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Research article

Hermite-Hadamard and Ostrowski type inequalities in $\mathfrak{h}$-calculus with applications

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Abstract: In this paper, we prove Hermite-Hadamard inequality for convex functions in the framework of $\mathfrak{h}$-calculus. We also use the notions of $\mathfrak{h}$-derivative and $\mathfrak{h}$-integral to prove Ostrowski's and trapezoidal type inequalities for bounded functions. It is also shown that the newly established inequalities are the generalization of the comparable inequalities in the literature. Finally, using some examples, we demonstrate the validity of newly formed inequalities and show how they can be used to special means of real numbers.

Keywords: Hermite-Hadamard inequality; Ostrowski inequality; $\mathfrak{h}$-integral; quantum calculus; convex functions
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1. Introduction

The modern name for the study of calculus without limits is quantum calculus (shortly, $\mathfrak{h}$-calculus and q-calculus). The fundamental concepts of q-calculus and $\mathfrak{h}$-calculus are given by Kac in [18]. It has a wide range of applications in mathematics, including combinatorics, simple hypergeometric functions, number theory, orthogonal polynomials, and other sciences, as well as mechanics, relativity theory, and quantum theory [18].

In convex functions theory, Hermite-Hadamard (H-H) inequality is very important which was discovered by C. Hermite and J. Hadamard independently (see, also [15], and [24, p.137]),

$$\mathfrak{f}\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int\limits_{\alpha}^{\beta}\mathfrak{f}(\varkappa)d\varkappa \leq \frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}, \tag{1.1}$$

where $\mathfrak{f}$ is a convex function. In the case of concave mappings, the above inequality is satisfied in reverse order. For recent variant of different integral inequalities for different kinds of functions, one can consult [25–27]. The quantum variants of the inequality (1.1) given by Alp et al [8] and Bermudo et al [10] using the q-integrals as follows:

$$\mathfrak{f}\left(\frac{q\alpha+\beta}{[2]_q}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{f}(\varkappa)\ {}_{\alpha}d_q\varkappa \leq \frac{q\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{[2]_q}, \tag{1.2}$$

and

$$\mathfrak{f}\left(\frac{\alpha+q\beta}{[2]_q}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{f}(\varkappa)\ {}^{\beta}d_q\varkappa \leq \frac{\mathfrak{f}(\alpha)+q\mathfrak{f}(\beta)}{[2]_q}, \tag{1.3}$$

where $\mathfrak{f}$ is convex mapping and $[2]_q = 1 + q$. It is obvious that the inequalities (1.2) and (1.3) turn into (1.1) as $q \to 1^-$.

Many integral inequalities have been studied using q-integrals for various types of functions. For example, in [2, 5, 8–12, 17, 21], the authors used q-derivatives and q-integrals to prove H-H integral inequalities and their left-right estimates for convex and coordinated convex functions. In [22], Noor et al presented a generalized version of q-H-H integral inequalities. For generalized quasi-convex functions, Nwaeze et al. proved certain parameterized q-integral inequalities in [23]. Khan et al proved q-H-H inequality using the green function in [19]. Budak et al [13], Ali et al [1, 3, 7] and Vivas-Cortez et al [28] developed new q-Simpson's and q-Newton's type inequalities for convex and coordinated convex functions. For q-Ostrowski's inequalities for convex and co-ordinated convex functions one can consult [4, 6, 14].

The main goal of this study is to discover $\mathfrak{h}$-calculus analogs of various classical integral inequalities. In particular, we prove $\mathfrak{h}$-generalization of Hölder's inequality, H-H inequality and related inequalities, Ostrowski's inequality and trapezoidal inequality.

To the best of our knowledge, this is the first research that uses the $\mathfrak{h}$-integral to study quantum integral inequalities.

The following is the structure of this paper: A brief overview of the concepts of $\mathfrak{h}$-calculus, as well as some related works, is given in Section 2. In Section 3, we use the $\mathfrak{h}$-integral and $\mathfrak{h}$-derivative to prove Hölder's inequality, H-H inequality for convex functions, H-H type inequalities for twice differentiable functions, Ostrowski's inequality, and trapezoidal inequality. In Section 4, we give application to special means of real numbers for newly established inequalities. Section 5 concludes with some applications of newly established results and recommendations for future studies.

2. Preliminaries of $\mathfrak{h}$-calculus

In this section, we recollect some formerly regarded concepts of $\mathfrak{h}$-calculus.

Definition 2.1. *[18] For a mapping* $\mathfrak{f} : [\alpha, \beta] \to \mathbb{R}$*, the* $\mathfrak{h}$*-derivative of* $\mathfrak{f}$ *at* $\varkappa \in [\alpha, \beta]$ *is defined as:*

$$D_{\mathfrak{h}} (\mathfrak{f} (\varkappa)) = \frac{\mathfrak{f} (\varkappa + \mathfrak{h}) - \mathfrak{f} (\varkappa)}{\mathfrak{h}}, \tag{2.1}$$

where $\mathfrak{h} \neq 0$.

The product and quotient rules for $\mathfrak{h}$-differentiation are simple to verify:

$$D_{\mathfrak{h}} (\mathfrak{f} (\varkappa) g (\varkappa)) = \mathfrak{f} (\varkappa) D_{\mathfrak{h}} (g (\varkappa)) + g (\varkappa + \mathfrak{h}) D_{\mathfrak{h}} (\mathfrak{f} (\varkappa)), \tag{2.2}$$

and

$$D_{\mathfrak{h}} \left(\frac{\mathfrak{f} (\varkappa)}{g (\varkappa)} \right) = \frac{g (\varkappa) D_{\mathfrak{h}} (\mathfrak{f} (\varkappa)) - \mathfrak{f} (\varkappa) D_{\mathfrak{h}} (g (\varkappa))}{g (\varkappa) g (\varkappa + \mathfrak{h})}. \tag{2.3}$$

Definition 2.2. *[18] The* $\mathfrak{h}$*-binomial* $(\varkappa - \alpha)_{\mathfrak{h}}^{n}$ *is expressed as:*

$$(\varkappa - \alpha)_{\mathfrak{h}}^{n} = (\varkappa - \alpha) (\varkappa - \alpha - \mathfrak{h}) ... (\varkappa - \alpha - (n - 1) \mathfrak{h}),$$

where $n \geq 1$ *and* $(\varkappa - \alpha)_{\mathfrak{h}}^{0} = 1$.

One can easily observe that

$$D_{\mathfrak{h}} (\varkappa - \alpha)_{\mathfrak{h}}^{n} = n (\varkappa - \alpha)_{\mathfrak{h}}^{n-1}.$$

Definition 2.3. *[18] For a mapping* $\mathfrak{f} : [\alpha, \beta] \to \mathbb{R}$*, the definite* $\mathfrak{h}$*-integral of* $\mathfrak{f}$ *is defined as:*

$$\int_{\alpha}^{\beta} \mathfrak{f} (\varkappa) d_{\mathfrak{h}} \varkappa = \begin{cases} \mathfrak{h} (\mathfrak{f} (\alpha) + \mathfrak{f} (\alpha + \mathfrak{h}) + ... + \mathfrak{f} (\beta - \mathfrak{h})), & \text{if } \alpha < \beta \\ 0, & \text{if } \alpha = \beta, \\ -\mathfrak{h} (\mathfrak{f} (\beta) + \mathfrak{f} (\beta + \mathfrak{h}) + ... + \mathfrak{f} (\alpha - \mathfrak{h})), & \text{if } \alpha > \beta, \end{cases} \tag{2.4}$$

where $\mathfrak{h} \neq 0$ *and* $\beta - \alpha \in \mathfrak{h}\mathbb{Z}$.

Remark 2.4. *If* $\mathfrak{f} (\varkappa) \leq g (\varkappa)$*, then the following* $\mathfrak{h}$*-integral relationship is true:*

$$\int_{\alpha}^{\beta} \mathfrak{f} (\varkappa) d_{\mathfrak{h}} \varkappa \leq \int_{\alpha}^{\beta} g (\varkappa) d_{\mathfrak{h}} \varkappa.$$

Theorem 2.5 (Fundemantal theorem of h-calculus)**.** *[18] If* $F (\varkappa)$ *is an* $\mathfrak{h}$*-antiderivative of* $\mathfrak{f} (\varkappa)$ *and* $\beta - \alpha \in \mathfrak{h}\mathbb{Z}$*, we have*

$$\int_{\alpha}^{\beta} \mathfrak{f} (\varkappa) d_{\mathfrak{h}} \varkappa = F (\beta) - F (\alpha). \tag{2.5}$$

From (2.2) and (2.5), we have the following $\mathfrak{h}$-variant of integration by parts:

$$\int_{\alpha}^{\beta} \mathfrak{f} (\varkappa) d_{\mathfrak{h}} g (\varkappa) = \mathfrak{f} (\beta) g (\beta) - \mathfrak{f} (\alpha) g (\alpha) - \int_{\alpha}^{\beta} g (\varkappa + \mathfrak{h}) d_{\mathfrak{h}} \mathfrak{f} (\varkappa).$$

For more details about the $\mathfrak{h}$-derivative and $\mathfrak{h}$-integral, one can consult [18].

3. $\mathfrak{h}$-integral inequalities

In this section, we prove different quantum integral inequalities via $\mathfrak{h}$-integral. Throughout the section, let $I = [\alpha,\beta] \subseteq \mathbb{R}$ and $\mathfrak{h} \neq 0$ be a constant.

Theorem 3.1 (Hölder's inequality). *Let $p_1, p_2 > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then, we have the following inequality*

$$\int_\alpha^\beta |\mathfrak{f}(t)|\,|g(t)|\,d_\mathfrak{h}t \leq \left(\int_\alpha^\beta |\mathfrak{f}(t)|^{p_1}\,d_\mathfrak{h}t\right)^{\frac{1}{p_1}} \left(\int_\alpha^\beta |g(t)|^{p_2}\,d_\mathfrak{h}t\right)^{\frac{1}{p_2}}.$$

Proof. From the Definition of $\mathfrak{h}$-integral and discrete Hölder's inequality, we have

$$\begin{aligned}
\int_\alpha^\beta |\mathfrak{f}(t)|\,|g(t)|\,d_\mathfrak{h}t &= \mathfrak{h}\sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} |\mathfrak{f}(\alpha+j\mathfrak{h})|\,|g(\alpha+j\mathfrak{h})| \\
&\leq \left(\mathfrak{h}\sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} |\mathfrak{f}(\alpha+j\mathfrak{h})|^{p_1}\right)^{\frac{1}{p_1}} \left(\mathfrak{h}\sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} |g(\alpha+j\mathfrak{h})|^{p_2}\right)^{\frac{1}{p_2}} \\
&= \left(\int_\alpha^\beta |\mathfrak{f}(t)|^{p_1}\,d_\mathfrak{h}t\right)^{\frac{1}{p_1}} \left(\int_\alpha^\beta |g(t)|^{p_2}\,d_\mathfrak{h}t\right)^{\frac{1}{p_2}},
\end{aligned}$$

and the proof is completed. □

Remark 3.2. *If we set the limit as $\mathfrak{h} \to 0$, then we obtain the classical integral Hölder's inequality.*

We needs the following lemma to prove the next result.

Lemma 3.3. *For any integrable mapping $\mathfrak{f} : [\alpha,\beta] \to \mathbb{R}$ and let φ be a convex mapping, the following inequality is true:*

$$\varphi\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(\varkappa)\,d_\mathfrak{h}\varkappa\right) \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(\mathfrak{f}(\varkappa))\,d_\mathfrak{h}\varkappa. \tag{3.1}$$

Proof. Since φ is a convex mapping, therefore for every point $(\varkappa_0, \varphi(\varkappa_0))$ on the graph of φ there is a line $y = \alpha(\varkappa - \varkappa_0) + \varphi(\varkappa_0)$ such that $\varphi(\varkappa) \geq \alpha(\varkappa - \varkappa_0) + \varphi(\varkappa_0)$ for all $\varkappa$ in the domain of φ. Now, we assume that $\varkappa_0 = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(\varkappa)\,d_\mathfrak{h}\varkappa$ and $\mathfrak{h}$-integrate the inequality

$$\varphi(\mathfrak{f}(\varkappa)) \geq \alpha(\mathfrak{f}(\varkappa) - \varkappa_0) + \varphi(\varkappa_0),$$

$$\int_\alpha^\beta \varphi(\mathfrak{f}(\varkappa))\,d_\mathfrak{h}\varkappa \geq \alpha(\varkappa_0 - \varkappa_0)(\beta-\alpha) + (\beta-\alpha)\varphi(\varkappa_0).$$

Thus,

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(\mathfrak{f}(\varkappa))\,d_\mathfrak{h}\varkappa \geq \varphi\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(\varkappa)\,d_\mathfrak{h}\varkappa\right),$$

and the proof is completed. □

Theorem 3.4 (H-H inequality)**.** *For a convex mapping $\mathfrak{f}: I \to \mathbb{R}$, the following inequality holds:*

$$\mathfrak{f}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{f}(\varkappa)\ d_{\mathfrak{h}}\varkappa \leq \frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}+\frac{\mathfrak{h}}{\beta-\alpha}\left(\frac{\mathfrak{f}(\alpha)-\mathfrak{f}(\beta)}{2}\right). \tag{3.2}$$

Proof. Since $\mathfrak{f}$ is convex, therefore

$$\mathfrak{f}\left(\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varkappa d_{\mathfrak{h}}\varkappa\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{f}(\varkappa)\, d_{\mathfrak{h}}\varkappa. \tag{3.3}$$

One can easily observe that

$$\begin{aligned} \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varkappa d_{\mathfrak{h}}\varkappa &= \frac{1}{\beta-\alpha}\sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1}(\alpha+j\mathfrak{h}) \\ &= \frac{\alpha+\beta-\mathfrak{h}}{2}. \end{aligned} \tag{3.4}$$

Thus the first inequality of (3.2) is proved.

To prove the second inequality, we again use the convexity and we have

$$\mathfrak{f}(\varkappa) \leq \mathfrak{f}(\alpha)+\frac{\mathfrak{f}(\beta)-\mathfrak{f}(\alpha)}{\beta-\alpha}(\varkappa-\alpha). \tag{3.5}$$

Applying $\mathfrak{h}$-integral on the both sides of (3.5), we have

$$\begin{aligned} \int_{\alpha}^{\beta}\mathfrak{f}(\varkappa)\, d_{\mathfrak{h}}\varkappa &\leq (\beta-\alpha)\,\mathfrak{f}(\alpha)+(\mathfrak{f}(\beta)-\mathfrak{f}(\alpha))\left(\frac{\alpha+\beta-\mathfrak{h}}{2}-\alpha\right) \\ &= \mathfrak{f}(\alpha)\left(\frac{\beta-\alpha+\mathfrak{h}}{2}\right)+\mathfrak{f}(\beta)\left(\frac{\beta-\alpha-\mathfrak{h}}{2}\right) \\ &= (\beta-\alpha)\,\frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}+\frac{\mathfrak{h}}{2}(\mathfrak{f}(\alpha)-\mathfrak{f}(\beta)), \end{aligned}$$

and the proof is completed. □

Remark 3.5. *If we take the limit as $\mathfrak{h} \to 0$ in Theorem 3.4, then we recapture the classical H-H inequality for convex functions.*

Example 3.6. *We define a convex mapping $\mathfrak{f}(\varkappa) = \varkappa^2$. Then, from inequality (3.2) for $\alpha = 1, \beta = 2$, and $\mathfrak{h} = \frac{1}{2}$, we have*

$$\mathfrak{f}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right) = \mathfrak{f}\left(\frac{5}{4}\right) = \left(\frac{5}{4}\right)^2 = \frac{25}{16},$$

$$\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{f}(\varkappa)\ d_{\mathfrak{h}}\varkappa = \int_{1}^{2}\varkappa^2 d_{\frac{1}{2}}\varkappa = \frac{13}{8},$$

and

$$\frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}-\mathfrak{h}\left(\frac{\mathfrak{f}(\alpha)-\mathfrak{f}(\beta)}{2}\right) = \frac{7}{4}.$$

It is obvious that

$$\frac{25}{16} < \frac{39}{24} < \frac{7}{4},$$

which shows that the inequality (3.2) is valid for convex functions.

We prove the following new refinements of H-H type inequality (3.2) for twice differentiable functions.

Theorem 3.7. *For a continuous mapping $\mathfrak{f} : I \to \mathbb{R}$, which is twice differentiable on I°, the following inequality holds:*

$$\begin{aligned} & \mathfrak{f}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)-\frac{m}{2}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)^2+\frac{m}{2}\left(\alpha(\beta-\mathfrak{h})+\frac{(\beta-\alpha-\mathfrak{h})(2(\beta-\alpha)-\mathfrak{h})}{6}\right) \qquad (3.6)\\ \leq\ & \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(\varkappa)\ d_{\mathfrak{h}}\varkappa \\ \leq\ & \frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}+\frac{\mathfrak{h}}{2(\beta-\alpha)}(\mathfrak{f}(\alpha)-\mathfrak{f}(\beta))-m\frac{\alpha^2+\beta^2}{4}+\mathfrak{h}m\left(\frac{\alpha+\beta}{4}\right) \\ & +\frac{m}{2}\left(\alpha(\beta-\mathfrak{h})+\frac{(\beta-\alpha-\mathfrak{h})(2(\beta-\alpha)-\mathfrak{h})}{6}\right), \end{aligned}$$

where $m = \inf_{\varkappa\in I^\circ} \mathfrak{f}''(\varkappa)$.

Proof. As $\mathfrak{f}$ is a twice differentiable function on I°. Set $g(\varkappa) = \mathfrak{f}(\varkappa) - \frac{m}{2}\varkappa^2$ and $g''(\varkappa) = \mathfrak{f}''(\varkappa) - m \geq 0$, then from inequality (3.2), we have

$$g\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right) \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta g(\varkappa)\ d_{\mathfrak{h}}\varkappa \leq \frac{g(\alpha)+g(\beta)}{2}+\frac{\mathfrak{h}}{\beta-\alpha}\left(\frac{g(\alpha)-g(\beta)}{2}\right). \qquad (3.7)$$

For $g(\varkappa) = \mathfrak{f}(\varkappa) - \frac{m}{2}\varkappa^2$, the inequality (3.7) becomes

$$\begin{aligned} & \mathfrak{f}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)-\frac{m}{2}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)^2 \qquad (3.8)\\ \leq\ & \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(\varkappa)\ d_{\mathfrak{h}}\varkappa-\frac{m}{2(\beta-\alpha)}\int_\alpha^\beta \varkappa^2 d_{\mathfrak{h}}\varkappa \\ \leq\ & \frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}+\frac{\mathfrak{h}}{2(\beta-\alpha)}(\mathfrak{f}(\alpha)-\mathfrak{f}(\beta))-m\frac{\alpha^2+\beta^2}{4}+\mathfrak{h}m\left(\frac{\alpha+\beta}{4}\right). \end{aligned}$$

Thus, we obtain the required inequality (3.6) by simple computations in (3.8). □

Remark 3.8. *If we take the limit as $\mathfrak{h} \to 0$ in Theorem 3.7, then Theorem 3.7 becomes [16, Theorem 1.1].*

Example 3.9. *We define a mapping $\mathfrak{f} : [0,1] \to \mathbb{R}$ by $\mathfrak{f}(\varkappa) = \varkappa^3 + \varkappa^2$. Then from inequality (3.6) for $\mathfrak{f}''(\varkappa) = 6\varkappa + 2$, for $\alpha = 0$, $\beta = 1$, $m = 2$ and $\mathfrak{h} = \frac{1}{2}$, we have*

$$\begin{aligned} & \mathfrak{f}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)-\frac{m}{2}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)^2+\frac{m}{2}\left(\alpha(\beta-\mathfrak{h})+\frac{(\beta-\alpha-\mathfrak{h})(2(\beta-\alpha)-\mathfrak{h})}{6}\right) \qquad (3.9)\\ =\ & \frac{9}{64}, \end{aligned}$$

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(\varkappa)\ d_{\mathfrak{h}}\varkappa = \frac{3}{16}, \qquad (3.10)$$

and

$$\begin{aligned}
& \frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}+\frac{\mathfrak{h}}{2(\beta-\alpha)}(\mathfrak{f}(\alpha)-\mathfrak{f}(\beta))-m\frac{\alpha^{2}+\beta^{2}}{4}+\mathfrak{h}m\left(\frac{\alpha+\beta}{4}\right) \\
& +\frac{m}{2}\left(\alpha(\beta-\mathfrak{h})+\frac{(\beta-\alpha-\mathfrak{h})(2(\beta-\alpha)-\mathfrak{h})}{6}\right) \\
= & \frac{3}{8}.
\end{aligned} \tag{3.11}$$

Finally, we conclude from the equalities (3.9)–(3.11) that the inequality (3.6) is valid and

$$\frac{9}{64}<\frac{3}{16}<\frac{3}{8}.$$

Theorem 3.10. *For a continuous mapping $\mathfrak{f}: I \to \mathbb{R}$, which is twice differentiable on I°, the following inequality holds:*

$$\begin{aligned}
& \mathfrak{f}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)-\frac{M}{2}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)^{2}+\frac{M}{2}\left(\alpha(\beta-\mathfrak{h})+\frac{(\beta-\alpha-\mathfrak{h})(2(\beta-\alpha)-\mathfrak{h})}{6}\right) \\
\geq & \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{f}(\varkappa)\ d_{\mathfrak{h}}\varkappa \\
\geq & \frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}+\frac{\mathfrak{h}}{2(\beta-\alpha)}(\mathfrak{f}(\alpha)-\mathfrak{f}(\beta))-M\frac{\alpha^{2}+\beta^{2}}{4}+\mathfrak{h}M\left(\frac{\alpha+\beta}{4}\right) \\
& +\frac{M}{2}\left(\alpha(\beta-\mathfrak{h})+\frac{(\beta-\alpha-\mathfrak{h})(2(\beta-\alpha)-\mathfrak{h})}{6}\right),
\end{aligned} \tag{3.12}$$

where $M=\max_{\varkappa\in I^{\circ}}\mathfrak{f}''(\varkappa)$.

Proof. As $\mathfrak{f}$ is a twice differentiable function on I°. Set $g(\varkappa)=-\mathfrak{f}(\varkappa)+\frac{M}{2}\varkappa^{2}$ and $g''(\varkappa)=-\mathfrak{f}''(\varkappa)+M\geq 0$, then from inequality (3.2), we have

$$g\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)\leq\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}g(\varkappa)\ d_{\mathfrak{h}}\varkappa\leq\frac{g(\alpha)+g(\beta)}{2}+\frac{\mathfrak{h}}{\beta-\alpha}\left(\frac{g(\alpha)-g(\beta)}{2}\right). \tag{3.13}$$

For $g(\varkappa)=-\mathfrak{f}(\varkappa)+\frac{M}{2}\varkappa^{2}$, the inequality (3.13) becomes

$$\begin{aligned}
& -\mathfrak{f}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)+\frac{M}{2}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)^{2} \\
\leq & -\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\mathfrak{f}(\varkappa)\ d_{\mathfrak{h}}\varkappa+\frac{M}{2(\beta-\alpha)}\int_{\alpha}^{\beta}\varkappa^{2}d_{\mathfrak{h}}\varkappa \\
\leq & -\frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2}-\frac{\mathfrak{h}}{2(\beta-\alpha)}(\mathfrak{f}(\alpha)-\mathfrak{f}(\beta))+M\frac{\alpha^{2}+\beta^{2}}{4}+\mathfrak{h}M\left(\frac{\alpha+\beta}{4}\right).
\end{aligned} \tag{3.14}$$

Thus, we obtain the required inequality (3.12) by simple computations in (3.14). □

Remark 3.11. *If we take the limit as $\mathfrak{h}\to 0$ in Theorem 3.10, then Theorem 3.10 becomes [16, Theorem 1.2].*

Example 3.12. *We define a mapping $\mathfrak{f} : [0,1] \to \mathbb{R}$ by $\mathfrak{f}(\varkappa) = \varkappa^3 + \varkappa^2$. Then from inequality (3.6) for $\mathfrak{f}''(\varkappa) = 6\varkappa + 2$, for $\alpha = 0$, $\beta = 1$, $M = 2$ and $\mathfrak{h} = \frac{1}{2}$, we have*

$$\begin{aligned} & \mathfrak{f}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right) - \frac{M}{2}\left(\frac{\alpha+\beta-\mathfrak{h}}{2}\right)^2 + \frac{M}{2}\left(\alpha(\beta-\mathfrak{h}) + \frac{(\beta-\alpha-\mathfrak{h})(2(\beta-\alpha)-\mathfrak{h})}{6}\right) \\ = \; & \frac{21}{64}, \end{aligned} \tag{3.15}$$

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(\varkappa)\, d_{\mathfrak{h}}\varkappa = \frac{3}{16}, \tag{3.16}$$

and

$$\begin{aligned} & \frac{\mathfrak{f}(\alpha)+\mathfrak{f}(\beta)}{2} + \frac{\mathfrak{h}}{2(\beta-\alpha)}(\mathfrak{f}(\alpha)-\mathfrak{f}(\beta)) - M\frac{\alpha^2+\beta^2}{4} + \mathfrak{h}M\left(\frac{\alpha+\beta}{4}\right) \\ & + \frac{M}{2}\left(\alpha(\beta-\mathfrak{h}) + \frac{(\beta-\alpha-\mathfrak{h})(2(\beta-\alpha)-\mathfrak{h})}{6}\right) \\ = \; & 0. \end{aligned} \tag{3.17}$$

Finally, we conclude from the equalities (3.15)–(3.17) that the inequality (3.12) is valid and

$$\frac{21}{64} > \frac{3}{16} > 0.$$

Now, we prove an Ostrowski inequality and trapezoidal inequality via $\mathfrak{h}$-integral.

Theorem 3.13 (Ostrowski inequality)**.** *For a Lipschitz mapping $\mathfrak{f} : I \to \mathbb{R}$ with*

$$\|\mathfrak{f}\|_{Lip} = \sup\left\{\left|\frac{\mathfrak{f}(\varkappa)-\mathfrak{f}(y)}{\varkappa - y}\right|;\ \varkappa \neq y\right\} = M < \infty.$$

The following inequality holds:

$$\begin{aligned} & \left|\mathfrak{f}(\varkappa) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(t)\, d_{\mathfrak{h}}t\right| \\ \leq \; & \frac{M}{2(\beta-\alpha)}\left[(\varkappa-\alpha)(\varkappa-\alpha+\mathfrak{h}) + (\beta-\varkappa)(\beta-\varkappa-\mathfrak{h})\right]. \end{aligned} \tag{3.18}$$

Proof. It is easy to note that

$$\begin{aligned} \left|\mathfrak{f}(\varkappa) - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathfrak{f}(t)\, d_{\mathfrak{h}}t\right| &= \left|\frac{1}{\beta-\alpha}\int_\alpha^\beta (\mathfrak{f}(\varkappa)-\mathfrak{f}(t))\, d_{\mathfrak{h}}t\right| \\ &\leq \frac{1}{\beta-\alpha}\int_\alpha^\beta |\mathfrak{f}(\varkappa)-\mathfrak{f}(t)|\, d_{\mathfrak{h}}t \\ &\leq \frac{M}{\beta-\alpha}\int_\alpha^\beta |\varkappa - t|\, d_{\mathfrak{h}}t \\ &= \frac{M}{2(\beta-\alpha)}\left[(\varkappa-\alpha)(\varkappa-\alpha+\mathfrak{h}) + (\beta-\varkappa)(\beta-\varkappa-\mathfrak{h})\right], \end{aligned}$$

and the proof is completed. □

Remark 3.14. *If we take the limit as* $\mathfrak{h} \to 0$ *in Theorem 3.13, then we have the following classical Ostrowski inequality*

$$\begin{aligned} & \left| \mathfrak{f}(\varkappa) - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \mathfrak{f}(t)\, dt \right| \\ \leq\ & \frac{M}{2(\beta - \alpha)} \left[(\varkappa - \alpha)^2 + (\beta - \varkappa)^2 \right]. \end{aligned}$$

See, also [20].

Theorem 3.15 (Trapezoidal inequality)**.** *For an* $\mathfrak{h}$*-differentiable mapping* $\mathfrak{f} : I \to \mathbb{R}$ *with* $\left| D_{\mathfrak{h}} \mathfrak{f}(\varkappa) \right| \leq M$*, the following inequality holds:*

$$\left| \int_{\alpha}^{\beta} \mathfrak{f}(t + \mathfrak{h})\, d_{\mathfrak{h}} t - (\beta - \alpha) \frac{\mathfrak{f}(\alpha) + \mathfrak{f}(\beta)}{2} \right| \leq \frac{M(\beta - \alpha)}{8} \left[(\beta - \alpha + 2\mathfrak{h}) + (\beta - \alpha - 2\mathfrak{h}) \right]. \tag{3.19}$$

Proof. Form $\mathfrak{h}$-integration by parts, it is easy to note that

$$\int_{\alpha}^{\beta} \left(t - \frac{\alpha + \beta}{2} \right) D_{\mathfrak{h}} \mathfrak{f}(t)\, d_{\mathfrak{h}} t = (\beta - \alpha) \frac{\mathfrak{f}(\alpha) + \mathfrak{f}(\beta)}{2} - \int_{\alpha}^{\beta} \mathfrak{f}(t + \mathfrak{h})\, d_{\mathfrak{h}} t. \tag{3.20}$$

Taking modulus on the both sides of (3.20), we have

$$\begin{aligned} \left| \int_{\alpha}^{\beta} \mathfrak{f}(t + \mathfrak{h})\, d_{\mathfrak{h}} t - (\beta - \alpha) \frac{\mathfrak{f}(\alpha) + \mathfrak{f}(\beta)}{2} \right| &\leq \int_{\alpha}^{\beta} \left| \left(t - \frac{\alpha + \beta}{2} \right) \right| \left| D_{\mathfrak{h}} \mathfrak{f}(t) \right| d_{\mathfrak{h}} t \\ &\leq M \int_{\alpha}^{\beta} \left| \left(t - \frac{\alpha + \beta}{2} \right) \right| d_{\mathfrak{h}} t \\ &= \frac{M(\beta - \alpha)}{8} \left[(\beta - \alpha + 2\mathfrak{h}) + (\beta - \alpha - 2\mathfrak{h}) \right], \end{aligned}$$

and the proof is finished. □

Remark 3.16. *If we set* $\mathfrak{h} \to 0$ *in Theorem 3.15, then we have the following classical trapezoidal inequality*

$$\left| \int_{\alpha}^{\beta} \mathfrak{f}(t)\, dt - (\beta - \alpha) \frac{\mathfrak{f}(\alpha) + \mathfrak{f}(\beta)}{2} \right| \leq \frac{M(\beta - \alpha)^2}{4}.$$

See, also [20].

4. Applications

For arbitrary positive numbers κ_1, κ_2 ($\kappa_1 \neq \kappa_2$), we consider the means as follows:

(1) The arithmatic mean

$$\mathcal{A} = \mathcal{A}(\kappa_1, \kappa_2) = \frac{\kappa_1 + \kappa_2}{2}.$$

(2) The geometric mean

$$\mathcal{G} = \mathcal{G}(\kappa_1, \kappa_2) = \sqrt{\kappa_1 \kappa_2}.$$

(3) The harmonic means

$$\mathcal{H} = \mathcal{H}(\kappa_1, \kappa_2) = \frac{2\kappa_1\kappa_2}{\kappa_1 + \kappa_2}.$$

Proposition 4.1. *For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, the following inequality is true:*

$$\left(\mathcal{A}(\alpha,\beta) - \frac{\mathfrak{h}}{2}\right)^2 \leq \Theta_1 \leq \mathcal{A}\left(\alpha^2, \beta^2\right) - \mathfrak{h}\mathcal{A}(\alpha,\beta),$$

where

$$\Theta_1 = \frac{1}{\beta-\alpha}\int_\alpha^\beta \varkappa^2 d_{\mathfrak{h}}\varkappa = \frac{\mathfrak{h}}{\beta-\alpha}\sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} (\alpha + j\mathfrak{h})^2.$$

Proof. The inequality (3.2) in Theorem 3.4 for mapping $\mathfrak{f}(\varkappa) = \varkappa^2$ leads to this conclusion. □

Proposition 4.2. *For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, the following inequality is true:*

$$\left(\mathcal{A}(\alpha,\beta) - \frac{\mathfrak{h}}{2}\right)^{-2} \leq \Theta_2 \leq \mathcal{H}^{-1}\left(\alpha^2, \beta^2\right) - \mathfrak{h}\mathcal{H}^{-1}(\alpha,\beta)\,\mathcal{G}^{-1}(\alpha,\beta)$$

where

$$\Theta_2 = \frac{1}{\beta-\alpha}\int_\alpha^\beta \varkappa^{-2} d_{\mathfrak{h}}\varkappa = \frac{\mathfrak{h}}{\beta-\alpha}\sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} (\alpha + j\mathfrak{h})^{-2}.$$

Proof. The inequality (3.2) in Theorem 3.4 for mapping $\mathfrak{f}(\varkappa) = \varkappa^{-2}$, $\varkappa \neq 0$ leads to this conclusion. □

Proposition 4.3. *For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, the following inequality is true:*

$$|\mathcal{A}^p(\alpha,\beta) - \Theta_3| \leq \frac{M(\beta-\alpha)}{4}\left[\left(\frac{\beta-\alpha}{2} + \mathfrak{h}\right) + \left(\frac{\beta-\alpha}{2} - \mathfrak{h}\right)\right],$$

$$\Theta_3 = \frac{1}{\beta-\alpha}\int_\alpha^\beta \varkappa^{p} d_{\mathfrak{h}}\varkappa = \frac{\mathfrak{h}}{\beta-\alpha}\sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} (\alpha + j\mathfrak{h})^{p}.$$

Proof. The inequality in Theorem 3.13 for $\varkappa = \frac{\alpha+\beta}{2}$ and mapping $\mathfrak{f}(\varkappa) = \varkappa^p$, $p \geq 2$ leads to this conclusion. □

Proposition 4.4. *For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, the following inequality is true:*

$$\left|\mathcal{A}^{-1}(\alpha,\beta) - \Theta_4\right| \leq \frac{M(\beta-\alpha)}{4}\left[\left(\frac{\beta-\alpha}{2} + \mathfrak{h}\right) + \left(\frac{\beta-\alpha}{2} - \mathfrak{h}\right)\right],$$

$$\Theta_4 = \frac{1}{\beta-\alpha}\int_\alpha^\beta \varkappa^{-1} d_{\mathfrak{h}}\varkappa = \frac{\mathfrak{h}}{\beta-\alpha}\sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} (\alpha + j\mathfrak{h})^{-1}.$$

Proof. The inequality in Theorem 3.13 for $\varkappa = \frac{\alpha+\beta}{2}$ and mapping $\mathfrak{f}(\varkappa) = \varkappa^{-1}$, $\varkappa \neq 0$ leads to this conclusion. □

Proposition 4.5. *For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, the following inequality is true:*

$$\left|\Theta_5 - (\beta - \alpha)\mathcal{A}(\alpha^p, \beta^p)\right| \leq \frac{M(\beta - \alpha)}{8}\left[(\beta - \alpha + 2\mathfrak{h}) + (\beta - \alpha - 2\mathfrak{h})\right],$$

where

$$\Theta_5 = \int_\alpha^\beta (t + \mathfrak{h})^p \, d_\mathfrak{h}\varkappa = \mathfrak{h} \sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} (\alpha + (j+1)\mathfrak{h})^p .$$

Proof. The inequality in Theorem 3.15 for the mapping $\mathfrak{f}(\varkappa) = \varkappa^p$, $p \geq 2$ leads to this conclusion. □

Proposition 4.6. *For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, the following inequality is true:*

$$\left|\Theta_6 - (\beta - \alpha)\ln \mathcal{G}(\alpha, \beta)\right| \leq \frac{M(\beta - \alpha)}{8}\left[(\beta - \alpha + 2\mathfrak{h}) + (\beta - \alpha - 2\mathfrak{h})\right],$$

where

$$\Theta_6 = \int_\alpha^\beta \ln(t + \mathfrak{h}) \, d_\mathfrak{h}\varkappa = \mathfrak{h} \sum_{j=0}^{\frac{\beta-\alpha}{\mathfrak{h}}-1} \ln(\alpha + (j+1)\mathfrak{h}) .$$

Proof. The inequality in Theorem 3.15 for the mapping $\mathfrak{f}(\varkappa) = \ln \varkappa$ leads to this conclusion. □

5. Conclusions

In this work, we used $\mathfrak{h}$-calculus to prove $\mathfrak{h}$-analogs of some classical inequalities, such as H-H inequality, Ostrowski inequality, and trapezoidal inequality. We have utilized several mathematical examples to demonstrate the correctness of the stated inequalities. Since the h-integral of the function is very difficult to calculate, therefore the newly established inequalities (3.2), (3.6) and (3.12) gave the bounds of h-integral of the function and with the help of these bounds we can approximate the value of the h-integral of functions. Moreover, the newly established inequalities (3.18) and (3.19) could be applied to find the error bounds of the h-integral formulas. Future researchers would be able to show similar inequalities for the other classes of functions and for co-ordinated convex functions in their future study.

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Conflict of interest

The authors declare no conflict of interest.

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