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Research article

Ricci curvature of semi-slant warped product submanifolds in generalized complex space forms

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Abstract: The objective of this paper is to achieve the inequality for Ricci curvature of a semi-slant warped product submanifold isometrically immersed in a generalized complex space form admitting a nearly Kaehler structure in the expressions of the squared norm of mean curvature vector and warping function. In addition, the equality case is likewise discussed. We provide numerous physical applications of the derived inequalities. Later, we proved that under a certain condition the base manifold $N_T^{n_1}$ is isometric to a n_1-dimensional sphere $S^{n_1}(\frac{\lambda_1}{n_1})$ with constant sectional curvature $\frac{\lambda_1}{n_1}$.

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1. Introduction

The realization of warped product manifolds came into existence after the approach of R. L. Bishop and B. O'Neill [7] on manifolds of negative curvature. Examining the fact that a Riemannian product of manifolds can not have negative curvature, they construct the model of warped product manifolds for the class of manifolds of negative (or non positive) curvature which is defined as follows:

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and ψ be a positive smooth function on N_1. If $\pi : N_1 \times N_2 \to N_1$ and $\eta : N_1 \times N_2 \to N_2$ are the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in N_1 \times N_2$, then the *warped product manifold* is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (\psi \circ \pi)^2 g_2(\eta_* X, \eta_* Y),$$

for all $X, Y \in TM$. The function ψ is called the *warping function* of the warped product manifold. If the warping function is constant, then the warped product is trivial, i.e., simply Riemannian product. On the grounds that warped product manifolds admit a number of applications in Physics and theory of relativity [5], this has been a topic of extensive research. Warped products provide many basic solutions to Einstein field equations [5]. The concept of modelling of space-time near black holes adopts the idea of warped product manifolds [19]. Schwartzschild space-time is an example of warped product $P \times_r S^2$, where the base $P = R \times R^+$ is a half plane $r > 0$ and the fibre S^2 is the unit sphere. Under certain conditions, the Schwartzchild space-time becomes the black hole. A cosmological model to model the universe as a space-time known as Robertson-Walker model is a warped product [30].

Some natural properties of warped product manifolds were studied in [7]. B. Y. Chen (see [9, 11]) performed an extrinsic study of warped product submanifolds in a Kaehler manifold. Since then, many geometers have explored warped product manifolds in different settings like almost complex and almost contact manifolds and various existence results have been investigated (see the survey article [13]).

In 1999, Chen [10] discovered a relationship between Ricci curvature and squared mean curvature vector for an arbitrary Riemannian manifold. On the line of Chen, a series of articles have been appeared to formulate the relationship between Ricci curvature and squared mean curvature in the setting of some important structures on Riemannian manifolds (see [3, 4, 16, 20–22, 26, 27, 34]). Recently Ali et al. [1] established a relationship between Ricci curvature and squared mean curvature for warped product submanifolds of a sphere and provide many physical applications.

In this paper our aim is to obtain a relationship between Ricci curvature and squared mean curvature for semi-slant warped product submanifolds in the setting of generalized complex space form admitting a nearly Kaehler structure. Further, we provide some applications in terms of Hamiltonians and Euler-Lagrange equation. In the last we also worked out some applications of Obata's differential equation.

2. Preliminaries

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and Riemannian metric g satisfying the following

$$J^2 = -I, \quad g(JX, JY) = g(X, Y), \tag{2.1}$$

for all vector fields X, Y on $\bar{M}$. If almost Hermitian manifold satisfies the following property

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0 \tag{2.2}$$

for all vector fields $X, Y \in T\bar{M}$, then $\bar{M}$ is called the nearly Kaehler manifold. The six dimensional sphere S^6 is an example of nearly Kaehler manifold which is not a Kaehler manifold. S^6 has an almost complex structure J defined by the vector cross product in the space of purely imaginary Cayley numbers which satisfies the tensorial equation of nearly Kaehler manifold. There is a more general class of almost Hermitian manifolds than nearly Kaehler manifold, this class is known as RK-manifold. A generalized complex space form is an RK-manifold of constant holomorphic sectional curvature c and of constant α and is denoted by $\bar{M}(c, \alpha)$. The sphere S^6 endowed with the standard nearly Kaehler structure is an example of generalized complex space form which is not a complex space form. The curvature tensor $\bar{R}$ of a generalized complex space form $\bar{M}(c, \alpha)$ is given by

$$\begin{aligned}\bar{R}(X,Y,Z,W) &= \frac{c+3\alpha}{4}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\ &+ \frac{c-\alpha}{4}[g(X,JZ)g(JY,W)-g(Y,JZ)g(JX,W) \\ &+ 2g(X,JY)g(JZ,W)],\end{aligned} \tag{2.3}$$

for any $X,Y,Z,W \in T\bar{M}$.

Let M be an n-dimensional Riemannian manifold isometrically immersed in a m-dimensional Riemannian manifold $\bar{M}$. Then the Gauss and Weingarten formulas are $\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$ respectively, for all $X, Y \in TM$ and $\xi \in T^\perp M$, where ∇ is the induced Levi-Civita connection on M, ξ is a vector field normal to M, h is the second fundamental form of M, $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_ξ is the shape operator of the second fundamental form. The second fundamental form h and the shape operator are associated by the following formula

$$g(h(X,Y),\xi) = g(A_\xi X, Y). \tag{2.4}$$

The equation of Gauss is given by

$$\begin{aligned}R(X,Y,Z,W) &= \bar{R}(X,Y,Z,W) + g(h(X,W),h(Y,Z)) \\ &- g(h(X,Z),h(Y,W)),\end{aligned} \tag{2.5}$$

for all $X,Y,Z,W \in TM$, where $\bar{R}$ and R are the curvature tensors of $\bar{M}$ and M respectively. For any $X \in TM$ and $N \in T^\perp M$, JX and JN can be decomposed as follows

$$JX = PX + FX \tag{2.6}$$

and

$$JN = tN + fN, \tag{2.7}$$

where PX (resp. tN) is the tangential and FX (resp. fN) is the normal component of JX (resp. JN).

For any orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of the tangent space T_xM, the mean curvature vector $H(x)$ and its squared norm are defined as follows

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i), \quad \|H\|^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} g(h(e_i,e_i),h(e_j,e_j)), \tag{2.8}$$

where n is the dimension of M. If $h = 0$ then the submanifold is said to be totally geodesic and minimal if $H = 0$. If $h(X,Y) = g(X,Y)H$ for all $X, Y \in TM$, then M is called totally umbilical.

The scalar curvature of $\bar{M}$ is denoted by $\bar{\tau}(\bar{M})$ and is defined as

$$\bar{\tau}(M^n) = \sum_{1\le p<q\le m} \bar{\kappa}_{pq}, \tag{2.9}$$

where $\bar{\kappa}_{pq} = \bar{\kappa}(e_p \wedge e_q)$ and m is the dimension of the Riemannian manifold $\bar{M}$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\bar{\tau}(M^n) = \sum_{1\leq p,q\leq m} \bar{\kappa}_{pq}. \tag{2.10}$$

In a similar way, the scalar curvature $\bar{\tau}(L_x)$ of a L-plane is given by

$$\bar{\tau}(L_x) = \sum_{1\leq p<q\leq m} \bar{\kappa}_{pq}. \tag{2.11}$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of the tangent space T_xM and if e_r belongs to the orthonormal basis $\{e_{n+1}, \cdots, e_m\}$ of the normal space $T^{\perp}M$, then we have

$$h^r_{pq} = g(h(e_p, e_q), e_r)) \tag{2.12}$$

and

$$||h||^2 = \sum_{p,q=1}^{n} g(h(e_p, e_q), h(e_p, e_q)). \tag{2.13}$$

Let κ_{pq} and $\bar{\kappa}_{pq}$ be the sectional curvatures of the plane sections spanned by e_p and e_q at x in the submanifold M^n and in the Riemannian space form $\bar{M}^m(c)$, respectively. Thus by Gauss equation, we have

$$\kappa_{pq} = \bar{\kappa}_{pq} + \sum_{r=n+1}^{m} (h^r_{pp}h^r_{qq} - (h^r_{pq})^2). \tag{2.14}$$

The global tensor field for orthonormal frame of vector field $\{e_1, \cdots, e_n\}$ on M^n is defined as

$$\bar{S}(X, Y) = \sum_{i=1}^{n}\{g(\bar{R}(e_i, X)Y, e_i)\}, \tag{2.15}$$

for all $X, Y \in T_xM^n$. The above tensor is called the Ricci tensor. If we fix a distinct vector e_u from $\{e_1, \cdots, e_n\}$ on M^n, which is governed by χ. Then the Ricci curvature is defined by

$$Ric(\chi) = \sum_{\substack{p=1\\ p\neq u}}^{n} K(e_p \wedge e_u). \tag{2.16}$$

Consider the warped product submanifold $N_1 \times_{\psi} N_2$. Let X be a vector field on M_1 and Z be a vector field on M_2, then from Lemma 7.3 of [7], we have

$$\nabla_X Z = \nabla_Z X = (\frac{X\psi}{\psi})Z \tag{2.17}$$

where ∇ is the Levi-Civita connection on M. For a warped product $M = M_1 \times_{\psi} M_2$ it is easy to observe that

$$\nabla_X Z = \nabla_Z X = (X \ln \psi)Z \tag{2.18}$$

for $X \in TM_1$ and $Z \in TM_2$.
$\nabla\psi$ is the gradient of ψ and is defined as

$$g(\nabla\psi, X) = X\psi, \tag{2.19}$$

for all $X \in TM$.

Let M be an n-dimensional Riemannian manifold with the Riemannian metric g and let $\{e_1, e_2, \cdots, e_n\}$ be an orthogonal basis of TM. Then as a result of (2.19), we get

$$||\nabla\psi||^2 = \sum_{i=1}^{n}(e_i(\psi))^2. \tag{2.20}$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{n}\{(\nabla_{e_i}e_i)\psi - e_ie_i\psi\}. \tag{2.21}$$

The Hessian tensor for a differentiable function ψ is a symmetric covariant tensor of rank 2 and is defined as

$$\Delta\psi = -trace\ H^{\psi}$$

For the warped product submanifolds, we have following well known result [14]

$$\sum_{p=1}^{n_1}\sum_{q=1}^{n_2}\kappa(e_p \wedge e_q) = \frac{n_2\Delta\psi}{\psi} = n_2(\Delta\ln\psi - ||\nabla\ln\psi||^2). \tag{2.22}$$

Now, we state the Hopf's Lemma.

Hopf's Lemma. [12] Let M be a m-dimensional connected compact Riemannian manifold. If ψ is a differentiable function on M such that $\Delta\psi \geq 0$ everywhere on M (or $\Delta\psi \leq 0$ everywhere on M), then ψ is a constant function.

For a compact orientable Riemannian manifold M with or without boundary and as a consequences of the integration theory of manifolds, we have

$$\int_M \Delta\psi\, dV = 0, \tag{2.23}$$

where ψ is a function on M and dV is the volume element of M.

3. Semi-slant warped product submanifolds of a nearly Kaehler manifold

The notion of semi-slant submanifolds of a Kaehler manifold is geometrically new and interesting. Infact, the study of differential geometry of semi-slant submanifolds as a generalization of CR-submanifolds and slant submanifolds of a Kaehlerian submanifolds was initiated by N. Papaghiuc [32]. In [23] V. A. Khan and M. A. Khan studied semi-slant submanifolds of a nearly Kaehler manifold and obtained some basic and interesting results. Further, B. Sahin [33] proved the non existence proper semi-slant warped product submanifolds in the setting of Kaehler manifold. So, it was natural to see the existence of semi-slant warped product submanifolds in a more general setting namely nearly Kaehler manifold and in this series V. A. Khan and K. A. Khan [24] studied different types of warped product submanifolds in nearly Kaehler manifolds. Suppose, N_T and N_θ be the holomorphic and slant submanifolds of an almost Hermitian manifold $\bar{M}$. Now, there are two possibilities of warped product submanifolds of $\bar{M}$, these warped products are $N_\theta \times_\psi N_T$ and $N_T \times_\psi N_\theta$. In [24] V. A. Khan and K. A.

Khan proved the non-existence of the first type of warped product $N_\theta \times_\psi N_T$ in nearly Kaehler manifolds and they studied the existence of the warped product $N_T \times_\psi N_\theta$, these warped product submanifolds are called semi-slant warped product submanifolds and studied extensively (see [2, 24, 25]). Throughout this study, we consider the warped product submanifolds $M = N_T^{n_1} \times_\psi N_\theta^{n_2}$ of a nearly Kaehler manifold, where n_1 and n_2 are the dimensions of the holomorphic and slant submanifolds.

Now, we have the following initial result:

Lemma 3.1. *Let $M = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a semi-slant warped product submanifold isometrically immersed in a nearly Kaehler manifold $\bar{M}$. Then*

(i) g(h(X, Y), JZ) = 0,
(ii) g(h(JX, JX), N) = - g(h(X, X), N),

for any $X, Y \in TN_T$, $Z \in TN_\theta$ and $N \in \mu$, where μ is the invariant subbundle of $T^\perp M$.

Proof. By using Gauss and Weingarten formulae in Eq (2.2), we have

$$\nabla_X PZ + h(X, PZ) - A_{FZ}X + \nabla^\perp_X FZ - J\nabla_X Z - Jh(X, Z) + \nabla_Z JX +$$

$$+h(JX, Z) - J\nabla_Z X - Jh(X, Z) = 0,$$

taking inner product with Y and using (2.4), we get the required result.
To prove (*ii*), for any $X \in TN_T$ we have

$$\bar{\nabla}_X JX = (\bar{\nabla}_X J)X + J\bar{\nabla}_X X.$$

Using Gauss formula and (2.2) in above, we get

$$\nabla_X JX + h(JX, X) = J\nabla_X X + Jh(X, X).$$

Taking inner product with JN, above equation yields

$$g(h(JX, X), JN) = g(h(X, X), N). \tag{3.1}$$

Interchanging X by JX the above equation gives

$$g(h(JX, X), JN) = -g(h(JX, JX), N). \tag{3.2}$$

From (3.1) and (3.2), we get the required result. □

From the above result it is evident that the isometric immersion $N_T^{n_1} \times_\psi N_\theta^{n_2}$ into a nearly Kaehler manifold is D^T-minimal. The D^T- minimality property provides us a useful relationship between the semi-slant warped product submanifold $N_T \times_\psi N_\theta$ and the equation of Gauss.

Definition 3.1 The warped product $N_1 \times_\psi N_2$ isometrically immersed in a Riemannian manifold $\bar{M}$ is called N_i totally geodesic if the partial second fundamental form h_i vanishes identically. It is called N_i-minimal if the partial mean curvature vector H^i becomes zero for $i = 1, 2$.

Let $\{e_1, \cdots, e_p, e_{p+1} = Je_1, \cdots, e_{n_1} = Je_p, e_{n_1+1} = e^1, \cdots, e_{n_1+q} = e^q, e_{n_1+q+1} = e^{q+1} = \sec\theta Pe^1, \cdots, e_{(n_2=2q)} = e^{n_2} = \sec\theta Pe^q\}$ be a local orthonormal frame of vector fields on the semi-slant warped product submanifold $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ such that the set $\{e_1, \cdots, e_p, e_{p+1} = Je_1, \cdots, e_{n_1} = Je_p\}$

is tangent to N_T and the set $\{e^1, \cdots, e^q, \cdots e^{n_2}\}$ is tangent to N_θ. Moreover, $\{e_{n+1} = \csc\theta F e^1, \cdots, e_{n+n_2} = \csc\theta F e^q, e_{n+n_2+1} = \bar{e}^1, \cdots, e_m = \bar{e}^k\}$ is a basis for the normal bundle $T^\perp M$, such that the set $\{e_{n+1} = \csc\theta F e^1, \cdots, e_{n+n_2} = \csc\theta F e^q\}$ is tangent to FD^θ and $\{\bar{e}^1, \cdots, \bar{e}^k\}$ is tangent to the complementary invariant subbundle μ with even dimension k.

From Lemma 3.1, it is easy to conclude that

$$\sum_{r=n+1}^{m}\sum_{i,j=1}^{n_1} g(h(e_i, e_j), Je_r) = 0. \tag{3.3}$$

Thus it follows that the trace of h due to N_T becomes zero. Hence in view of the Definition 3.1, we obtain the following important result.

Theorem 3.1. *Let $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a semi-slant warped product submanifold isometrically immersed in a nearly Kaehler manifold. Then M^n is D^T-minimal.*

So, it is easy to conclude the following

$$||H||^2 = \frac{1}{n^2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1} + \cdots + h^r_{nn})^2, \tag{3.4}$$

where $||H||^2$ is the squared mean curvature.

4. Ricci curvature for semi-slant warped product submanifold

In this section, we investigate Ricci curvature in terms of the squared norm of mean curvature and the warping function as follows:

Theorem 4.1. *Let $M = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a semi-slant warped product submanifold isometrically immersed in a generalized complex space form $\bar{M}(c, \alpha)$ admitting nearly Kaehler structure. Then for each orthogonal unit vector field $\chi \in T_xM$, either tangent to N_T or N_θ, we have*

(1) The Ricci curvature satisfy the following inequality

(i) If χ is tangent to $N_T^{n_1}$, then

$$Ric(\chi) \leq \frac{1}{4}n^2||H||^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n + n_1n_2 - 1) + \frac{3(c-\alpha)}{8}. \tag{4.1}$$

(ii) If χ is tangent to $N_\theta^{n_2}$, then

$$Ric(\chi) \leq \frac{1}{4}n^2||H||^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n + n_1n_2 - 1) + \frac{3(c-\alpha)}{8}\cos^2\theta. \tag{4.2}$$

(2) If $H(x) = 0$, then for each point $x \in M^n$ there is a unit vector field X which satisfies the equality case of (1) if and only if M^n is mixed totally geodesic and χ lies in the relative null space N_x at x.

(3) For the equality case we have

(a) The equality case of (4.1) holds identically for all unit vector fields tangent to $N_T^{n_1}$ at each $x \in M^n$ if and only if M^n is mixed totally geodesic and D^T-totally geodesic semi-slant warped product submanifold in $\bar{M}^m(c, \alpha)$.

(b) *The equality case of (4.2) holds identically for all unit vector fields tangent to $N_\theta^{n_2}$ at each $x \in M^n$ if and only if M is mixed totally geodesic and either M^n is D^θ- totally geodesic semi-slant warped product or M^n is a D^θ totally umbilical in $\bar{M}^m(c,\alpha)$ with dim $D^\theta = 2$.*

(c) *The equality case of (1) holds identically for all unit tangent vectors to M^n at each $x \in M^n$ if and only if either M^n is totally geodesic submanifold or M^n is a mixed totally geodesic totally umbilical and D^T-totally geodesic submanifold with dim $N_\theta^{n_2} = 2$.*

where n_1 and n_2 are the dimensions of N_T and N_θ respectively.

Proof. Suppose that $M = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a semi-slant warped product submanifold of a generalized complex space form. From Gauss equation, we have

$$n^2\|H\|^2 = 2\tau(M^n) + \|h\|^2 - 2\bar{\tau}(M^n). \tag{4.3}$$

Let $\{e_1, \cdots, e_{n_1}, e_{n_1+1}, \cdots, e_n\}$ be a local orthonormal frame of vector fields on M^n such that $\{e_1, \cdots, e_{n_1}\}$ are tangent to N_T and $\{e_{n_1+1}, \cdots, e_n\}$ are tangent to N_θ. So, the unit tangent vector $\chi = e_A \in \{e_1, \cdots, e_n\}$ can be expanded (4.3) as follows

$$\begin{aligned} n^2\|H\|^2 &= 2\tau(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \cdots + h_{nn}^r - h_{AA}^r)^2 + (h_{AA}^r)^2\} - \sum_{r=n+1}^{m}\sum_{1\le p\ne q\le n} h_{pp}^r h_{qq}^r \\ &- 2\bar{\tau}(M^n). \end{aligned} \tag{4.4}$$

The above expression can be written as follows

$$\begin{aligned} n^2\|H\|^2 = 2\tau(M^n) &+ \frac{1}{2}\sum_{r=n+1}^{m}\{(h_{11}^r + \cdots + h_{nn}^r)^2 + (2h_{AA}^r - (h_{11}^r + \cdots + h_{nn}^r))^2\} \\ &+ 2\sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(h_{pq}^r)^2 - 2\sum_{r=n+1}^{m}\sum_{1\le p<q\le n} h_{pp}^r h_{qq}^r - 2\bar{\tau}(M^n). \end{aligned}$$

In view of the Lemma 3.1, the preceding expression takes the form

$$\begin{aligned} n^2\|H\|^2 = &\sum_{r=n+1}^{m}\{(h_{n_1+1 n_1+1}^r + \cdots + h_{nn}^r)^2 + +(2h_{AA}^r - (h_{n_1+1 n_1+1}^r + \cdots + h_{nn}^r))^2\} \\ &+ 2\tau(M^n) + \sum_{r=n+1}^{m}\sum_{1\le p<q\le n}(h_{pq}^r)^2 - \sum_{r=n+1}^{m}\sum_{1\le p<q\le n} h_{pp}^r h_{qq}^r + \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\ne A}}(h_{aA}^r)^2 \\ &+ \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}}(h_{pq}^r)^2 - \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}} h_{pp}^r h_{qq}^r - 2\bar{\tau}(M^n). \end{aligned} \tag{4.5}$$

By Eq (2.14), we have

$$\sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}}(h_{pq}^r)^2 - \sum_{r=n+1}^{m}\sum_{\substack{1\le p<q\le n\\ p,q\ne A}} h_{pp}^r h_{qq}^r = \sum_{\substack{1\le p<q\le n\\ p,q\ne A}} \bar{\kappa}_{pq} - \sum_{\substack{1\le p<q\le n\\ p,q\ne A}} \kappa_{pq} \tag{4.6}$$

Substituting the values of Eq (4.6) in (4.5), we discover

$$\begin{aligned} n^2\|H\|^2 =& 2\tau(M^n) + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2 + \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(h^r_{pq})^2 \\ & - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n} h^r_{pp}h^r_{qq} - 2\bar{\tau}(M^n) + \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\neq A}}(h^r_{aA})^2 + \sum_{\substack{1\leq p<q\leq n\\ p,q\neq A}}\bar{\kappa}_{p,q} - \sum_{\substack{1\leq p<q\leq n\\ p,q\neq A}}\kappa_{pq}. \end{aligned} \tag{4.7}$$

Since, $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$, then from (2.11), the scalar curvature of M^n can be defined as follows

$$\tau(M^n) = \sum_{1\leq p<q\leq n}\kappa(e_p\wedge e_q) = \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n}\kappa(e_i\wedge e_j) + \sum_{1\leq r<k\leq n_1}\kappa(e_r\wedge e_k) + \sum_{n_1+1\leq l<o\leq n}\kappa(e_l\wedge e_o) \tag{4.8}$$

The usage of (2.11) and (2.22), we derive

$$\tau(M^n) = \frac{n_2\Delta\psi}{\psi} + \tau(N_T^{n_1}) + \tau(N_\theta^{n_2}) \tag{4.9}$$

Utilizing (4.9) together with (2.14) and (2.3) in (4.7), we have

$$\begin{aligned} \frac{1}{2}n^2\|H\|^2 =& \frac{n_2\Delta\psi}{\psi} + \sum_{\substack{1\leq p<q\leq n\\ p,q\neq A}}\bar{\kappa}_{p,q} + \bar{\tau}(N_T^{n_1}) + \bar{\tau}(N_\theta^{n_2}) + \sum_{r=n+1}^{m}\{\sum_{1\leq p<q\leq n}(h^r_{pq})^2 - \sum_{\substack{1\leq p<q\leq n\\ p,q\neq A}}h^r_{pp}h^r_{qq}\} \\ & + \sum_{r=n+1}^{m}\sum_{\substack{a=1\\ a\neq A}}(h^r_{aA})^2 + \sum_{r=n+1}^{m}\sum_{1\leq i\neq j\leq n_1}(h^r_{ii}h^r_{jj} - (h^r_{ij})^2) + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s\neq t\leq n}(h^r_{ss}h^r_{tt} - (h^r_{st})^2) \\ & + \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{AA} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2 - \frac{c+3\alpha}{4}(n(n-1)) - \frac{(c-\alpha)}{4}(3n_1 + 3n_2\cos^2\theta). \end{aligned} \tag{4.10}$$

Considering unit tangent vector $\chi = e_a$, we have two choices: χ is either tangent to the base manifold $N_T^{n_1}$ or to the fibre $N_\theta^{n_2}$.
Case i: If e_a is tangent to $N_T^{n_1}$, then fix a unit tangent vector from $\{e_1, \cdots, e_{n_1}\}$ and suppose $\chi = e_a = e_1$. Then from (4.10) and (2.16), we find

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} - \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \cdots h^r_{nn}))^2 \\ & - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n_1}(h^r_{pq})^2 + \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2 - \sum_{1\leq i<j\leq n_1}h^r_{ii}h^r_{jj}] \\ & + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h^r_{st})^2 + \sum_{r=n+1}^{m}[\sum_{n_1+1\leq s<t\leq n}(h^r_{ij})^2 - \sum_{n_1+1\leq s<t\leq n}h^r_{ss}h^r_{tt}] \\ & + \sum_{r=n+1}^{m}\sum_{2\leq p<q\leq n}h^r_{pp}h^r_{qq} + \frac{c+3\alpha}{4}(n(n-1)) + \frac{(c-\alpha)}{4}(3n_1 \\ & + 3n_2\cos^2\theta) - \sum_{2\leq p<q\leq n}\bar{\kappa}_{pq} - \bar{\tau}(N_T^{n_1}) - \bar{\tau}(N_\theta^{n_2}). \end{aligned} \tag{4.11}$$

From (2.3), (2.11) and (2.12), we have

$$\sum_{2\leq p<q\leq n} \bar{\kappa}_{p,q} = \frac{c+3\alpha}{8}(n-1)(n-2) + \frac{c-\alpha}{8}[3(n_1-1)+3n_2\cos^2\theta], \tag{4.12}$$

$$\bar{\tau}(N_T^{n_1}) = \frac{c+3\alpha}{8}n_1(n_1-1) + \frac{c-\alpha}{8}3n_1, \tag{4.13}$$

$$\bar{\tau}(N_\theta^{n_2}) = \frac{c+3\alpha}{8}n_2(n_2-1) + \frac{c-\alpha}{8}3n_2\cos^2\theta. \tag{4.14}$$

Using (4.19)–(4.21) in (4.11), we have

$$\begin{aligned} Ric(\chi) \leq & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n+n_1n_2-1) + \frac{3(c-\alpha)}{8} - \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11} - (h^r_{n_1+1n_1+1} + \cdots \\ & + h^r_{nn}))^2 - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(h^r_{pq})^2 + \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2 + \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h^r_{st})^2] \\ & - \sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1} h^r_{ii}h^r_{jj} + \sum_{n_1+1\leq s<t\leq n} h^r_{ss}h^r_{tt}] + \sum_{r=n+1}^{m}\sum_{2\leq p<q\leq n} h^r_{pp}h^r_{qq}. \end{aligned} \tag{4.15}$$

Further, the sixth and seventh terms on right hand side of the last inequality can be written as

$$\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h^r_{ij})^2 + \sum_{n_1+1\leq s<t\leq n}(h^r_{st})^2] - \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n}(h^r_{pq})^2 = -\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2.$$

Similarly, we have

$$\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1} h^r_{ii}h^r_{jj} + \sum_{n_1+1\leq s\neq t\leq n} h^r_{ss}h^r_{tt} - \sum_{2\leq p<q\leq n} h^r_{pp}h^r_{qq}] = \sum_{r=n+1}^{m}[\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pp}h^r_{qq} - \sum_{j=2}^{n_1} h^r_{11}h^r_{jj}].$$

Utilizing above two values in (4.15), we get

$$\begin{aligned} Ric(\chi) \ \leq \ & \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n+n_1n_2-1) + \frac{3(c-\alpha)}{8} - \frac{1}{2}\sum_{r=n+1}^{m}(2h^r_{11} \\ - \ & (h^r_{n_1+1n_1+1} + \cdots h^r_{nn}))^2 - \sum_{r=n+1}^{m}\left[\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2 + \sum_{b=2}^{n_1} h^r_{11}h^r_{bb} - \sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pp}h^r_{qq}\right]. \end{aligned} \tag{4.16}$$

Since $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ is $N_T^{n_1}$-minimal then we can observe the following

$$\sum_{r=n+1}^{m}\sum_{p=2}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pp}h^r_{qq} = -\sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n} h^r_{11}h^r_{qq} \tag{4.17}$$

and

$$\sum_{r=n+1}^{m}\sum_{b=2}^{n_1} h^r_{11}h^r_{bb} = -\sum_{r=n+1}^{m}(h^r_{11})^2. \tag{4.18}$$

Simultaneously, we can conclude

$$\frac{1}{2}\sum_{r=n+1}^{m}(2h_{11}^r-(h_{n_1+1n_1+1}^r+\cdots+h_{nn}^r))^2 \quad + \quad \sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n}h_{11}^rh_{qq}^r=2\sum_{r=n+1}^{m}(h_{11}^r)^2+\frac{1}{2}n^2\|H\|^2. \tag{4.19}$$

Using (4.17) and (4.18) in (4.16), after the assessment of (4.19), we finally get

$$\begin{aligned} Ric(\chi) \quad &\leq \quad \frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta\psi}{\psi}+\frac{c+3\alpha}{4}(n+n_1n_2-1)+\frac{3(c-\alpha)}{8} \\ &- \quad \frac{1}{4}\sum_{r=n+1}^{m}\sum_{q=n_1+1}^{n}(h_{qq}^r)^2-\sum_{r=n+1}^{m}\{(h_{11}^r)^2-\sum_{q=n_1+1}^{n}h_{11}^rh_{qq}^r+\frac{1}{4}(h_{n_1+1n_1+1}^r+\cdots+h_{nn}^r)^2\}. \end{aligned} \tag{4.20}$$

Further, using the fact that $\sum_{r=n+1}^{m}(h_{n_1+1n_1+1}^r+\cdots+h_{nn}^r)=n^2\|H\|^2$, we get

$$\begin{aligned} Ric(\chi) \quad &\leq \quad \frac{1}{4}n^2\|H\|^2-\frac{n_2\Delta\psi}{\psi}+\frac{c+3\alpha}{4}(n+n_1n_2-1)+\frac{3(c-\alpha)}{8} \\ &- \quad \frac{1}{4}\sum_{r=n+1}^{m}(2h_{11}^r-\sum_{q=n_1+1}^{n}h_{qq}^r)^2. \end{aligned} \tag{4.21}$$

From the above inequality, we can conclude the inequality (4.1).

Case ii: If e_a is tangent to $N_\theta^{n_2}$, then we choose the unit vector from $\{e_{n_1+1},\cdots,e_n\}$, suppose that the unit vector is e_n, i.e. $\chi=e_n$. Then from (2.3), (2.11) and (2.12), we have

$$\sum_{1\leq p<q\leq n-1}\bar{\kappa}_{pq}=\frac{c+3\alpha}{8}(n-1)(n-2)-\frac{c-\alpha}{8}(3n_1+3(n_2-1)\cos^2\theta), \tag{4.22}$$

$$\bar{\tau}(N_T^{n_1})=\frac{c+3\alpha}{8}n_1(n_1-1)+\frac{c-\alpha}{8}(3n_1), \tag{4.23}$$

$$\bar{\tau}(N_\theta^{n_2})=\frac{c+3\alpha}{8}n_2(n_2-1)+\frac{c-\alpha}{8}(3n_2\cos^2\theta). \tag{4.24}$$

Now, in a similar way as in Case *i*, using (4.22)–(4.24), we have

$$\begin{aligned} Ric(\chi) \quad &\leq \quad \frac{1}{2}n^2\|H\|^2-\frac{n_2\Delta\psi}{\psi}-\frac{1}{2}\sum_{r=n+1}^{m}((h_{n_1+1n_1+1}^r+\cdots h_{nn}^r)-2h_{nn}^r)^2 \\ &- \quad \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n_1}(h_{pq}^r)^2+\sum_{r=n+1}^{m}[\sum_{1\leq i<j\leq n_1}(h_{ij}^r)^2-\sum_{1\leq i<j\leq n_1}h_{ii}^rh_{jj}^r] \\ &+ \quad \sum_{r=n+1}^{m}\sum_{n_1+1\leq s<t\leq n}(h_{st}^r)^2+\sum_{r=n+1}^{m}[\sum_{n_1+1\leq s<t\leq n}(h_{ij}^r)^2-\sum_{n_1+1\leq s<t\leq n}h_{ss}^rh_{tt}^r] \\ &+ \quad \sum_{r=n+1}^{m}\sum_{1\leq p<q\leq n-1}h_{pp}^rh_{qq}^r+\frac{c+3\alpha}{4}(n+n_1n_2-1)+\frac{3(c-\alpha)}{8}\cos^2\theta. \end{aligned} \tag{4.25}$$

Using similar steps of Case i, the above inequality takes the form

$$\begin{aligned} Ric(\chi) \leq& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n+n_1n_2-1) + \frac{3(c-\alpha)}{8}\cos^2\theta - \frac{1}{2}\sum_{r=n+1}^{m}\{(h^r_{n_1+1n_1+1}+\cdots \\ &+ h^r_{nn}) - 2h^r_{nn}\}^2 - \sum_{r=n+1}^{m}[\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2 + \sum_{b=n_1+1}^{n-1}h^r_{nn}h^r_{bb} - \sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n-1}h^r_{pp}h^r_{qq}]. \end{aligned} \tag{4.26}$$

By the Lemma 3.1, one can observe that

$$\sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n-1}h^r_{pp}h^r_{qq} = 0. \tag{4.27}$$

Utilizing this in (4.26), we get

$$\begin{aligned} Ric(\chi) \leq& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n+n_1n_2-1) + \frac{3(c-\alpha)}{8}\cos^2\theta \\ &- \frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1}+\cdots h^r_{nn}) - 2h^r_{nn})^2 - \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n-1}h^r_{nn}h^r_{bb}. \end{aligned} \tag{4.28}$$

The last term of the above inequality can be written as

$$-\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n-1}h^r_{nn}h^r_{bb} = -\sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n}h^r_{nn}h^r_{bb} + \sum_{r=n+1}^{m}(h^r_{nn})^2$$

Moreover, the fifth term of (4.28) can be expanded as

$$\begin{aligned} -\frac{1}{2}\sum_{r=n+1}^{m}((h^r_{n_1+1n_1+1}+\cdots+h^r_{nn}) - 2h^r_{nn})^2 =& -\frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\cdots+h^r_{nn})^2 \\ &- 2\sum_{r=n+1}^{m}(h^r_{nn})^2 + \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj}. \end{aligned} \tag{4.29}$$

Using last two values in (4.28), we have

$$\begin{aligned} Ric(\chi) \leq& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n+n_1n_2-1) + \frac{3(c-\alpha)}{8}\cos^2\theta \\ &- \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\cdots h^r_{nn})^2 - 2\sum_{r=n+1}^{m}(h^r_{nn})^2 + 2\sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj} \\ &- \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2 - \sum_{r=n+1}^{m}\sum_{b=n_1+1}^{n}h^r_{nn}h^r_{bb} + \sum_{r=n+1}^{m}(h^r_{nn})^2, \end{aligned} \tag{4.30}$$

or equivalently

$$\begin{aligned} Ric(\chi) \leq& \frac{1}{2}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n+n_1n_2-1) + \frac{3(c-\alpha)}{8}\cos^2\theta \\ &- \frac{1}{2}\sum_{r=n+1}^{m}(h^r_{n_1+1n_1+1}+\cdots h^r_{nn})^2 - \sum_{r=n+1}^{m}(h^r_{nn})^2 + \sum_{r=n+1}^{m}\sum_{j=n_1+1}^{n}h^r_{nn}h^r_{jj} - \sum_{r=n+1}^{m}\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n}(h^r_{pq})^2. \end{aligned} \tag{4.31}$$

On applying similar techniques as in the proof of Case i, we arrive

$$Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 - \frac{n_2\Delta\psi}{\psi} + \frac{c+3\alpha}{4}(n + n_1n_2 - 1) + \frac{3(c-\alpha)}{8}\cos^2\theta - \frac{1}{4}\sum_{r=n+1}^{m}(2h^r_{nn} - (h^r_{n_1+1n_1+1} + \cdots + h^r_{nn}))^2, \tag{4.32}$$

which gives the inequality (4.2).
Next, we explore the equality cases of the inequality (4.1). First, we redefine the notion of the relative null space $\mathcal{N}_x$ of the submanifold M^n in the generalized complex space form $\bar{M}^m(c,\alpha)$ at any point $x \in M^n$, the relative null space was defined by B. Y. Chen [10], as follows

$$\mathcal{N}_x = \{X \in T_xM^n : h(X,Y) = 0,\ \forall\ Y \in T_xM^n\}.$$

For $A \in \{1,\cdots,n\}$ a unit vector field e_A tangent to M^n at x satisfies the equality sign of (4.1) identically if and only if

$$(i)\ \sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ (ii)\ \sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n} h^r_{bA} = 0\ (iii)\ 2h^r_{AA} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.33}$$

such that $r \in \{n+1,\cdots m\}$ the condition (i) implies that M^n is mixed totally geodesic semi-slant warped product submanifold. Combining statements (ii) and (iii) with the fact that M^n is semi-slant warped product submanifold, we get that the unit vector field $\chi = e_A$ belongs to the relative null space N_x. The converse is trivial, this proves statement (2).

For a semi-slant warped product submanifold, the equality sign of (4.1) holds identically for all unit tangent vector belong to $N_T^{n_1}$ at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ \ (ii)\sum_{b=1}^{n}\sum_{\substack{A=1\\ b\neq A}}^{n_1} h^r_{bA} = 0\ \ (iii)\ 2h^r_{pp} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.34}$$

where $p \in \{1,\cdots,n_1\}$ and $r \in \{n+1,\cdots,m\}$. Since M^n is semi-slant warped product submanifold, the third condition implies that $h^r_{pp} = 0,\ \ p \in \{1,\cdots,n_1\}$. Using this in the condition (ii), we conclude that M^n is D^T-totally geodesic semi-slant warped product submanifold in $\bar{M}^m(c,\alpha)$ and mixed totally geodesicness follows from the condition (i), which proves (a) in the statement (3).

For a semi-slant warped product submanifold, the equality sign of (4.2) holds identically for all unit tangent vector fields tangent to $N_\theta^{n_2}$ at x if and only if

$$(i)\sum_{p=1}^{n_1}\sum_{q=n_1+1}^{n} h^r_{pq} = 0\ (ii)\sum_{b=1}^{n}\sum_{\substack{A=n_1+1\\ b\neq A}}^{n} h^r_{bA} = 0\ (iii)\ 2h^r_{KK} = \sum_{q=n_1+1}^{n} h^r_{qq}, \tag{4.35}$$

such that $K \in \{n_1+1,\cdots,n\}$ and $r \in \{n+1,\cdots,m\}$. From the condition (iii) two cases emerge, that is

$$h^r_{KK} = 0,\ \ \forall K \in \{n_1+1,\cdots,n\}\ \text{ and }\ r \in \{n+1,\cdots,m\}\ \text{ or }\ dim\, N_\theta^{n_2} = 2. \tag{4.36}$$

If the first case of (4.35) satisfies, then by virtue of condition (*ii*), it is easy to conclude that M^n is a D^θ-totally geodesic semi-slant warped product submanifold in $\bar{M}^m(c,\alpha)$. This is the first case of part (*b*) of statement (3).

For the other case, assume that M^n is not D^θ-totally geodesic semi-slant warped product submanifold and $\dim N_\theta^{n_2} = 2$. Then condition (*ii*) of (4.35) implies that M^n is D^θ-totally umbilical semi-slant warped product submanifold in $\bar{M}(c,\alpha)$, which is second case of this part. This verifies part (*b*) of (3).

To prove (*c*) using parts (*a*) and (*b*) of (3), we combine (4.34) and (4.35). For the first case of this part, assume that $dimN_\theta^{n_2} \neq 2$. Since from parts (*a*) and (*b*) of statement (3) we conclude that M^n is D^T-totally geodesic and D^θ-totally geodesic submanifold in $\bar{M}^m(c,\alpha)$. Hence M^n is a totally geodesic submanifold in $\bar{M}^m(c,\alpha)$.

For another case, suppose that first case does not satisfy. Then parts (*a*) and (*b*) provide that M^n is mixed totally geodesic and D^T-totally geodesic submanifold of $\bar{M}^m(c,\alpha)$ with $dimN_\theta^{n_2} = 2$. From the condition (*b*) it follows that M^n is D^θ-totally umbilical semi-slant warped product submanifold and from (*a*) it is D^T-totally geodesic, which is part (*c*). This proves the theorem. □

In view of (2.22), we have another version of the Theorem 4.1 as follows:

Theorem 4.2. *Let $M = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a semi-slant warped product submanifold isometrically immersed in a generalized complex space form $\bar{M}(c,\alpha)$ admitting nearly Kaehler structure. Then for each orthogonal unit vector field $\chi \in T_xM$, either tangent to N_T or N_θ, then the Ricci curvature satisfy the following inequalities:*

(i) If χ is tangent to N_T, then

$$Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 - n_2\Delta \ln\psi + n_2\|\nabla \ln\psi\|^2 + \frac{c+3\alpha}{4}(n + n_1n_2 - 1) + \frac{3(c-\alpha)}{8}. \tag{4.37}$$

(ii) If χ is tangent to N_θ, then

$$Ric(\chi) \leq \frac{1}{4}n^2\|H\|^2 - n_2\Delta \ln\psi + n_2\|\nabla \ln\psi\|^2 + \frac{c+3\alpha}{4}(n + n_1n_2 - 1) + \frac{3(c-\alpha)}{8}\cos^2\theta. \tag{4.38}$$

The equality cases are similar as Theorem 4.1.

Since, CR-warped product submanifolds are semi-slant submanifolds with the slant angle $\theta = \frac{\pi}{2}$. Therefore, as an example of CR-warped product submanifold, we compile some result of [18] as follows.

Example 4.1. *Let $\{e_0, e_i(1 \leq i \leq 7)\}$ be the canonical basis of Cayley division algebra on R^8 over R, and R^7 is the subspace of R^8 generated by the purely imaginary Cayley numbers $e_i(1 \leq i \leq 7)$. Then*

$$S^6 = \{x_1e_1 + x_2e_2 + \cdots + x_7e_7 : x_1^2 + x_2^2 + \cdots + x_7^2 = 1\}$$

is an unit sphere admitting nearly Kaehler structure $(\bar{\nabla}, J, g)$. Now suppose that $S^2 = \{x = (x_2, x_4, x_6) \in R^3 : x_2^2 + x_4^2 + x_6^2 = 1\}$ is an unit sphere. For a real triple $P = \{p_1, p_2, p_3) : p_1 + p_2 + p_3 = 0$ and $p_1p_2p_3 \neq 0\}$, let $F_P : S^2 \times R \to S^5 \subset S^6$ be a mapping, which is define as follows

$$\begin{aligned} F_P(x_1,x_2,x_3,t) =& x_1(\cos(tp_1)e_1 + \sin(tp_1)e_5) + x_2(\cos(tp_2)e_2 + \sin(tp_2)e_6) \\ &+ x_3(\cos(tp_3)e_3 + \sin(tp_3)e_7), \end{aligned} \tag{4.39}$$

where $x_1^2 + x_2^2 + x_3^2 = 1$ *and* $t \in R$. *Then it is clear that* F_P *is an isometric immersion of CR-warped product submanifold* $S^2 \times_f R$ *in to* S^6. *Moreover, induced warped product metric* $\bar{g}$ *on* $S^2 \times_f R$ *is given by*

$$\bar{g} = \pi_1^* g_0 + (\sum_{i=1}^{3} (x_i p_i)^2) \pi_2^* dt^2,$$

where $\pi_1 : S^2 \times_f R \to S^2$ *and* $\pi_2 : S^2 \times R \to R$ *are the natural projections and* g_0 *is the Riemannian metric on* S^2 *and the warping function is given by* $f = \sqrt{\sum_{i=1}^{3}(x_i p_i)^2}$.

5. Some geometric applications in Mechanics

In this section, we investigate some applications of our attained inequalities, this section is divided in different subsections as follows:

5.1. Application of Hopf's Lemma

In this subsection, we shall consider that the submanifold M^n is a compact such that $\partial M = \phi$. In the next theorem, we will see the application of Hopf's lemma for semi-slant warped product submanifold.

Theorem 5.1. *Let* $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ *be a semi-slant warped product submanifold isometrically immersed in a generalized complex space form* $\bar{M}(c, \alpha)$ *admitting nearly Kaehler structure. If the unit tangent vector* χ *is tangent to either* N_T *or* N_θ, *then* M^n *is simply Riemannian product submanifold if the Ricci curvature satisfy one of the following inequalities:*

(i) the unit vector field χ *is tangent to* N_T *and*

$$Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 + \frac{c + 3\alpha}{4}(n + n_1 n_2 - 1) + \frac{3(c - \alpha)}{8}. \tag{5.1}$$

(ii) the unit vector field χ *is tangent to* N_θ *and*

$$Ric(\chi) \leq \frac{1}{4} n^2 \|H\|^2 + \frac{c + 3\alpha}{4}(n + n_1 n_2 - 1) + \frac{3(c - \alpha)}{8} \cos^2\theta. \tag{5.2}$$

Proof. Suppose that inequality (5.1) holds. Then from (4.1), we get $\frac{\Delta\psi}{\psi} \leq 0$, which implies $\Delta\psi \leq 0$, on using Hopf's Lemma, we observe that the warping function is constant and the submanifold M^n is Riemannian product. Similar result can be proved by using inequality (5.2). □

5.2. First eigenvalue of the warping function

The lower bound of Ricci curvature contains numerous geometric properties. Suppose the submanifold M^n is complete non-compact and x be a any arbitrary point on M^n. For the Riemannian manifold M^n, $\lambda_1(M^n)$ denotes the first eigenvalue of the following Dirichlet boundary value problem.

$$\Delta\phi = \lambda\phi \ \text{ in } \ M^n \ \text{ and } \ \phi = 0 \ \text{ on } \ \partial M^n, \tag{5.3}$$

where Δ denotes the Laplacian on M^n and defined as $\Delta\phi = -div(\nabla\phi)$. By the principle of monotonicity one has $r < t$ which indicates that $\lambda_1(M_r^n) > \lambda_1(M_t^n)$ and $\lim_{r\to\infty} \lambda_1(D_r)$ exists and first eigenvalue is defined as

$$\lambda_1(M) = \lim_{r\to\infty} \lambda_1(D_r).$$

Several geometers have been worked on the analysis of first eigenvalue of the Laplacian operator (see [15, 17, 31]). For a non-constant warping function the maximum (minimum) principle on the eigenvalue λ_1, we have ([6, 10])

$$\lambda_1 \int_{M^n} \phi^2 dv \leq \int_{M^n} \|\nabla\phi\|^2 dV. \tag{5.4}$$

The equality holds if and only if $\Delta\phi = \lambda_1\phi$.

The relation between Ricci curvature and first eigenvalue is derived in the following theorem:

Theorem 5.2. *Let $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a semi-slant warped product submanifold isometrically immersed in a generalized complex space form $\bar{M}(c, \alpha)$ admitting nearly Kaehler structure. Suppose that the warping function $\ln\psi$ is an eigenfunction of the Laplacian of M^n associated to the first eigenvalue $\lambda_1(M^n)$ of the problem* (5.3)*, then the following inequalities hold:*

(i) If the unit vector field χ is tangent to N_T then

$$\begin{aligned}\int_{M^n} Ric(\chi) dV \leq \frac{1}{4}n^2 \int_{M^n} \|H\|^2 dV + n_2\lambda_1 \int_{M^n} (\ln\psi)^2 dV +\\ + [\frac{c+3\alpha}{4}(n + n_1 n_2 - 1) + \frac{3(c-\alpha)}{8}] Vol(M^n).\end{aligned} \tag{5.5}$$

(ii) If the unit vector field χ is tangent to N_θ then

$$\begin{aligned}\int_{M^n} Ric(\chi) dV \leq \frac{1}{4}n^2 \int_{M^n} \|H\|^2 dV + n_2\lambda_1 \int_{M^n} (\ln\psi)^2 dV +\\ + [\frac{c+3\alpha}{4}(n + n_1 n_2 - 1) + \frac{3(c-\alpha)}{8}\cos^2\theta] Vol(M^n).\end{aligned} \tag{5.6}$$

The equality cases are same as in Theorem 4.1.

Proof. Since M^n is compact that mean it has lower and upper bounds. Let $\lambda_1 = \lambda_1(M)$ and $\ln\psi$ be a solution of Dirichlet boundary problem corresponding to the first eigenvalue $\lambda_1(M^n)$. Suppose $\chi \in TN_T$, then the inequality (4.37) can be written as follows

$$Ric(\chi) - n_2\|\nabla\ln\psi\|^2 \leq \frac{1}{4}n^2\|H\|^2 - n_2\Delta\ln\psi + \frac{c+3\alpha}{4}(n + n_1 n_2 - 1) + \frac{3(c-\alpha)}{8}. \tag{5.7}$$

Integrating above inequality with respect to volume element dV, we find

$$\begin{aligned}\int_{M^n} Ric(\chi) dV - n_2 \int_{M^n} \|\nabla\ln\psi\|^2 dv \leq \frac{n^2}{4} \int_{M^n} \|H\|^2 dV + \frac{3(c-\alpha)}{8} Vol(M^n)\\ + \frac{c+3\alpha}{4}(n + n_1 n_2 - 1) Vol(M^n).\end{aligned} \tag{5.8}$$

Since λ_1 is an eigenvalue of the eigenfunction $\ln\psi$, such that $\Delta\ln\psi = \lambda_1\ln\psi$, then equality in (5.4) holds for $\phi = \ln\psi$,

$$\int_{M^n} \|\nabla\ln\psi\|^2 dV = \lambda_1 \int_{M^n} (\ln\psi)^2 dV, \tag{5.9}$$

using in (5.8), we obtain

$$\begin{aligned}\int_{M^n} Ric(\chi)dV - n_2\lambda_1 \int_{M^n} (\ln\psi)^2 dV \leq & \frac{n^2}{4}\int_{M^n} \|H\|^2 dV + \frac{3(c-\alpha)}{8})Vol(M^n) \\ & + \frac{c+3\alpha}{4}(n+n_1n_2-1)Vol(M^n),\end{aligned} \tag{5.10}$$

which proves the part (i). Similarly, one can proves the part (ii). □

5.3. Dirichlet energy and Euler-Lagrangian equation for the warping function

Let M^n be a compact Riemannian manifold and ϕ be a positive differentiable function on M^n. Then formula for Dirichlet energy of a function ϕ is given by [8]

$$E(\phi) = \frac{1}{2}\int_{M^n} \|\nabla\phi\|^2 dV, \tag{5.11}$$

where dV is the volume element of M^n and formula for Lagrangian of the function ϕ on M^n is given in [8]

$$L_\phi = \frac{1}{2}\|\nabla\phi\|^2. \tag{5.12}$$

The Euler-Lagrange equation for L_ϕ is given by

$$\Delta\phi = 0. \tag{5.13}$$

Considering that the semi-slant warped product submanifold $M^n = N_T^{n_1} \times_\psi N_\theta^{n_1}$ is a compact orientable without boundary such that $\partial M^n = \phi$. Then in the following theorem we have a relation between Dirichlet energy, Ricci curvature and mean curvature vector.

Theorem 5.3. *Let $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a semi-slant warped product submanifold of a generalized complex space form admitting nearly Kaehler manifold. Then we have the following inequalities for the Dirichlet energy of the warping function* $\ln\psi$*:*

(i) If the unit vector field χ is tangent to N_T then

$$\begin{aligned}E(\ln\psi) \geq & \frac{1}{2n_2}\int_{M^n} Ric(\chi)dV - \frac{n^2}{8n_2}\int_{M^n} \|H\|^2 dV - [\frac{(c+3\alpha)}{8}.\frac{(n+n_1n_2-1)}{n_2} \\ & + \frac{3(c-\alpha)}{16n_2}]Vol(M^n).\end{aligned} \tag{5.14}$$

(ii) If the unit vector field χ is tangent to N_θ then

$$\begin{aligned}E(\ln\psi) \geq & \frac{1}{2n_2}\int_{M^n} Ric(\chi)dV - \frac{n^2}{8n_2}\int_{M^n} \|H\|^2 dV - [\frac{(c+3\alpha)}{8}.\frac{(n+n_1n_2-1)}{n_2} \\ & + \frac{3(c-\alpha)}{16n_2}\cos^2\theta)]Vol(M^n).\end{aligned} \tag{5.15}$$

The equality cases are similar as in Theorem 4.1.

Proof. For a positive valued differentiable function ϕ defined on a compact orientable Riemannian manifold without boundary, by theory of integration on Riemannian manifold we have $\int_{M^n} \Delta\phi dV = 0$. On applying this fact for the warping function $\ln\psi$, we have

$$\int_{M^n} \Delta \ln\psi dV = 0. \tag{5.16}$$

Integrating inequality (4.1) with respect to volume element dV on semi-slant warped product submanifold M^n, which is compact and orientable without boundary, we get

$$\begin{aligned}\int_{M^n} Ric(\chi)dV \leq & \frac{n^2}{4}\int_{M^n} \|H\|^2 dV + n_2 \int_{M^n} \|\nabla \ln\psi\|^2 dV - n_2 \int_{M^n} \Delta \ln\psi dV \\ & + [\frac{c+3\alpha}{4}(n + n_1 n_2 - 1) + \frac{3(c-\alpha)}{8}]Vol(M^n).\end{aligned} \tag{5.17}$$

□

Using the formula (5.11) and after some computation, the required inequality is derived. In a similar method, we can prove the inequality (5.15)

Further, in the following theorem we will compute the Lagrangian for the warping function $\ln\psi$.

Theorem 5.4. *Let $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a compact orientable semi-slant warped submanifold isometrically immersed in a generalized complex space form admitting nearly Kaehler manifold such that the warping function $\ln\psi$ satisfies the Euler-Lagrangian equation, then*

(i) If the unit vector field χ is tangent to N_T, then

$$L_{\ln\psi} \geq \frac{1}{2n_2}Ric(\chi) - \frac{n^2}{8n_2}\|H\|^2 - \frac{(c+3\alpha)}{8}.\frac{(n + n_1 n_2 - 1)}{n_2} - \frac{3(c-\alpha)}{16n_2}. \tag{5.18}$$

(ii) If the unit vector field χ is tangent to N_θ, then

$$L_{\ln\psi} \geq \frac{1}{2n_2}Ric(\chi) - \frac{n^2}{8n_2}\|H\|^2 - \frac{(c+3\alpha)}{8}.\frac{(n + n_1 n_2 - 1)}{n_2} - \frac{3(c-\alpha)}{16n_2}\cos^2\theta, \tag{5.19}$$

where $L_{\ln\psi}$ is the Lagrangian of the warping function defined in (5.12). The equality cases are same as Theorem 4.1.

Proof. The proof follows immediately on using (5.12) and (5.13) in Theorem 4.1. □

Further, the Hamiltonian for a local orthonormal frame at any point $x \in M^n$ is expressed as follows [8]

$$H(p, x) = \frac{1}{2}\sum_{i=1}^{n} p(e_i)^2. \tag{5.20}$$

On replacing p by a differential operator $d\phi$, then from (2.20), we get

$$H(d\phi, x) = \frac{1}{2}\sum_{i=1}^{n} d\phi(e_i)^2 = \frac{1}{2}\sum_{i=1}^{n} e_i(\phi)^2 = \frac{1}{2}\|\nabla\phi\|^2. \tag{5.21}$$

In the next result we obtain a relation between Hamiltonian of warping function, Ricci curvature and squared norm of mean curvature vector.

Theorem 5.5. *Let $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ be a semi-slant warped product submanifold isometrically immersed in a generalized complex space form $\bar{M}(c,\alpha)$ admitting nearly Kaehler structure then the Hamiltonian of the warping function satisfy the following inequalities*

(i) If $\chi \in TN_T$, then

$$H(d\ln\psi, x) \geq \frac{1}{2n_2}\{Ric(\chi) + n_2\Delta\ln\psi - \frac{n^2}{4}||H||^2 - \frac{c+3\alpha}{4}(n + n_1n_2 - 1) - \frac{3(c-\alpha)}{8}\} \tag{5.22}$$

(ii) If $\chi \in TN_\theta$, then

$$H(d\ln\psi, x) \geq \frac{1}{2n_2}\{Ric(\chi) + n_2\Delta\ln\psi - \frac{n^2}{4}||H||^2 - \frac{c+3\alpha}{4}(n + n_1n_2 - 1) - \frac{3(c-\alpha)}{8}\cos^2\theta\} \tag{5.23}$$

Proof. By the application of (5.21) in theorem 4.1, we get the required results. □

5.4. Applications of Obata's differential equation

This subsection is based on the study of Obata [29]. Basically, Obata characterized a Riemannian manifolds by a specific ordinary differential equation and derived that an n-dimensional complete and connected Riemannian manifold (M^n, g) to be isometric to the n-sphere S^n if and only if there exists a non-constant smooth function ϕ on M^n that is the solution of the differential equation $H^\phi = -c\phi g$, where H^ϕ is the Hessian of ϕ. Inspired by the work of Obata [29], we obtain the following characterization

Theorem 5.6. *Suppose $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ is a compact orientable warped product submanifold isometrically immersed in a generalized complex space form $M^m(c,\alpha)$ admitting nearly Kaehler structure with positive Ricci curvature and satisfying one of the following relation*

(i) $\chi \in TN_T$ and

$$||Hess\phi||^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}||H||^2 - \frac{3\lambda_1}{4n_1n_2}[(c+3\alpha)(n+n_1n_2-1) + \frac{3(c-\alpha)}{2}], \tag{5.24}$$

(ii) $\chi \in TN_\theta$ and

$$||Hess\phi||^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}||H||^2 - \frac{3\lambda_1}{4n_1n_2}[(c+3\alpha)(n+n_1n_2-1) - \frac{3(c-\alpha)}{2}\cos^2\theta)], \tag{5.25}$$

where $\lambda_1 > 0$ is an eigenvalue of the warping function $\phi = \ln\psi$. Then the base manifold $N_T^{n_1}$ is isometric to the sphere $S^{n_1}(\frac{\lambda_1}{n_1})$ with constant sectional curvature $\frac{\lambda_1}{n_1}$.

Proof. Let $\chi \in TN_T$. Consider that $\phi = \ln\psi$ and define the following relation as

$$||Hess\phi - t\phi I||^2 = ||Hess\phi||^2 + t^2\phi^2||I||^2 - 2t\phi g(Hess\phi, I). \tag{5.26}$$

But we know that $||I||^2 = trace(II^*) = p$ and

$$g(Hess(\phi), I^*) = trace(Hess\phi, I^*) = traceHess(\phi).$$

Then Eq (5.26) transform to

$$||Hess\phi - t\phi I||^2 = ||Hess\phi||^2 + pt^2\phi^2 - 2t\phi\Delta\phi. \tag{5.27}$$

Assuming λ_1 is an eigenvalue of the eigenfunction ϕ then $\Delta\phi = \lambda_1\phi$. Thus we get

$$||Hess\phi - t\phi I||^2 = ||Hess\phi||^2 + (pt^2 - 2t\lambda)\phi^2. \tag{5.28}$$

On the other hand, we obtain $\Delta\phi^2 = 2\phi\Delta\phi + ||\nabla\phi||^2$ or $\lambda_1\phi^2 = 2\lambda_1\phi^2 + ||\nabla\phi||^2$ which implies that $\phi^2 = -\frac{1}{\lambda_1}||\nabla\phi||^2$, using this in Eq (5.28), we have

$$||Hess\phi - t\phi I||^2 = ||Hess\phi||^2 + (2t - \frac{pt^2}{\lambda_1})||\nabla\phi||^2. \tag{5.29}$$

In particular $t = -\frac{\lambda_1}{n_1}$ on (5.29) and integrating with respect to dV, we get

$$\int_{M^n} ||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 dV = \int_{M^n} ||Hess\phi||^2 dV - \frac{3\lambda_1}{n_1}\int_{M^n} ||\nabla\phi||^2 dV. \tag{5.30}$$

Integrating the inequality (4.37) and using the fact $\int_{M^n} \Delta\phi dV = 0$, we have

$$\begin{aligned}\int_{M^n} Ric(\chi)dV \leq & \frac{n^2}{4}\int_{M^n} ||H||^2 dV + n_2\int_{M^n} ||\nabla\phi||^2 dV + \\ & + \frac{c+3\alpha}{4}(n + n_1 n_2 - 1)Vol(M^n) + \frac{3(c-\alpha)}{8}Vol(M^n).\end{aligned} \tag{5.31}$$

From (5.30) and (5.31) we derive

$$\begin{aligned}\frac{1}{n_2}\int_{M^n} Ric(\chi)dV \leq & \frac{n^2}{4n_2}\int_{M^n} ||H||^2 dV - \frac{n_1}{3\lambda_1}\int_{M^n} ||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 dV \\ & + \frac{n_1}{3\lambda_1}\int_{M^n} ||Hess\phi||^2 dV + \frac{c+3\alpha}{4}\frac{(n + n_1 n_2 - 1)}{n_2}Vol(M^n) \\ & + \frac{3(c-\alpha)}{8n_2}Vol(M^n).\end{aligned} \tag{5.32}$$

According to assumption $Ric(\chi) \geq 0$, the above inequality gives

$$\begin{aligned}\int_{M^n} ||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 dV \leq & \frac{3n^2\lambda_1}{4n_1 n_2}\int_{M^n} ||H||^2 dV + \int_{M^n} ||Hess\phi||^2 dV \\ & + \frac{c+3\alpha}{4}\frac{3\lambda_1(n + n_1 n_2 - 1)}{n_1 n_2}Vol(M^n) + \frac{9\lambda_1(c-\alpha)}{8n_1 n_2}Vol(M^n).\end{aligned} \tag{5.33}$$

From (5.24), we get

$$\int_{M^n} ||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 dV \leq 0, \tag{5.34}$$

but we know that

$$\int_{M^n} ||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 dV \geq 0. \tag{5.35}$$

Combining last two statements, we get

$$\int_{M^n} ||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 dV = 0 \Rightarrow Hess\phi = -\frac{\lambda_1}{n_1}\phi I. \tag{5.36}$$

Since the warping function $\phi = \ln\psi$ is not constant function on M^n so Eq (5.36) is Obata's [29] differential equation with constant $c = \frac{\lambda_1}{n_1} > 0$. As $\lambda_1 > 0$ and therefore the base submanifold $N_T^{n_1}$ is isometric to the sphere $S^{n_1}(\frac{\lambda_1}{n_1})$ with constant sectional curvature $\frac{\lambda_1}{n_1}$. Similarly, we can prove the theorem by using part (ii). □

In [17] Rio et al. studied another version of Obata's differential equation in the characterization of Euclidean sphere. Basically, they proved that if ϕ is a real valued non constant function on a Riemannian manifold satisfying $\Delta\phi + \lambda_1\phi = 0$ such that $\lambda < 0$, then M^n is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function ϕ is the solution of the following differential equation

$$\frac{d^2\phi}{dt^2} + \lambda_1\phi = 0. \tag{5.37}$$

Motivated by the study of Rio et al. [17] and Ali et al. [1] we obtain the following characterization.

Theorem 5.7. *Suppose $M^n = N_T^{n_1} \times_\psi N_\theta^{n_2}$ is a compact orientable semi-slant warped product submanifold isometrically immersed in generalized complex space form $\bar{M}(c,\alpha)$ admitting nearly Kaehler structure with positive Ricci curvature and satisfying one of the following statement:*

(i) $\chi \in TN_T$ and

$$||Hess\phi||^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}||H||^2 - \frac{3\lambda_1}{4n_1n_2}[(c+3\alpha)(n+n_1n_2-1) + \frac{3(c-\alpha)}{2}] \tag{5.38}$$

(ii) $\chi \in TN_\theta$ and

$$||Hess\phi||^2 = -\frac{3\lambda_1 n^2}{4n_1n_2}||H||^2 - \frac{3\lambda_1}{4n_1n_2}[(c+3\alpha)(n+n_1n_2-1) + \frac{3(c-\alpha)}{2}\cos^2\theta)], \tag{5.39}$$

where $\lambda_1 < 0$ is a negative eigenvalue of the eigenfunction $\phi = \ln\psi$. Then $N_T^{n_1}$ is isometric to a warped product of the Euclidean line and a complete Riemannian manifold whose warping function $\phi = \ln\psi$ satisfies the differential equation

$$\frac{d^2\phi}{dt^2} + \lambda_1\phi = 0. \tag{5.40}$$

Proof. Since we assumed that the Ricci curvature is positive then by the Myers's theorem according to which, a complete Riemannian manifold with positive Ricci curvature is compact that mean M^n is compact semi-slant warped product submanifold with free boundary [28]. Then by (5.32) we get

$$\begin{aligned}\frac{1}{n_2}\int_{M^n} Ric(\chi)dV \leq \frac{n^2}{4n_2}\int_{M^n}||H||^2dV - \frac{n_1}{3\lambda_1}\int_{M^n}||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 dV \\ + \frac{n_1}{3\lambda_1}\int_{M^n}||Hess\phi||^2dV + \frac{c+3\alpha}{4}\frac{(n+n_1n_2-1)}{n_2}Vol(M^n) + \frac{3(c-\alpha)}{8n_2}Vol(M^n).\end{aligned} \tag{5.41}$$

According to hypothesis, Ricci curvature is positive $Ric(\chi) > 0$, then we have

$$\begin{aligned}\int_{M^n} ||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 dV < \frac{3n^2\lambda_1}{4n_1n_2}\int_{M^n} ||H||^2 dV + \int_{M^n} ||Hess\phi||^2 dV \\ + \frac{c+3\alpha}{4}.\frac{3\lambda_1(n+n_1n_2-1)}{n_1n_2} Vol(M^n) + \frac{9\lambda_1(c-\alpha)}{8n_2} Vol(M^n).\end{aligned} \tag{5.42}$$

If Eq (5.38) holds, then from above inequality we get $||Hess\phi + \frac{\lambda_1}{n_1}\psi I||^2 < 0$, which is not possible hence $||Hess\phi + \frac{\lambda_1}{n_1}\phi I||^2 = 0$. Since $\lambda < 0$, then by result of [17], the submanifold $N_T^{n_1}$ is isometric to a warped product of the Euclidean line and a complete Riemannian manifold, where the warping function on R is the solution of the differential equation (5.40). This proves the theorem. Similarly by assuming (5.39), we can also prove the theorem.

□

6. Conclusions

In this paper firstly we have obtained a Ricci curvature inequality of a semi-slant warped product submanifold isometrically immersed in a generalized complex space form admitting a nearly Kaehler structure. Then we have given some applications on Hopf's Lemma, Dirichlet energy, Euler-Lagrangian equation, Hamiltonian of warping functions of a semi-slant warped product submanifold isometrically immersed in a generalized complex space form admitting a nearly Kaehler structure. In last we have characterized semi-slant warped product submanifold isometrically immersed in a generalized complex space form admitting a nearly Kaehler structure in the basis of Obata differential equation.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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