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Research article

Generalizations of Ostrowski type inequalities via F-convexity

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Abstract: The aim of this article is to give new generalizations of both the Ostrowski's inequality and some of its new variants with the help of the F-convex function class, which is a generalization of the strongly convex functions. Young's inequality, which is well known in the literature, as well as Hölder's inequality, was used to obtain the new results. Also we obtain some results for convex and strongly convex functions by utilizing these inequalities.

Keywords: convex functions; F-convex functions; strongly convex functions; Ostrowski's inequality; Young's inequality
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1. Introduction

Let $f: I \to \mathbb{R}$ be continuous on (a,b), whose derivative $f': (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e.,

$$\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty.$$

Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le (b-a)\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$. The inequality can also be written in another form as follow:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right],$$

where $|f'(t)| \leq M$. The constant $\frac{1}{4}$ is the best and cannot be replaced by a smaller one.

In 1938, Ukrainian mathematician Alexander Markowich Ostrowski established the above inequality which is called Ostrowski's inequality in the literature. It has attracted the attention of many researchers since the day it was introduced and one of its strengths is that it can be used to estimate the deviation of functional value from its integral mean. We would like to bring some papers to the attention of interested readers who want more information about Ostrowski-type inequalities. In [1], Anastassiou presented classical inequalities of this type for outside of the convexity concept. Besides, in [14], the authors have proved some Ostrowski's type inequalities for differentiable mappings that do not satisfy the properties of convexity. To provide more information related to Ostrowski type inequalities for convex and different kinds of convex functions, we suggest the papers [4–6]. Besides these references, for a collection of important results dealing with Ostrowski's inequality, we refer to the paper [9].

Convexity is one of the most frequently used concepts to obtain new Ostrowski type inequalities. Let us remind the definition of this class that we will use in this article. A function $f : I \to \mathbb{R}$ is convex on I where $I \subseteq \mathbb{R}$ is an interval if the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality (1.2) changes the direction then f is called concave function.

The concept of convexity has been perhaps the most popular subject of the inequality theory. Undoubtedly, the fact that this concept has many interesting applications by taking an active role. In recent years, many generalizations of convexity have been established by several researchers. We will mention some of them briefly as follow. In [15], the authors have given some general inequalities that involve convexity. In [7], the readers can find some integral inequalities that obtained by using the concept of concave functions. In [8, 10], the authors have mentioned some different kinds of convex functions and associated inequalities.

The following class of functions that Poljak introduced is providing a stronger condition than convexity [16].

If the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

is valid for all $x, y \in I$ and $t \in [0,1]$, the function $f : I \to \mathbb{R}$ is called strongly convex with modulus $c > 0$.

After the emergence of this concept, many articles have been published on this subject. For example, we can see interesting basic results for this class in [2, 12]. In [13], Nikodem and Pales presented inequalities for inner product spaces. Besides, we have Ostrowski type [17] and majorization type [11] results for this function class in the literature. Since strongly convex functions are important strengthening of the classical convex functions, we can expect better estimates related to deviation of functional value from its integral mean.

Take into consideration from this thought, Set et al. gave the following Ostrowski type inequalites.

Theorem 1.1. *[17] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$, with $a < b$. If $|f'|$ is strongly convex on $[a,b]$ with respect to $c > 0$, $|f'| \leq M$ and*

$M \geq \max\left\{\frac{c(x-a)^2}{6}, \frac{c(b-x)^2}{6}\right\}$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{(x-a)^2}{2(b-a)}\left(M - \frac{c(x-a)^2}{6}\right) + \frac{(b-x)^2}{2(b-a)}\left(M - \frac{c(b-x)^2}{6}\right) \tag{1.3}$$

for all $x, y \in [a, b]$.

Theorem 1.2. *[17] Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a, b]$, *where* $a, b \in I$, *with* $a < b$. *If* $|f'|^q$ *is strongly convex on* $[a, b]$ *with respect to* $c > 0$, $|f'| \leq M$ *and* $M \geq \max\left\{\frac{c(x-a)^2}{6}, \frac{c(b-x)^2}{6}\right\}$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\frac{(x-a)^2}{b-a}\left(M^q - \frac{c(x-a)^2}{6}\right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a}\left(M^q - \frac{c(b-x)^2}{6}\right)^{\frac{1}{q}}\right] \tag{1.4}$$

for all $x, y \in [a, b]$, $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

In [2], Adamek defined a generalization of strong convexity which is called F-convex.

Definition 1.1. *[2] Let* $F : \mathbb{R} \to \mathbb{R}$ *be a fixed function. A function* $f : I \to \mathbb{R}$ *is called* F*-convex if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)F(x-y)$$

for all $x, y \in I$ *and* $t \in [0, 1]$.

Readers can find other interesting results for F-convex functions in [3]. This class of functions is also a generalization of the approximate convex and semiconvex function classes.

2. Lemmas

To prove our main results, we use the following two lemmas introduced in [5, 6] respectively.

Lemma 2.1. *[6] Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°, *where* $a, b \in I$, *with* $a < b$. *If* $f' \in L[a, b]$, *then*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx + (1-t)a)\,dt - \frac{(b-x)^2}{b-a}\int_0^1 tf'(tx + (1-t)b)\,dt$$

for each $x \in [a, b]$.

Lemma 2.2. *[5] Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°, *where* $a, b \in I$, *with* $a < b$. *If* $f' \in L[a, b]$, *then*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = (b-a)\int_0^1 p(t)\,f'(ta + (1-t)b)\,dt$$

for each $t \in [0,1]$, *where*

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{b-x}{b-a}\right] \\ t-1, & t \in \left(\frac{b-x}{b-a}, 1\right] \end{cases}$$

for all $x \in (a,b)$.

3. Main results

In this section, we obtained new Ostrowski-type results for F-convex functions using Lemmas 2.1 and 2.2.

Theorem 3.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I°, *where* $a, b \in I$, *with* $a < b$. *If* $|f'|$ *is* F*-convex on* $[a,b]$ *with* $|f'| \leq M$ *and* $M \geq \max\left\{\frac{F(x-a)}{6}, \frac{F(b-x)}{6}\right\}$, *where* $F : \mathbb{R} \to \mathbb{R}$ *continous on* $[a,b]$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{(x-a)^2}{2(b-a)}\left(M - \frac{1}{6}F(x-a)\right) + \frac{(b-x)^2}{2(b-a)}\left(M - \frac{1}{6}F(b-x)\right) \tag{3.1}$$

for all $x, y \in [a,b]$.

Proof. First, if we modify the right hand side of Lemma 2.1, we get

$$\begin{aligned} & \frac{(x-a)^2}{b-a}\int_0^1 t f'(tx+(1-t)a)\,dt - \frac{(b-x)^2}{b-a}\int_0^1 t f'(tx+(1-t)b)\,dt \\ = & \frac{(x-a)^2}{b-a}\int_0^1 t f'(tx+(1-t)a)\,dt - \frac{(b-x)^2}{b-a}\int_0^1 (1-t) f'(tb+(1-t)x)\,dt, \end{aligned}$$

then we have

$$\begin{aligned} \left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq & \frac{(x-a)^2}{b-a}\int_0^1 t\,|f'(tx+(1-t)a)|\,dt \\ & + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)\,|f'(tb+(1-t)x)|\,dt. \end{aligned} \tag{3.2}$$

Since $|f'|$ is F-convex on $[a,b]$ and $|f'| \leq M$, we get

$$\begin{aligned} \int_0^1 t\,|f'(tx+(1-t)a)|\,dt & \leq \int_0^1 \left[t^2\,|f'(x)| + t(1-t)\,|f'(a)| - t^2(1-t)F(x-a)\right] dt \\ & \leq \frac{M}{2} - \frac{1}{12}F(x-a) \end{aligned}$$

and

$$\int_0^1 (1-t)\left|f'(tb+(1-t)x)\right|dt \le \int_0^1 \left[t(1-t)\left|f'(b)\right| + (1-t)^2\left|f'(x)\right| - t(1-t)^2 F(b-x)\right]dt$$
$$\le \frac{M}{2} - \frac{1}{12}F(x-b).$$

By using these results we can easily see that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{(x-a)^2}{2(b-a)}\left(M - \frac{1}{6}F(x-a)\right) + \frac{(b-x)^2}{2(b-a)}\left(M - \frac{1}{6}F(b-x)\right),$$

which completes the proof. □

Remark 3.1. *Choosing* $F(x) = 0$ *in* (3.1), *then we have the inequality* (1.1).

Remark 3.2. *If we choose* $F(x) = cx^2$ *in* (3.1)*, then we obtain* (1.3).

Corollary 3.1. *If we take* $x = \frac{a+b}{2}$ *in* (3.1), *then we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \le \frac{b-a}{4}\left(M - \frac{1}{6}F\left(\frac{b-a}{2}\right)\right).$$

Theorem 3.2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$*, where* $a, b \in I$*, with* $a < b$*. If* $|f'|^q$ *is* F*-convex on* $[a,b]$ *with* $|f'| \le M$ *and* $M^q \ge \max\left\{\frac{F(x-a)}{6}, \frac{F(b-x)}{6}\right\}$*, where* $F : \mathbb{R} \to \mathbb{R}$ *continous on* $[a,b]$*, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2}{b-a}\left(M^q - \frac{1}{6}F(x-a)\right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a}\left(M^q - \frac{1}{6}F(b-x)\right)^{\frac{1}{q}}\right] \tag{3.3}$$

for all $x, y \in [a,b]$, $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using (3.2) and the well-known Hölder inequality for $q > 1$, we can write

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2}{b-a}\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tb+(1-t)x)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is F-convex on $[a,b]$ and $|f'| \leq M$, we get

$$\int_0^1 |f'(tx+(1-t)a)|^q\,dt \leq \int_0^1 \left[|f'(x)|^q + (1-t)|f'(a)|^q - t(1-t)F(x-a)\right]dt \leq M^q - \frac{1}{6}F(x-a)$$

and

$$\int_0^1 |f'(tb+(1-t)x)|\,dt \leq \int_0^1 \left[|f'(b)| + (1-t)|f'(x)| - t(1-t)F(b-x)\right]dt \leq M^q - \frac{1}{6}F(b-x).$$

Also we have

$$\int_0^1 t^p dt = \int_0^1 (1-t)^p\,dt = \frac{1}{p+1}.$$

Combining these results, the proof is completed. □

Remark 3.3. *Choosing* $F(x) = cx^2$ *in* (3.3) *we obtain* (1.4).

Corollary 3.2. *Choosing* $F(x) = 0$ *in* (3.3), *then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{M}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(x-a)^2 + (b-x)^2\right].$$

Corollary 3.3. *If we take* $x = \frac{a+b}{2}$ *in* (3.3), *then we have*

$$\left|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{b-a}{4}\left(M^q - \frac{1}{6}F\left(\frac{b-a}{2}\right)\right)^{\frac{1}{q}}\right].$$

Theorem 3.3. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$, *with* $a < b$. *If* $|f'|^q$ *is* F*-convex on* $[a,b]$ *with* $|f'| \leq M$ *and* $M^q \geq \max\left\{\frac{F(x-a)}{6}, \frac{F(b-x)}{6}\right\}$, *where* $F : \mathbb{R} \to \mathbb{R}$ *continous on* $[a,b]$, *then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{3.4}$$
$$\leq \frac{(x-a)^2}{b-a}\left[\frac{1}{p(p+1)} + \frac{1}{q}\left(M^q - \frac{1}{6}F(x-a)\right)\right] + \frac{(b-x)^2}{b-a}\left[\frac{1}{p(p+1)} + \frac{1}{q}\left(M^q - \frac{1}{6}F(b-x)\right)\right]$$

for all $x, y \in [a,b]$, $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using (3.2) and applying Young's inequality, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{3.5}$$
$$\leq \frac{(x-a)^2}{b-a}\left(\frac{1}{p}\int_0^1 t^p dt+\frac{1}{q}\int_0^1 |f'(tx+(1-t)a)|^q\,dt\right)$$
$$+\frac{(b-x)^2}{b-a}\left(\frac{1}{p}\int_0^1 (1-t)^p\,dt+\frac{1}{q}\int_0^1 |f'(tb+(1-t)x)|^q\,dt\right).$$

Since $|f'|^q$ is F-convex on $[a,b]$ and $|f'|\leq M$, we know from the proof of the last theorem that

$$\int_0^1 |f'(tx+(1-t)a)|^q\,dt\leq M^q-\frac{1}{6}F(x-a),$$

$$\int_0^1 |f'(tb+(1-t)x)|^q\,dt\leq M^q-\frac{1}{6}F(b-x)$$

and

$$\int_0^1 t^p dt=\int_0^1 (1-t)^p\,dt=\frac{1}{p+1}.$$

If we write the last three results in (3.5), then we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{(x-a)^2}{b-a}\left[\frac{1}{p(p+1)}+\frac{1}{q}\left(M^q-\frac{1}{6}F(x-a)\right)\right]$$
$$+\frac{(b-x)^2}{b-a}\left[\frac{1}{p(p+1)}+\frac{1}{q}\left(M^q-\frac{1}{6}F(b-x)\right)\right],$$

so that the desired inequality is achieved. □

Corollary 3.4. *Choosing* $F(x)=0$ *in* (3.4), *then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq \frac{1}{b-a}\left(\frac{1}{p(p+1)}+\frac{M^q}{q}\right)\left[(x-a)^2+(b-x)^2\right].$$

Corollary 3.5. *If we take* $x=\frac{a+b}{2}$ *in* (3.4), *then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq \frac{b-a}{4}\left[\frac{1}{p(p+1)}+\frac{1}{q}\left(M^q-\frac{1}{6}F\left(\frac{b-a}{2}\right)\right)\right].$$

Corollary 3.6. *Choosing $F(x) = cx^2$ in* (3.4)*, then the following Ostrowski type inequality for strongly convex functions holds:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ \leq\ & \frac{(x-a)^2}{b-a}\left[\frac{1}{p(p+1)} + \frac{1}{q}\left(M^q - \frac{1}{6}c(x-a)^2\right)\right] \\ &+ \frac{(b-x)^2}{b-a}\left[\frac{1}{p(p+1)} + \frac{1}{q}\left(M^q - \frac{1}{6}c(b-x)^2\right)\right]. \end{aligned}$$

Theorem 3.4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$, with $a < b$. If $|f'|^q$ is F-convex on $[a,b]$ with $|f'| \leq M$ and $M^q \geq \frac{1}{6}F(b-a)$, where $F : \mathbb{R} \to \mathbb{R}$ continous on $[a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \left(\frac{(x-a)^{p+1} + (b-x)^{p+1}}{(b-a)(p+1)}\right)^{\frac{1}{p}} \left(M^q - \frac{1}{6}F(b-a)\right)^{\frac{1}{q}} \tag{3.6}$$

for all $x, y \in [a,b]$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.2, we have

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = (b-a)\int_0^1 p(t)\, f'(tb + (1-t)a)\,dt, \tag{3.7}$$

where

$$p(t) = \begin{cases} t, & t \in \left[0, \frac{x-a}{b-a}\right] \\ t-1, & t \in \left(\frac{x-a}{b-a}, 1\right] \end{cases}.$$

By use of Hölder inequality for $q > 1$, in (3.7) we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq (b-a)\left(\int_0^1 |p(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_0^1 |f'(tb + (1-t)a)|^q\,dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is F-convex on $[a,b]$ and $|f'| \leq M$, we get

$$\begin{aligned} \int_0^1 |f'(tb + (1-t)a)|^q\,dt &\leq \int_0^1 \left[t|f'(b)|^q + (1-t)|f'(a)|^q - t(1-t)F(b-a)\right]dt \\ &\leq M^q - \frac{1}{6}F(b-a). \end{aligned}$$

Also we have

$$\int_0^1 |p(t)|^p\,dt = \int_0^{\frac{x-a}{b-a}} t^p dt + \int_{\frac{x-a}{b-a}}^1 (1-t)^p\,dt = \frac{(x-a)^{p+1} + (b-x)^{p+1}}{(b-a)^{p+1}(p+1)}.$$

Combining these, the proof is completed. □

Corollary 3.7. *If we choose $F(x) = 0$ in* (3.6), *then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq M \left(\frac{(x-a)^{p+1} + (b-x)^{p+1}}{(b-a)(p+1)} \right)^{\frac{1}{p}}.$$

Corollary 3.8. *If we take $x = \frac{a+b}{2}$ in* (3.6), *then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq \frac{b-a}{2} \left(M^q - \frac{1}{6} F(b-a) \right)^{\frac{1}{q}}.$$

Corollary 3.9. *If we choose $F(x) = cx^2$ in* (3.6)*, then we obtain the following inequality for strongly convex functions:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq \left(\frac{(x-a)^{p+1} + (b-x)^{p+1}}{(b-a)(p+1)} \right)^{\frac{1}{p}} \left(M^q - \frac{1}{6} c (b-a)^2 \right)^{\frac{1}{q}}.$$

Theorem 3.5. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a, b \in I$, with $a < b$. If $|f'|^q$ is F-convex on $[a,b]$ with $|f'| \leq M$ and $M^q \geq \frac{1}{6} F(b-a)$, where $F : \mathbb{R} \to \mathbb{R}$ continous on $[a,b]$, then*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \\ \leq\ & (b-a) \left[\left(\frac{(x-a)^{p+1} + (b-x)^{p+1}}{p(p+1)(b-a)^p} \right) + \frac{1}{q} \left(M^q - \frac{1}{6} F(b-a) \right) \right] \end{aligned} \tag{3.8}$$

for all $x, y \in [a,b]$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using (3.7) and applying Young's inequality, we obtain

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| &\leq (b-a) \int_0^1 |p(t) f'(tb + (1-t)a)\, dt| \\ &\leq (b-a) \left(\frac{1}{p} \int_0^1 |p(t)|^p\, dt + \frac{1}{q} \int_0^1 |f'(tb + (1-t)a)|^q\, dt \right). \end{aligned}$$

Since $|f'|^q$ is F-convex on $[a,b]$ and $|f'| \leq M$, we have from the previous results:

$$\int_0^1 |f'(tb + (1-t)a)|^q\, dt \leq M^q - \frac{1}{6} F(b-a),$$

also

$$\int_0^1 |p(t)|^p \, dt = \int_0^{\frac{x-a}{b-a}} t^p dt + \int_{\frac{x-a}{b-a}}^1 (1-t)^p \, dt = \frac{(x-a)^{p+1} + (b-x)^{p+1}}{(b-a)^{p+1}(p+1)}.$$

Eventually we get

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a) \left[\left(\frac{(x-a)^{p+1} + (b-x)^{p+1}}{p(p+1)(b-a)^p} \right) + \frac{1}{q} \left(M^q - \frac{1}{6} F(b-a) \right) \right].$$

□

Corollary 3.10. *If we take* $x = \frac{a+b}{2}$ *in* (3.8) , *then we have*

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a) \left[\left(\frac{b-a}{p(p+1) 2^p} \right) + \frac{1}{q} \left(M^q - \frac{1}{6} F(b-a) \right) \right].$$

Corollary 3.11. *If we choose* $F(x) = cx^2$ *in* (3.8)*, then we have the following inequality for strongly convex functions:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq (b-a) \left[\left(\frac{(x-a)^{p+1} + (b-x)^{p+1}}{p(p+1)(b-a)^p} \right) + \frac{1}{q} \left(M^q - \frac{1}{6} c (b-a)^2 \right) \right].$$

4. Conclusions

In studies on Ostrowski type inequalities, the main purpose is to try to obtain the best possible upper limits. For this purpose, in this article, some ostrowski type inequalities were obtained with the help of a new generalization of strongly convex functions, which is an important strengthening of convexity. Some of the results obtained are generalizations of the existing inequalities in the literature, while others are the most general Ostrowski type inequalities obtained for the strongly convex functions class.

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Conflict of interest

The authors declare that they have no competing interests.

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