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Research article

Unicity of solution for a semi-infinite inverse heat source problem

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Abstract: A semi-infinite inverse source problem in heat conduction equations is considered, where the source term is assumed to be compactly supported in the region. After introducing a suitable artificial boundary, the semi-infinite problem is transformed into a bounded one and the corresponding exact expression of the boundary condition is derived. Then we rigorously prove the uniqueness of the solution of original problem, together with the stability of the corresponding optimal control solution.

Keywords: semi-infinite domain; heat conduction equation; inverse source problem; uniqueness; stability
Mathematics Subject Classification: 35R30, 49J20

1. Introduction

Parameter identification is a major branch of inverse problem research. Its main task is to utilize all or part of the measured data to retrieve the unknown parameters in the system (see [7, 10, 14, 21, 22]). These unknown parameters often have extremely important impact on the evolution process of the system.

In this paper, we consider the following semi-infinite heat conduction equation (see [8, 23]):

$$\begin{cases} u_t - (a(x)u_x)_x = f(x), & (x,t) \in (-1,+\infty) \times (0,T], \\ u|_{x=-1} = 0, & t \in (0,T], \\ u|_{t=0} = \varphi(x), & x \in [-1,+\infty), \\ u \to 0, & x \to +\infty, \end{cases} \tag{1.1}$$

where $a(x)$ is a given smooth function satisfying

$$a(x) \geq a_0 > 0, \quad x \in [-1,+\infty); \qquad a(x) \equiv a_0, \quad x \in [0,+\infty). \tag{1.2}$$

The support sets of $f(x)$ and $\varphi(x)$ are bounded and satisfy

$$\operatorname{supp}\{\varphi(x)\} \subset \{\, x \mid -1 \leq x \leq 0 \,\},$$

$$\mathrm{supp}\{f(x)\} \subset \{\, x \mid -1 \le x \le 0 \,\}.$$

Without loss of generality, we assume $f(0) = 0$. The initial value function $\varphi(x)$ is known and satisfies $\varphi(-1) = 0$, while the source function $f(x)$ is unknown. In this paper, we are interested in the inverse problem of identifying the heat source $f(x)$.

Model (1.1) comes from the heat conduction problem on some semi-unbounded domains. In practical engineering problems, people often encounter heat transfer problems of large-scale component, such as a long metal rod (rail), but the heat source in the components is completely unknown. The only prior information we know is that the heat source only acts on a small area of the component, and keeps to be zero in other areas. We need to use some additional conditions to reconstruct the heat source in the system, which is the inverse problem we shall study.

The inverse source problem in heat conduction equation (see [15, 18]) is a classical problem in the field of inverse problems. At present, most of the existing results focus on the case of bounded domain, while the documents regarding the unbounded case are quite few. Model (1.1) describes the inverse heat conduction problem on a semi-infinite rod. Theoretically, in order to guarantee the uniqueness of the solution, we need to give all the temperature data of the system at $t = T$, i.e.,

$$u(x, T) = g(x), \quad x \in [-1, +\infty).$$

However, considering the feature of this paper, that is, the source function has compact support, it seems that the measurement data on interval $[-1, 0]$ is enough, because $f(x) \equiv 0,\ \ x \in [0, +\infty)$. So an interesting question arises: *Is the measurement data on* $[-1, 0]$ *sufficient to guarantee the uniqueness of the solution?*

This problem is non-trivial. It should be noted that $f(x) \equiv 0,\ \ x \in [0, +\infty)$ does not ensure $g(x) \equiv 0,\ \ x \in [0, +\infty)$. In fact, due to the thermal diffusion effect, it is impossible for g to be always zero on $[0, +\infty)$. So why the measurement data on $[0, +\infty)$ is not necessary? Can the amount of measurement data be greatly reduced? We try to give a affirmative answer to this problem.

The main ideas are as follows: Firstly, using the idea of artificial boundary, introducing artificial boundary at $x = 0$, deriving the corresponding boundary conditions, transforming the original unbounded problem into a bounded one, and then proving the uniqueness of the solution. The result to be shown in the article is very interesting and significant, which lays a substantial theoretical foundation for the design of related algorithms.

The inverse source problems of parabolic equations and parabolic coupled systems have attracted much interest in these last years (see e.g., [1–3, 5, 6, 9, 11, 12, 16, 20, 24]). The main arguments in these papers rely on the development of the maximum principle, and suitable Carleman estimates. Regarding numerical methods for such inverse problems, we refer to [4, 13, 19, 25–27]. The above systems are considered to be defined in a bounded region. However, the documents treated with the unbounded case are quite few regardless of theoretical or numerical analysis. To the best of our knowledge, we have not find yet the works on uniqueness in determination of source term for semi-infinite heat conduction equations.

This article is organized as follows: In Section 2, the uniqueness of the solution for the original problem is proved. In Section 3, the inverse problem is transformed to an optimal control problem and the necessary condition of the optimal solution is derived. The stability of the minimizer is proved in Section 4.

2. Uniqueness

In order to turn the original problem into a bounded one, we will use the artificial boundary method (see [8]). The main idea is to introduce appropriate artificial boundary first, and then transform the calculation on unbounded domain into that on bounded domain.

In this paper, we choose the following line segment

$$\Sigma = \{ (x,t)|\ x = 0,\ 0 < t < T \},$$

as the artificial boundary. So the unbounded region $Q := (-1, +\infty) \times (0, T)$ is divided by Σ into two parts: The unbounded region

$$Q_1 = \{ (x,t)|\, 0 < x < +\infty,\ 0 < t < T \},$$

and the bounded part

$$Q_0 = \{ (x,t)|\ -1 < x < 0,\ 0 < t < T \}.$$

Due to the fact that φ and f are compactly supported in Q_0, it is easy to see that on Q_1, $u(x,t)$ satisfies the following semi-infinite problem:

$$\begin{cases} u_t - a_0 u_{xx} = 0, & (x,t) \in Q_1, \\ u|_{t=0} = 0, & 0 \le x < \infty, \\ u \to 0, & x \to +\infty. \end{cases} \tag{2.1}$$

When $u(0,t)$ is given, the solution of Eq (2.1) can be expressed as follows (see [23]):

$$u(x,t) = \frac{x}{2\sqrt{a_0\pi}} \int_0^t u(0,\xi) \frac{1}{(t-\xi)^{\frac{3}{2}}} \exp\left(-\frac{x^2}{4a_0(t-\xi)}\right) d\xi. \tag{2.2}$$

Letting $\eta = \frac{x}{2\sqrt{a_0(t-\xi)}}$, then (2.2) is transformed to

$$u(x,t) = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{a_0 t}}}^{\infty} u(0, t - \frac{x^2}{4a_0\eta^2}) \mathrm{e}^{-\eta^2} d\eta. \tag{2.3}$$

Differentiating (2.3) with respect to x, we have

$$\begin{aligned} \frac{\partial u}{\partial x}(x,t) &= \frac{-u(0,0)}{\sqrt{a_0\pi t}} \mathrm{e}^{-\frac{x^2}{4a_0 t}} + \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{a_0 t}}}^{\infty} \frac{\partial u}{\partial t}\left(0, t - \frac{x^2}{4a_0\eta^2}\right) \mathrm{e}^{-\eta^2} \left(-\frac{2x}{4a_0\eta^2}\right) d\eta \\ &= \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{a_0 t}}}^{\infty} \frac{\partial u}{\partial t}\left(0, t - \frac{x^2}{4a_0\eta^2}\right) \mathrm{e}^{-\eta^2} \left(-\frac{2x}{4a_0\eta^2}\right) d\eta, \end{aligned} \tag{2.4}$$

where we have used $u(0,0) = \varphi(0) = 0$. Letting $\eta = \frac{x}{2\sqrt{a_0(t-\xi)}}$, i.e., $\xi = t - \frac{x^2}{4a_0\eta^2}$, we obtain

$$\frac{\partial u}{\partial x}(x,t) = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \frac{\partial u}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} \mathrm{e}^{-\frac{x^2}{4a_0(t-\xi)}} d\xi. \tag{2.5}$$

Letting $x \to 0^+$, we get

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -\frac{1}{\sqrt{a_0 \pi}} \int_0^t \frac{\partial u}{\partial \xi}(0, \xi) \frac{1}{\sqrt{t-\xi}} d\xi, \tag{2.6}$$

which is the exact artificial boundary condition we are looking for.

Combining (1.1) and (2.6), the forward problem can be stated as follows:

$$\begin{cases} u_t - (a(x)u_x)_x = f(x), & (x,t) \in Q_0, \\ u|_{x=-1} = 0, & t \in (0,T], \\ u|_{t=0} = \varphi(x), & x \in [-1,0], \\ \left.\frac{\partial u}{\partial x}\right|_{x=0} = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \frac{\partial u}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} d\xi, & t \in (0,T]. \end{cases} \tag{2.7}$$

Remark 2.1. *Recall the definition of Caputo fractional derivative (see [17]), that is,*

$${}_0^C D_t^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha-n+1}},$$

where n is a positive integer, and $n-1 < \alpha \le n$ is a positive real number. The Eq (2.7) *can be rewritten as*

$$\begin{cases} u_t - (a(x)u_x)_x = f(x), & (x,t) \in Q_0, \\ u|_{x=-1} = 0, & t \in (0,T], \\ u|_{t=0} = \varphi(x), & x \in [-1,0], \\ \left.\frac{\partial u}{\partial x}\right|_{x=0} = -\frac{1}{\sqrt{a_0}} {}_0^C D_t^{\frac{1}{2}} u(0,t), & t \in (0,T], \end{cases}$$

where we have used $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. So the forward problem can be viewed as a heat conduction equation associated with fractional order boundary condition.

Now, we consider the following inverse problem. Hereafter, we use the notation D to denote the interval $(-1,0)$.

Problem P: Given the following additional condition:

$$u(x,T) = g(x), \quad x \in \bar{D}, \tag{2.8}$$

where $g(x)$ is a known function, how to determine a pair of function (u, f) satisfying (2.7) and (2.8).

Lemma 2.1. For any $p(x) \in H^1(D)$ which satisfies $p(-1) = 0$, we have the following estimate

$$\int_D p^2 dx \le \int_D |p_x|^2 dx.$$

Proof. Notice that

$$\begin{aligned} p^2(x) &= \left(\int_{-1}^x \frac{dp}{d\xi} d\xi \right)^2 \\ &\le \left(\int_{-1}^x \left| \frac{dp}{d\xi} \right| \cdot 1 d\xi \right)^2 \\ &\le \int_{-1}^x 1 d\xi \cdot \int_{-1}^x \left| \frac{dp}{d\xi} \right|^2 d\xi \le \int_D |p_x|^2 \, dx. \end{aligned}$$

Integrating on D, one can easily get the conclusion. □

Suppose that $(u_i, f_i),\ i = 1, 2$ are two pairs solution of system (2.7) which satisfy the condition (2.8). Setting

$$U = u_1 - u_2, \quad F = f_1 - f_2,$$

then U satisfies

$$\begin{cases} U_t - (a(x)U_x)_x = F(x), & (x,t) \in Q_0, \\ U|_{x=-1} = 0, & t \in (0,T], \\ U|_{t=0} = 0, & x \in \bar{D}, \\ \frac{\partial U}{\partial x}\Big|_{x=0} = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \frac{\partial U}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} d\xi, & t \in (0,T], \end{cases} \tag{2.9}$$

and the homogeneous terminal condition

$$U(x,T) = 0, \quad x \in \bar{D}. \tag{2.10}$$

It can be easily seen that $F(x) \equiv 0,\quad U(x,t) \equiv 0$ is a possible solution. If the homogeneous systems (2.9) and (2.10) has only zero solution, then the uniqueness is proved.

Theorem 2.2. Any solution pair (F, U) to the inverse problems (2.9) and (2.10) is identically zero, i.e., the solution pair is unique.

Proof. Let $V = U\mathrm{e}^{-\lambda t}$, where λ is a parameter to be chosen later. Then V satisfies the following equation:

$$\begin{cases} V_t - (a(x)V_x)_x + \lambda V = F(x)\mathrm{e}^{-\lambda t} := G(x,t), & (x,t) \in Q_0, \\ V|_{x=-1} = 0, & t \in (0,T], \\ V|_{t=0} = 0, & x \in \bar{D}, \\ \frac{\partial V}{\partial x}\Big|_{x=0} = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \left[\lambda V(0,\xi) + \frac{\partial V}{\partial \xi}(0,\xi)\right] \mathrm{e}^{\lambda(\xi - t)} \frac{1}{\sqrt{t-\xi}} d\xi, & t \in (0,T], \\ V|_{t=T} = 0, & x \in \bar{D}. \end{cases} \tag{2.11}$$

Denote by

$$I_1 = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \left[\lambda V(0,\xi) + \frac{\partial V}{\partial \xi}(0,\xi)\right] \mathrm{e}^{\lambda(\xi-t)} \frac{1}{\sqrt{t-\xi}} d\xi.$$

Notice that

$$V(0,0) = 0, \quad V_t(0,0) = F(0) = 0.$$

So, one can easily get the following equality:

$$\frac{dI_1}{dt}\Big|_{x=0} = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \left[\lambda \frac{\partial V}{\partial \xi}(0,\xi) + \frac{\partial^2 V}{\partial \xi^2}(0,\xi)\right] \mathrm{e}^{\lambda(\xi-t)} \frac{1}{\sqrt{t-\xi}} d\xi. \tag{2.12}$$

Differentiating Eq (2.11) with respect to t, denoting by $W = V_t$ and using (2.12), we obtain the following equation:

$$\begin{cases} W_t - (a(x)W_x)_x + \lambda W = G_t, & (x,t) \in Q_0, \\ W|_{x=-1} = 0, & t \in (0,T], \\ W|_{t=0} = G(x,0) = F(x), & x \in \bar{D}, \\ \frac{\partial W}{\partial x}\Big|_{x=0} = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \left[\lambda W(0,\xi) + \frac{\partial W}{\partial \xi}(0,\xi)\right] \mathrm{e}^{\lambda(\xi-t)} \frac{1}{\sqrt{t-\xi}} d\xi, & t \in (0,T], \\ W|_{t=T} = G(x,T) = F(x)\mathrm{e}^{-\lambda T}, & x \in \bar{D}. \end{cases} \tag{2.13}$$

Multiplying on both sides of (2.13) by W and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_D W^2dx + \int_D a|W_x|^2dx - \int_D (aW_xW)_xdx + \lambda\int_D W^2dx = \int_D WG_tdx. \tag{2.14}$$

Denote the third term in (2.14) by I_2. Using the boundary conditions of (2.13), we have

$$\begin{aligned} I_2 &= -\int_D (aW_xW)_xdx \\ &= -a(x)W_xW|_{x=0} \\ &= a(0)\frac{1}{\sqrt{a_0\pi}}W(0,t)\int_0^t\left[\lambda W(0,\xi) + \frac{\partial W}{\partial \xi}(0,\xi)\right]\mathrm{e}^{\lambda(\xi-t)}\frac{1}{\sqrt{t-\xi}}d\xi. \end{aligned} \tag{2.15}$$

We assert

$$\int_0^T I_2dt \geq 0. \tag{2.16}$$

To prove this assertion, we introduce the following auxiliary function $P(x,t)$ which satisfies the following semi-infinite equation:

$$\begin{cases} P_t - a_0P_{xx} + \lambda P = 0, & (x,t)\in Q_1, \\ P|_{t=0} = 0, & 0 \leq x < \infty, \\ P|_{x=0} = W(0,t), & t\in(0,T], \\ P \to 0, & x\to+\infty. \end{cases} \tag{2.17}$$

Similar to the previous derivation, one can easily check that P satisfies the following Neumann boundary conditions

$$\left.\frac{\partial P}{\partial x}\right|_{x=0} = -\frac{1}{\sqrt{a_0\pi}}\int_0^t\left[\lambda W(0,\xi) + \frac{\partial W}{\partial \xi}(0,\xi)\right]\mathrm{e}^{\lambda(\xi-t)}\frac{1}{\sqrt{t-\xi}}d\xi. \tag{2.18}$$

Let us make some a-priori estimates for P. Let

$$P_1 = P - \mathrm{e}^{-x}W(t) \tag{2.19}$$

(*For convenience, we abbreviate* $W(0,t)$ *with* $W(t)$). Then P_1 satisfies the following equation:

$$\begin{cases} P_{1t} - a_0P_{1xx} + \lambda P_1 = \mathrm{e}^{-x}[(a_0-\lambda)W - W_t] := h(x,t), & (x,t)\in Q_1, \\ P_1|_{t=0} = 0, & 0\leq x<\infty, \\ P_1|_{x=0} = 0, & t\in(0,T], \\ P_1 \to 0, & x\to+\infty, \end{cases} \tag{2.20}$$

where we have used the fact $W(0) = F(0) = 0$. Let us multiply the Eq (2.20) by P_1 and integrate by parts using the Cauchy inequality

$$\begin{aligned} &\frac{1}{2}\int_0^{+\infty} P_1^2(\cdot,t)dx + \int_0^t\int_0^{+\infty} a_0|P_{1x}|^2dxdt \\ \leq\ &(|\lambda|+1)\int_0^t\int_0^{+\infty} P_1^2dxdt + \int_0^T\int_0^{+\infty} h^2dxdt. \end{aligned} \tag{2.21}$$

Using the Gronwall inequality, we obtain

$$\max_{0\le t\le T}\int_0^{+\infty}P_1^2(\cdot,t)dx+\int_0^T\int_0^{+\infty}a_0|P_{1x}|^2dxdt\le C\int_0^T\int_0^{+\infty}h^2dxdt. \tag{2.22}$$

From (2.19) and (2.22), we know

$$\int_0^T\int_0^{+\infty}P^2dxdt\le C\int_0^T\int_0^{+\infty}h^2dxdt+\int_0^T\int_0^{+\infty}W^2(t)\mathrm{e}^{-2x}dxdt\le C. \tag{2.23}$$

Multiplying the Eq (2.17) by P and integrating on Q_1, we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^{+\infty}P^2(\cdot,T)dx+\int_0^T\int_0^{+\infty}a_0|P_x|^2dxdt\\&\qquad-\int_0^T\int_0^{+\infty}(a_0P_xP)_xdxdt+\lambda\int_0^T\int_0^{+\infty}P^2dxdt=0.\end{aligned} \tag{2.24}$$

Combining (2.17)–(2.24), we obtain

$$\begin{aligned}&\int_0^T I_2dt\\=\;&\int_0^T\left\{a(0)\frac{1}{\sqrt{a_0\pi}}W(0,t)\int_0^t\left[\lambda W(0,\xi)+\frac{\partial W}{\partial\xi}(0,\xi)\right]\mathrm{e}^{\lambda(\xi-t)}\frac{1}{\sqrt{t-\xi}}d\xi\right\}dt\\=\;&\frac{1}{2}\int_0^{+\infty}P^2(\cdot,T)dx+\int_0^T\int_0^{+\infty}a_0|P_x|^2dxdt+\lambda\int_0^T\int_0^{+\infty}P^2dxdt.\end{aligned} \tag{2.25}$$

If $\int_0^T\int_0^{+\infty}a_0|P_x|^2dxdt=0$, then $P_x(\cdot,t)\equiv 0,\ t\in[0,T]$ which implies $P(x,\cdot)\equiv W(0,\cdot),\ x\in[0,+\infty)$. Using the last asymptotic condition in (2.17), we get $P(x,t)\equiv 0,\ (x,t)\in\bar{Q}_1$. So $\int_0^T I_2dt\ge 0$ is obvious. If $\int_0^T\int_0^{+\infty}a_0|P_x|^2dxdt>0$, then from (2.23) we can always choose $\lambda<0$ to be sufficiently close to zero such that

$$\int_0^T\int_0^{+\infty}a_0|P_x|^2dxdt+\lambda\int_0^T\int_0^{+\infty}P^2dxdt\ge 0. \tag{2.26}$$

Therefore, in either case, the assertion (2.16) is true.

Now, let's continue the proof of Theorem 2.2. Integrating the Eq (2.14) with respect to t and using (2.16), we have

$$\begin{aligned}&\frac{1}{2}\int_D G^2(\cdot,T)dx-\frac{1}{2}\int_D G^2(\cdot,0)dx+a_0\int_0^T\int_D|W_x|^2dxdt+\lambda\int_0^T\int_D W^2dxdt\\\le\;&\frac{1}{2}\int_D G^2(\cdot,T)dx-\frac{1}{2}\int_D G^2(\cdot,0)dx+\int_0^T\int_D a|W_x|^2dxdt+\lambda\int_0^T\int_D W^2dxdt\\\le\;&\int_0^T\int_D|WG_t|dxdt.\end{aligned} \tag{2.27}$$

Using the Cauchy-Schwarz inequality and Lemma 2.1, we derive from (2.27)

$$
\begin{aligned}
& \frac{1}{2}\int_D G^2(\cdot,T)dx - \frac{1}{2}\int_D G^2(\cdot,0)dx + a_0\int_0^T\int_D W^2 dxdt + \lambda\int_0^T\int_D W^2 dxdt \\
\leq\ & \frac{1}{2}\int_D G^2(\cdot,T)dx - \frac{1}{2}\int_D G^2(\cdot,0)dx + a_0\int_0^T\int_D |W_x|^2 dxdt + \lambda\int_0^T\int_D W^2 dxdt \\
\leq\ & \int_0^T\int_D |WG_t| dxdt \\
\leq\ & (a_0+\lambda)\int_0^T\int_D W^2 dxdt + \frac{1}{4(a_0+\lambda)}\int_0^T\int_D |G_t|^2 dxdt,
\end{aligned}
$$

i.e.,

$$
\frac{1}{2}\int_D G^2(\cdot,T)dx - \frac{1}{2}\int_D G^2(\cdot,0)dx \leq \frac{1}{4(a_0+\lambda)}\int_0^T\int_D |G_t|^2 dxdt. \tag{2.28}
$$

The above inequality can be rewritten as

$$
\int_0^T\int_D \left[GG_t - \frac{1}{4(a_0+\lambda)}|G_t|^2\right] dxdt \leq 0. \tag{2.29}
$$

Noticing that

$$
G = F(x)\mathrm{e}^{-\lambda t}, \quad G_t = -\lambda \mathrm{e}^{-\lambda t}F(x),
$$

one can easily get from (2.29)

$$
\int_0^T\int_D (4a_0\lambda + 5\lambda^2)F^2(x)\mathrm{e}^{-2\lambda t} dxdt \geq 0. \tag{2.30}
$$

Choosing $-\frac{4}{5}a_0 < \lambda < 0$ which satisfies (2.26), we have

$$
4a_0\lambda + 5\lambda^2 < 0. \tag{2.31}
$$

Combining (2.30) and (2.31), we get

$$
F(x) = f_1(x) - f_2(x) \equiv 0, \quad \text{a.e. in } D.
$$

This completes the proof of Theorem 2.2. □

Remark 2.2. *The assertion* (2.16) *is very useful, which illustrates the well-posedness of the solution of truncated problem* (2.13)*. In fact, from* (2.14) *and* (2.16)*, one can easily derive the following estimate:*

$$
\max_{0\leq t\leq T}\int_D W^2 dx + \iint_{Q_0} |W_x|^2 dxdt \leq C\left(\int_D F^2 dx + \iint_{Q_0} |G_t|^2 dxdt\right),
$$

where C is a constant.

Remark 2.3. *The boundedness of* $\|P\|_{L^2(Q_1)}$ *is indispensable in the derivation of* (2.16)*. Since the parameter* λ *may be negative, it is difficult to guarantee the validity of* (2.16) *without* (2.23)*. In fact, if* $\|P\|_{L^2(Q_1)} \to +\infty$*, we can't find a non-zero negative* λ *to make* (2.16) *hold. Moreover, it should be pointed out that the Poincaré type inequality such as Lemma 2.1 can not be applied to such case because the domain in x-direction is infinite.*

Remark 2.4. *It is quite difficult to prove* $\int_0^T I_2 dt \geq 0$ *directly without constructing a suitable auxiliary function. So far, we do not know whether there are other ways to prove the assertion.*

3. Optimal control problem

Although the solution of the original problem is unique, the problem **P** is ill-posed, that is, the solution does not stably depend on the measurement data. Next, we consider the following optimal control problem, which is indeed the regularized problem of the original one.

Construct the following control functional:

$$J(f) = \frac{1}{2}\int_D |u(x,T;f) - g(x)|^2 dx + \frac{\sigma}{2}\int_D |f(x)|^2 dx, \tag{3.1}$$

where the observation data $g(x)$ satisfies

$$g(x) \in L^2(D). \tag{3.2}$$

Problem P1: Consider the following optimal control problem:

$$\hat{f} = \underset{f\in L^2(D)}{\operatorname{argmin}} J(f), \tag{3.3}$$

where $u(x,t;f)$ is the solution to (2.7) for a given source $f(x) \in L^2(D)$, and $\sigma > 0$ is a regularization parameter.

Theorem 3.1. Let $\hat{f}$ be the solution of the optimal control problem (3.3). Then for any $k \in L^2(D)$, there exists a triple of functions $(u, v; \hat{f})$ satisfying the following system:

$$\begin{cases} u_t - (a(x)u_x)_x = \hat{f}(x), & (x,t) \in Q_0, \\ u|_{x=-1} = 0, & t \in (0,T], \\ u|_{t=0} = \varphi(x), & x \in \bar{D}, \\ \left.\frac{\partial u}{\partial x}\right|_{x=0} = -\frac{1}{\sqrt{a_0\pi}}\int_0^t \frac{\partial u}{\partial \xi}(0,\xi)\frac{1}{\sqrt{t-\xi}}d\xi, & t \in (0,T], \end{cases} \tag{3.4}$$

$$\begin{cases} v_t - (a(x)v_x)_x = k(x) - \hat{f}(x), & (x,t) \in Q_0, \\ v|_{x=-1} = 0, & t \in (0,T], \\ v|_{t=0} = 0, & x \in \bar{D}, \\ \left.\frac{\partial v}{\partial x}\right|_{x=0} = -\frac{1}{\sqrt{a_0\pi}}\int_0^t \frac{\partial v}{\partial \xi}(0,\xi)\frac{1}{\sqrt{t-\xi}}d\xi, & t \in (0,T], \end{cases} \tag{3.5}$$

and

$$\int_D [u(x,T;\hat{f}) - g(x)]v(x,T)dx + \sigma\int_D \hat{f}(x)(k(x) - \hat{f}(x))dx \geq 0. \tag{3.6}$$

Proof. For any $k \in L^2(D),\ 0 \leq \delta \leq 1$, we have

$$f_\delta(x) := (1-\delta)\hat{f}(x) + \delta k(x) \in L^2(D).$$

Then

$$J_\delta := J(f_\delta) = \frac{1}{2}\int_D |u(x,T;f_\delta) - g(x)|^2 dx + \frac{\sigma}{2}\int_D |f_\delta(x)|^2 dx. \tag{3.7}$$

Let u_δ be the solution to the Eq (2.7) with given $f = f_\delta$. Since $\hat{f}$ is an optimal solution, we have

$$\left.\frac{dJ_\delta}{d\delta}\right|_{\delta=0} = \int_D [u(x,T;\hat{f}) - g(x)]\left.\frac{\partial u_\delta}{\partial \delta}\right|_{\delta=0} dx + \sigma\int_D \hat{f}(k-\hat{f})dx \geq 0. \tag{3.8}$$

Let $\tilde{u}_\delta := \frac{\partial u_\delta}{\partial \delta}$, direct calculations lead to the following equation:

$$\begin{cases} (\tilde{u}_\delta)_t - (a(x)(\tilde{u}_\delta)_x)_x = k - \hat{f}, & (x,t) \in Q_0, \\ \tilde{u}_\delta|_{x=-1} = 0, & t \in (0,T], \\ \tilde{u}_\delta|_{t=0} = 0, & x \in \bar{D}, \\ \frac{\partial \tilde{u}_\delta}{\partial x}\Big|_{x=0} = -\frac{1}{\sqrt{a_0 \pi}} \int_0^t \frac{\partial \tilde{u}_\delta}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} d\xi, & t \in (0,T]. \end{cases} \tag{3.9}$$

Let $v = \tilde{u}_\delta|_{\delta=0}$, then v satisfies

$$\begin{cases} v_t - (a(x)v_x)_x = k - \hat{f}, & (x,t) \in Q_0, \\ v|_{x=-1} = 0, & t \in (0,T], \\ v|_{t=0} = 0, & x \in \bar{D}, \\ \frac{\partial v}{\partial x}\Big|_{x=0} = -\frac{1}{\sqrt{a_0 \pi}} \int_0^t \frac{\partial v}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} d\xi, & t \in (0,T]. \end{cases} \tag{3.10}$$

Combining (3.8) and (3.10), one can easily obtain that

$$\int_D [u(x,T;\hat{f}) - g(x)]v(x,T)dx + \sigma \int_D \hat{f}(x)(k(x) - \hat{f}(x))dx \geq 0.$$

This completes the proof of Theorem 3.1. □

4. Stability

In this section, we would like to discuss the stability of the optimal solutions.

Theorem 4.1. Suppose that $g_1(x)$ and $g_2(x)$ are two given functions which satisfy (3.2). Let $f_1(x)$ and $f_2(x)$ be the minimizers of the optimal control problem **P1** corresponding to g_1 and g_2, respectively. Then we have the following stability estimate:

$$\int_D |f_1 - f_2|^2 dx \leq \frac{1}{2\sigma} \int_D |g_1 - g_2|^2 dx.$$

Proof. By taking $k = f_2$ when $\hat{f} = f_1$ and taking $k = f_1$ when $\hat{f} = f_2$ in (3.6), we have

$$\int_D [u_1(x,T) - g_1(x)]v_1(x,T)dx + \sigma \int_D f_1(x)(f_2(x) - f_1(x))dx \geq 0, \tag{4.1}$$

$$\int_D [u_2(x,T) - g_2(x)]v_2(x,T)dx + \sigma \int_D f_2(x)(f_1(x) - f_2(x))dx \geq 0. \tag{4.2}$$

where $\{u_i, v_i\}(i = 1,2)$ are solutions of systems (3.4)/(3.5) with $f = f_i$ $(i = 1,2)$, respectively. From (4.1) and (4.1), one can derive

$$\begin{aligned} & \sigma \int_D |f_1 - f_2|^2 dx \\ \leq\ & \int_D [u_1(x,T) - g_1(x)]v_1(x,T)dx + \int_D [u_2(x,T) - g_2(x)]v_2(x,T)dx \\ =\ & \int_D [u_1(x,T) - u_2(x,T)]v_1(x,T)dx + \int_D [u_2(x,T) - g_2][v_1(x,T) + v_2(x,T)]dx \end{aligned}$$

$$+\int_D (g_2 - g_1)v_1(x,T)dx. \tag{4.3}$$

Setting

$$u_1 - u_2 = U, \;\; v_1 + v_2 = V,$$

then U and V satisfy

$$\begin{cases} U_t - (aU_x)_x = f_1 - f_2, \\ U|_{x=-1} = 0, \\ U|_{t=0} = 0, \\ \frac{\partial U}{\partial x}\Big|_{x=0} = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \frac{\partial U}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} d\xi, \end{cases} \tag{4.4}$$

$$\begin{cases} V_t - (aV_x)_x = 0, \\ V|_{x=-1} = 0, \\ V|_{t=0} = 0, \\ \frac{\partial V}{\partial x}\Big|_{x=0} = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \frac{\partial V}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} d\xi. \end{cases} \tag{4.5}$$

We assert that Eq (4.5) has only zero solution. In fact, multiplying on both sides of (4.5) with v and integrating by parts, we have

$$\frac{1}{2}\int_D V^2(x,T)dx + \int_0^T \int_D a|V_x|^2 dxdt$$
$$+ \frac{a(0)}{\sqrt{a_0\pi}} \int_0^T V(0,t) \int_0^t \frac{\partial V}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} d\xi dt = 0. \tag{4.6}$$

Similar to the previous proof, it is easy to see that the third integral in the above equality is greater than or equal to zero. So we get from (4.6) $V_x \equiv 0, \;\; (x,t) \in D \times [0,T]$. Noticing that $V|_{x=-1} = 0$, one can easily get

$$V(x,t) \equiv 0, \;\; (x,t) \in D \times [0,T],$$

i.e.,

$$v_1(x,t) = -v_2(x,t), \;\; (x,t) \in D \times [0,T]. \tag{4.7}$$

Moreover, v_1 satisfies the following equation

$$\begin{cases} v_{1t} - (av_{1x})_x = f_2 - f_1, \\ v_1|_{x=-1} = 0, \\ v_1|_{t=0} = 0, \\ \frac{\partial v_1}{\partial x}\Big|_{x=0} = -\frac{1}{\sqrt{a_0\pi}} \int_0^t \frac{\partial v_1}{\partial \xi}(0,\xi) \frac{1}{\sqrt{t-\xi}} d\xi. \end{cases} \tag{4.8}$$

By noticing (4.4) and (4.8) we have

$$U(x,t) = -v_1(x,t). \tag{4.9}$$

Combining (4.3), (4.7) and (4.9) we have

$$\sigma \int_D |f_1 - f_2|^2 dx$$

$$
\begin{aligned}
&\leq \int_D U(x,T)v_1(x,T)dx + \int_D (g_2 - g_1)v_1(x,T)dx \\
&\leq -\int_D |v_1(x,T)|^2 dx + \frac{1}{2}\int_D |v_1(x,T)|^2 dx + \frac{1}{2}\int_D |g_1 - g_2|^2 dx \\
&= -\frac{1}{2}\int_D |v_1(x,T)|^2 dx + \frac{1}{2}\int_D |g_1 - g_2|^2 dx, \qquad (4.10)
\end{aligned}
$$

which implies

$$
\int_D |f_1 - f_2|^2 dx \leq \frac{1}{2\sigma}\int_D |g_1 - g_2|^2 dx. \qquad (4.11)
$$

This completes the proof of Theorem 4.1. □

Remark 4.1. *It should be mentioned that the regularization parameter plays a major role in the numerical simulation of ill-posed problems. From Theorem 4.1 we can obtain that if there exists a constant δ such that*

$$
\|g_1 - g_2\| \leq \delta, \quad \textit{and} \quad \frac{\delta^2}{\sigma} \to 0,
$$

then the reconstructed heat source is unique and stable.

5. Conclusions

The calculation of parameter identification on unbounded domain is always a difficult problem in the field of inverse problems. In the past, justified by the asymptotic property of the solution at infinity, artificial boundary conditions are added, and then the calculation problem on unbounded domain is transformed into the bounded case. It should be pointed out that such artificial boundary is often transferred directly and has large error. For example, if $u(x,t) \to 0,\ x \to +\infty$, then the artificial boundary condition is often chosen as $u(L,t) = 0$, where $L > 0$ is a relatively large constant. If high precision is needed, the truncation area, i.e., the parameter L should not be too small.

In this paper, we consider the inverse source problem on unbounded domains. Based on the priori property of the source term, we introduce an appropriate artificial boundary and derive the exact expression of the boundary condition. Compared with the common Dirichlet, Nuemann or Robin boundary conditions (usually local), our boundary condition is nonlocal, i.e., it depends on all previous states of the system. Then, we rigorously prove the uniqueness of the solution of the original problem and the stability of the solution of the optimal control problem, which lays a solid theoretical foundation for the future numerical simulation.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. I. Bushuyev, Global uniqueness for inverse parabolic problems with final observation, *Inverse Probl.*, **11** (1995), L11.
2. P. Cannarsa, J. Tort, M. Yamamoto, Determination of source terms in a degenerate parabolic equation, *Inverse Probl.*, **26** (2010), 105003.
3. J. R. Cannon, Y. Lin, An inverse problem of finding a parameter in a semilinear heat equation, *J. Math. Anal. Appl.*, **145** (1990), 470–484. https://doi.org/10.1016/0022-247X(90)90414-B
4. M. Dehghan, An inverse problems of finding a source parameter in a semilinear parabolic equation, *Appl. Math. Model.*, **25** (2001), 743–754. https://doi.org/10.1016/S0307-904X(01)00010-5
5. Z. C. Deng, K. Qian, X. B. Rao, L. Yang, G. W. Luo, An inverse problem of identifying the source coefficient in a degenerate heat equation, *Inverse Probl. Sci. Eng.*, **23** (2015), 498–517. https://doi.org/10.1080/17415977.2014.922079
6. P. Duchateau, W. Rundell, Unicity in an inverse problem for an unknown reaction term in a reaction-diffusion equation, *J. Differ. Equations*, **59** (1985), 155–164. https://doi.org/10.1016/0022-0396(85)90152-4
7. H. W. Engl, M. Hanke, A. Neubauer, *Regularization of inverse problems*, Dordrecht: Kluwer Academic Publishers, 1996.
8. H. D. Han, X. N. Wu, *Artificial boundary method–Numerical solutions of partial differential equations on unbounded domains*, Beijing: Tsinghua University Press, 2009.
9. A. Hasanov, Simultaneous determination of source terms in a linear parabolic problem from the final overdetermination: Weak solution approach, *J. Math. Anal. Appl.*, **330** (2007), 766–779. https://doi.org/10.1016/j.jmaa.2006.08.018
10. V. Isakov, *Inverse problems for partial differential equations*, New York: Springer, 1998. https://doi.org/10.1007/978-1-4899-0030-2
11. V. Isakov, *Inverse source problems*, American Mathematical Society, 1990.
12. V. Isakov, Inverse parabolic problems with the final overdetermination, *Commun. Pure Appl. Math.*, **44** (1991), 185–209. https://doi.org/10.1002/cpa.3160440203
13. T. Johansson, D. Lesnic, Determination of a spacewise dependent heat source, *J. Comput. Appl. Math.*, **209** (2007), 66–80. https://doi.org/10.1016/j.cam.2006.10.026
14. A. Kirsch, *An introduction to the mathematical theory of inverse problem*, New York: Springer, 1999.
15. C. S. Liu, L. Qiu, J. Lin, Simulating thin plate bending problems by a family of two-parameter homogenization functions, *Appl. Math. Model.*, **79** (2020), 284–299. https://doi.org/10.1016/j.apm.2019.10.036
16. M. S. Pilant, W. Rundell, An inverse problem for a nonlinear parabolic equation, *Commun. Partial Differ. Equations*, **11** (1986), 445–457. https://doi.org/10.1080/03605308608820430
17. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.

18. L. Qiu, J. Lin, F. Wang, Q. Qin, C. Liu, A homogenization function method for inverse heat source problems in 3D functionally graded materials, *Appl. Math. Model.*, **91** (2021), 923–933. https://doi.org/10.1016/j.apm.2020.10.012
19. X. B. Rao, Y. X. Wang, K. Qian, Z. C. Deng, L. Yang, Numerical simulation for an inverse source problem in a degenerate parabolic equation, *Appl. Math. Model.*, **39** (2015), 7537–7553. https://doi.org/10.1016/j.apm.2015.03.016
20. W. Rundell, The determination of a parabolic equation from initial and final data, *Proc. Amer. Math. Soc.*, **99** (1987), 637–642. https://doi.org/10.1090/S0002-9939-1987-0877031-4
21. A. A. Samarskii, P. N. Vabishchevich, *Numerical methods for solving inverse problems of mathematical physics*, De Gruyter, 2007. https://doi.org/10.1515/9783110205794
22. A. Tikhonov, V. Arsenin, *Solutions of ill-posed problems*, Beijing: Geology Press, 1979.
23. A. Tikhonov, A. Samarsky, *Equations of mathematical physics*, Beijing: Higher Education Press, 1956.
24. L. Yang, Z. C. Deng, Uniqueness for an inverse source problem in degenerate parabolic equations, *J. Math. Anal. Appl.*, **488** (2020), 124095. https://doi.org/10.1016/j.jmaa.2020.124095
25. L. Yang, M. Dehghan, J. N. Yu, G. W. Luo, Inverse problem of time-dependent heat sources numerical reconstruction, *Math. Comput. Simul.*, **81** (2011), 1656–1672. https://doi.org/10.1016/j.matcom.2011.01.001
26. L. Yang, Z. C. Deng, J. N. Yu, G. W. Luo, Two regularization strategies for an evolutional type inverse heat source problem, *J. Phys. A: Math. Theor.*, **42** (2009), 365203.
27. L. Yang, J. N. Yu, G. W. Luo, Z. C. Deng, Reconstruction of a space and time dependent heat source from finite measurement data, *Int. J. Heat Mass Transfer*, **55** (2012), 6573–6581. https://doi.org/10.1016/j.ijheatmasstransfer.2012.06.064

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