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Research article

The generalization of Hermite-Hadamard type Inequality with exp-convexity involving non-singular fractional operator

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Abstract: The theory of convex function has a lot of applications in the field of applied mathematics and engineering. The Caputo-Fabrizio non-singular operator is the most significant operator of fractional theory which permits to generalize the classical theory of differentiation. This study consider the well known Hermite-Hadamard type and associated inequalities to generalize further. To full fill this mileage, we use the exponential convexity and fractional-order differential operator and also apply some existing inequalities like Holder, power mean, and Holder-Iscan type inequalities for further extension. The generalized exponential type fractional integral Hermite-Hadamard type inequalities establish involving the global integral. The applications of the developed results are displayed to verify the applicability. The establish results of this paper can be considered an extension and generalization of the existing results of convex function and inequality in literature and we hope that will be more helpful for the researcher in future work.

Keywords: Caputo-Fabrizio fractional integral; Hermite-Hadamard inequality; exp-convexity
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1. Introduction

The fractional theory of calculus is one of the significant appealing subjects to researchers because of its intensive range of applications in diverse fields of sciences. Nowadays, fractional calculus has a lot of application in many scientific disciplines such as economy [1], geophysics [2], bio-engineering [3], biology [4], medicine [5], demography [6], and also in signal processing. From the last three decades, many scholars and researchers have focused their attention on fractional calculus and studied many real problems in sort of fractional theory [7–11]. Some scholars deduced that fractional theory is necessary to establish new fractional operators with different singular and non-singular kernels to provide an effective area to systematize the real problems in various area of engineering and science [12–16]. Because integer order operator is the local operator that can't measure the effect of neighbors where heavy tails occurs. The fractional-order operator is not local that's why it contrives to tackle sensitive situations where classical operators bear singularity.

Some recent applications of fractional operators are presenting in different scientific areas. Zhang et al. [17] have established numerous the sufficient criteria of the quasi-uniform synchronization for Caputo fractional neural networks by using Laplace transformation and Gronwall inequality. Wang et al. [18] developed the algebraic criterion for the globally projective synchronization of fractional Caputo quantum valued neural networks with the aid of inequality scheme and the Lyapunov direct technique. Zhang et al. [19] have applied Lyapunov function associated to fractional operators to prove the delay-independent asymptotic stability of Riemann¢Liouville fractional-order neutral-type delayed neural networks. Sun established a new integral identity by taking local fractional integral and generalized midpoint inequalities, trapezoidal inequalities and Simpson inequalities [20]. Sun has used harmonically convex to develop a general identity and impart the generalization of Hermite-Hadamard, Ostrowski and Simpson type inequalities [21]. Sun et al. [22] constructed a identity by harmonically convex function on Yang's fractal sets to generalize the two Hermite-Hadamard type inequalities with local fractional integral. Guessab et al. [23,24] analyzed the error between barycentric approximation and convex function and illustrated best possible point wise error estimates of function, also provided a scheme to examine the new differential quasi-interpolation operator. Guessab have presented the Jensen type inequalities on convex polytopes and examined the error in the approximation of a convex function [25,26]. Guessan et al. [27] developed a Korovkin-type theorems which illustrates that it is not necessary a sequence of operators have identity limit. Rajesh et al. [28] considered deformation Due to Thermal Source in elastic body and obtained stress components and temperature distribution with the fractional order. Hobiny et al. [29, 30] investigated the estimate the variation of temperature, the components of stress by generalized thermoelastic theory with fractional operator and examined the influence of time relaxation and fractional order on fiber-reinforced medium. Saeed et al. [31, 32] inspected the impact of fractional order on the two-dimension porous materials and acquired the solution of fractional bio-heat model.

In this study we used the Caputo-Fabrizio fractional integral operator to develop more generalized results. The reason behind this is that this fractional operator contains a non-singular kernel. The significant property of the Caputo-Fabrizio fractional operator is that it turns to an integer order employing Laplace transformation and can find the exact solutions of the real phenomenon.

The fractional theory of calculus has vital applications in the development of fractional inequalities. The Hermite-Hadamard inequality is one of the fundamental definitions in the study of convexity and its generalization. The generalization of Hermite-Hadamard has been presented by many scholars with various fractional derivatives [33–35].

One our best knowledge, Wang et al. [36] used modified h-convex function and find some new results. They established results for Hermite-Hadamard type and associated inequalities involving and h-convexity but the developed results does not contain any exponential characteristic. This study inspired and motivated to construct further generalized inequalities involving the exponential convexity via non-singular fractional-order Caputo-Fabrizio operator.

This article is organized as, some useful and important definitions, theorems, and other notations are discussed in Section 2, a generalization of Hermite-Hadamard type inequalities is depicted in Section 3, Section 4 is dedicated to some new results and at last concluding remarks of this paper are presented.

2. Preliminaries

Here, we will discuss some important notations that will help us further.

Definition 1. (See [37]). Let $\Lambda : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ is a function. Λ is said to be convex function, if the following inequality holds:

$$\Lambda(t\psi + (1-t)\varpi) \leq t\Lambda(\psi) + (1-t)\Lambda(\varpi),$$

where $t \in [0, 1]$.

Definition 2. (See [39]). Let $\Lambda : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ is a function. Λ is said to be exponential convex function, if the following inequality holds:

$$\Lambda(t\psi + (1-t)\varpi) \leq (e^t - 1)\Lambda(\psi) + (e^{1-t} - 1)\Lambda(\varpi),$$

where $t \in [0, 1]$.

Definition 3. (See [38]). Let $F \in H^1(a, b),\ b > a$, Then the left and right Caputo-Fabrizio fractional derivatives are, respectively:

$$\left({}_{\psi}^{CF}D^{\varepsilon}F\right)(t) = \frac{CF(\varepsilon)}{1-\varepsilon}\int_{\psi}^{t} F'(\tau)e^{-\varepsilon\frac{(\tau-t)^{\varepsilon}}{1-\varepsilon}}d\tau,$$

$$\left({}^{CF}D_{\varpi}^{\varepsilon}F\right)(t) = \frac{CF(\varepsilon)}{1-\varepsilon}\int_{t}^{\varpi} F'(\tau)e^{-\varepsilon\frac{(\tau-t)^{\varepsilon}}{1-\varepsilon}}d\tau,$$

and the associated left and right integrals are, respectively,

$$\left({}_{\psi}^{CF}I^{\varepsilon}F\right)(t) = \frac{1-\varepsilon}{CF(\varepsilon)}F(t) + \frac{\varepsilon}{CF(\varepsilon)}\int_{\psi}^{t} F'(\tau)d\tau,$$

$$\left({}^{CF}I_{\varpi}^{\varepsilon}F\right)(t) = \frac{1-\varepsilon}{CF(\varepsilon)}F(t) + \frac{\varepsilon}{CF(\varepsilon)}\int_{t}^{\varpi} F'(\tau)d\tau,$$

where, $CF(\varepsilon) > 0$ and $CF(0) = CF(1) = 1$.

Theorem 2.1. *(See [39]). Let $\Lambda : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function. If $\psi < \varpi$ and $\Lambda \in L[\psi, \varpi]$ then the following Hermite-Hadamard double inequality holds:*

$$\frac{1}{2(\sqrt{e}-1)}\Lambda\left(\frac{\psi+\varpi}{2}\right) \leq \frac{1}{\varpi-\psi}\int_{\psi}^{\varpi}\Lambda(x)dx \leq \frac{\Lambda(\psi)+\Lambda(\varpi)}{2}.$$

Lemma 1. (See [40]). Let $\Lambda : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable function. If $\psi < \varpi$ and $\Lambda \in L[\psi, \varpi]$ then the following equality holds:

$$\frac{\Lambda(\psi) + \Lambda(\varpi)}{2} - \frac{1}{\varpi - \psi} \int_{\psi}^{\varpi} \Lambda(x)dx = \frac{\varpi - \psi}{2} \int_{0}^{1} (1 - 2t)\Lambda'(t\psi - (1 - t)\varpi)dt.$$

Lemma 2. (See [41]). Let $\Lambda : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable function. If $\psi < \varpi$, $\varepsilon \in [0, 1]$ and $\Lambda \in L_1[\psi, \varpi]$ then following equality holds:

$$\begin{aligned} &\frac{\Lambda(\psi) + \Lambda(\varpi)}{2} - \frac{B(\varepsilon)}{\varepsilon(\varpi - \psi)} \left[({}_{\psi}^{CF} I^{\varepsilon} \Lambda)(x) + ({}^{CF} I_{\varpi}^{\varepsilon} \Lambda)(x) \right] \\ &= \frac{\varpi - \psi}{2} \int_{0}^{1} (1 - 2t)\Lambda'(t\psi - (1 - t)\varpi)dt - \frac{2(1 - \varepsilon)}{\varepsilon(\varpi - \psi)} \Lambda(x), \end{aligned}$$

where $x \in [\psi, \varpi]$ and $B(\varepsilon) > 0$ is a normalization function.

Theorem 2.2. *(Hölder-Iscan integral inequality see [42]). Let* $\Lambda, \daleth : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ *are real mappings. Consider* $p > 0$, $p^{-1} + q^{-1} = 1$ *and* $|\Lambda|^q |\daleth|^q$ *are integrable functions so:*

$$\begin{aligned} \int_{\psi}^{\varpi} |\Lambda(x)\daleth(x)|dx \leq & \frac{1}{\varpi - \psi} \left(\int_{\psi}^{\varpi} (\varpi - x)|\Lambda(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\psi}^{\varpi} (\varpi - x)|\daleth(x)|^q dx \right)^{\frac{1}{q}} \\ & + \frac{1}{\varpi - \psi} \left(\int_{\psi}^{\varpi} (\psi - x)|\Lambda(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\psi}^{\varpi} (\psi - x)|\daleth(x)|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

Theorem 2.3. *(Improved power-mean integral inequality see [43]). Let* $\Lambda, \daleth : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ *are real mappings. Consider* $q > 0$, *and* $|\Lambda|^q |\daleth|^q$ *are integrable functions so:*

$$\begin{aligned} \int_{\psi}^{\varpi} |\Lambda(x)\daleth(x)|dx \leq & \frac{1}{\varpi - \psi} \left(\int_{\psi}^{\varpi} (\varpi - x)|\Lambda(x)|dx \right)^{1-\frac{1}{q}} \left(\int_{\psi}^{\varpi} (\varpi - x)|\Lambda(x)||\daleth(x)|^q dx \right)^{\frac{1}{q}} \\ & + \frac{1}{\varpi - \psi} \left(\int_{\psi}^{\varpi} (\psi - x)|\Lambda(x)|dx \right)^{1-\frac{1}{q}} \left(\int_{\psi}^{\varpi} (\psi - x)|\Lambda(x)||\daleth(x)|^q dx \right)^{\frac{1}{q}}. \end{aligned}$$

3. Generalization of Hermite-Hadamard inequality

Here, we will discuss some novel results that will be further useful.

Theorem 3.1. *Consider that* $\Lambda : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ *is an exponential convex function and* $\Lambda \in L_1[\psi, \varpi]$. *If* $\varepsilon \in [0, 1]$ *then we have:*

$$\begin{aligned} \frac{1}{2(\sqrt{e} - 1)} \Lambda\left(\frac{\psi + \varpi}{2}\right) &\leq \frac{B(\varepsilon)}{\varepsilon(\varpi - \psi)} \left[({}_{\psi}^{CF} I^{\varepsilon} \Lambda)(t) + ({}^{CF} I_{\varpi}^{\varepsilon} \Lambda)(t) - \frac{2(1 - \varepsilon)}{B(\varepsilon)} \Lambda(t) \right] \\ &\leq (e - 2) \left(\Lambda(\psi) + \Lambda(\varpi) \right). \end{aligned} \tag{3.1}$$

Where $B(\varepsilon) > 0$ *is normalization function and* $t \in [\psi, \varpi]$.

Proof. Since Λ is exp-convex function then (see [39]):

$$\begin{aligned}\frac{1}{2(\sqrt{e}-1)}\Lambda\left(\frac{\psi+\varpi}{2}\right) &\leq \frac{1}{\varpi-\psi}\int_{\psi}^{\varpi}\Lambda(x)dx\\ &= \frac{1}{\varpi-\psi}\left[\int_{\psi}^{t}\Lambda(x)dx+\int_{t}^{\varpi}\Lambda(x)dx\right].\end{aligned} \tag{3.2}$$

Multiplying both sides by $\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}$ and adding $\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)$ in (3.2):

$$\begin{aligned}&\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}\frac{1}{2(\sqrt{e}-1)}\Lambda\left(\frac{\psi+\varpi}{2}\right)+\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\\ &\quad\leq \frac{(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)+\frac{\varepsilon}{B(\varepsilon)}\int_{\psi}^{t}\Lambda(x)dx+\frac{(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)+\frac{\varepsilon}{B(\varepsilon)}\int_{t}^{\varpi}\Lambda(x)dx.\end{aligned} \tag{3.3}$$

$$\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}\frac{1}{2(\sqrt{e}-1)}\Lambda\left(\frac{\psi+\varpi}{2}\right)+\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\leq\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda)(t)+({}^{CF}I^{\varepsilon}_{\varpi}\Lambda)(t)\right]. \tag{3.4}$$

On the suitable rearrangements of (3.4), we will get our required result.

For right hand side, we substitute $x = t\psi + (1-t)\varpi$:

$$\frac{1}{\varpi-\psi}\int_{\psi}^{\varpi}\Lambda(x)dx=\int_{0}^{1}\Lambda(t\psi+(1-t)\varpi)dt. \tag{3.5}$$

Since Λ is an exp-function so:

$$\frac{1}{\varpi-\psi}\left[\int_{\psi}^{t}\Lambda(x)dx+\int_{t}^{\varpi}\Lambda(x)dx\right]=\int_{0}^{1}\left[(e^{t}-1)\Lambda(\psi)+(e^{1-t}-1)\Lambda(\varpi)\right]dt. \tag{3.6}$$

Multiplying both sides by $\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}$ and adding $\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)$ in (3.6), we will get:

$$\begin{aligned}&\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda)(t)+({}^{CF}I^{\varepsilon}_{\varpi}\Lambda)(t)\right]\\ &\leq\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}(e-2)[\Lambda(\psi)+\Lambda(\varpi)]+\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t).\end{aligned} \tag{3.7}$$

On the suitable rearrangements of (3.7), we will get our required result.

Remark 1. If we put $e^{t}-1=h(t)$ and $e^{1-t}-1=1-h(t)$ then we get a result of [36].

Theorem 3.2. *Consider that $\Lambda, \daleth : j=[\psi,\varpi]\subseteq\mathbb{R}\to\mathbb{R}$ be an exponential convex functions and $\Lambda\daleth\in L_1[\psi,\varpi]$. If $\varepsilon\in[0,1]$ then we have:*

$$\begin{aligned}&\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda\daleth)(t)+({}^{CF}I^{\varepsilon}_{\varpi}\Lambda\daleth)(t)-\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\daleth(t)\right]\\ &\leq\frac{5-4e+e^{2}}{2}M(\psi,\varpi)+(3-e)N(\psi,\varpi),\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}M(\psi,\varpi)&=\Lambda(\psi)\daleth(\psi)+\Lambda(\varpi)\daleth(\varpi),\\ N(\psi,\varpi)&=\Lambda(\psi)\daleth(\varpi)+\Lambda(\varpi)\daleth(\psi),\end{aligned}$$

$B(\varepsilon)>0$ is normalization function and $t\in[\psi,\varpi]$.

Proof. Since $\Lambda, \urcorner$ are exp-convex function so:

$$\Lambda(t\psi + (1-t)\varpi) \leq (e^t - 1)\Lambda(\psi) + (e^{1-t} - 1)\Lambda(\varpi). \tag{3.9}$$

$$\urcorner(t\psi + (1-t)\varpi) \leq (e^t - 1)\urcorner(\psi) + (e^{1-t} - 1)\urcorner(\varpi). \tag{3.10}$$

Multiplying (3.9) and (3.10):

$$\begin{aligned}&\Lambda(t\psi + (1-t)\varpi) \times \urcorner(t\psi + (1-t)\varpi)\\ &\quad\leq (e^t - 1)^2\Lambda(\psi)\urcorner(\psi) + (e^{1-t} - 1)^2\Lambda(\varpi)\urcorner(\varpi) + (e - e^t - e^{1-t} + 1)N(\psi, \varpi).\end{aligned} \tag{3.11}$$

Integrating over $[0, 1]$ and by make change of variable:

$$\begin{aligned}&\frac{1}{\varpi - \psi}\int_\psi^\varpi \Lambda(x)\urcorner(x)dx\\ \leq&\left(\frac{5 - 4e - e^2}{2}\right)M(\psi, \varpi) + (3 - e)N(\psi, \varpi).\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\frac{1}{\varpi - \psi}\left[\int_\psi^t \Lambda(x)\urcorner(x)dx + \int_t^\varpi \Lambda(x)\urcorner(x)dx\right]\\ \leq&\left(\frac{5 - 4e - e^2}{2}\right)M(\psi, \varpi) + (3 - e)N(\psi, \varpi).\end{aligned} \tag{3.13}$$

Multiplying both sides by $\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}$ and adding $\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t)$ in (3.13), we will get:

$$\begin{aligned}&\frac{(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t) + \frac{\varepsilon}{B(\varepsilon)}\int_\psi^t \Lambda(x)\urcorner(x)dx + \frac{(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t) + \frac{\varepsilon}{B(\varepsilon)}\int_t^\varpi \Lambda(x)\urcorner(x)dx\\ &\quad\leq \frac{\varepsilon(\varpi - \psi)}{B(\varepsilon)}\left[\left(\frac{5 - 4e - e^2}{2}\right)M(\psi, \varpi) + (3 - e)N(\psi, \varpi)\right] + \frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t).\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\left[({}_\psi^{CF}I^\varepsilon\Lambda\urcorner)(t) + ({}^{CF}I_\varpi^\varepsilon\Lambda\urcorner)(t)\right]\\ &\quad\leq \frac{\varepsilon(\varpi - \psi)}{B(\varepsilon)}\left[\left(\frac{5 - 4e - e^2}{2}\right)M(\psi, \varpi) + (3 - e)N(\psi, \varpi)\right] + \frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t).\end{aligned} \tag{3.15}$$

On the suitable rearrangements of (3.15), we will get our required result.

Remark 2. If we put $e^t - 1 = h(t)$ and $e^{1-t} - 1 = 1 - h(t)$ then we get a result of [36].

Theorem 3.3. *Consider that $\Lambda, \urcorner : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ be an exponential convex functions and $\Lambda\urcorner \in L_1[\psi, \varpi]$, the set of integrable functions. If $\varepsilon \in [0, 1]$ then we have:*

$$\begin{aligned}&\frac{1}{2(e^{\frac{1}{2}} - 1)^2}\Lambda\left(\frac{\psi + \varpi}{2}\right)\urcorner\left(\frac{\psi + \varpi}{2}\right)\\ &-\frac{B(\varepsilon)}{\varepsilon(\varpi - \psi)}\left[({}_\psi^{CF}I^\varepsilon\Lambda\urcorner)(t) + ({}^{CF}I_\varpi^\varepsilon\Lambda\urcorner)(t)\right] - \frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t)\\ \leq&\frac{5 - 4e + e^2}{2}N(\psi, \varpi) + (3 - e)M(\psi, \varpi),\end{aligned} \tag{3.16}$$

where

$$M(\psi,\varpi)=\Lambda(\psi)\urcorner(\psi)+\Lambda(\varpi)\urcorner(\varpi),$$
$$N(\psi,\varpi)=\Lambda(\psi)\urcorner(\varpi)+\Lambda(\varpi)\urcorner(\psi),$$

$B(\varepsilon)>0$ *is normalization function and* $t\in[\psi,\varpi]$.

Proof. Firstly, utilizing the property of the exponential type convex functions $\Lambda,\urcorner$ are exp-convex functions (see [39]) and for $t=\frac{1}{2}$, so:

$$\Lambda\left(\frac{\psi+\varpi}{2}\right)=\Lambda\left(\frac{(t\psi+(1-t)\varpi)+((1-t)\psi+t\varpi)}{2}\right)$$
$$\leq(e^{\frac{1}{2}}-1)\Lambda(t\psi+(1-t)\varpi)+(e^{1-\frac{1}{2}}-1)\Lambda(1-t)\psi+t\varpi), \tag{3.17}$$

and

$$\urcorner\left(\frac{\psi+\varpi}{2}\right)=\urcorner\left(\frac{(t\psi+(1-t)\varpi)+((1-t)\psi+t\varpi)}{2}\right).$$
$$\leq(e^{\frac{1}{2}}-1)\urcorner(t\psi+(1-t)\varpi)+(e^{1-\frac{1}{2}}-1)\urcorner(1-t)\psi+t\varpi). \tag{3.18}$$

Multiplying (3.17) and (3.18),

$$\begin{aligned}&\Lambda\left(\frac{\psi+\varpi}{2}\right)\urcorner\left(\frac{\psi+\varpi}{2}\right)\\ \leq&(e^{\frac{1}{2}}-1)^2\left[\Lambda(t\psi+(1-t)\varpi)\urcorner(t\psi+(1-t)\varpi)+\Lambda((1-t)\psi+t\varpi)\urcorner((1-t)\psi+t\varpi)\right]\\ +&(e^{\frac{1}{2}}-1)^2\left[(e^{1-t}-1)^2N(\psi,\varpi)+(e^t-1)^2N(\psi,\varpi)+2(e-e^t-e^{1-t}+1)M(\psi,\varpi)\right].\end{aligned} \tag{3.19}$$

Integrating (3.19) over $[0,1]$ and making the change of variables,

$$\begin{aligned}&\frac{1}{2(e^{\frac{1}{2}}-1)^2}\Lambda\left(\frac{\psi+\varpi}{2}\right)\urcorner\left(\frac{\psi+\varpi}{2}\right)\\ \leq&\frac{1}{\varpi-\psi}\int_{\psi}^{\varpi}\Lambda(x)\urcorner(x)dx+\frac{5-4e+e^2}{2}N(\psi,\varpi)+(3-e)M(\psi,\varpi).\end{aligned} \tag{3.20}$$

Multiplying both sides by $\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}$ and adding $\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t)$ in (3.20), we will get:

$$\begin{aligned}&\frac{1}{2(e^{\frac{1}{2}}-1)^2}\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}\Lambda\left(\frac{\psi+\varpi}{2}\right)\urcorner\left(\frac{\psi+\varpi}{2}\right)\\ -&\left[\frac{(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t)+\frac{\varepsilon}{B(\varepsilon)}\int_{\psi}^{t}\Lambda(x)\urcorner(x)dx+\frac{(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t)+\frac{\varepsilon}{B(\varepsilon)}\int_{t}^{\varpi}\Lambda(x)\urcorner(x)dx\right]\\ \leq&\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}\left[\left(\frac{5-4e-e^2}{2}\right)N(\psi,\varpi)+(3-e)M(\psi,\varpi)\right]+\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t).\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\frac{1}{2(e^{\frac{1}{2}}-1)^2}\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}\Lambda\left(\frac{\psi+\varpi}{2}\right)\urcorner\left(\frac{\psi+\varpi}{2}\right)-\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda\urcorner)(t)+({}^{CF}I^{\varepsilon}_{\varpi}\Lambda\urcorner)(t)\right]\\ \leq&\frac{\varepsilon(\varpi-\psi)}{B(\varepsilon)}\left[\left(\frac{5-4e-e^2}{2}\right)N(\psi,\varpi)+(3-e)M(\psi,\varpi)\right]+\frac{2(1-\varepsilon)}{B(\varepsilon)}\Lambda(t)\urcorner(t).\end{aligned} \tag{3.22}$$

On the suitable rearrangements of (3.22), we will get our required result.

Remark 3. If we put $e^t-1=h(t)$ and $e^{1-t}-1=1-h(t)$ then we get a result of [36].

4. Some new results

Theorem 4.1. *Consider that $\Lambda : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function. $|\Lambda'|$ is an exponential convex function on interval j with $\psi < \varpi$ and $\Lambda \in L_1[\psi, \varpi]$. If $\varepsilon \in [0, 1]$ then we have:*

$$\left|\frac{\Lambda(\psi) + \Lambda(\varpi)}{2} - \frac{B(\varepsilon)}{\varepsilon(\varpi - \psi)}\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda)(t) + ({}^{CF}I^{\varepsilon}_{\varpi}\Lambda)(t)\right] + \frac{2(1-\varepsilon)}{\varepsilon(\varpi - \psi)}\Lambda(t)\right| \leq \frac{(8\sqrt{e} - 2e - 7)(\varpi - \psi)}{4}[\Lambda'(\psi) - \Lambda'(\varpi))], \tag{4.1}$$

where $B(\varepsilon) > 0$ is normalization function and $t \in [\psi, \varpi]$.

Proof. By using the Lemma 2,

$$\left|\frac{\Lambda(\psi) + \Lambda(\varpi)}{2} - \frac{B(\varepsilon)}{\varepsilon(\varpi - \psi)}\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda)(t) + ({}^{CF}I^{\varepsilon}_{\varpi}\Lambda)(t)\right] + \frac{2(1-\varepsilon)}{\varepsilon(\varpi - \psi)}\Lambda(t)\right| = \frac{\varpi - \psi}{2}\int_0^1 |1 - 2t||\Lambda'(t\psi + (1-t)\varpi)|dt. \tag{4.2}$$

Since Λ' is exp-convex function so:

$$\left|\frac{\Lambda(\psi) + \Lambda(\varpi)}{2} - \frac{B(\varepsilon)}{\varepsilon(\varpi - \psi)}\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda)(t) + ({}^{CF}I^{\varepsilon}_{\varpi}\Lambda)(t)\right] + \frac{2(1-\varepsilon)}{\varepsilon(\varpi - \psi)}\Lambda(t)\right| \leq \frac{\varpi - \psi}{2}\int_0^1 |1 - 2t|[(e^t - 1)|\Lambda'(\psi)| + (e^{1-t} - 1)|\Lambda'(\varpi)|]dt. \tag{4.3}$$

So,

$$\left|\frac{\Lambda(\psi) + \Lambda(\varpi)}{2} - \frac{B(\varepsilon)}{\varepsilon(\varpi - \psi)}\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda)(t) + ({}^{CF}I^{\varepsilon}_{\varpi}\Lambda)(t)\right] + \frac{2(1-\varepsilon)}{\varepsilon(\varpi - \psi)}\Lambda(t)\right| \leq \frac{(8\sqrt{e} - 2e - 7)(\varpi - \psi)}{4}[\Lambda'(\psi) - \Lambda'(\varpi))]. \tag{4.4}$$

Hence, this is complete proof.

Remark 4. If we put $e^t - 1 = h(t)$ and $e^{1-t} - 1 = 1 - h(t)$ then we get a result of [36].

Theorem 4.2. *Consider that $\Lambda : j = [\psi, \varpi] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable positive function. $|\Lambda'|^q$ is an exponential convex function on interval j with $\psi < \varpi$, where $p > 1, p^{-1} + q^{-1} = 1$ and $\Lambda \in L_1[\psi, \varpi]$. If $\varepsilon \in [0, 1]$ then we have:*

$$\left|\frac{\Lambda(\psi) + \Lambda(\varpi)}{2} - \frac{B(\varepsilon)}{\varepsilon(\varpi - \psi)}\left[({}^{CF}_{\psi}I^{\varepsilon}\Lambda)(t) + ({}^{CF}I^{\varepsilon}_{\varpi}\Lambda)(t)\right] + \frac{2(1-\varepsilon)}{\varepsilon(\varpi - \psi)}\Lambda(t)\right| \leq \frac{(e-2)^{\frac{1}{q}}(\varpi - \psi)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(|\Lambda'(\psi)|^q + |\Lambda'(\varpi)|^q\right)^{\frac{1}{q}}, \tag{4.5}$$

where $B(\varepsilon) > 0$ is normalization function and $t \in [\psi, \varpi]$.

Proof. We will use similar arguments as in previous theorem, by using the Lemma 2,

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[\left({}_{\psi}^{CF}I^{\varepsilon}\Lambda\right)(t)+\left({}^{CF}I_{\varpi}^{\varepsilon}\Lambda\right)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right| = \frac{\varpi-\psi}{2}\int_0^1 |1-2t||\Lambda'(t\psi+(1-t)\varpi)|dt. \tag{4.6}$$

Applying the Holder's inequality:

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[\left({}_{\psi}^{CF}I^{\varepsilon}\Lambda\right)(t)+\left({}^{CF}I_{\varpi}^{\varepsilon}\Lambda\right)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right| \leq \frac{\varpi-\psi}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\int_0^1 ((|\Lambda'(t\psi+(1-t)\varpi)|)^q dt)\right]^{\frac{1}{q}}. \tag{4.7}$$

Applying the exp-convexity:

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[\left({}_{\psi}^{CF}I^{\varepsilon}\Lambda\right)(t)+\left({}^{CF}I_{\varpi}^{\varepsilon}\Lambda\right)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right| \leq \frac{(\varpi-\psi)(e-2)^{\frac{1}{q}}}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(|\Lambda'(\psi)|^q+|\Lambda'(\varpi)|^q)^{\frac{1}{q}}. \tag{4.8}$$

Hence, this is complete proof.

Remark 5. If we put $e^t-1=h(t)$ and $e^{1-t}-1=1-h(t)$ then we get a result of [36].

Theorem 4.3. *Consider that $\Lambda: j=[\psi,\varpi]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable positive function. $|\Lambda'|^q$ is an exponential convex function on interval j with $\psi<\varpi$, where $q>1$, and $\Lambda\in L_1[\psi,\varpi]$. If $\varepsilon\in[0,1]$ then we have:*

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[\left({}_{\psi}^{CF}I^{\varepsilon}\Lambda\right)(t)+\left({}^{CF}I_{\varpi}^{\varepsilon}\Lambda\right)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right| \leq \frac{(\varpi-\psi)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{8\sqrt{e}-2e-7}{2}(|\Lambda'(\psi)|^q+|\Lambda'(\varpi)|^q)\right]^{\frac{1}{q}}, \tag{4.9}$$

where $B(\varepsilon)>0$ is normalization function and $t\in[\psi,\varpi]$.

Proof. Now, we will also use similar arguments as in previous theorem, by using the Lemma 2,

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[\left({}_{\psi}^{CF}I^{\varepsilon}\Lambda\right)(t)+\left({}^{CF}I_{\varpi}^{\varepsilon}\Lambda\right)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right| = \frac{\varpi-\psi}{2}\int_0^1 |1-2t||\Lambda'(t\psi+(1-t)\varpi)|dt. \tag{4.10}$$

Since $q>1$, applying the power mean inequality:

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[({}_{\psi}^{CF}I^{\varepsilon}\Lambda)(t)+({}^{CF}I_{\varpi}^{\varepsilon}\Lambda)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right|$$
$$\leq\frac{\varpi-\psi}{2}\left[\left(\int_0^1|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t||\Lambda'(t\psi+(1-t)\varpi)|^q dt\right)^{\frac{1}{q}}\right]. \tag{4.11}$$

Applying the exp-convexity:

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[({}_{\psi}^{CF}I^{\varepsilon}\Lambda)(t)+({}^{CF}I_{\varpi}^{\varepsilon}\Lambda)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right|$$
$$\leq\frac{\varpi-\psi}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[|\Lambda'(\psi)|^q\int_0^1|1-2t|(e^t-1)dt+|\Lambda'(\varpi)|^q\int_0^1|1-2t|(e^{1-t}-1)dt\right]^{\frac{1}{q}}. \tag{4.12}$$

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[({}_{\psi}^{CF}I^{\varepsilon}\Lambda)(t)+({}^{CF}I_{\varpi}^{\varepsilon}\Lambda)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right|$$
$$\leq\frac{\varpi-\psi}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{8\sqrt{e}-2e-7}{2}\right)(|\Lambda'(\psi)|^q+|\Lambda'(\varpi)|^q)\right]^{\frac{1}{q}}. \tag{4.13}$$

Hence, this is complete proof.

Remark 6. If we put $e^t-1=h(t)$ and $e^{1-t}-1=1-h(t)$ then we get a result of [36].

Theorem 4.4. *Suppose that $\Lambda: j=[\psi,\varpi]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable positive function. $|\Lambda'|^q$ is an exponential convex function on interval j with $\psi<\varpi$, where $p>1, p^{-1}+q^{-1}=1$ and $\Lambda\in L_1[\psi,\varpi]$. If $\varepsilon\in[0,1]$ then we have:*

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[({}_{\psi}^{CF}I^{\varepsilon}\Lambda)(t)+({}^{CF}I_{\varpi}^{\varepsilon}\Lambda)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right|$$
$$\leq\frac{(\varpi-\psi)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left[\left(e|\Lambda'(\psi)|^q+\frac{1}{2}(|\Lambda'(\varpi)|^q-|\Lambda'(\psi)|^q)\right)^{\frac{1}{q}}+\left(e|\Lambda'(\varpi)|^q+\frac{1}{2}(|\Lambda'(\psi)|^q-|\Lambda'(\varpi)|^q)\right)^{\frac{1}{q}}\right], \tag{4.14}$$

where $B(\varepsilon)>0$ is normalization function and $t\in[\psi,\varpi]$.

Proof. Now, we will use similar arguments as in previous theorem, by using the Lemma 2,

$$\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[({}_{\psi}^{CF}I^{\varepsilon}\Lambda)(t)+({}^{CF}I_{\varpi}^{\varepsilon}\Lambda)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right|$$
$$=\frac{\varpi-\psi}{2}\int_0^1|1-2t||\Lambda'(t\psi+(1-t)\varpi)|dt. \tag{4.15}$$

Applying the Hölder-Iscan inequality:

$$\begin{aligned}
&\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[\left({}_{\psi}^{CF}I^{\varepsilon}\Lambda\right)(t)+\left({}^{CF}I_{\varpi}^{\varepsilon}\Lambda\right)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right|\\
&\leq\frac{\varpi-\psi}{2}\left(\int_0^1(1-t)|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)|\Lambda'(t\psi+(1-t)\varpi)|^q dt\right)^{\frac{1}{q}}\\
&\quad+\frac{\varpi-\psi}{2}\left(\int_0^1 t|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t|\Lambda'(t\psi+(1-t)\varpi)|^q dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{4.16}$$

Applying the exp-convexity:

$$\begin{aligned}
&\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[\left({}_{\psi}^{CF}I^{\varepsilon}\Lambda\right)(t)+\left({}^{CF}I_{\varpi}^{\varepsilon}\Lambda\right)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right|\\
&\leq\frac{\varpi-\psi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(|\Lambda'(\psi)|^q\int_0^1(1-t)(e^t-1)dt+|\Lambda'(\varpi)|^q\int_0^1(1-t)(e^{1-t}-1)dt\right)^{\frac{1}{q}}\\
&\quad+\frac{\varpi-\psi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(|\Lambda'(\psi)|^q\int_0^1 t(e^t-1)dt+|\Lambda'(\varpi)|^q\int_0^1 t(e^{1-t}-1)dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{4.17}$$

$$\begin{aligned}
&\left|\frac{\Lambda(\psi)+\Lambda(\varpi)}{2}-\frac{B(\varepsilon)}{\varepsilon(\varpi-\psi)}\left[\left({}_{\psi}^{CF}I^{\varepsilon}\Lambda\right)(t)+\left({}^{CF}I_{\varpi}^{\varepsilon}\Lambda\right)(t)\right]+\frac{2(1-\varepsilon)}{\varepsilon(\varpi-\psi)}\Lambda(t)\right|\leq\frac{(\varpi-\psi)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\\
&\left[\left(e|\Lambda'(\psi)|^q+\frac{1}{2}(|\Lambda'(\varpi)|^q-|\Lambda'(\psi)|^q)\right)^{\frac{1}{q}}+\left(e|\Lambda'(\varpi)|^q+\frac{1}{2}(|\Lambda'(\psi)|^q-|\Lambda'(\varpi)|^q)\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{4.18}$$

Hence, this is complete proof.

Remark 7. If we put $e^t-1=h(t)$ and $e^{1-t}-1=1-h(t)$ then we get a result of [36].

5. Applications to special means

This part is presenting the applications to the secured results.

(1) The arithmetic mean is:

$$A=A(\psi,\varpi)=\frac{\psi+\varpi}{2},\qquad \psi,\varpi\geq 0.$$

(2) The geometric mean is:

$$G=G(\psi,\varpi)=\sqrt{\psi\varpi},\qquad \psi,\varpi\geq 0.$$

(3) The harmonic mean is:

$$H=H(\psi,\varpi)=\frac{2\psi\varpi}{\psi+\varpi},\qquad \psi,\varpi>0.$$

(4) The logarithmic mean is:

$$L = L(\psi, \varpi) = \begin{cases} \frac{\varpi - \psi}{\ln \varpi - \ln \psi}, & \psi \neq \varpi, \\ \varpi, & \psi = \varpi. \end{cases}$$

(5) The p-logarithmic mean is:

$$L_p = L_p(\psi, \varpi) = \begin{cases} \left(\frac{\varpi^{p+1} - \psi^{p+1}}{(p+1)(\varpi - \psi)}\right)^{\frac{1}{p}}, & \psi \neq \varpi,\ p \in \mathbb{R}\,[-1, 0], \\ \psi, & \psi = \varpi,\ \psi, \varpi > 0. \end{cases}$$

(6) The Identric mean is:

$$I = I(\psi, \varpi) = \frac{1}{e}\left(\frac{\varpi^{\varpi}}{\psi^{\psi}}\right)^{\frac{1}{\varpi - \psi}}, \qquad \psi, \varpi > 0.$$

It is obvious that L_p is monotonically increasing over $p \in \mathbb{R}$, $L_0 = I$, $L_{-1} = L$.

Proposition 1. *Let $\psi, \varpi \in \mathbb{R}^+$ with $\psi < \varpi$ and $n \in (-\infty, 0) \cup [1, \infty) - [-1]$. Then, the following inequality holds:*

$$\frac{A^n(\psi, \varpi)}{2(\sqrt{e} - 1)} \leq L_n^n(\psi, \varpi) \leq 2(e-2)A(\psi^n, \varpi^n). \tag{5.1}$$

Proof. *We can easily prove the above defined inequality by putting the $\Lambda(x) = x^n$, $x \in (0, \infty)$, $\varepsilon = 1$, $B(\varepsilon) = 1$, in (3.1).*

Proposition 2. *Let $\psi, \varpi \in \mathbb{R}^+$ with $\psi < \varpi$ and $n \in (-\infty, 0) \cup [1, \infty) - [-1]$. Then, the following inequality holds:*

$$\frac{A^{-1}(\psi, \varpi)}{2(\sqrt{e} - 1)} \leq L^{-1}(\psi, \varpi) \leq 2(e-2)H^{-1}(\psi, \varpi). \tag{5.2}$$

Proof. *We can easily prove the above defined inequality by putting the $\Lambda(x) = x^{-1}$, $x \in (0, \infty)$, $\varepsilon = 1$, $B(\varepsilon) = 1$, in (3.1).*

Proposition 3. *Let $\psi, \varpi \in \mathbb{R}^+$ with $\psi < \varpi$ and $n \in (-\infty, 0) \cup [1, \infty) - [-1]$. Then, the following inequality holds:*

$$2(e-2)\ln G(\psi, \varpi) \leq \ln I(\psi, \varpi) \leq \frac{\ln A(\psi, \varpi)}{2(\sqrt{e}) - 1}. \tag{5.3}$$

Proof. *We can easily prove the above defined inequality by putting the $\Lambda(x) = -\ln x$, $x \in (0, 1]$, $\varepsilon = 1$, $B(\varepsilon) = 1$, in (3.1).*

6. Conclusions

The Caputo-Fabrizio integral operator is utilized to establish some new Hermite-Hadmard type inequalities with the setting of exponential convexity. Some fascinating consequences are acquired involving Caputo-Fabrizio fractional integral operator for the exponential convex function. The acquired results are novel and hope that this research work will be useful and helpful for scholars to develop new ideas in different scientific disciplines.

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Conflict of interest

The authors declare that they have no competing interests.

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