\\ =&\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla v|^2 + V(|x|)g^2(v)\right) + \frac{q}{2}C(g(v)) + \frac{q}{4}\kappa\mathcal{D}(g(v)) - \frac{1}{4}\int_{\mathbb{R}^2}(I_\alpha * F(g(v)))F(g(v)).\end{aligned}$$

Arguing as in [4, 9], we consider $\xi \in C_0^\infty(\mathbb{R}^2)$ which satisfies $0 \leq \xi(x) \leq 1$, $\xi(x) = 0$ for $|x| \geq 2$ and $\xi(x) = 1$ for $|x| \leq 1$. By (g_3), we can deduce that $g(t\xi(x)) \geq g(t)\xi(x)$ for $t \geq 0$. According to Yang et al. [25], from (f_3) and (g_4) that for $t > \frac{1}{\|\xi\|}$, we have

$$\int_\Omega (I_\alpha * F(g(t\xi)))F(g(t\xi)) \geq \int_\Omega \left(I_\alpha * F\left(g\left(\frac{\xi}{\|\xi\|}\right)\right)\right) F\left(g\left(\frac{\xi}{\|\xi\|}\right)\right) t^\vartheta \|\xi\|^\vartheta.$$

Thus from (g_1), one has

$$\begin{aligned}\mathcal{J}_\lambda(t\xi) \leq& \frac{t^2}{2}\int_{\mathbb{R}^2}\Big(|\nabla v|^2 + V(|x|)v^2\Big) + \frac{t^6}{2}q\mathcal{C}(g(v)) + \frac{t^8}{4}q\kappa\mathcal{D}(g(v))\\ &- \frac{1}{4}\int_{\mathbb{R}^2}(I_\alpha * F(g(t\xi)))F(g(t\xi))\\ \leq& \frac{t^2}{2}\int_{\mathbb{R}^2}\Big(|\nabla v|^2 + V(|x|)v^2\Big) + \frac{t^6}{2}q\mathcal{C}(g(v)) + \frac{t^8}{4}q\kappa\mathcal{D}(g(v))\\ &- \frac{1}{4}\int_{\mathbb{R}^2}\left(I_\alpha * F\left(g\left(\frac{\xi}{\|\xi\|}\right)\right)\right)F\left(g\left(\frac{\xi}{\|\xi\|}\right)\right)t^\vartheta\|\xi\|^\vartheta,\end{aligned}$$

for all $t > 0$. By $\vartheta > 8$, we deduce that $\mathcal{J}_\lambda(t\xi) \to -\infty$ as $t \to +\infty$. Thus, there exists a $t_0 > 0$ such that $\mathcal{J}_\lambda(t_0\xi) < 0$. Then taking a function $v = t_0\xi$, we have $\mathcal{J}_\lambda(v) < 0$ for all $\lambda \in \mathbb{I}$.

(ii) By Chen et al. [4] and Fang et al. [10], there exists $\rho' > 0$ such that

$$C\|v\|^2 \leq \int_{\mathbb{R}^2}(|\nabla v|^2 + V(|x|)g^2(v))$$

for all $\|v\| \leq \rho'$. From (g_2), Lemma 2.3 and Sobolev imbedding inequality, for $\varepsilon > 0$ sufficiently small, one has

$$\begin{aligned}\mathcal{J}_\lambda(v) \geq& \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla v|^2 + V(|x|)g^2(v)\Big) + \frac{q}{2}\mathcal{C}(g(v)) + \frac{q}{4}\kappa\mathcal{D}(g(v))\\ &- \frac{1}{2}\int_{\mathbb{R}^2}(I_\alpha * F(g(v)))F(g(v))\\ \geq& \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla v|^2 + V(|x|)g^2(v)\Big) - \frac{C}{2}\Big(\int_{\mathbb{R}^2}(\varepsilon|f(v)|^2 + C_\varepsilon|f(v)|^p)^{\frac{4}{2+\alpha}}\Big)^{\frac{2+\alpha}{2}}\\ \geq& \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla v|^2 + V(|x|)g^2(v)\Big) - C\varepsilon^2\Big(\int_{\mathbb{R}^2}|v|^{\frac{4}{2+\alpha}}\Big)^{\frac{2+\alpha}{2}} - CC_\varepsilon^2\Big(\int_{\mathbb{R}^2}|v|^{\frac{2p}{2+\alpha}}\Big)^{\frac{2+\alpha}{2}}\\ \geq& C(\|v\|^2 - \|v\|^p), \quad \text{for all } \|v\| \leq \rho'.\end{aligned}$$

Since $p > 2 + \alpha$, we get $\mathcal{J}_\lambda(v) > 0$ if ρ' is small enough. Hence, $\mathcal{J}_\lambda(0)$ is strict local minimum, $c_\lambda > 0$.

By Theorem 3.1, it is easy to know that for any almost everywhere $\lambda \in \mathbb{I}$, there exists a bounded sequence $\{w_n\} \subset H_r^1(\mathbb{R}^2)$ such that $\mathcal{J}'_\lambda(w_n) \to 0$ and $\mathcal{J}_\lambda(w_n) \to c_\lambda$, which is called (PS) sequence.

Lemma 3.3. *Assume that $\{w_n\} \subset H_r^1(\mathbb{R}^2)$ is a sequence of obtain above. Then, for almost $\lambda \in \mathbb{I}$ there exists $w_\lambda \in H_r^1(\mathbb{R}^2)\backslash\{0\}$, such that $\mathcal{J}'_\lambda(w_\lambda) = 0$ and $\mathcal{J}_\lambda(w_\lambda) = c_\lambda$.*

Proof. By Theorem 3.1 and Lemma 3.2, we know that $\{w_n\} \subset H_r^1(\mathbb{R}^2)$ is bounded, then up to a subsequence, there exists $w_\lambda \in H_r^1(\mathbb{R}^2)\backslash\{0\}$ such that $w_n \rightharpoonup w_\lambda$ in $H_r^1(\mathbb{R}^2)$, $w_n \to w_\lambda$ in $L^s(\mathbb{R}^2)$ $(s > 2)$ and $w_n \to w_\lambda$ a.e. in $\mathbb{R}^2$. By the Lebesgue-dominated convergence theorem, it is easy to check that $\mathcal{J}'_\lambda(w_\lambda) = 0$. Similar to [9, 10, 22, 28], we get

$$C\|w_n - w_\lambda\|^2 \leq \int_{\mathbb{R}^2}\Big[|\nabla(w_n - w_\lambda)|^2 + V(|x|)(g(w_n)g'(w_n) - g(w_\lambda)g'(w_\lambda))(w_n - w_\lambda)\Big]. \tag{3.1}$$

(f_1) and (f_2) imply that for each $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|f(x, w_n)| \leq \varepsilon|w_n| + C_\varepsilon|w_n|^{p-1} \leq \varepsilon|w_n|^{\frac{\alpha}{2}} + C_\varepsilon|w_n|^{p-1} \quad \text{for all } w_n \in \mathbb{R}. \tag{3.2}$$

Furthermore, using Lemma 2.3, the Hölder inequality and (g_2), (g_5), one obtains

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}(I_\alpha * F(g(w_n)))f(g(w_n))g'(w_n)(w_n-w_\lambda)\Big|\\
&\le \int_{\mathbb{R}^2}(I_\alpha*(\varepsilon|g(w_n)|^2+C_\varepsilon|g(w_n)|^p))(\varepsilon|g(w_n)|+C_\varepsilon|g(w_n)|^{p-1})|g'(w_n)||w_n-w_\lambda|\\
&\le C\Big(\int_{\mathbb{R}^2}\Big[\varepsilon|w_n|+C_\varepsilon|w_n|^{\frac{p}{2}}\Big]^{\frac{4}{2+\alpha}}\Big)^{\frac{2+\alpha}{4}}\Big(\int_{\mathbb{R}^2}\Big[\varepsilon|w_n-w_\lambda|+C_\varepsilon|w_n|^{\frac{p-2}{2}}|w_n-w_\lambda|\Big]^{\frac{4}{2+\alpha}}\Big)^{\frac{2+\alpha}{4}}\\
&\le C\bigg(\varepsilon\Big(\int_{\mathbb{R}^2}|w_n|^{\frac{4}{2+\alpha}}\Big)^{\frac{2+\alpha}{4}}+C_\varepsilon\Big(\int_{\mathbb{R}^2}|w_n|^{\frac{2p}{2+\alpha}}\Big)^{\frac{2+\alpha}{4}}\bigg)\\
&\quad\times\bigg(\varepsilon\Big(\int_{\mathbb{R}^2}|w_n-w_\lambda|^{\frac{4}{2+\alpha}}\Big)^{\frac{2+\alpha}{4}}\bigg)+C_\epsilon\Big(\int_{\mathbb{R}^2}|w_n|^{\frac{2p-4}{2+\alpha}}|w_n-w_\lambda|^{\frac{4}{2+\alpha}}\Big)^{\frac{2+\alpha}{4}}\bigg)\\
&\le CC_\varepsilon\bigg(\Big(\int_{\mathbb{R}^2}|w_n|^{\frac{2p}{2+\alpha}}\Big)^{\frac{(p-2)}{p}}\Big(\int_{\mathbb{R}^2}|w_n-w_\lambda|^{\frac{2p}{2+\alpha}}\Big)^{\frac{2}{p}}\bigg)^{\frac{2+\alpha}{4}}\\
&\le CC_\varepsilon\Big(\int_{\mathbb{R}^2}|w_n-w_\lambda|^{\frac{2p}{2+\alpha}}\Big)^{\frac{2+\alpha}{2p}}\to 0.
\end{aligned}\tag{3.3}$$

In the same way, we can prove that

$$\Big|\int_{\mathbb{R}^2}(I_\alpha * F(g(w_\lambda)))f(g(w_\lambda))g'(w_\lambda)(w_n-w_\lambda)\Big|\to 0.\tag{3.4}$$

Thus, it follows from (1.7), (3.1)–(3.4) and Lemma 2.4 that

$$\begin{aligned}
0\leftarrow&\langle\mathcal{J}'_\lambda(w_n)-\mathcal{J}'_\lambda(w_\lambda),w_n-w_\lambda\rangle\\
=&\int_{\mathbb{R}^2}\Big[|\nabla(w_n-w_\lambda)|^2+V(|x|)(g(w_n)g'(w_n)-g(w_\lambda)g'(w_\lambda))(w_n-w_\lambda)\Big]\\
&+\frac{q}{2}\langle\mathcal{C}'(g(w_n))-\mathcal{C}'(g(w_\lambda)),w_n-w_\lambda\rangle+\frac{q}{4}\kappa\langle\mathcal{D}'(g(w_n))-\mathcal{D}'(g(w_\lambda)),w_n-w_\lambda\rangle\\
&-\lambda\int_{\mathbb{R}^2}\Big[(I_\alpha*F(g(w_n)))f(g(w_n))g'(w_n)\\
&-(I_\alpha*F(g(w_\lambda)))f(g(w_\lambda))g'(w_\lambda)\Big](w_n-w_\lambda)\\
\ge&\ C\|w_n-w_\lambda\|^2+o_n(1),
\end{aligned}$$

then, we deduce that $w_n\to w_\lambda$ in $H^1_r(\mathbb{R}^2)$. Thus, w_λ is a nontrivial critical point of $\mathcal{J}_\lambda$ with $\mathcal{J}_\lambda(w_\lambda)=c_\lambda$. This completes the proof.

Proof of Theorem 1.1. At first, by Theorem 3.1, for a.e. $\lambda\in\mathbb{I}$, there exists $w_\lambda\in H^1_r(\mathbb{R}^2)$ such that $w_n\rightharpoonup w_\lambda\neq 0$ in $H^1_r(\mathbb{R}^2)$, $J'_\lambda(w_n)\to 0$ and $\mathcal{J}_\lambda(w_n)\to c_\lambda$. By Lemma 3.3, one obtains $\mathcal{J}'_\lambda(w_\lambda)=0$, $\mathcal{J}_\lambda(w_\lambda)=c_\lambda$. Then, take $\{\lambda_n\}\subset\mathbb{I}$ such that $\lim\limits_{n\to+\infty}\lambda_n=1$, $w_{\lambda_n}\in H^1_r(\mathbb{R}^2)$ and $\mathcal{J}'_{\lambda_n}(w_{\lambda_n})=0$, $\mathcal{J}_{\lambda_n}(w_{\lambda_n})=c_{\lambda_n}$. Next, we claim that $\|w_{\lambda_n}\|\le C$. From (f_3), (1.6), (1.7) and Lemma 3.2 and $\mathcal{J}_{\lambda_n}(w_{\lambda_n})\le c_{\frac{1}{2}}$, $J'_{\lambda_n}(w_{\lambda_n})=0$, it follows that

$$\begin{aligned}
c_{\frac{1}{2}}\ge&\ \mathcal{J}_{\lambda_n}(w_{\lambda_n})-\frac{1}{2\vartheta}\langle\mathcal{J}'_{\lambda_n}(w_{\lambda_n}),g(w_{\lambda_n})/g'(w_{\lambda_n})\rangle\\
=&\Big(\frac{1}{2}-\frac{1}{2\vartheta}\cdot\frac{1+4g^2(w_{\lambda_n})}{1+2g^2(w_{\lambda_n})}\Big)\int_{\mathbb{R}^2}|\nabla(w_{\lambda_n})|^2+\Big(\frac{1}{2}-\frac{1}{2\vartheta}\Big)\int_{\mathbb{R}^2}V(|x|)g^2(w_{\lambda_n})
\end{aligned}\tag{3.5}$$

$$
\begin{aligned}
&+\Big(\frac{1}{2}-\frac{3}{2\vartheta}\Big)qC(g(w_{\lambda_n}))+\Big(\frac{1}{4}-\frac{1}{\vartheta}\Big)q\kappa\mathcal{D}(g(w_{\lambda_n}))\\
&+\frac{1}{2}\Big(\int_{\mathbb{R}^2}(I_\alpha * F(g(w_n)))\Big(\frac{1}{\vartheta}f(g(w_{\lambda_n}))g(w_{\lambda_n})-F(g(w_{\lambda_n}))\Big)\Big)\\
\geq & C\Big(\int_{\mathbb{R}^2}(|\nabla w_{\lambda_n}|^2+g^2(w_{\lambda_n})\Big).
\end{aligned}
$$

(3.5) infer that $\int_{\mathbb{R}^2}|\nabla w_{\lambda_n}|^2\leq C$. From (g_1) and (g_6), it holds

$$
\int_{\mathbb{R}^2}|w_{\lambda_n}|^2=\int_{|w_{\lambda_n}|>1}|w_{\lambda_n}|^2+\int_{|w_{\lambda_n}|\leq 1}|w_{\lambda_n}|^2\leq C\Big(\int_{\mathbb{R}^2}|g(w_{\lambda_n})|^4+\int_{\mathbb{R}^2}|g(w_{\lambda_n})|^2\Big).
$$

Then by (1.8), Proposition 2.2 and (3.5), we deduce that $\int_{\mathbb{R}^2}|w_{\lambda_n}|^2\leq C$. Hence, there is a constant $C>0$ independent of n such that $\|w_n\|=\int_{\mathbb{R}^2}(|\nabla w_n|^2+w_n^2)\leq C$. Next, we can suppose that the limit of $\mathcal{J}_{\lambda_n}(w_{\lambda_n})$ exists. By Theorem 3.1, we have $c_{\lambda_n}\to c_1$ is continuous from the left. So, we get $0\leq \lim\limits_{n\to+\infty}\mathcal{J}_{\lambda_n}(w_{\lambda_n})=c_{\lambda_n}\leq c_{\frac{1}{2}}$. Thus, using the fact that

$$
\mathcal{J}(w_{\lambda_n})=\mathcal{J}_{\lambda_n}(w_{\lambda_n})+\frac{(\lambda_n-1)}{2}\int_{\mathbb{R}^2}(I_\alpha * F(g(w_{\lambda_n})))F(g(w_{\lambda_n}))=c_{\lambda_n}+o(1)=c_1,
$$

and for any $\psi\in H_r^1(\mathbb{R}^2)\backslash\{0\}$, there holds

$$
\langle\mathcal{J}'(w_{\lambda_n}),\psi\rangle=\langle\mathcal{J}'_{\lambda_n}(w_{\lambda_n}),\psi\rangle+(\lambda_n-1)\int_{\mathbb{R}^2}(I_\alpha * F(g(w_{\lambda_n})))f(g(w_{\lambda_n}))g'(w_{\lambda_n})\psi=o(1).
$$

Then, up to a subsequence, $\{w_{\lambda_n}\}$ is a bounded $(PS)_{c_1}$ sequence of $\mathcal{J}$. Preceding the same method as Lemma 3.3, we get that the existence of a nontrivial solution $v_0\in H_r^1(\mathbb{R}^2)$ for $\mathcal{J}$ satisfying $\mathcal{J}(v_0)=c_1$, $\mathcal{J}'(v_0)=0$.

To seek the ground state solutions, we need to define $m_0:=\inf\{\mathcal{J}(v_0):v_0\neq 0,\mathcal{J}'(v_0)=0\}$. From (3.5), we have $m_0\geq 0$. Set $\{v_n'\}$ be a sequence satisfies $\mathcal{J}'(v_n')=0$, $\mathcal{J}(v_n')\to m_0$. Similar to the discussed above, one obtains $\{v_n'\}$ is a bounded $(PS)_{m_0}$ sequence of $\mathcal{J}$. Arguing as in Lemma 3.3, one obtains there exists a $\bar{v}_0\in H_r^1(\mathbb{R}^2)$ such that $\mathcal{J}'(\bar{v}_0)=0$, $\mathcal{J}(\bar{v}_0)=m_0$, which implies that $\bar{u}_0=g(\bar{v}_0)$ is a ground state solution of (1.1). This completes the proof.

4. Conclusions

In this paper, we have considered the modified Chern-Simons-Schrödinger equation involving radially symmetric variable potential V and general Choquard type nonlinearity. By using a change of variable and variational argument, we obtain the existence of ground state solutions. It is hoped that the results obtained in this paper may be a new starting point for further research in this field.

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Conflict of interest

The authors declare that they have no competing interests.

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