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Research article

Involvement of the fixed point technique for solving a fractional differential system

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Abstract: Some physical phenomena were described through fractional differential equations and compared with integer-order differential equations which have better results, which is why researchers of different areas have paid great attention to study this direction. So, in this manuscript, we discuss the existence and uniqueness of solutions to a system of fractional deferential equations (FDEs) under Riemann-Liouville (R-L) integral boundary conditions. The solution method is obtained by two basic rules, the first rule is the Leray-Schauder alternative and the second is the Banach contraction principle. Finally, the theoretical results are supported by an illustrative example.

Keywords: Caputo fractional derivatives; R-L integrals; fixed point methodology; Leray-Schauder alternative
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1. Background materials

Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The continuum of order in the fractional calculus allows the order of the fractional operator to be considered as a variable, see [1–4]. The fractional calculus has allowed to formulate the operations of integration and differentiation to for fractional order. The order may take on any real or imaginary value. This fact enables us to consider the order of the fractional integrals and derivatives to be a function of time or of some other variable.

Recently, many authors have addressed differential equations with fractional derivatives for a different category of problems. FDEs arise in various engineering and scientific areas, where they are found in the mathematical modeling of systems and processes in the specialists of aerodynamics and electrodynamics of complex medium, physics, biophysics, chemistry, economics, blood flow phenomena, quantum theory, signal and image processing, polymer functional science, see, for example [5–9].

It should be noted that FDEs are better prepared to depict the genetic characteristics of different materials and processes than proper differential equations. Based on this advantage, fractional differential models become more realistic, practical and precise in obtaining the objective of classical models in differential. For better understanding of some real world problems, some researchers suggested recently discovered fractional operators. Among these operators, we mention the ones considered in [10–21].

Fixed point theory (FPT) is an important pillar of non-linear analysis due to its many applications in various mathematical disciplines. The fixed-point style shined after Banach launched his famous principle, known as the Banach contraction principle. It is mainly involved in fractional differential equations by which the existence and uniqueness of solutions of many differential and integral equations with initial and boundary stipulations can be studied. For more details, see [22–27].

One of the most important theorem of nonlinear functional analysis is the Leray-Schauder alternative, proved in 1934 by the topological degree [28]. A lot of authors proved several kinds of Leray-Schauder type alternatives by different methods, not based on topological degree and applied this methodology in many applications to ordinary differential equations, for more details, see [29, 30].

Similar to earlier, by the standard fixed-point principle and Leray-Schauder alternative, the existence and unique solutions for a tripled system of FDEs via R-L integral boundary stipulations of different order are studied. The system takes the form:

$$\begin{cases} {}^cD^{\omega}_{0^+}a(s) = X(s, a(s), b(s), c(s)), & 0 \le s \le 1, \\ {}^cD^{\varkappa}_{0^+}b(s) = Y(s, a(s), b(s), c(s)), & 0 \le s \le 1, \\ {}^cD^{\varrho}_{0^+}c(s) = Z(s, a(s), b(s), c(s)), & 0 \le s \le 1, \\ a(0) = \rho I^e a(\eta) = \rho \int_0^{\eta} \frac{(\eta-\hbar)^{e-1}}{\Gamma(e)} a(\hbar)d\hbar, & \eta \in (0, 1), \\ b(0) = \sigma I^f b(\theta) = \sigma \int_0^{\theta} \frac{(\theta-\hbar)^{f-1}}{\Gamma(f)} b(\hbar)d\hbar, & \theta \in (0, 1), \\ c(0) = \varsigma I^g c(\vartheta) = \varsigma \int_0^{\vartheta} \frac{(\vartheta-\hbar)^{g-1}}{\Gamma(g)} c(\hbar)d\hbar, & \vartheta \in (0, 1), \end{cases} \tag{1.1}$$

where ${}^cD^{\omega}_{0^+}$, ${}^cD^{\varkappa}_{0^+}$ and ${}^cD^{\varrho}_{0^+}$ represent the Caputo Fractional Differentials (CFDs), $0 < \omega, \varkappa, \varrho \le 1$, $X, Y, Z \in C\left([0, 1] \times \mathbb{R}^3, \mathbb{R}\right)$, and $\rho, \sigma, \varsigma, e, f, g \in \mathbb{R}$. Ultimately, an example to support the results is given. Further, classical derivatives are in local nature, i.e., using classical derivatives we can describe changes in a neighborhood of a point but using fractional derivatives (our system) we can describe changes in an interval. Namely, the fractional derivative is in non-local nature. This property makes these derivatives suitable to simulate more physical phenomena such as earthquake vibrations, polymers, etc. Moreover, it is explain time delay and some fractal properties. Therefore, the search for solutions to these systems has received great attention from researchers.

2. Fundamental facts

Assume that $\Xi = \left\{a(s) : a(s) \in C^1([0, 1])\right\}$ is equipped with $\|a\| = \max_{s\in[0,1]} \{|a(s)|\}$. Clearly $(\Xi, \|.\|)$ is a Banach space (BS).

Again, let $\Lambda = \left\{b(s) : b(s) \in C^1([0, 1])\right\}$ be endowed with $\|b\| = \max_{s\in[0,1]}\{|b(s)|\}$. It is clear that the product $(\Xi \times \Lambda, \|(a, b)\|)$ is also a BS with $\|(a, b)\| = \|a\| + \|b\|$.

Also, consider $\wp = \left\{c(s) : c(s) \in C^1([0,1])\right\}$ under the norm $\|c\| = \max_{s\in[0,1]}\{|c(s)|\}$. Then $(\Xi \times \Lambda \times \wp, \|(a,b,c)\|)$ is a BS too with $\|(a,b,c)\| = \|a\| + \|b\| + \|c\|$.

The following definitions and lemmas are follows immediately from [5,8].

Definition 2.1. The standard CFD of order f for continuously differentiable function $L : [0,\infty) \to \mathbb{R}$ is described by

$$ {}^cD^f L(s) = \frac{1}{\Gamma(n-f)} \int_0^s (s-\hbar)^{n-f-1} L^n(\hbar) d\hbar,\ n-1 < f < n,\ n = [f]+1, $$

where $[f]$ represents the integer part of the real number f.

Definition 2.2. The R-L fractional integral of order f is described by

$$ I^f L(s) = \frac{1}{\Gamma(f)} \int_0^s \frac{L(\hbar)}{(s-\hbar)^{1-f}} d\hbar,\ f > 0, $$

provided the integral exists.

The lemmas below illustrate some properties of CFDs and R-L fractional integrals [5].

Lemma 2.3. *Suppose that* $f, g \geq 0$, $X \in L_1[a,b]$. *Then* $I^f I^g X(s) = I^{f+g} X(s)$ *and* ${}^cD^f I^f X(s) = X(s)$, $\forall s \in [0,1]$.

Lemma 2.4. *Assume that* $\varkappa > \omega > 0$, $X \in L_1[a,b]$. *Then* ${}^cD^\omega I^\varkappa X(s) = I^{\varkappa-\omega}(s)$, $\forall s \in [0,1]$.

Lemma 2.5. *Let* $\rho \neq \frac{\Gamma(1+e)}{\eta^e}$, *then for* $X \in C([0,1],\mathbb{R})$ *be a given function, the solution of the FDE*

$$ {}^cD^\omega l(s) = X(s),\ \omega \in (0,1], $$

under the boundary stipulation

$$ l(0) = \rho I^e l(\eta) = \rho \int_0^\eta \frac{(\eta-\hbar)^{e-1}}{\Gamma(e)} l(\hbar) d\hbar,\ \eta \in (0,1), $$

is constructed by

$$ l(s) = \frac{1}{\Gamma(\omega)} \int_0^s (s-\hbar)^{\omega-1} X(\hbar) d\hbar + \frac{\rho\Gamma(1+e)}{\Gamma(1+e) - \rho\eta^\omega} \int_0^\eta \frac{(\eta-\hbar)^{e+\omega-1}}{\Gamma(\omega+e)} X(\hbar) d\hbar,\ s \in [0,1]. $$

3. Main theorems

We will start our results with the following assumptions:

$$ A_1 = \frac{1}{\Gamma(\omega+1)} + \frac{|\rho|\,\eta^{e+\omega}\Gamma(1+e)}{\Gamma(e+\omega+1)\,|\Gamma(1+e) - \rho\eta^e|}, \tag{3.1} $$

$$ A_2 = \frac{1}{\Gamma(\varkappa+1)} + \frac{|\sigma|\,\theta^{f+\varkappa}\Gamma(1+f)}{\Gamma(f+\varkappa+1)\left|\Gamma(1+f) - \sigma\theta^f\right|}, \tag{3.2} $$

$$A_3 = \frac{1}{\Gamma(\varrho+1)} + \frac{|\varsigma|\,\vartheta^{g+\varrho}\Gamma(1+g)}{\Gamma(g+\varrho+1)\,|\Gamma(1+g) - \varsigma\vartheta^{g}|}, \tag{3.3}$$

and

$$A_0 = \min\left\{ \begin{array}{c} 1 - (A_1\alpha_1 + A_2\beta_1 + A_3\gamma_1), \\ 1 - (A_1\alpha_2 + A_2\beta_2 + A_3\gamma_2), \\ 1 - (A_1\alpha_3 + A_2\beta_3 + A_3\gamma_3) \end{array} \right\}, \tag{3.4}$$

where $\alpha_i, \beta_i, \gamma_i \geq 0$, $(i = 1, 2, 3)$.

Assume that $\Omega : \Xi \times \Lambda \times \wp \to \Xi \times \Lambda \times \wp$ be an operator described by:

$$\begin{aligned} &\Omega(a,b,c)(s) \\ &= \begin{pmatrix} \Omega_1(a,b,c)(s) \\ \Omega_2(a,b,c)(s) \\ \Omega_3(a,b,c)(s) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\Gamma(\omega)}\int_0^s (s-\hbar)^{\omega-1} X(\hbar, a(\hbar), b(\hbar), c(\hbar))d\hbar + \frac{\rho\Gamma(1+e)}{\Gamma(1+e)-\rho\eta^{\omega}}\int_0^{\eta} \frac{(\eta-\hbar)^{e+\omega-1}}{\Gamma(\omega+e)} X(\hbar, a(\hbar), b(\hbar), c(\hbar))d\hbar \\ \frac{1}{\Gamma(\varkappa)}\int_0^s (s-\hbar)^{\varkappa-1} Y(\hbar, a(\hbar), b(\hbar), c(\hbar))d\hbar + \frac{\sigma\Gamma(1+f)}{\Gamma(1+f)-\sigma\theta^{\varkappa}}\int_0^{\theta} \frac{(\theta-\hbar)^{f+\varkappa-1}}{\Gamma(\varkappa+f)} Y(\hbar, a(\hbar), b(\hbar), c(\hbar))d\hbar \\ \frac{1}{\Gamma(\varrho)}\int_0^s (s-\hbar)^{\varrho-1} Z(\hbar, a(\hbar), b(\hbar), c(\hbar))d\hbar + \frac{\varsigma\Gamma(1+g)}{\Gamma(1+g)-\varsigma\vartheta^{\varrho}}\int_0^{\vartheta} \frac{(\vartheta-\hbar)^{g+\varrho-1}}{\Gamma(\varrho+g)} Z(\hbar, a(\hbar), b(\hbar), c(\hbar))d\hbar \end{pmatrix}. \end{aligned}$$

The results of this part are based on two rules: The first rule based on Leray-Schauder alternative.

Lemma 3.1. *[31] Assume that ∇ is a normed linear spaces and the mapping $\varphi : \nabla \to \nabla$ is a completely continuous mapping if*

$$\mho(\varphi) = \{\delta \in \nabla : \delta = \beta\varphi(\delta), \text{ for some } \beta \in (0,1)\}.$$

Then either $\mho(\varphi)$ is at the boundary, or φ has at least one fixed point.

Theorem 3.2. *Assume that $\rho \neq \frac{\Gamma(1+e)}{\eta^{e}}$, $\sigma \neq \frac{\Gamma(1+f)}{\theta^{f}}$ and $\varsigma \neq \frac{\Gamma(1+g)}{\vartheta^{g}}$. Suppose that there are real constants $\alpha_i, \beta_i, \gamma_i \geq 0$ $(i = 1, 2, 3)$ and $\alpha_0 > 0, \beta_0 > 0, \gamma_0 > 0$ so that for each $\delta_i \in \mathbb{R}$ $(i = 1, 2, 3)$, we get*

$$\begin{aligned} |X(s, \delta_1, \delta_2, \delta_3)| &\leq \alpha_0 + \alpha_1|\delta_1| + \alpha_2|\delta_2| + \alpha_3|\delta_3|, \\ |Y(s, \delta_1, \delta_2, \delta_3)| &\leq \beta_0 + \beta_1|\delta_1| + \beta_2|\delta_2| + \beta_3|\delta_3|, \end{aligned}$$

and

$$|Z(s, \delta_1, \delta_2, \delta_3)| \leq \gamma_0 + \gamma_1|\delta_1| + \gamma_2|\delta_2| + \gamma_3|\delta_3|.$$

Furthermore, suppose

$$A_1\alpha_1 + A_2\beta_1 + A_3\gamma_1 < 1,\ A_1\alpha_2 + A_2\beta_2 + A_3\gamma_2 < 1 \text{ and } A_1\alpha_3 + A_2\beta_3 + A_3\gamma_3 < 1,$$

where A_1–A_3 are described in (3.1)–(3.3). Then the boundary value problem (BVP) (1.1) has at least one solution.

Proof. In the beginning, it must be proved the completely continuous for $\Omega : \Xi \times \Lambda \times \wp \to \Xi \times \Lambda \times \wp$. Because the functions X, Y and Z are continuous, then Ω is continuous too. Suppose that $\psi \subset \Xi \times \Lambda \times \wp$ is a bounded set, then there exists positive coefficients ℓ_1, ℓ_2 and ℓ_3 so that, for all $(a, b, c) \in \psi$.

$$|X(s, a(s), b(s), c(s))| \leq \ell_1,\ \ |Y(s, a(s), b(s), c(s))| \leq \ell_2 \text{ and } |Z(s, a(s), b(s), c(s))| \leq \ell_3.$$

Then for any $(a,b,c)\in\psi$, we can get

$$
\begin{aligned}
|\Omega_1(a,b,c)(s)| &\le \frac{1}{\Gamma(\omega)}\int_0^s (s-\hbar)^{\omega-1}|X(\hbar,a(\hbar),b(\hbar),c(\hbar))|\,d\hbar \\
&\quad + \frac{|\rho|\,\Gamma(1+e)}{|\Gamma(1+e)-\rho\eta^{\omega}|}\int_0^{\eta}\frac{(\eta-\hbar)^{e+\omega-1}}{\Gamma(\omega+e)}|X(\hbar,a(\hbar),b(\hbar),c(\hbar))|\,d\hbar \\
&\le \ell_1\left[\frac{1}{\Gamma(\omega+1)}+\frac{|\rho|\,\Gamma(1+e)}{\Gamma(\omega+e+1)\,|\Gamma(1+e)-\rho\eta^{\omega}|}\right]=\ell_1A_1. \qquad (3.5)
\end{aligned}
$$

Similarly, one can obtain that

$$
|\Omega_2(a,b,c)(s)|\le \ell_2\left[\frac{1}{\Gamma(\varkappa+1)}+\frac{|\sigma|\,\theta^{f+\varkappa}\Gamma(1+f)}{\Gamma(f+\varkappa+1)\left|\Gamma(1+f)-\sigma\theta^{f}\right|}\right]=\ell_2A_2, \qquad (3.6)
$$

and

$$
|\Omega_3(a,b,c)(s)|\le \ell_3\left[\frac{1}{\Gamma(\varrho+1)}+\frac{|\varsigma|\,\vartheta^{g+\varrho}\Gamma(1+g)}{\Gamma(g+\varrho+1)\,|\Gamma(1+g)-\varsigma\vartheta^{g}|}\right]=\ell_3A_3. \qquad (3.7)
$$

It follows from (3.5)–(3.7) that Ω is uniformly bounded.

Thereafter, we prove that Ω is equi-continuous. Consider $0\le s_1\le s_2\le 1$, so, we get

$$
\begin{aligned}
&|\Omega_1(a(s_2),b(s_2),c(s_2))-\Omega_1(a(s_1),b(s_1),c(s_1))| \\
\le\ & \left|\int_0^{s_2}\frac{(s_2-\hbar)^{\omega-1}}{\Gamma(\omega)}X(\hbar,a(\hbar),b(\hbar),c(\hbar))d\hbar-\int_0^{s_1}\frac{(s_1-\hbar)^{\omega-1}}{\Gamma(\omega)}X(\hbar,a(\hbar),b(\hbar),c(\hbar))d\hbar\right| \\
\le\ & \frac{\ell_1}{\Gamma(\omega)}\left|\int_0^{s_1}\left[(s_2-\hbar)^{\omega-1}-(s_1-\hbar)^{\omega-1}\right]d\hbar+\int_{s_1}^{s_2}(s_2-\hbar)^{\omega-1}\,d\hbar\right| \\
\le\ & \frac{\ell_1}{\Gamma(\omega+1)}\left(s_2^{\omega}-s_1^{\omega}\right),
\end{aligned}
$$

analogously, we see that

$$
\begin{aligned}
&|\Omega_2(a(s_2),b(s_2),c(s_2))-\Omega_2(a(s_1),b(s_1),c(s_1))| \\
\le\ & \frac{\ell_2}{\Gamma(\varkappa)}\left|\int_0^{s_1}\left[(s_2-\hbar)^{\varkappa-1}-(s_1-\hbar)^{\varkappa-1}\right]d\hbar+\int_{s_1}^{s_2}(s_2-\hbar)^{\varkappa-1}\,d\hbar\right| \\
\le\ & \frac{\ell_2}{\Gamma(\varkappa+1)}\left(s_2^{\varkappa}-s_1^{\varkappa}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&|\Omega_3(a(s_2),b(s_2),c(s_2))-\Omega_3(a(s_1),b(s_1),c(s_1))| \\
\le\ & \frac{\ell_3}{\Gamma(\omega)}\left|\int_0^{s_1}\left[(s_2-\hbar)^{\varrho-1}-(s_1-\hbar)^{\varrho-1}\right]d\hbar+\int_{s_1}^{s_2}(s_2-\hbar)^{\varrho-1}\,d\hbar\right| \\
\le\ & \frac{\ell_3}{\Gamma(\varrho+1)}\left(s_2^{\varrho}-s_1^{\varrho}\right).
\end{aligned}
$$

This proves that $\Omega(a,b,c)$ is equicontinuous, and thus the operator $\Omega(a,b,c)$ is completely continuous.

Ultimately, we shall check the set $\mho = \{(a,b,c) \in \Xi \times \Lambda \times \wp : (a,b,c) = \beta\Omega(a,b,c),\ \beta \in [0,1]\}$ is bounded. Consider $(a,b,c) \in \mho$, then $(a,b,c) = \beta\Omega(a,b,c)$. For each $0 \le s \le 1$, we get

$$a(s) = \beta\Omega_1(a,b,c)(s),\ b(s) = \beta\Omega_2(a,b,c)(s) \text{ and } c(s) = \beta\Omega_3(a,b,c)(s).$$

Then

$$\begin{aligned} |a(s)| &\le \left[\frac{1}{\Gamma(\omega+1)} + \frac{|\rho|\,\eta^{e+\omega}\Gamma(1+e)}{\Gamma(e+\omega+1)\,|\Gamma(1+e)-\rho\eta^{e}|}\right] \\ &\quad \times (\alpha_0 + \alpha_1 |a(s)| + \alpha_2 |b(s)| + \alpha_3 |c(s)|), \\ |b(s)| &\le \left[\frac{1}{\Gamma(\varkappa+1)} + \frac{|\sigma|\,\theta^{f+\varkappa}\Gamma(1+f)}{\Gamma(f+\varkappa+1)\left|\Gamma(1+f)-\sigma\theta^{f}\right|}\right] \\ &\quad \times (\beta_0 + \beta_1 |a(s)| + \beta_2 |b(s)| + \beta_3 |c(s)|), \end{aligned}$$

and

$$\begin{aligned} |c(s)| &\le \left[\frac{1}{\Gamma(\varrho+1)} + \frac{|\varsigma|\,\vartheta^{g+\varrho}\Gamma(1+g)}{\Gamma(g+\varrho+1)\,|\Gamma(1+g)-\varsigma\vartheta^{g}|}\right] \\ &\quad \times (\gamma_0 + \gamma_1 |a(s)| + \gamma_2 |b(s)| + \gamma_3 |c(s)|). \end{aligned}$$

The above three inequalities can be written as

$$\begin{aligned} \|a\| &\le A_1 (\alpha_0 + \alpha_1 \|a\| + \alpha_2 \|b\| + \alpha_3 \|c\|), \\ \|b\| &\le A_2 (\beta_0 + \beta_1 \|a\| + \beta_2 \|b\| + \beta_3 \|c\|), \end{aligned}$$

and

$$\|c\| \le A_3 (\gamma_0 + \gamma_1 \|a\| + \gamma_2 \|b\| + \gamma_3 \|c\|),$$

which implies that

$$\begin{aligned} \|a\| + \|b\| + \|c\| &\le (A_1\alpha_0 + A_2\beta_0 + A_3\gamma_0) + (A_1\alpha_1 + A_2\beta_1 + A_3\gamma_1)\|a\| \\ &\quad + (A_1\alpha_2 + A_2\beta_2 + A_3\gamma_2)\|b\| + (A_1\alpha_3 + A_2\beta_3 + A_3\gamma_3)\|c\|, \end{aligned}$$

this leads to

$$\|(a,b,c)\| \le \frac{A_1\alpha_0 + A_2\beta_0 + A_3\gamma_0}{A_0}, \text{ for each } s \in [0,1],$$

where A_0 is given by (3.4), which illustrates that $\mho$ is bounded. Hence according to Lemma 3.1 there is at least one FP for the operator Ω, which is a solution to the BVP (1.1). This finishes the proof. □

The second rule based on Banach's FP theorem [32]. By using it, we prove the existence and uniqueness of solutions to the BVP (1.1).

Theorem 3.3. *Let the functions* $X, Y, Z : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ *be continuous and there are coefficients* p_i, q_i, r_i, $i = 1,2,3$ *so that for each* $s \in [0,1]$ *and* $a_i, b_i \in \mathbb{R}$, $i = 1,2,3$,

$$\begin{aligned} |X(s,a_1,a_2,a_3) - X(s,b_1,b_2,b_3)| &\le p_1|a_1-b_1| + p_2|a_2-b_2| + p_3|a_3-b_3|, \\ |Y(s,a_1,a_2,a_3) - Y(s,b_1,b_2,b_3)| &\le q_1|a_1-b_1| + q_2|a_2-b_2| + q_3|a_3-b_3|, \end{aligned}$$

and

$$|Z(s,a_1,a_2,a_3)-Z(s,b_1,b_2,b_3)|\le r_1|a_1-b_1|+r_2|a_2-b_2|+r_3|a_3-b_3|.$$

In addition, suppose that

$$A_1(p_1+p_2+p_3)+A_2(q_1+q_2+q_3)+A_3(r_1+r_2+r_3)<1,$$

where A_1–A_3 are described in (3.1)–(3.3). Then there exists a unique solution for the BVP (1.1).

Proof. Consider

$$\begin{aligned}\sup_{s\in[0,1]} X(s,0,0,0) &= \chi_1<\infty,\\ \sup_{s\in[0,1]} Y(s,0,0,0) &= \chi_2<\infty,\end{aligned}$$

and

$$\sup_{s\in[0,1]} Z(s,0,0,0)=\chi_3<\infty,$$

so that

$$\xi\ge\frac{\chi_1A_1+\chi_2A_2+\chi_3A_3}{1-A_1(p_1+p_2+p_3)-A_2(q_1+q_2+q_3)-A_3(r_1+r_2+r_3)}.$$

Now, we shall show that $\Omega F_\xi\subset F_\xi$, where $F_\xi=\{(a,b,c)\in\Xi\times\Lambda\times\wp:\|(a,b,c)\|\le\xi\}$.
For $(a,b,c)\in F_\xi$, we get

$$\begin{aligned}&|\Omega_1(a,b,c)(s)|\\
\le\ &\frac{1}{\Gamma(\omega)}\int_0^s(s-\hbar)^{\omega-1}|X(\hbar,a(\hbar),b(\hbar),c(\hbar))|\,d\hbar\\
&+\frac{|\rho|\Gamma(1+e)}{|\Gamma(1+e)-\rho\eta^{\omega}|}\int_0^\eta\frac{(\eta-\hbar)^{e+\omega-1}}{\Gamma(\omega+e)}|X(\hbar,a(\hbar),b(\hbar),c(\hbar))|\,d\hbar\\
\le\ &\frac{1}{\Gamma(\omega)}\int_0^s(s-\hbar)^{\omega-1}(|X(\hbar,a(\hbar),b(\hbar),c(\hbar))|-|X(\hbar,0,0,0)|+|X(\hbar,0,0,0)|)\,d\hbar\\
&+\frac{|\rho|\Gamma(1+e)}{|\Gamma(1+e)-\rho\eta^{\omega}|}\int_0^\eta\frac{(\eta-\hbar)^{e+\omega-1}}{\Gamma(\omega+e)}(|X(\hbar,a(\hbar),b(\hbar),c(\hbar))|-|X(\hbar,0,0,0)|+|X(\hbar,0,0,0)|)\,d\hbar\\
\le\ &\left(\frac{1}{\Gamma(\omega+1)}+\frac{|\rho|\eta^{e+\omega}\Gamma(1+e)}{\Gamma(e+\omega+1)|\Gamma(1+e)-\rho\eta^{e}|}\right)(p_1\|a\|+p_2\|b\|+p_3\|c\|+\chi_1)\\
\le\ &A_1\left[(p_1+p_2+p_3)\xi+\chi_1\right].\end{aligned}$$

Hence

$$\|\Omega_1(a,b,c)(s)\|\le A_1\left[(p_1+p_2+p_3)\xi+\chi_1\right].$$

By the same manner, we can get

$$\|\Omega_2(a,b,c)(s)\|\le A_2\left[(q_1+q_2+q_3)\xi+\chi_2\right],$$

and

$$\|\Omega_3(a,b,c)(s)\|\le A_3\left[(r_1+r_2+r_3)\xi+\chi_3\right].$$

Thus, $\|\Omega(a,b,c)(s)\| \leq \xi$.

Finally, we show that the operator Ω is a contraction. Indeed for $(a_2,b_2,c_2),(a_1,b_1,c_1) \in \Xi \times \Lambda \times \wp$ and for any $s \in [0,1]$, we can write

$$\begin{aligned}
& |\Omega_1(a_2,b_2,c_2)(s) - \Omega_1(a_1,b_1,c_1)(s)| \\
\leq\ & \frac{1}{\Gamma(\omega)} \int_0^s (s-\hbar)^{\omega-1} |X(\hbar, a_2(\hbar), b_2(\hbar), c_2(\hbar)) - X(\hbar, a_1(\hbar), b_1(\hbar), c_1(\hbar))| \, d\hbar \\
& + \frac{|\rho|\Gamma(1+e)}{|\Gamma(1+e) - \rho\eta^{\omega}|} \int_0^{\eta} \frac{(\eta-\hbar)^{e+\omega-1}}{\Gamma(\omega+e)} |X(\hbar, a_2(\hbar), b_2(\hbar), c_2(\hbar)) - X(\hbar, a_1(\hbar), b_1(\hbar), c_1(\hbar))| \, d\hbar \\
\leq\ & \left(\frac{1}{\Gamma(\omega+1)} + \frac{|\rho|\eta^{e+\omega}\Gamma(1+e)}{\Gamma(e+\omega+1)|\Gamma(1+e) - \rho\eta^{e}|} \right) (p_1\|a_2 - a_1\| + p_2\|b_2 - b_1\| + p_3\|c_2 - c_1\|) \\
\leq\ & A_1 (p_1\|a_2 - a_1\| + p_2\|b_2 - b_1\| + p_3\|c_2 - c_1\|) \\
\leq\ & A_1 (p_1 + p_2 + p_3)(\|a_2 - a_1\| + \|b_2 - b_1\| + \|c_2 - c_1\|),
\end{aligned}$$

consequently, we get

$$\|\Omega_1(a_2,b_2,c_2) - \Omega_1(a_1,b_1,c_1)\| \leq A_1(p_1 + p_2 + p_3)(\|a_2 - a_1\| + \|b_2 - b_1\| + \|c_2 - c_1\|). \tag{3.8}$$

Analogously, we obtain

$$\|\Omega_2(a_2,b_2,c_2) - \Omega_2(a_1,b_1,c_1)\| \leq A_2(q_1 + q_2 + q_3)(\|a_2 - a_1\| + \|b_2 - b_1\| + \|c_2 - c_1\|), \tag{3.9}$$

and

$$\|\Omega_3(a_2,b_2,c_2) - \Omega_3(a_1,b_1,c_1)\| \leq A_3(r_1 + r_2 + r_3)(\|a_2 - a_1\| + \|b_2 - b_1\| + \|c_2 - c_1\|). \tag{3.10}$$

Inequalities (3.8)–(3.10) implies that

$$\begin{aligned}
\|\Omega(a_2,b_2,c_2) - \Omega(a_1,b_1,c_1)\| \leq\ & (A_1(p_1 + p_2 + p_3) + A_2(q_1 + q_2 + q_3) + A_3(r_1 + r_2 + r_3)) \\
& \times (\|a_2 - a_1\| + \|b_2 - b_1\| + \|c_2 - c_1\|).
\end{aligned}$$

Because

$$(A_1(p_1 + p_2 + p_3) + A_2(q_1 + q_2 + q_3) + A_3(r_1 + r_2 + r_3)) < 1,$$

then Ω is a contraction. So, according to Banach's contraction principle, there is a unique FP of the operator Ω, which is a unique solution of Problem (1.1). This complete the required. □

The example below support the theoretical results.

Example 3.4. Assume that the system of fractional BVP below:

$$\begin{cases}
{}^{c}D^{\frac{1}{3}}a(s) = \frac{1}{20(1+s)^2}\frac{|a(s)|}{1+|a(s)|} + 1 + \frac{1}{25}\cos b(s) + \frac{1}{30}\sin c(s), & s \in [0,1], \\
{}^{c}D^{\frac{1}{3}}b(s) = \frac{1}{30\pi}\cos\left(\frac{\pi}{2}a(s)\right) + \frac{1}{20}\sin b(s) + \frac{1}{32(1+s)^2}\frac{|c(s)|}{1+|a(s)|} + \frac{1}{2}, & s \in [0,1], \\
{}^{c}D^{\frac{1}{3}}c(s) = \frac{1}{20}\cos a(s) + \frac{1}{25(1+s)^2}\frac{|b(s)|}{1+|b(s)|} + \frac{1}{3} + \frac{1}{32\pi}\sin(2\pi c(s)), & s \in [0,1], \\
a(0) = \sqrt{5}I^{\frac{5}{2}}a(\frac{1}{3}),\ b(0) = \sqrt{3}I^{\frac{3}{2}}b(\frac{1}{2}),\ c(0) = \sqrt{2}I^{\frac{1}{2}}b(\frac{3}{4}).
\end{cases} \tag{3.11}$$

Here, $\omega = \varkappa = \varrho = \frac{1}{3}$, $\rho = \sqrt{5}$, $\sigma = \sqrt{3}$, $\varsigma\sqrt{2}$, $\eta = \frac{1}{3}$, $Z = \frac{1}{2}$, $\vartheta = \frac{3}{4}$, $e = \frac{5}{2}$, $f = \frac{3}{2}$, $g = \frac{1}{2}$,

$$\begin{aligned} X(s, a(s), b(s), c(s)) &= \frac{1}{20\,(1+s)^2}\frac{|a(s)|}{1+|a(s)|} + 1 + \frac{1}{25}\cos b(s) + \frac{1}{30}\sin c(s), \\ Y(s, a(s), b(s), c(s)) &= \frac{1}{30\pi}\cos\left(\frac{\pi}{2}a(s)\right) + \frac{1}{20}\sin b(s) + \frac{1}{32\,(1+s)^2}\frac{|c(s)|}{1+|a(s)|} + \frac{1}{2}, \end{aligned}$$

and

$$Z(s, a(s), b(s), c(s)) = \frac{1}{20}\cos a(s) + \frac{1}{25\,(1+s)^2}\frac{|b(s)|}{1+|b(s)|} + \frac{1}{3} + \frac{1}{32\pi}\sin(2\pi c(s)).$$

It should be noted that

$$\rho = \sqrt{5} \neq \frac{\Gamma(\frac{7}{2})}{\frac{\sqrt{3}}{27}} = \frac{\Gamma(\frac{5}{2}+1)}{\left(\frac{1}{3}\right)^{\frac{5}{2}}} = \frac{\Gamma(e+1)}{\eta^e},$$

$$\sigma = \sqrt{3} \neq \frac{\Gamma(\frac{5}{2})}{\frac{1}{2\sqrt{2}}} = \frac{\Gamma(\frac{5}{2})}{\left(\frac{1}{2}\right)^{\frac{3}{2}}} = \frac{\Gamma(f+1)}{Z^f},$$

and

$$\varsigma = \sqrt{2} \neq \frac{\Gamma(\frac{3}{2})}{\frac{\sqrt{3}}{2}} = \frac{\Gamma(1+\frac{1}{2})}{(\frac{3}{4})^{\frac{1}{2}}} = \frac{\Gamma(g+1)}{\vartheta^g}.$$

Furthermore,

$$\begin{aligned} &|X(s, a_1(s), a_2(s), a_3(s)) - X(s, b_2(s), b_2(s), b_2(s))| \\ \leq\ & \frac{1}{25}|a_1 - b_1| + \frac{1}{25}|a_2 - b_2| + \frac{1}{25}|a_3 - b_3|, \end{aligned}$$

$$\begin{aligned} &|Y(s, a_1(s), a_2(s), a_3(s)) - Y(s, b_2(s), b_2(s), b_2(s))| \\ \leq\ & \frac{1}{25}|a_1 - b_1| + \frac{1}{25}|a_2 - b_2| + \frac{1}{25}|a_3 - b_3|, \end{aligned}$$

and

$$\begin{aligned} &|Z(s, a_1(s), a_2(s), a_3(s)) - Z(s, b_2(s), b_2(s), b_2(s))| \\ \leq& \frac{1}{25}|a_1 - b_1| + \frac{1}{25}|a_2 - b_2| + \frac{1}{25}|a_3 - b_3|. \end{aligned}$$

Moreover,

$$\begin{aligned} & A_1(p_1 + p_2 + p_3) + A_2(q_1 + q_2 + q_3) + A_3(r_1 + r_2 + r_3) \\ \approx\ & \frac{3}{25}\left(\frac{3}{\Gamma\left(\frac{1}{3}\right)} + \frac{0.3305}{15.5379}\right) + \frac{3}{25}\left(\frac{3}{\Gamma\left(\frac{1}{3}\right)} + \frac{0.3305}{1.2364}\right) + \frac{3}{25}\left(\frac{3}{\Gamma\left(\frac{1}{3}\right)} + \frac{0.9861}{0.3185}\right) \\ \approx\ & 0.8093004 < 1. \end{aligned}$$

Thus, all requirements of Theorem 3.3 are fulfilled, hence Problem (3.11) has a unique solution.

4. Conclusions

Fractional derivatives do not take into account only local characteristics of the dynamics but considers the global evolution of the system; for that reason, when dealing with certain phenomena, they provide more accurate models of real-world behavior than standard derivatives. Nonlinear systems describing different phenomena can be modeled with fractional derivatives. Chaotic behavior has also been reported in some fractional models. There exist theoretical results related to existence and uniqueness of solutions to initial and boundary value problems with fractional differential equations; for the nonlinear case, there are still few of them. So, in this manuscript, we were able to study existence of a unique solution to a system of FDEs with nonlocal integral boundary conditions using Banach contraction principle. Ultimately, theoretical results were supported by an illustrative example. As a future work, our method can be applied to obtain existence of solutions for two fractional $q-$differential inclusions under some integral boundary value conditions as the work of [33, 34]. Moreover, the kernel can be taken as a singular one to solve partial integro-differential equations and to study Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel motivated by the work of [35–37]. In addition, we can replace Caputo fractional derivatives with conformable derivative functions to obtain a solution to fractional-order differential equations. These new investigations and applications would enhance the impact of the new setup.

Abbreviations

- FDEs Fractional derivative equations
- CFDs Caputo fractional derivatives
- FPT Fixed point technique
- BS Banach space
- R-L Riemann-Liouville
- BVP Boundary value problem

Availability of data and material

The data used to support the findings of this study are available from the corresponding author upon request.

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

References

1. G. S. Samko, B. Ross, Integration and differentiation to a variable fractional order, *Integr. Transf. Spec. F.*, **1** (2007), 277–300. https://doi.org/10.1080/10652469308819027
2. C. F. Lorenzo, T. T. Hartley, Variable order and distributed order fractional operators, *Nonlinear Dyn.*, **29** (2002), 57–98. https://doi.org/10.1023/A:1016586905654
3. X. Zheng, H. Wang, A hidden-memory variable-order time-fractional optimal control model: Analysis and approximation, *SIAM J. Control Optim.*, **59** (2021), 1851–1880. https://doi.org/10.1137/20M1344962
4. X. Zheng, H. Wang, An error estimate of a numerical approximation to a hidden-memory variable-order space-time fractional diffusion equation, *SIAM J. Numer. Anal.*, **58** (2020), 2492–2514. https://doi.org/10.1137/20M132420X
5. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, Elsevier, 2006.
6. V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of fractional dynamic systems*, Cambridge: Cambridge Scientific Publishers, 2009.
7. V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.: Theory Methods Appl.*, **69** (2008), 2677–2682. https://doi.org/10.1016/j.na.2007.08.042
8. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.
9. J. Sabatier, O. P. Agrawal, J. A. Tenreiro Machado, *Advances in fractional calculus*, Theoretical developments and applications in physics and engineering, Dordrecht: Springer, 2007. https://doi.org/10.1007/978-1-4020-6042-7
10. B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett.*, **23** (2010), 390–394. https://doi.org/10.1016/j.aml.2009.11.004
11. X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, *Appl. Math. Lett.*, **22** (2009), 64–69. https://doi.org/10.1016/j.aml.2008.03.001
12. J. Wang, H. Xiang, Z. Liu, Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations, *Int. J. Differ. Equ.*, **2010** (2010), 186928. https://doi.org/10.1155/2010/186928
13. D. Băleanu, O. G. Mustafa, R. P. Agarwal, An existence result for a superlinear fractional differential equation, *Appl. Math. Lett.*, **23** (2010), 1129–1132. https://doi.org/10.1016/j.aml.2010.04.049
14. J. Sun, Y. Liu, G. Liu, Existence of solutions for fractional differential systems with antiperiodic boundary conditions, *Comput. Math. Appl.*, **64** (2012), 1557–1566. https://doi.org/10.1016/j.camwa.2011.12.083
15. C. Bai, Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative, *J. Math. Anal. Appl.*, **384** (2011), 211–231. https://doi.org/10.1016/j.jmaa.2011.05.082

16. M. Cichoń, H. A. H. Salem, On the lack of equivalence between differential and integral forms of the Caputo-type fractional problems, *J. Pseudo-Differ. Oper. Appl.*, **11** (2020), 1869–1895. https://doi.org/10.1007/s11868-020-00345-z
17. A. Shah, R. A. Khan, A. Khan, H. Khan, J. F. Gómez-Aguilar, Investigation of a system of nonlinear fractional order hybrid differential equations under usual boundary conditions for existence of solution, *Math. Methods Appl. Sci.*, **44** (2020), 1628–1638. https://doi.org/10.1002/mma.6865
18. Kamran, G. Ali, J. F. Gómez-Aguilar, Approximation of partial integro differential equations with a weakly singular kernel using local meshless method, *Alex. Eng. J.*, **59** (2020), 2091–2100. https://doi.org/10.1016/j.aej.2020.01.010
19. H. Khan, J. F. Gómez-Aguilar, T. Abdeljwad, A. Khan, Existence results and stability criteria for ABC-fuzzy-Volterra integro-differential equation, *Fractals*, **28** (2020), 2040048. https://doi.org/10.1142/S0218348X20400484
20. O. Martínez-Fuentes, F. Meléndez-Vázquez, G. Fernández-Anaya, J. F. Gómez-Aguilar, Analysis of fractional-order nonlinear dynamic systems with general analytic kernels: Lyapunov stability and inequalities, *Mathematics*, **9** (2021), 2084. https://doi.org/10.3390/math9172084
21. Asma, J. F. Gómez-Aguilar, G. ur Rahman, M. Javed, Stability analysis for fractional order implicit Ψ-Hilfer differential equations, *Math. Methods Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7948
22. H. A. Hammad, H. Aydi, N. Mlaiki, Contributions of the fixed point technique to solve the 2D Volterra integral equations, Riemann-Liouville fractional integrals, and Atangana-Baleanu integral operators, *Adv. Differ. Equ.*, **2021** (2021), 97. https://doi.org/10.1186/s13662-021-03255-6
23. H. A. Hammad, M. De la Sen, Tripled fixed point techniques for solving system of tripled fractional differential equations, *AIMS Math.*, **6** (2020), 2330–2343. https://doi.org/10.3934/math.2021141
24. H. A. Hammad, H. Aydi, M. De la Sen, Solutions of fractional differential type equations by fixed point techniques for multi-valued contractions, *Complexity*, **2021** (2021), 5730853. https://doi.org/10.1155/2021/5730853
25. H. A. Hammad, W. Chaolamjiak, Solving singular coupled fractional differential equations with integral boundary constraints by coupled fixed point methodology, *AIMS Math.*, **6** (2021), 13370–13391. https://doi.org/10.3934/math.2021774
26. H. A. Hammad, P. Agarwal, S. Momani, F. Alsharari, Solving a fractional-order differential equation using rational symmetric contraction mappings, *Fractal Fract.*, **5** (2021), 159. https://doi.org/10.3390/fractalfract5040159
27. N. Fabiano, N. Nikolić, S. Thenmozhi, S. Radenović, N. Čıtaković, Tenth order boundary value problem solution existence by fixed point theorem, *J. Inequal. Appl.*, **2020** (2020), 166. https://doi.org/10.1186/s13660-020-02429-2
28. J. Leray, J. Schauder, Topologie et équations fonctionnelles, *Ann. Sci. É. N. S.*, **51** (1934), 45–78.
29. H. Ben-El-Mechaiekh, A. Idzik, A Leray-Schauder type theorem for approximable maps, *Proc. Amer. Math. Soc.*, **122** (1994), 105–109.
30. S. Park, Generalized Leray-Schauder principles for compact admissible multifuctions, *Topol. Methods Nonl. An.*, **5** (1995), 271–277.

31. A. Granas, J. Dugundji, *Fixed point theory*, New York: Springer, 2003. https://doi.org/10.1007/978-0-387-21593-8

32. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.*, **3** (1922), 133–181.

33. S. M. Aydojan, J. F. Gómez-Aguilar, D. Baleanu, S. Rezapour, M. E. Sami, Approximate endpoint solutions for a class of fractional q-differential inclusions by computational results, *Fractals*, **28** (2020), 2040029. https://doi.org/10.1142/S0218348X20400290

34. P. Pedi, A. Kumar, T. Abdeljwad, A. Khan, J. F. Gómez-Aguilar, Mild solutions of coupled hybrid fractional order system with Caputo-Hadamard derivatives, *Fractals*, **29** (2021), 2150158. https://doi.org/10.1142/S0218348X21501589

35. Kamran, G. Ali, J. F. Gómez-Aguilar, Approximation of partial integro differential equations with a weakly singular kernel using local meshless method, *Alex. Eng. J.*, **59** (2020), 2091–2100. https://doi.org/10.1016/j.aej.2020.01.010

36. A. Khan, H. Khan, J. F. Gómez-Aguilar, T. Abdeljwad, Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel, *Chaos Solitons Fractals*, **127** (2019), 422–427. https://doi.org/10.1016/j.chaos.2019.07.026

37. H. Khan, T. Abdeljwad, J. F. Gómez-Aguilar, H. Tajadodi, A. Khan, Fractional order Volterra integro-differential equation with Mittag-Leffler kernel, *Fractals*, **29** (2021), 2150154. https://doi.org/10.1142/S0218348X21501541