Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 7143–7165.
DOI: 10.3934/math.2022398
Received: 28 October 2021
Revised: 16 January 2022
Accepted: 26 January 2022
Published: 09 February 2022

Research article

Mathematical assessment of the dynamics of the tobacco smoking model: An application of fractional theory

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Abstract: In this paper we consider fractional-order mathematical model describing the spread of the smoking model in the sense of Caputo operator with tobacco in the form of snuffing. The threshold quantity $\mathcal{R}_0$ and equilibria of the model are determined. We prove the existence of the solution via fixed-point theory and further examine the uniqueness of of the solution of the considered model. The new version of numerical approximation's framework for the approximation of Caputo operator is used. Finally, the numerical results are presented to justify the significance of the arbitrary fractional order derivative. The analysis shows fractional-order model of tobacco smoking in Caputo sense gives useful information as compared to the classical integer order tobacco smoking model.

Keywords: fractional order model; tobacco smoking disease; Caputo fractional derivatives; fixed point theorem; numerical simulation
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

Mathematical biology is a field of great concern both for mathematicians and biologists in the current century. This field has many applications. The main focus of a researcher is to describe the dynamics of infectious diseases and their control aspects in terms of mathematical language. It was Brownlee [1] who developed the initial framework for mathematical biology and provide a concrete base for the subject. He used a probabilistic approach and within three years and proposed a law about the spread of infection [2]. A detailed mathematical touch to the subject was given in the work of Kermark and McKendrik [3]. Following the approach of Kermark-McKendrik, different types of infections were modelled and analyzed in a more sophisticated way (see [4–14]). Similar to infectious diseases, we have a very dangerous social habit common in all part of the world which not causes the severe diseases but kills. This social habit is nothing but smoking. It may be defined as the process in which an individual inhales the smoke of tobacco or the process in which tobacco smoke is taken by the mouth and then released through pipes or cigars. The habit of smoking in Europe was initially introduced with the entrance of Columbus in the 16th-century [15]. However, before this habit, other species of strange nature had adversely affected the human habitat and the whole ecosystem. It was Nicot who widely spread the use of tobacco in England as money yield and promoted it as a business. Due to this connection, the word "Nicotine" was derived from his name. Late in the nineteenth century, cigarette manufacturing equipment was invented whose production speed was 200 units per minute and currently it is 9000. The habit of smoking causes many deadly diseases which include mouth, lung and throat cancer [16–19].

In the 18th century, Fourier and Reimann-Liouville did remarkable work in ordinary calculus. Then many other researchers started to work in fractional calculus (FC) [20–23]. Fractional calculus is a generalized form and we can obtain the equation of ordinary calculus by substituting specific values for the chosen parameters. Fractional calculus has many applications in-memory processes, mathematical modeling and in different hereditary process. The related application can be found in [24–38]. This field attracts more researchers due to their vast applications and use of the fractional order of the differential as well as the integral calculus.

The Caputo derivative is very useful when dealing with real-world problem because, it allows traditional initial and boundary conditions to be included in the formulation of the problem and in addition the derivative of a constant is zero; however, functions that are not differentiable do not have fractional derivative. In this paper we study the fractional tobacco smoking model. We calculate the stability and equilibrium for the disease free point of the proposed model. We also give the theoretical results to validate our results obtained via fractional order method with numerical results. The main focus of this work is to construct fractional operator for the tobacco smoking system under Caputo derivative with fractional operator $\zeta \in (0, 10)$. The ordinary differential equations which contains the integer order derivative contains the integer order derivative can be globalize to the fractional differential equations by fractional order (FDEs). The FDEs with parameter $0 < \alpha < 1$ may be found in [20–23, 27].

The outline of this paper is organized as follows. In Section 2, we give the essential definitions. Section 3, presents the formulation of the smoking model in classical integer and fractional case. The threshold quantity and equilibria of the proposed model are determined in Section 4. The existence theory of the proposed model given in Section 5. Also, the numerical methods to solve the considered problems are given in Section 6. Indeed, numerical results of the proposed models under different values of fractional orders are given in Section 7. Finally, the conclusion regarding the present finding is given in Section 8.

2. Preliminaries

In this section we provide some basic definitions, theorems and results that will be used in this article.

Definition 1. [23] For a integrable function g, the Caputo derivative of fractional order $\zeta \in (0,1)$ is given by

$$^{c}D^{\zeta}g(t) = \frac{1}{\Gamma(m-\zeta)}\int_0^t \frac{g^{(m)}(v)}{(t-v)^{\zeta-m+1}}dv, \quad m = [\zeta]+1.$$

Also, the corresponding fractional integral of order ζ with $\mathrm{Re}(\zeta) > 0$ is given by

$$^{c}I^{\zeta}g(t) = \frac{1}{\Gamma(\zeta)}\int_0^t (t-v)^{\zeta-1}g(v)dv.$$

Lemma 1. *[26] Let $S(\theta) \in C(0,T)$, then the solution of fractional differential equation*

$$\begin{cases} {}^{C}D_0^{p}\daleth(\theta) = S(\theta), \theta \in [0,T], \\ \daleth(0) = S_0 \end{cases}$$

is given by

$$I(\theta) = \sum_{j=0}^{p} N_j\theta^j + \frac{1}{\Gamma(\mu)}\int_0^{\theta}(\theta-\xi)^{\mu-1}S(\xi)d\xi.$$

For $N_j \in R$, j = 0, 1, 2, 3, . . . , p.

Lemma 2. *[22, 23, 27] "The following result holds for fractional differential equations*

$$I^{\zeta}[^{c}D^{\zeta}\theta(t)] = \theta(t) + \alpha_0 + \alpha_1 t + \alpha_2 t^2 + ... + \alpha_{m-1}t^{m-1},$$

for arbitrary $\alpha_i \in R$, $i = 0, 1, 2, 3, ..., m-1$, where $m = [\zeta]+1$ and $[\zeta]$ symbolizes the integer part of ζ".

Lemma 3. *[27] Let $\theta \in AC^n[0,T], \zeta > 0$ and $n = [\zeta]$, then the following result holds*

$$\mathbb{I}^{\zeta}[^{c}D^{\zeta}\theta(\vartheta)] = \theta(\vartheta) - \sum_{j=0}^{n-1}\frac{D^j\theta(a)}{j!}(t-a)^j.$$

Lemma 4. *[23, 27] In view of Lemma (3), the solution of $\mathbb{D}^{\zeta}\theta(t) = y(t)$, $n-1 < \zeta < n$ is given by*

$$\theta(t) = \mathbb{I}^{\zeta}y(t) + c_0 + c_1(t) + c_2t^2 + \ldots + c_{n-1}t^{n-1}$$

where $c_j \in R$.

Definition 2. [23] Suppose we have Caputo's fractional differential equation of order ζ,

$$^{c}D^{\zeta}\theta(t) = f(t,\theta(t))$$

Then the solution is given as

$$\theta(t_{n+}) = \theta(t_n) + \frac{f(t_n, \theta_n)}{h\Gamma(\zeta)}\left\{\frac{2h}{\zeta}t_{n+1}^{\zeta} - \frac{t_{n+1}^{\zeta+1}}{\zeta+1} + \frac{h}{\zeta}t_n^{\zeta} - \frac{t^{\zeta+1}}{\zeta}\right\}$$
$$\frac{f(t_{n-1}, \theta_{n-1})}{h\Gamma(\zeta}\left\{\frac{h}{\zeta}t_{n+1}^{\zeta} - \frac{t_{n+1}^{\zeta+1}}{\zeta+1} + \frac{t_n^{\zeta}}{\zeta+1}\right\} + R_n^{\zeta}(t).$$

Where $R_n^{\zeta}(t)$ represent the remainder term. For the study of convergence and uniqueness of the solution of the scheme, we refer to [23].

Theorem 1. [22, 23] *"Let G be a Banach space and $\mathfrak{P} : X \to G$ is compact and continuous, if the set*

$$E = \{\theta \in G : \theta = m\mathfrak{P}\theta,\ m \in (0, 1)\},$$

is bounded, then $\mathfrak{P}$ has a unique fixed point".

3. Model formulation

According to the report of the center for disease control and prevention (CDC) smoking is the leading cause of premature and preventable deaths worldwide. By the above report approximately 440000 deaths occurs in the United State annually and the United Kingdom 105000 [37]. Further half of the smokers die due to smoking related diseases. Also smoking reduce the life expectancy by ten to twelve years. Moreover the ratio of heart attack is more than 70% as compare to non-smokers. Also this cause the lung cancer and it is also ten time higher than the non-smokers. The other cancer diseases are also linked with smoking. This includes pancreas, breast, cervix, stomach, mouth and throat cancers. These all diseases are occurring due to cigarette smoking, because one cigarette include aroung 4000 toxins and chemical compounds. Recently, Din et al. [39] proposed a stochastic smoking model, in which the total population is divided into five classes; susceptible individuals $\mathbb{V}(t)$, snuffing individuals $\mathbb{Y}(t)$, casual smokers $\mathbb{X}(t)$, chain $\mathbb{W}(t)$ and quit smokers $\mathbb{Z}(t)$ at any time t, so total population $N(t) = \mathbb{V}(t) + \mathbb{Y}(t) + \mathbb{X}(t) + \mathbb{W}(t) + \mathbb{Z}(t)$, and the model is expressed as follows:

$$\begin{aligned}
d\mathbb{V}(t) &= \left[\Pi - \frac{\beta\mathbb{V}(t)\mathbb{Y}(t)}{N} - d\mathbb{V}(t) + \lambda\mathbb{Z}(t)\right]dt + \alpha_1\mathbb{V}(t)dB_1(t),\\
d\mathbb{Y}(t) &= \left[\frac{\beta\mathbb{V}(t)\mathbb{Y}(t)}{N} - \frac{\delta\mathbb{Y}(t)\mathbb{X}(t)}{N} - (\gamma + d)\mathbb{Y}(t)\right]dt + \alpha_2\mathbb{Y}(t)dB_2(t),\\
d\mathbb{X}(t) &= \left[\frac{\delta\mathbb{Y}(t)\mathbb{X}(t)}{N} - (\mu + \omega + d)\mathbb{X}(t)\right]dt + \alpha_3\mathbb{X}(t)dB_3(t),\\
d\mathbb{W}(t) &= \left[\omega\mathbb{X}(t) - (\kappa + d)\mathbb{W}(t)\right]dt + \alpha_4\mathbb{W}(t)dB_4(t),\\
d\mathbb{Z}(t) &= \left[\kappa\mathbb{W}(t) - (\lambda + d)\mathbb{Z}(t)\right]dt + \alpha_5\mathbb{Z}(t)dB_5(t).
\end{aligned} \tag{3.1}$$

The biological interpretation of parameters used in the model are presented in Table 1.

Table 1. Parameters description.

Symbols	Description
Π	Inflow rate (either by birth or through migration)
β	Rate at which susceptible starts snuffing
λ	Relapse rate
γ	Tobacco related death rate in snuffing compartment
d	Natural death rate
ω	Rate at which occasional smokers tend to be chain smokers
μ	Death due to tobacco related diseases
δ	Rate through which snuffing population become casual smokers
κ	Quitting rate

Where $B_i(t)$ for $i = 1 \cdots 5$ stand for the independent Brownian motions and α_1, $\alpha_2, \alpha_3, \alpha_4$, and α_5 are the intensities of the white noises. In this work, we put Brownian motions intensities zero in model (3.1), then we obtain the following ordinary differential equations system:

$$\begin{aligned}
\mathbb{V}'(t) &= \Pi - \frac{\beta \mathbb{V}(t)\mathbb{Y}(t)}{N} - d\mathbb{V}(t) + \lambda \mathbb{Z}(t),\\
\mathbb{Y}'(t) &= \frac{\beta \mathbb{V}(t)\mathbb{Y}(t)}{N} - \frac{\delta \mathbb{Y}(t)\mathbb{X}(t)}{N} - (\gamma + d)\mathbb{Y}(t),\\
\mathbb{X}'(t) &= \frac{\delta \mathbb{Y}(t)\mathbb{X}(t)}{N} - (\mu + \omega + d)\mathbb{X}(t),\\
\mathbb{W}'(t) &= \omega \mathbb{X}(t) - (\kappa + d)\mathbb{W}(t),\\
\mathbb{Z}'(t) &= \kappa \mathbb{W}(t) - (\lambda + d)\mathbb{Z}(t),
\end{aligned} \tag{3.2}$$

with initial conditions

$$\mathbb{V}(0) > 0, \quad \mathbb{Y}(0) > 0, \quad \mathbb{X}(0) > 0, \quad \mathbb{W}(0) > 0, \quad \mathbb{Z}(0) > 0. \tag{3.3}$$

For the better understanding the dynamics of proposed model, the author's generalized the aforementioned model to fractional order, due to its great degree of freedom, glob in nature and more reliable. We generalize the model (3.2) under the corresponding Caputo fractional order derivative as

$$\begin{aligned}
{}^cD^{\zeta}\mathbb{V}(t) &= \Pi - \frac{\beta \mathbb{V}(t)\mathbb{Y}(t)}{N} - d\mathbb{V}(t) + \lambda \mathbb{Z}(t),\\
{}^cD^{\zeta}\mathbb{Y}(t) &= \frac{\beta \mathbb{V}(t)\mathbb{Y}(t)}{N} - \frac{\delta \mathbb{Y}(t)\mathbb{X}(t)}{N} - (\gamma + d)\mathbb{Y}(t),\\
{}^cD^{\zeta}\mathbb{X}(t) &= \frac{\delta \mathbb{Y}(t)\mathbb{X}(t)}{N} - (\mu + \omega + d)\mathbb{X}(t),\\
{}^cD^{\zeta}\mathbb{W}(t) &= \omega \mathbb{X}(t) - (\kappa + d)\mathbb{W}(t),\\
{}^cD^{\zeta}\mathbb{Z}(t) &= \kappa \mathbb{W}(t) - (\lambda + d)\mathbb{Z}(t).
\end{aligned} \tag{3.4}$$

With given initial conditions,

$$\mathbb{V}(0) > 0, \quad \mathbb{Y}(0) > 0, \quad \mathbb{X}(0) > 0, \quad \mathbb{W}(0) > 0, \quad \mathbb{Z}(0) > 0. \tag{3.5}$$

4. Stability analysis of the system (3.2)

In this section, we will interrogate the stability analysis of system (3.2) [18, 24, 25].

4.1. Smoking free equilibrium point

First, we interrogate the disease-free steady-state of the proposed system (3.2) which is indicated by E_0. We use $\mathbb{Y}(t) = \mathbb{X}(t) = \mathbb{W}(t) = \mathbb{Z}(t) = 0$, so the smoking free equilibrium point E_0 is

$$E_0 = (\mathbb{V}_0, \mathbb{Y}_0, \mathbb{X}_0.\mathbb{W}_0, \mathbb{Z}_0) = \left(\frac{\Pi}{d}, 0, 0, 0, 0\right) \quad and \quad N_0 = \mathbb{V}_0.$$

4.2. Basic reproduction

Next-generation technique [12, 16, 18] is utilized to determined the reproduction value of the system (3.2). We calculated the reproduction parameter of the system (3.2) in the following next-generation method

$$F = \begin{bmatrix} \frac{\beta\Pi}{dN_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} (\gamma + d) & 0 & 0 \\ 0 & (\mu + \omega + d) & 0 \\ 0 & -\omega & -(\kappa + d) \end{bmatrix}.$$

The dominant eigenvalue of $FV^{-1} = \frac{\beta\Pi}{dN_0(\gamma+d)}$, so

$$\mathcal{R}_0 = \frac{\beta\Pi}{dN_0(\gamma + d)}.$$

4.3. Smoking present equilibrium point

The system (3.2) is also have equilibrium point (EE) which denoted by $E_1 = (\mathbb{V}_1, \mathbb{Y}_1, \mathbb{X}_1.\mathbb{W}_1, \mathbb{Z}_1)$, using the left side of system (3.2) is equal to zero, as follows:

$$\begin{aligned}
\mathbb{V}_1 &= \frac{N_1(\delta W_1 + (\gamma + d))}{\beta}, \\
\mathbb{Y}_1 &= \frac{(\mu + \omega + d)}{\delta}, \\
\mathbb{X}_1 &= \frac{(\kappa + d)(\gamma + d)[\delta d(\mathcal{R}_0 - 1) - \beta(\mu + \omega + d)]}{(\kappa + d)\beta\delta\omega + (\mu + \omega + d)(\beta\delta((\kappa + d) + \delta^2 d))}, \\
\mathbb{W}_1 &= \frac{\omega\mathbb{X}_1}{(\kappa + d)}, \\
\mathbb{Z}_1 &= \frac{\kappa\mathbb{W}_1}{\lambda + d}, \\
N_1 &= \mathbb{V}_1 + \mathbb{Y}_1 + \mathbb{X}_1 + \mathbb{W}_1 + \mathbb{Z}_1.
\end{aligned}$$

Theorem 2. *If $\mathcal{R}_0 < 1$, the infection-free steady-state (E_0) of system (3.2) is*
(i) Locally asymptotically stable.
(ii) Globally asymptotically stable.
Otherwise unstable.

Proof (i). For the local stability at E_0, the Jacobian of system (3.2) is

$$J(E_0) = \begin{pmatrix} -d & \frac{-\beta\Pi}{dN_0} & 0 & 0 & \lambda \\ 0 & \frac{\beta\Pi}{dN_0} - (\gamma + d) & 0 & 0 & 0 \\ 0 & 0 & -(\mu + \omega + d) & 0 & 0 \\ 0 & 0 & \omega & -(\kappa + d) & 0 \\ 0 & 0 & 0 & \kappa & -(d + \lambda) \end{pmatrix},$$

the above matrix has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5,

$$\begin{aligned} \lambda_1 &= -d < 0, \\ \lambda_2 &= (\gamma + d)(\mathcal{R}_0 - 1), \\ \lambda_3 &= -(\mu + \omega + d) < 0, \\ \lambda_4 &= -(\kappa + d) < 0, \\ \lambda_4 &= -(d + \lambda) < 0, \end{aligned}$$

implying that $\lambda_2 < 0$ for $\mathcal{R}_0 < 1, \lambda_2 = 0$ for $\mathcal{R}_0 = 1$ and $\lambda_2 > 0$ for $\mathcal{R}_0 > 1$. Thus, the eigenvalues are negative which proves the conclusion.
Proof (ii). For the proof of condition (ii), first we construct the Lyapunov function L as

$$L = \ln \frac{\mathbb{V}}{\mathbb{V}_0} + \ln \frac{\mathbb{Y}}{\mathbb{Y}_0} + \mathbb{X} + \mathbb{W}. \tag{4.1}$$

Differentiating Eq (4.1) with respect to time

$$L' = \frac{\Pi}{\mathbb{V}} - \frac{\beta\mathbb{Y}}{N} + \frac{\kappa\mathbb{W}}{\mathbb{V}} - d + \frac{\beta\mathbb{V}}{N} - \frac{\delta\mathbb{X}}{N} - (\kappa + d) - (\mu + \omega + d)\mathbb{X},$$

using the values of E_0 in the above equation,

$$\begin{aligned} L' &= d - d + \frac{\beta\Pi}{dN_0} - (\gamma + d), \\ &= \mathcal{R}_0(\gamma + d) - (\gamma + d), \\ &= (\gamma + d)(\mathcal{R}_0 - 1), \end{aligned} \tag{4.2}$$

therefore, if $\mathcal{R}_0 < 1$, then $L' < 0$, which implies that the system (3.2) is globally stable around E_0.

Theorem 3. *The Smoking present equilibrium point E_1 of system (3.2) is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. For the local stability at E_1 the Jacobian of system (3.2) is

$$J(E_1)=\begin{pmatrix} -\frac{\beta\mathbb{Y}_1-d}{N_1} & -\frac{\beta\mathbb{V}_1}{N_1} & 0 & 0 & \lambda \\ \frac{\beta\mathbb{Y}_1}{N_1} & \frac{\beta\mathbb{V}_1}{N_1}-\frac{\delta\mathbb{X}_1}{N_1}-(\gamma+d) & -\frac{\delta\mathbb{Y}_1}{N_1} & 0 & 0 \\ 0 & \frac{\delta\mathbb{X}_1}{N_1} & \frac{\delta\mathbb{Y}_1}{N_1}-(d+\omega+d) & 0 & 0 \\ 0 & 0 & \omega & -(\kappa+d) & 0 \\ 0 & 0 & 0 & \kappa & -(\lambda+d) \end{pmatrix},$$

easily we can get one eigenvalue of the above matrix is

$$\lambda_5=-(\lambda+d)<0, \tag{4.3}$$

and

$$J(E_1)=\begin{pmatrix} -\frac{\beta\mathbb{Y}_1-d}{N_1} & -\frac{\beta\mathbb{V}_1}{N_1} & 0 & 0 \\ \frac{\beta\mathbb{Y}_1}{N_1} & \frac{\beta\mathbb{V}_1}{N}-\frac{\delta\mathbb{X}_1}{N_1}-(\gamma+d) & -\frac{\delta\mathbb{X}_1}{N_1} & 0 \\ 0 & \frac{\delta\mathbb{X}_1}{N_1} & 0 & 0 \\ 0 & 0 & \omega & -(\kappa+d) \end{pmatrix},$$

using the basic matrix properties then we can get the following matrix

$$J(E_1)=\begin{pmatrix} -\frac{\beta\mathbb{Y}_1}{N_1}-d & -\frac{\beta\mathbb{V}_1}{N_1} & \frac{\omega}{(\mu+\omega+d)} & 0 \\ 0 & -\frac{\beta\mathbb{V}}{N_1}+\frac{d\delta\mathbb{X}}{\beta\mathbb{Y}_1} & -\frac{\delta\mathbb{Y}_1}{N_1}-\frac{d\delta\mathbb{Y}_1}{\beta\mathbb{Y}_1}+\frac{\omega}{(\mu+\omega+d)} & 0 \\ \frac{\beta\mathbb{Y}_1}{N_1} & \frac{\delta\mathbb{X}_1}{N_1} & -\frac{\delta W_1}{N_1} & 0 \\ 0 & 0 & \omega & -(\kappa+d) \end{pmatrix}.$$

For simplification, this matrix can also be written as

$$J(E_1)=\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here,

$$A=\begin{pmatrix} -\frac{\beta\mathbb{Y}_1}{N_1}-d & -\frac{\beta\mathbb{V}_1}{N_1} \\ 0 & -\frac{\beta\mathbb{V}_1}{N_1}+\frac{d\delta\mathbb{X}}{\beta\mathbb{Y}_1} \end{pmatrix},\quad B=\begin{pmatrix} \frac{\omega}{(\mu+\omega+d)} & 0 \\ -\frac{\delta\mathbb{Y}_1}{N_1}-\frac{d\delta\mathbb{Y}_1}{\beta\mathbb{Y}_1}+\frac{\omega}{(\mu+\omega+d)} & 0 \end{pmatrix},$$
$$C=\begin{pmatrix} 0 & -\frac{\beta\mathbb{V}_1}{N_1}+\frac{d\delta\mathbb{X}_1}{\beta\mathbb{Y}_1} \\ 0 & 0 \end{pmatrix},\quad D=\begin{pmatrix} -\frac{\delta W_1}{N_1} & 0 \\ \omega & -(\kappa+d) \end{pmatrix}.$$

Since the eigenvalues of $J(E^*)$ depend on the eigenvalues of A and D, the eigenvalues of A are given as follows:

$$\lambda_1=-\frac{\beta\mathbb{Y}_1}{N_1}-d<0,$$
$$\lambda_2=-(\gamma+d)+\frac{d\delta\mathbb{X}_1}{\beta\mathbb{Y}_1\Pi}\left(\Pi-\mathbb{Y}_1(\gamma+d)\mathcal{R}_0\right),$$

if $R_0>\frac{\delta\Pi}{(\mu+\omega+d)(\gamma+d)}$, then $\lambda_2<0$.

Now, the eigenvalues of D are

$$\lambda_3=-\delta\frac{\mathbb{Y}_1}{N_1}<0,$$
$$\lambda_4=-(\kappa+d)<0,$$

which is the required proof.

5. Existence of the solution

In this section we construct the conditions for the existence and uniqueness of the solution, to get the desired results, we construct the following function:

$$\begin{cases} \vartheta_1(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) = \Pi - \dfrac{\beta\mathbb{V}(t)\mathbb{Y}(t)}{N} - d\mathbb{V}(t) + \lambda\mathbb{Z}(t), \\ \vartheta_2(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) = \dfrac{\beta\mathbb{V}(t)\mathbb{Y}(t)}{N} - \dfrac{\delta\mathbb{Y}(t)\mathbb{X}(t)}{N} - (\gamma + d)\mathbb{Y}(t), \\ \vartheta_3(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) = \dfrac{\delta\mathbb{Y}(t)\mathbb{X}(t)}{N} - (\mu + \omega + d)\mathbb{X}(t), \\ \vartheta_4(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) = \omega\mathbb{X}(t) - (\kappa + d)\mathbb{W}(t), \\ \vartheta_5(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) = \kappa\mathbb{W}(t) - (\lambda + d)\mathbb{Z}(t). \end{cases} \tag{5.1}$$

Suppose that the considered space $\mathbb{C}[0,T] = \mathbb{B}$ be a Banach space with norm

$$\|\theta(t)\| = \sup_{t\in[0,T]} \Big[|\mathbb{V}(t)| + |\mathbb{Y}(t)| + |\mathbb{X}(t)| + |\mathbb{W}(t)| + |\mathbb{Z}(t)|\Big],$$

where

$$\theta(t) = \begin{cases} \mathbb{V}(t) \\ \mathbb{Y}(t) \\ \mathbb{X}(t) \\ \mathbb{W}(t) \\ \mathbb{Z}(t) \end{cases}, \theta_0(t) = \begin{cases} \mathbb{V}^0 \\ \mathbb{Y}^0 \\ \mathbb{X}^0 \\ \mathbb{W}^0 \\ \mathbb{Z}^0 \end{cases}, \mathfrak{T}(t,\theta(t)) = \begin{cases} \vartheta_1(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) \\ \vartheta_2(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) \\ \vartheta_3(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) \\ \vartheta_4(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) \\ \vartheta_5(t,\mathbb{V},\mathbb{Y},\mathbb{X},\mathbb{W},\mathbb{Z}) \end{cases}. \tag{5.2}$$

With the help of (5.2), the system (3.4) can be written in as

$$\begin{aligned} &{}^cD^\zeta\theta(t) = \mathfrak{T}(t,\theta(t)), t \in [0,T], \\ &\theta(0) = \theta_0. \end{aligned} \tag{5.3}$$

By Lemma (1), Eq (5.3) converts into the following form

$$\theta(t) = \theta_0 + \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\mathfrak{T}(s,\theta(s))ds, \ \ t \in J = [0,T]. \tag{5.4}$$

To prove the existence of the solution we make the following assumptions
$(P1)$ $\exists$ constants $K_1^*, M_1^* \ni$

$$|\mathfrak{T}(t,\theta(t))| \le K_1^*|\theta|^q + M_1^*.$$

$(P2)$ $\exists$ $L_* > 0$, $\ni$ for each $\theta, \bar{\theta}$

$$|\mathfrak{T}(t,\theta) - \mathfrak{T}(t,\bar{\theta})| \le L_*\|\theta - \bar{\theta}\|.$$

And let an operator $\mathfrak{P} : \mathbb{B} \to \mathbb{B}$ be defined as:

$$\mathfrak{P}\theta(t) = \theta_0 + \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\mathfrak{T}(s,\theta(s))ds. \tag{5.5}$$

Theorem 4. *When the assumptions* ($P1$) *and* ($P2$) *are true it verifies that the problem* (5.3)*, has at least one fixed point which implies that the problem of our study has also at least one solution.*

Proof. Furthermore we proceed as:
Step I. First we have to show that $\mathfrak{P}$ is continuous. To acquire the results we suppose that $\mathfrak{T}_j$ is continuous for $j = 1, 2, 3, 4, 5, 6$. Which implies that $\mathfrak{T}(s, \theta(s))$ is also continuous. Assume $\theta_n.\theta \in G \ni \theta_n \to \theta$, we must have $\mathfrak{P}\theta_n \to \mathfrak{P}\theta$.
For this we consider

$$\begin{aligned}
\|\mathfrak{P}\theta_n - \mathfrak{P}\theta\| &= \max_{t\in J=[0,T]}\left|\int_0^t \frac{(t-s)^{\zeta-1}}{}\Gamma(\zeta)\mathfrak{T}_n(s,\theta_n(s))ds - \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\mathfrak{T}(s,\theta(s))ds\right|,\\
&\leq \max_{t\in J=[0,T]}\int_0^t \left|\frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\right| |\mathfrak{T}_n(s,\theta_n(s)) - \mathfrak{T}(s,\theta(s))|ds, \qquad (5.6)\\
&\leq \frac{T^t}{\Gamma(\zeta+1)}\|\mathfrak{T}_n - \mathfrak{T}\| \to 0 \ \ as \ \ n\to\infty.
\end{aligned}$$

As $\mathfrak{T}$ is continuous, therefore $\mathfrak{P}\theta_n \to \mathfrak{P}\theta$, yields that $\mathfrak{P}$ is continuous.
Step II. Now to prove that $\mathfrak{P}$ is bounded for any $\theta \in G$, we make of the supposition that $\mathfrak{P}$ satisfies the growth condition:

$$\begin{aligned}
\|\mathfrak{P}\theta\| &= \max_{t\in[0,T]}\left|\theta_0 + \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\mathfrak{T}(s,\theta(s))ds\right|,\\
&\leq |\theta_0| + \max_{t\in[0,T]}\frac{1}{\Gamma(\zeta)}\int_0^t \left|(t-s)^{\zeta-1}\right| |\mathfrak{T}(s,\theta(s))|ds, \qquad (5.7)\\
&\leq |\theta_0| + \frac{T^\zeta}{\Gamma(\zeta+1)}[K_1^*\|\theta\|^q + M_1^*].
\end{aligned}$$

Here we assume a $\mathscr{S}$, the subset of G with the property of boundedness and we need to prove that $\mathfrak{P}(\mathscr{S})$ is also bounded. To reach our destination, we assume that for any $\theta \in \mathscr{S}$, now as $\mathscr{S}$ is bounded so $\exists\ K_q \geq 0 \ni$

$$\|\theta\| \leq K_q, \forall \ \ \theta \in \mathbb{S}. \qquad (5.8)$$

Further for any $\theta \in \mathscr{S}$ by using the growth condition, we have

$$\begin{aligned}
\|\mathfrak{P}\theta\| &\leq |\theta_0| + \frac{T^\zeta}{\Gamma(\zeta+1)}[K_1^*\|\theta^q\| + M_1^*]\\
&\leq |\theta_0| + \frac{T^\zeta}{\Gamma(\zeta+1)}[K_1^* K_q + M_1^*].
\end{aligned} \qquad (5.9)$$

Therefore, $\mathfrak{P}(\mathscr{S})$ is bounded.
Step III. Here we attempt to prove that operator we defined is equi continuous, for this we assume that $t_2 \leq t_1 \in J = [0, T]$, then

$$\begin{aligned}
|\mathfrak{P}\theta(t_1) - \mathfrak{P}\theta(t_2)| &= \left|\frac{1}{\Gamma(\zeta)}\int_0^t (t_1-s)^{\zeta-1}\mathfrak{T}(s,\theta(s))ds - \frac{1}{\Gamma(\zeta)}\int_0^t (t_2-s)^{\zeta-1}\mathfrak{T}(s,\theta(s))ds\right|,\\
&\leq \left|\frac{1}{\Gamma(\zeta)}\int_0^t (t_1-s)^{\zeta-1} - \frac{1}{\Gamma(\zeta)}\int_0^t (t_2-s)^{\zeta-1}\right| |\mathfrak{T}(s,\theta(s))|ds, \qquad (5.10)\\
&\leq \frac{T^\zeta}{\Gamma(\zeta+1)}[K_1^*\|\theta\|^q + M_1^*][t_1 - t_2].
\end{aligned}$$

By taking advantage of Arzel$\acute{a}$-Ascoli theorem, we can say that $\mathfrak{P}(\mathscr{S})$ is relative compact.
Step IV. In this step we need to prove that the set defined below is bounded

$$\mathbf{E} = \{\theta \in G : \theta = m\mathfrak{P}\theta, \;\; m \in (0, 1)\}. \tag{5.11}$$

To prove this we suppose that $\theta \in \mathbf{E}$, $\ni$ for each $t \in J$ where $J = [0, T]$ we have

$$\|\theta\| = m\|\mathfrak{P}\theta\| \leq m\left[|\theta_0| + \frac{T^{\zeta}}{\Gamma(\zeta + 1)} K_1^* \|\theta\|^q + M_1^*\right]. \tag{5.12}$$

From here we can claim that the set defined above is bounded. By using Schaefer's FPT the operator we defined i.e., $\mathfrak{P}$ has atleast one fixed point, and hence the model we studied in this paper has atleast one solution. □

Theorem 5. *The problem* (5.3) *is unique solution, if* $\frac{T^{\zeta} K_1^*}{\Gamma(\zeta+1)} < 1$.

Proof. Let $\theta, \bar{\theta} \in G$, then

$$\begin{aligned} \|\mathfrak{P}\theta - \mathfrak{P}\bar{\theta}\| &\leq \max_{t \in J=[0,T]} \int_0^t \left|\frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)}\right| |\mathfrak{T}(s, \theta(s)) - \mathfrak{T}(s, \bar{\theta}(s))| ds, \\ &\leq \frac{T^{\zeta} L_{\mathfrak{T}}}{\Gamma(\zeta + 1)} \|\theta - \bar{\theta}\|. \end{aligned} \tag{5.13}$$

Hence we can say that the fixed point is unique and therefor our solution is unique. □

6. Numerical solution

This section is devoted to the numerical solution of the proposed model. For this we will use the well-known two steps fractional order Adam's Bashforth method. The considered model is given as

$$\begin{aligned} {}^{c}D^{\zeta}\mathbb{V}(t) &= \Pi - \frac{\beta \mathbb{V}(t)\mathbb{Y}(t)}{N} - d\mathbb{V}(t) + \lambda \mathbb{Z}(t), \\ {}^{c}D^{\zeta}\mathbb{Y}(t) &= \frac{\beta \mathbb{V}(t)\mathbb{Y}(t)}{N} - \frac{\delta \mathbb{Y}(t)\mathbb{X}(t)}{N} - (\gamma + d)\mathbb{Y}(t), \\ {}^{c}D^{\zeta}\mathbb{X}(t) &= \frac{\delta \mathbb{Y}(t)\mathbb{X}(t)}{N} - (\mu + \omega + d)\mathbb{X}(t), \\ {}^{c}D^{\zeta}\mathbb{W}(t) &= \omega \mathbb{X}(t) - (\kappa + d)\mathbb{W}(t), \\ {}^{c}D^{\zeta}\mathbb{Z}(t) &= \kappa \mathbb{W}(t) - (\lambda + d)\mathbb{Z}(t). \end{aligned} \tag{6.1}$$

To obtain the desired results, we apply the fundamental theorem of fractional calculus to system (3.4) gives

$$
\begin{aligned}
\mathbb{V}(t) &= \mathbb{V}(0) + \frac{1}{\Gamma(\zeta)} \int_0^t \mathfrak{G}_1(\beta, \mathbb{V}(\beta))(t-\beta)^{\zeta-1} d\beta, \\
\mathbb{Y}(t) &= \mathbb{Y}(0) + \frac{1}{\Gamma(\zeta)} \int_0^t \mathfrak{G}_2(\beta, \mathbb{Y}(\beta))(t-\beta)^{\zeta-1} d\beta, \\
\mathbb{X}(t) &= \mathbb{X}(0) + \frac{1}{\Gamma(\zeta)} \int_0^t \mathfrak{G}_3(\beta, \mathbb{X}(\beta))(t-\beta)^{\zeta-1} d\beta, \\
\mathbb{W}(t) &= \mathbb{W}(0) + \frac{1}{\Gamma(\zeta)} \int_0^t \mathfrak{G}_4(\beta, \mathbb{W}(\beta))(t-\beta)^{\zeta-1} d\beta, \\
\mathbb{Z}(t) &= \mathbb{Z}(0) + \frac{1}{\Gamma(\zeta)} \int_0^t \mathfrak{G}_5(\beta, \mathbb{Z}(\beta))(t-\beta)^{\zeta-1} d\beta.
\end{aligned}
$$

The unknown terms $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4, \mathfrak{G}_5$ are given below. Now for $t = t_{n+1}$, we get

$$
\begin{aligned}
\mathbb{V}(t_{n+1}) &= \mathbb{V}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_1(t, \mathbb{V}(t))(t_{n+1}-t)^{\zeta-1} dt, \\
\mathbb{Y}(t_{n+1}) &= \mathbb{Y}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_2(t, \mathbb{Y}(t))(t_{n+1}-t)^{\zeta-1} dt, \\
\mathbb{X}(t_{n+1}) &= \mathbb{X}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_3(t, \mathbb{X}(t))(t_{n+1}-t)^{\zeta-1} dt, \\
\mathbb{W}(t_{n+1}) &= \mathbb{W}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_4(t, \mathbb{W}(t))(t_{n+1}-t)^{\zeta-1} dt, \\
\mathbb{Z}(t_{n+1}) &= \mathbb{Z}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_5(t, \mathbb{Z}(t))(t_{n+1}-t)^{\zeta-1} dt.
\end{aligned} \tag{6.2}
$$

For $t = t_n$ we get the following

$$\begin{aligned}
\mathbb{V}(t_n) &= \mathbb{V}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_n} \mathfrak{G}_1(t, \mathbb{V}(t))(t_n - t)^{\zeta-1} dt, \\
\mathbb{Y}(t_n) &= \mathbb{Y}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_n} \mathfrak{G}_2(t, \mathbb{Y}(t))(t_n - t)^{\zeta-1} dt, \\
\mathbb{X}(t_n) &= \mathbb{X}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_n} \mathfrak{G}_3(t, \mathbb{X}(t))(t_n - t)^{\zeta-1} dt, \qquad (6.3) \\
\mathbb{W}(t_n) &= \mathbb{W}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_n} \mathfrak{G}_4(t, \mathbb{W}(t))(t_n - t)^{\zeta-1} dt, \\
\mathbb{Z}(t_n) &= \mathbb{Z}(0) + \frac{1}{\Gamma(\zeta)} \int_0^{t_n} \mathfrak{G}_5(t, \mathbb{Z}(t))(t_n - t)^{\zeta-1} dt.
\end{aligned}$$

By $\mathbb{V}(t_{n+1})-\mathbb{V}(t_n)$, $\mathbb{Y}(t_{n+1})-\mathbb{Y}(t_n)$, $\mathbb{X}(t_{n+1})-\mathbb{X}(t_n)$, $\mathbb{W}(t_{n+1})-\mathbb{W}(t_n)$, $\mathbb{Z}(t_{n+1})-\mathbb{Z}(t_n)$ in (6.2) and (6.3), we obtain

$$\begin{aligned}
\mathbb{V}(t_{n+1}) &= \mathbb{V}(t_n) + \mathscr{G}^1_{\zeta,1} + \mathscr{G}^1_{\eta,2}, \\
\mathbb{Y}(t_{n+1}) &= \mathbb{Y}(t_n) + \mathscr{G}^2_{\zeta,1} + \mathscr{G}^2_{\eta,2}, \\
\mathbb{X}(t_{n+1}) &= \mathbb{X}(t_n) + \mathscr{G}^3_{\zeta,1} + \mathscr{G}^3_{\eta,2}, \qquad (6.4) \\
\mathbb{W}(t_{n+1}) &= \mathbb{W}(t_n) + \mathscr{G}^4_{\zeta,1} + \mathscr{G}^4_{\eta,2}, \\
\mathbb{Z}(t_{n+1}) &= \mathbb{Z}(t_n) + \mathscr{G}^5_{\zeta,1} + \mathscr{G}^5_{\eta,2}.
\end{aligned}$$

Where

$$\begin{aligned}
\mathscr{G}^1_{\zeta,1} &= \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_1(t, \mathbb{V}(t))(t_{n+1} - t)^{\zeta-1} dt, \\
A^2_{\zeta,1} &= \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_2(t, \mathbb{Y}(t))(t_{n+1} - t)^{\zeta-1} dt, \\
\mathscr{G}^3_{\zeta,1} &= \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_3(t, \mathbb{X}(t))(t_{n+1} - t)^{\zeta-1} dt, \\
\mathscr{G}^4_{\zeta,1} &= \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_4(t, \mathbb{W}(t))(t_{n+1} - t)^{\zeta-1} dt, \\
\mathscr{G}^5_{\zeta,1} &= \frac{1}{\Gamma(\zeta)} \int_0^{t_{n+1}} \mathfrak{G}_5(t, \mathbb{Z}(t))(t_{n+1} - t)^{\zeta-1} dt.
\end{aligned}$$

and

$$\mathscr{G}^{1}_{\zeta,2} = \frac{1}{\Gamma(\zeta)} \int_{0}^{t_n} \mathfrak{G}_1(t, \mathbb{V}(t))(t_n - t)^{\zeta-1} dt,$$

$$\mathscr{G}^{2}_{\zeta,2} = \frac{1}{\Gamma(\zeta)} \int_{0}^{t_n} \mathfrak{G}_2(t, \mathbb{Y}(t))(t_n - t)^{\zeta-1} dt,$$

$$\mathscr{G}^{3}_{\zeta,2} = \frac{1}{\Gamma(\zeta)} \int_{0}^{t_n} \mathfrak{G}_3(t, \mathbb{X}(t))(t_n - t)^{\zeta-1} dt,$$

$$\mathscr{G}^{4}_{\zeta,2} = \frac{1}{\Gamma(\zeta)} \int_{0}^{t_n} \mathfrak{G}_4(t, \mathbb{W}(t))(t_n - t)^{\zeta-1} dt,$$

$$\mathscr{G}^{5}_{\zeta,2} = \frac{1}{\Gamma(\zeta)} \int_{0}^{t_n} \mathfrak{G}_5(t, \mathbb{Z}(t))(t_n - t)^{\zeta-1} dt.$$

By approximating $A^1_{\zeta,1}$, $A^1_{\zeta,2}$, $A^2_{\zeta,1}$, $A^2_{\zeta,2}$, $A^3_{\zeta,1}$, $A^3_{\zeta,2}$, $A^4_{\zeta,1}$, $A^4_{\zeta,2}$, $A^5_{\zeta,1}$, $A^5_{\zeta,2}$ with the help of Lagrange's polynomials and the plugging back in (6.4) we get the following solution

$$\begin{aligned}
\mathbb{V}(t_{n+1}) &= \mathbb{V}(t_n) + \frac{\mathfrak{G}_1(t_n, \mathbb{V}(t_n))}{\hbar\Gamma(\zeta)}\left\{\frac{2\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{\hbar}{\zeta}t^{\zeta}_{n} - \frac{t^{\zeta+1}}{\zeta}\right\} \\
&\quad + \frac{\mathfrak{G}_1(t_{n-1}, \mathbb{V}_{n-1})}{\hbar\Gamma(\zeta)}\left\{\frac{\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{t^{\zeta}_{n}}{\zeta+1}\right\} + \mathfrak{R}^{\zeta}_{1,n}(t), \\
\mathbb{Y}(t_{n+1}) &= \mathbb{Y}(t_n) + \frac{\mathfrak{G}_2(t_n, \mathbb{Y}(t_n))}{\hbar\Gamma(\zeta)}\left\{\frac{2\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{\hbar}{\zeta}t^{\zeta}_{n} - \frac{t^{\zeta+1}}{\zeta}\right\} \\
&\quad + \frac{\mathfrak{G}_2(t_{n-1}, \mathbb{Y}_{n-1})}{\hbar\Gamma(\zeta)}\left\{\frac{\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{t^{\zeta}_{n}}{\zeta+1}\right\} + \mathfrak{R}^{\zeta}_{2,n}(t), \\
\mathbb{X}(t_{n+1}) &= \mathbb{X}(t_n) + \frac{\mathfrak{G}_3(t_n, \mathbb{X}_{\hbar}(t_n))}{\hbar\Gamma(\zeta)}\left\{\frac{2\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{\hbar}{\zeta}t^{\zeta}_{n} - \frac{t^{\zeta+1}}{\zeta}\right\} \\
&\quad + \frac{\mathfrak{G}_3(t_{n-1}, \mathbb{X}_{n-1})}{\hbar\Gamma(\zeta)}\left\{\frac{\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{t^{\zeta}_{n}}{\zeta+1}\right\} + \mathfrak{R}^{\zeta}_{3,n}(t), \\
\mathbb{W}(t_{n+1}) &= \mathbb{W}(t_n) + \frac{\mathfrak{G}_4(t_n, \mathbb{W}(t_n))}{\hbar\Gamma(\zeta)}\left\{\frac{2\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{\hbar}{\zeta}t^{\zeta}_{n} - \frac{t^{\zeta+1}}{\zeta}\right\} \\
&\quad + \frac{\mathfrak{G}_4(t_{n-1}, \mathbb{W}_{n-1})}{\hbar\Gamma(\zeta)}\left\{\frac{\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{t^{\zeta}_{n}}{\zeta+1}\right\} + \mathfrak{R}^{\zeta}_{4,n}(t), \\
\mathbb{Z}(t_{n+1}) &= \mathbb{Z}(t_n) + \frac{\mathfrak{G}_5(t_n, \mathbb{Z}(t_n))}{\hbar\Gamma(\zeta)}\left\{\frac{2\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{\hbar}{\zeta}t^{\zeta}_{n} - \frac{t^{\zeta+1}}{\zeta}\right\} \\
&\quad + \frac{\mathfrak{G}_5(t_{n-1}, \mathbb{V}_{n-1})}{\hbar\Gamma(\zeta)}\left\{\frac{\hbar}{\zeta}t^{\zeta}_{n+1} - \frac{t^{\zeta+1}_{n+1}}{\zeta+1} + \frac{t^{\zeta}_{n}}{\zeta+1}\right\} + \mathfrak{R}^{\zeta}_{5,n}(t).
\end{aligned} \tag{6.5}$$

Where

$$\begin{aligned}
\mathfrak{G}_1 &= \Pi - \frac{\beta\mathbb{V}(t)\mathbb{Y}(t)}{N} - d\mathbb{V}(t) + \lambda\mathbb{Z}(t),\\
\mathfrak{G}_2 &= \frac{\beta\mathbb{V}(t)\mathbb{Y}(t)}{N} - \frac{\delta\mathbb{Y}(t)\mathbb{X}(t)}{N} - (\gamma + d)\mathbb{Y}(t),\\
\mathfrak{G}_3 &= \frac{\delta\mathbb{Y}(t)\mathbb{X}(t)}{N} - (\mu + \omega + d)\mathbb{X}(t),\\
\mathfrak{G}_4 &= \omega\mathbb{X}(t) - (\kappa + d)\mathbb{W}(t),\\
\mathfrak{G}_5 &= \kappa\mathbb{W}(t) - (\lambda + d)\mathbb{Z}(t).
\end{aligned}$$

And $\mathfrak{R}^{\zeta}_{1,n}(t)$, $\mathfrak{R}^{\zeta}_{2,n}(t)$, $\mathfrak{R}^{\zeta}_{3,n}(t)$, $\mathfrak{R}^{\zeta}_{4,n}(t)$, $\mathfrak{R}^{\zeta}_{5,n}(t)$ are the remainder's terms.

7. Numerical simulations

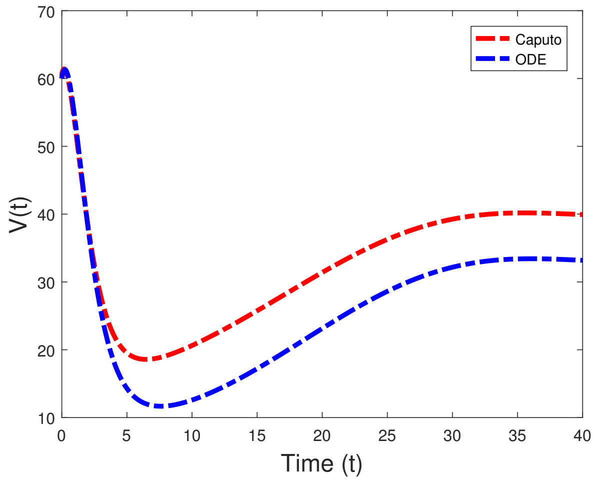
This section aims to present the numerical simulations of the fractional-order model (3.4) with Caputo fractional derivative. The time level is taken up to 40 units. The parameters and initial values are given in Table 2 for a biologically feasible way.

Table 2. The parameter values used in the simulations.

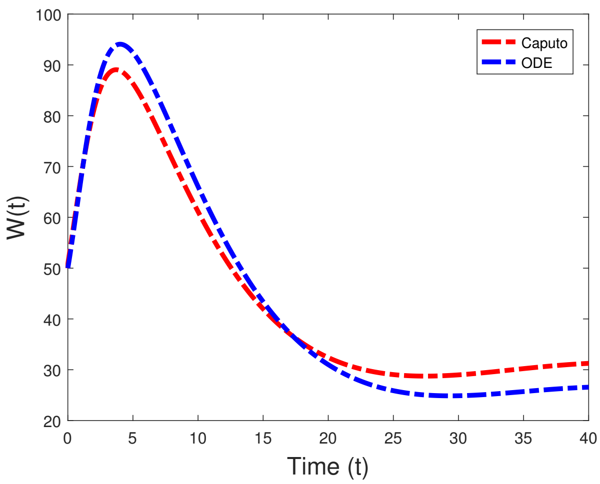
Parameters	value	Source
Π	0.80	[39]
d	0.01	[39]
β	0.70	assumed
γ	0.07	[39]
μ	0.01	[39]
κ	0.50	assumed
λ	0.002	[39]
δ	0.010	[39]
ω	0.001	[39]
$\mathbb{V}(0)$	60	[39]
$\mathbb{Y}(0)$	50	[39]
$\mathbb{X}(0)$	40	[39]
$\mathbb{W}(0)$	35	[39]
$\mathbb{Z}(0)$	15	[39]

We use the numerical solution introduced for the approximation of Caputo fractional derivative to model (3.4). For this we used the values given in Table 2. We took the interval for time between 0 and 40, i.e., [0−40]. The initial data for the population has been taken from the Table 2 for all the classes considers in this article. These includes five different classes namely susceptible class $\mathbb{V}(t)$, snuffing population $\mathbb{Y}(t)$, casual smokers $\mathbb{X}(t)$, chain $\mathbb{W}(t)$ and quit smokers $\mathbb{Z}(t)$. The numerical results obtained for the above classes are plotted in Figures 1–3. The Figures 1(a)–1(e) representing the comparison of model (3.4) when $\zeta = 0.90$, and Figures 2(a)–2(e) representing the comparison of the system (3.4) and its deterministic version, when $\zeta = 1.0$. We approximate the numerical solution of the governing model (3.4) under Caputo non-integer order operator by iterative scheme. All the compartment, i.e., $(\mathbb{V}(t), \mathbb{Y}(t), \mathbb{X}(t), \mathbb{W}(t), \mathbb{Z}(t))$ of the governing model against the parameters values given in Table 2 for different values of $\zeta = 0.95, 0.85, 0.75, 0.65, 0.55$. We show the dynamical behavior of the different compartments for the system (3.4) in Figures 3(a)–3(e).

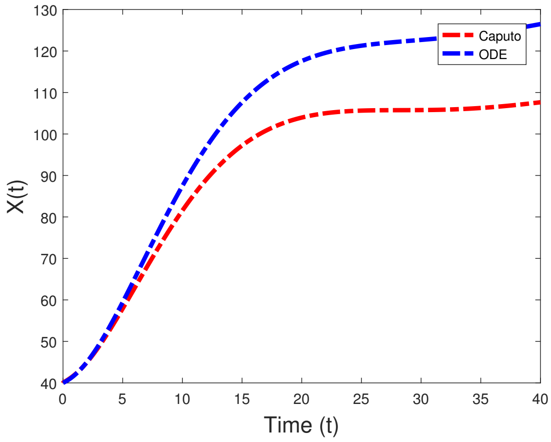
First of all it is observed that the decay in susceptible class is very rapid and then become stable with the passage of time. Analogously the infection cases is also decreasing at various fractional order of ζ. In this case of chain and quit smokers has been achieved for their maximum peak. From these results it is clearly seen that the the governing model depends on the fractional order ζ, This gives more flexible information about the behavior of the model that cannot be obtained with the classical integer-order model.



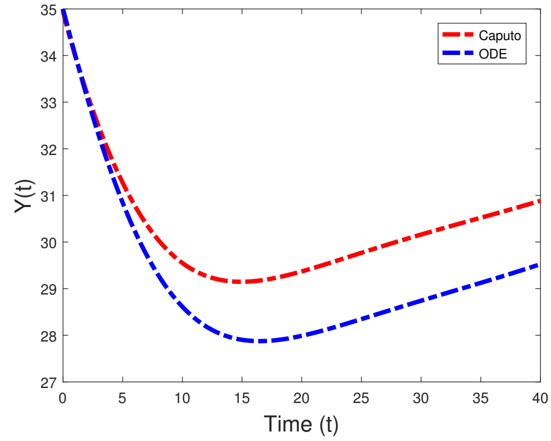
(a) $\mathbb{V}(t)$−Susceptible Individuals.



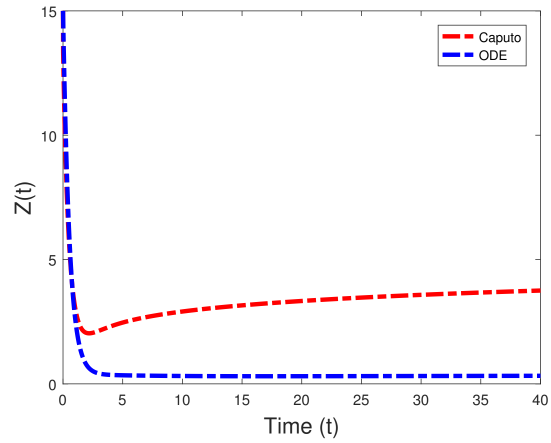
(b) $\mathbb{Y}(t)$−Snuffing Individuals.



(c) $\mathbb{X}(t)$−Casual smokers Individuals.

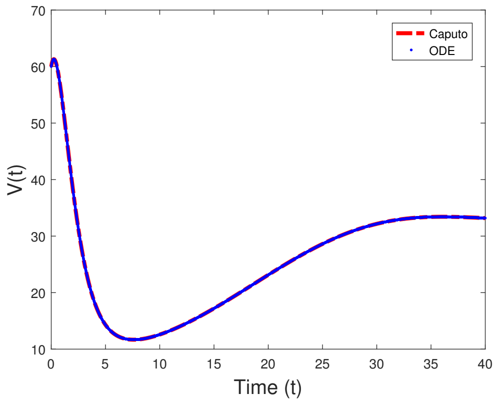


(d) $\mathbb{W}(t)$−Chain Individuals.

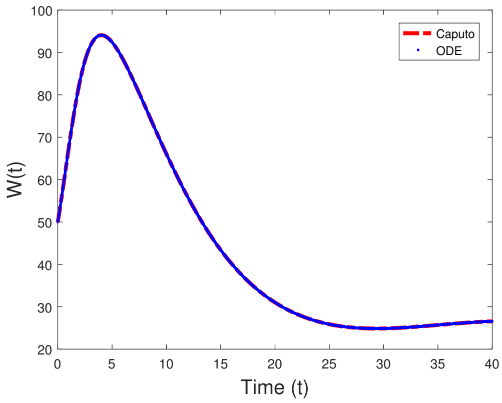


(e) $\mathbb{Z}(t)$−Quit Individuals.

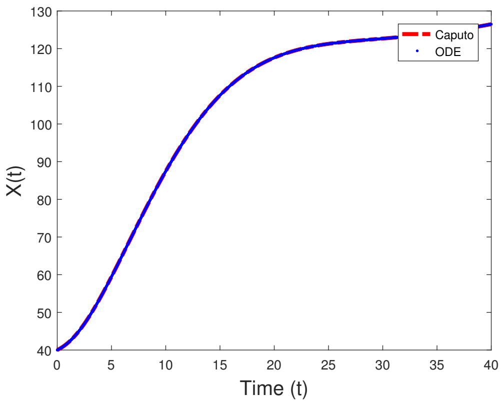
Figure 1. Simulations of susceptible, snuffing population, casual smokers, chain and quit smokers individuals of model (3.4), when $\zeta = 90.0$.



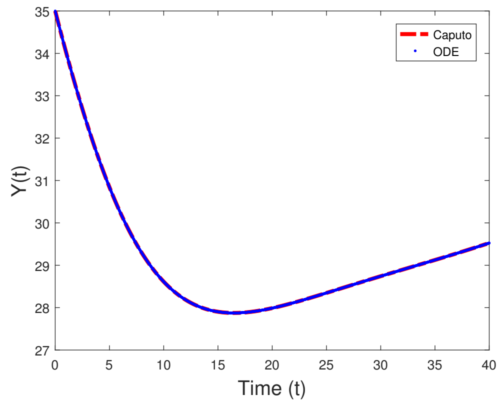
(a) $\mathbb{V}(t)$−Susceptible Individuals.



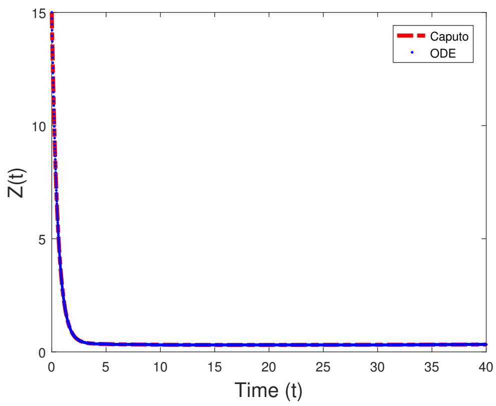
(b) $\mathbb{Y}(t)$−Snuffing Individuals.



(c) $\mathbb{X}(t)$−Casual smokers Individuals.

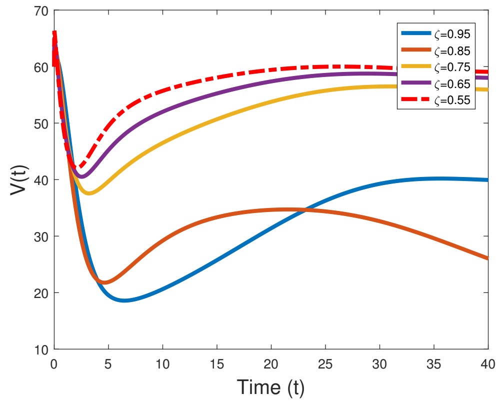


(d) $\mathbb{W}(t)$−Chain Individuals.

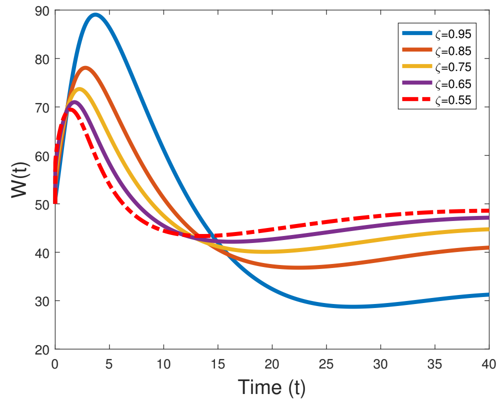


(e) $\mathbb{Z}(t)$−Quit Individuals.

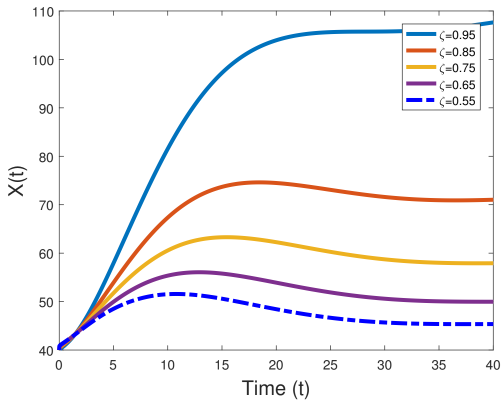
Figure 2. Simulations of susceptible, snuffing population, casual smokers, chain and quit smokers individuals of model (3.4), when $\zeta = 1.0$.



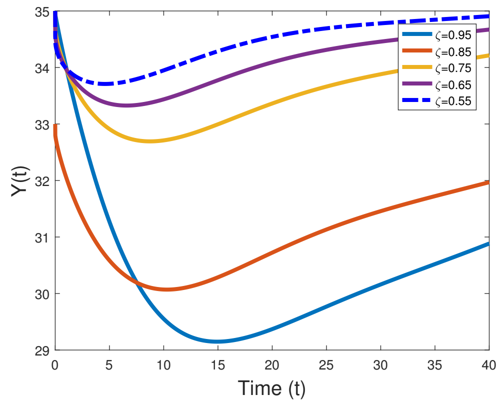
(a) $\mathbb{V}(t)$−Susceptible Individuals.



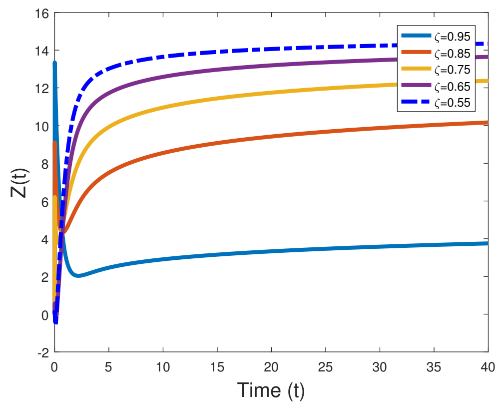
(b) $\mathbb{Y}(t)$−Snuffing Individuals.



(c) $\mathbb{X}(t)$−Casual smokers Individuals.



(d) $\mathbb{W}(t)$−Chain Individuals.



(e) $\mathbb{Z}(t)$−Quit Individuals.

Figure 3. Simulations of susceptible, snuffing population, casual smokers, chain and quit smokers individuals of model (3.4) when $\zeta = 0.95, 0.85, 0.75, 0.65, 0.55$.

8. Conclusions

In this work, the formulation of a model containing a snuffing class is presented; then the equilibrium points that are smoking free and smoking positive are discussed. Further, we formulated a Caputo-fractional derivative to describe the transmission dynamics of a mathematical model describing the dynamics of tobacco smoking disease. We have discussed various fundamental properties of the model. The existence and uniqueness property of proposed model of the disease discussed. With the help of fractional adam's bashforth scheme, an approximate solution of the model has been determined. For the numerical solution of the proposed model are performed and we assumed the initial populations greater than zero for $t > 0$. This can be clearly seen from the numerical results that our solution satisfy the initial data if the right hand side of the proposed system vanishes only at some specific conditions. The obtained results for various values of the fractional-order have been discussed. Our simulations show that the results obtained for the fractional model under the Caputo-derivatives provide a more realistic analysis than the classical integer-order tobacco smoking disease. The proposed method we can also study for different epidemics.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (Nos. 11901114, 62002068), and Guangzhou Science and technology innovation general project (No. 201904010010), Young innovative talents project of Guangdong Provincial Department of Education (No. 2017KQNCX081), Natural Science Foundation of Guangdong Province (Grant No. 2017A030310598), and Finance and Accounting Innovation Research Team under Guangdong, Hong Kong and Macau Greater Bay Area Capital Market (No. 2020WCXTD009).

Conflict of interest

The authors declare that they have no conflict of interest.

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