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Research article

Asymptotic behavior of a generalized functional equation

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Abstract: In this paper, we investigate the Hyers-Ulam stability problem of the following functional equation

$$f(x+y)+g(x-y)=h(x)+k(y),$$

on an unbounded restricted domain, which generalizes some of the results already obtained by other authors (for example [9, Theorem 2], [11, Theorem 5] and [21, Theorem 2]). Particular cases of this functional equation are Cauchy, Jensen, quadratic and Drygas functional equations. As a consequence, we obtain asymptotic behaviors of this functional equation.

Keywords: Hyers-Ulam stability; quadratic functional equation; asymptotic behavior
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1. Introduction

Assume that V and W are linear spaces over the field $\mathbb{F}$. Let us recall that a function $f : V \to W$ satisfies the *quadratic functional equation* provided

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x,y\in V. \tag{1.1}$$

In this case f is called a quadratic function. It is well known that a function $f : V \to W$ between real vector spaces V and W satisfies (1.1) if and only if there exists a unique symmetric bi-additive function $B : V\times V \to W$ such that $f(x) = B(x,x)$ for all $x \in V$ (see [1, 7, 13]). For the case $V = W = \mathbb{R}$, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^2$ satisfies (1.1). Indeed, each continuous quadratic function $f : \mathbb{R} \to \mathbb{R}$ has this form. The functional Eq (1.1) plays an important role in the characterization of inner product spaces [8]. We notice that if $\|.\|$ is a norm the parallelogram law is specifically true for norms derived from inner products.

In this paper, we deal with the stability of the functional equation

$$f(x+y)+g(x-y)=h(x)+k(y), \tag{1.2}$$

on restricted domains, where $f, g, h, k : \mathcal{X} \to \mathcal{Y}$ are unknown functions from normed linear space $\mathcal{X}$ to Banach space $\mathcal{Y}$. This functional equation is a generalization of the quadratic functional Eq (1.1). Special cases of this functional equation include the additive functional equation $f(x+y) = f(x)+f(y)$, the Jensen functional equation $f\left(\frac{x+y}{2}\right) = f(x)+f(y)$, the Pexider Cauchy functional equation $f(x+y) = g(x) + h(y)$, and many more. The general solutions of (1.2) were given in [4] without any regularity assumptions on functions f, g, h, k when (1.2) holds for all $x, y \in V$ (see also [14]).

The stability of the quadratic functional Eq (1.1) was first investigated by Skof [23]. Czerwik [2] generalized Skof's result. For more detailed information on the stability results of the functional Eq (1.1) and other quadratic functional equations, we refer the readers to [5,6,9,15–22,25]. We also refer the readers to the books [1,3,7,12,14].

In this paper, stability results of the functional Eq (1.2) on an unbounded restricted domain and its applications are introduced.

2. Stabillity of pexiderized quadratic functional equation

Let f be any function between two linear spaces. The symbols f_e and f_o denote the even and odd parts of f, respectively. Notice that $f_o(0) = 0$ and $f_e(0) = f(0)$.

The following theorem generalizes some of the results already obtained by other authors (for example [9, Theorem 2], [11, Theorem 5] and [21, Theorem 2]).

Theorem 2.1. *Let $\mathcal{X}$ be a linear normed space and $\mathcal{Y}$ be a Banach space, and let $d > 0$ and $\varepsilon \geqslant 0$. Suppose that $f, g, h, k : \mathcal{X} \to \mathcal{Y}$ satisfy*

$$\|f(x+y)+g(x-y)-h(x)-k(y)\| \leqslant \varepsilon, \tag{2.1}$$

for all $x, y \in \mathcal{X}$ with $\|x\|+\|y\| \geqslant d$. Then there are a unique quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ and exactly two additive functions $A_1, A_2 : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x)-Q(x)-A_1(x)-f(0)\| \leqslant 46\varepsilon, \tag{2.2}$$

$$\|g(x)-Q(x)-A_2(x)-g(0)\| \leqslant 46\varepsilon, \tag{2.3}$$

$$\|h(x)-2Q(x)-(A_1+A_2)(x)-h(0)\| \leqslant 29\varepsilon, \tag{2.4}$$

$$\|k(x)-2Q(x)-(A_1-A_2)(x)-k(0)\| \leqslant 29\varepsilon, \tag{2.5}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x by $-x$ and y by $-y$ in (2.1), and adding (subtracting) the resulting inequality to (from) (2.1), we obtain

$$\|f_e(x+y)+g_e(x-y)-h_e(x)-k_e(y)\| \leqslant \varepsilon, \tag{2.6}$$

$$\|f_o(x+y)+g_o(x-y)-h_o(x)-k_o(y)\| \leqslant \varepsilon, \tag{2.7}$$

for all $x, y \in \mathcal{X}$ with $\|x\| + \|y\| \geqslant d$. Putting $x = 0, y = 0, y = x$ and $y = -x$ in (2.6), respectively, we get

$$\|f_e(y) + g_e(y) - h(0) - k_e(y)\| \leqslant \varepsilon, \quad \|y\| \geqslant d, \tag{2.8}$$
$$\|f_e(x) + g_e(x) - h_e(x) - k(0)\| \leqslant \varepsilon, \quad \|x\| \geqslant d, \tag{2.9}$$
$$\|f_e(2x) + g(0) - h_e(x) - k_e(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d, \tag{2.10}$$
$$\|f(0) + g_e(2x) - h_e(x) - k_e(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{2.11}$$

It follows from (2.8)–(2.10) that

$$\|f_e(2x) - 2f_e(x) - 2g_e(x) + g(0) + h(0) + k(0)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d. \tag{2.12}$$

By using (2.8), (2.9) and (2.11), we have

$$\|g_e(2x) - 2f_e(x) - 2g_e(x) + f(0) + h(0) + k(0)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d. \tag{2.13}$$

Hence, (2.12) and (2.13) imply

$$\|f_e(2x) - g_e(2x) + g(0) - f(0)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant d.$$

Then

$$\|f_e(x) - g_e(x) + g(0) - f(0)\| \leqslant 6\varepsilon, \quad \|x\| \geqslant 2d. \tag{2.14}$$

In view of (2.12) and (2.14), we obtain

$$\|f_e(2x) - 4f_e(x) + \alpha\| \leqslant 15\varepsilon, \quad \|x\| \geqslant 2d, \tag{2.15}$$

where $\alpha := 2f(0) - g(0) + h(0) + k(0)$. If we replace x by $2^n x$ in (2.15), and divide the resulting inequality by 4^{n+1}, then we obtain

$$\left\| \frac{f_e(2^{n+1}x)}{4^{n+1}} - \frac{f_e(2^n x)}{4^n} + \frac{\alpha}{4^{n+1}} \right\| \leqslant \frac{15\varepsilon}{4^{n+1}}, \quad \|x\| \geqslant 2d, \ \ n \geqslant 0.$$

So

$$\left\| \frac{f_e(2^{n+1}x)}{4^{n+1}} - \frac{f_e(2^m x)}{4^m} + \sum_{k=m}^{n} \frac{\alpha}{4^{k+1}} \right\| \leqslant \sum_{k=m}^{n} \frac{15\varepsilon}{4^{k+1}}, \quad \|x\| \geqslant 2d, \ \ n \geqslant m \geqslant 0. \tag{2.16}$$

Therefore, $\{\frac{f_e(2^n x)}{4^n}\}_n$ is a Cauchy sequence for each fixed $x \in \mathcal{X}$ with $\|x\| \geqslant 2d$. Thus, by the completeness of $\mathcal{Y}$, the sequence $\{\frac{f_e(2^n x)}{4^n}\}_n$ is convergent for each fixed $x \in \mathcal{X}$ with $\|x\| \geqslant 2d$. Then it is easy to see that the sequence $\{\frac{f_e(2^n x)}{4^n}\}_n$ is convergent for each fixed $x \in \mathcal{X}$. We define the function $Q : \mathcal{X} \to \mathcal{Y}$ by

$$Q(x) = \lim_{n\to\infty} \frac{f_e(2^n x)}{4^n}, \quad x \in \mathcal{X}.$$

It follows from (2.14) that $Q(x) = \lim_{n\to\infty} \frac{g_e(2^n x)}{4^n}$ for all $x \in \mathcal{X}$. In view of (2.8) and (2.9) we have

$$2Q(x) = \lim_{n\to\infty} \frac{k_e(2^n x)}{4^n} \quad \text{and} \quad 2Q(x) = \lim_{n\to\infty} \frac{h_e(2^n x)}{4^n}, \quad x \in \mathcal{X}.$$

Let $x, y \in \mathcal{X} \setminus \{0\}$ and choose $m \in \mathbb{N}$ such that $\|2^n x\|, \|2^n y\| \geqslant d$ for all $n \geqslant m$. Writing $2^n x$ instead of x and $2^n y$ instead of y in (2.6) (for $n \geqslant m$), and dividing the resultant inequality by 4^n, and then letting n approach infinity, we obtain

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \quad x, y \in \mathcal{X} \setminus \{0\}.$$

Since $Q(0) = 0$ and Q is even, we infer that Q is quadratic. Putting $m = 0$ and taking the limit as $n \to \infty$ in (2.16), we get

$$\left\| f_e(x) - Q(x) - \frac{1}{3}\alpha \right\| \leqslant 5\varepsilon, \quad \|x\| \geqslant 2d. \tag{2.17}$$

Replacing y by $-y$ in (2.6), we have

$$\|g_e(x+y) + f_e(x-y) - h_e(x) - k_e(y)\| \leqslant \varepsilon, \quad \|x\| + \|y\| \geqslant d.$$

This inequality is similar to inequality (2.6). By a similar argument, we have

$$\left\| g_e(x) - Q(x) - \frac{1}{3}\beta \right\| \leqslant 5\varepsilon, \quad \|x\| \geqslant 2d, \tag{2.18}$$

where $\beta := 2g(0) - f(0) + h(0) + k(0)$. Adding (2.17) to (2.18), we get

$$\left\| f_e(x) + g_e(x) - 2Q(x) - \frac{1}{3}(\alpha + \beta) \right\| \leqslant 10\varepsilon, \quad \|x\| \geqslant 2d. \tag{2.19}$$

In view of (2.8), (2.9) and (2.19), we obtain

$$\left\| k_e(y) - 2Q(y) + h(0) - \frac{1}{3}(\alpha + \beta) \right\| \leqslant 11\varepsilon, \quad \|y\| \geqslant 2d, \tag{2.20}$$

$$\left\| h_e(x) - 2Q(x) + k(0) - \frac{1}{3}(\alpha + \beta) \right\| \leqslant 11\varepsilon, \quad \|x\| \geqslant 2d. \tag{2.21}$$

Now we extend inequalities (2.17), (2.18), (2.20) and (2.21) to $\mathcal{X}$. Let $z \in \mathcal{X}$, choose $y \in \mathcal{X}$ such that $\|y\| \geqslant 2d + \|z\|$ and let $x = z - y$. Then $\min\{\|x\|, \|x-y\|, \|y\|\} \geqslant 2d$. By (2.18), we have

$$\left\| g_e(x-y) - Q(x-y) - \frac{1}{3}\beta \right\| \leqslant 5\varepsilon. \tag{2.22}$$

It follows from (2.6) and (2.20)–(2.22) that

$$\|f_e(x+y) + Q(x-y) - 2Q(x) - 2Q(y) - f(0)\| \leqslant 28\varepsilon.$$

Since $z = x + y$ and Q is quadratic, we get

$$\|f_e(z) - Q(z) - f(0)\| \leqslant 28\varepsilon. \tag{a}$$

Similarly, for an arbitrary $z \in \mathcal{X}$, we conclude that

$$\|g_e(z) - Q(z) - g(0)\| \leqslant 28\varepsilon. \tag{b}$$

Now, let $x \in \mathcal{X}$ and choose $y \in \mathcal{X}$ such that $\|y\| \geqslant 2d + \|x\|$. It is clear that $\|x \pm y\| \geqslant 2d$. Then by (2.17) and (2.18), we have

$$\left\| f_e(x+y) - Q(x+y) - \frac{1}{3}\alpha \right\| \leqslant 5\varepsilon, \quad \left\| g_e(x-y) - Q(x-y) - \frac{1}{3}\beta \right\| \leqslant 5\varepsilon. \tag{2.23}$$

It follows from (2.6), (2.20) and (2.23) that

$$\|Q(x+y) + Q(x-y) - 2Q(y) - h_e(x) + h(0)\| \leqslant 22\varepsilon.$$

Since Q is quadratic, we obtain

$$\|h_e(x) - 2Q(x) - h(0)\| \leqslant 22\varepsilon. \tag{c}$$

Similarly, for an arbitrary $x \in \mathcal{X}$, we conclude that

$$\|k_e(x) - 2Q(x) - k(0)\| \leqslant 22\varepsilon. \tag{d}$$

Letting $x = 0, y = 0, y = x$ and $y = -x$ in (2.7), respectively, we get

$$\|f_o(y) - g_o(y) - k_o(y)\| \leqslant \varepsilon, \quad \|y\| \geqslant d, \tag{2.24}$$
$$\|f_o(x) + g_o(x) - h_o(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d, \tag{2.25}$$
$$\|f_o(2x) - h_o(x) - k_o(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d, \tag{2.26}$$
$$\|g_o(2x) - h_o(x) + k_o(x)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{2.27}$$

It follows from (2.24) and (2.25) that

$$\|2f_o(x) - h_o(x) - k_o(x)\| \leqslant 2\varepsilon, \quad \|2g_o(x) - h_o(x) + k_o(x)\| \leqslant 2\varepsilon, \quad \|x\| \geqslant d. \tag{2.28}$$

In view of (2.26)–(2.28), we obtain

$$\|f_o(2x) - 2f_o(x)\| \leqslant 3\varepsilon, \quad \|g_o(2x) - 2g_o(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d.$$

It is easy to see that

$$\left\| \frac{f_o(2^{n+1}x)}{2^{n+1}} - \frac{f_o(2^m x)}{2^m} \right\| \leqslant \sum_{i=m}^{n} \frac{3\varepsilon}{2^{i+1}}, \quad \left\| \frac{g_o(2^{n+1}x)}{2^{n+1}} - \frac{g_o(2^m x)}{2^m} \right\| \leqslant \sum_{i=m}^{n} \frac{3\varepsilon}{2^{i+1}}, \tag{2.29}$$

for all $\|x\| \geqslant d$. So, we can define $A_1, A_2 : \mathcal{X} \to \mathcal{Y}$ by

$$A_1(x) := \lim_{n\to\infty} \frac{f_o(2^n x)}{2^n} \quad \text{and} \quad A_2(x) := \lim_{n\to\infty} \frac{g_o(2^n x)}{2^n}, \quad x \in \mathcal{X}.$$

In view of (2.24) and (2.25), we conclude that

$$(A_1 - A_2)(x) = \lim_{n\to\infty} \frac{k_o(2^n x)}{2^n} \quad \text{and} \quad (A_1 + A_2)(x) = \lim_{n\to\infty} \frac{h_o(2^n x)}{2^n}, \quad x \in \mathcal{X}.$$

Let $x \in \mathcal{X}$ and $, y \in \mathcal{X} \setminus \{0\}$. We can choose $m \in \mathbb{N}$ such that $\|2^n y\| \geqslant d$ for all $n \geqslant m$. Writing $2^n x$ instead of x and $2^n y$ instead of y in (2.7) (for $n \geqslant m$), and dividing the resultant inequality by 2^n, and then letting n approach infinity, we obtain

$$A_1(x+y) + A_2(x-y) = (A_1 + A_2)(x) + (A_1 - A_2)(y). \tag{2.30}$$

Since $A_1(0) = A_2(0) = 0$, we get (2.30) for all $x, y \in \mathcal{X}$. For convenience, we set $A = A_1 + A_2$ and $L = A_1 - A_2$. Then A and L are odd because A_1 and A_2 are. By (2.30), we have

$$\begin{aligned} A(x+y) + A(x-y) - 2A(x) &= A_1(x+y) + A_2(x-y) - A(x) - L(y) \\ &\quad + A_1(x-y) + A_2(x+y) - A(x) + L(y) \\ &= 0, \quad x, y \in \mathcal{X}. \end{aligned}$$

Hence A is additive. Thus

$$A_1(x+y) - A_1(x) - A_1(y) = A_2(x) + A_2(y) - A_2(x+y), \quad x, y \in \mathcal{X}.$$

Using (2.30), we obtain

$$A_2(x+y) - A_2(x-y) = 2A_2(y), \quad x, y \in \mathcal{X}.$$

So A_2 is additive, and consequently A_1 is additive.

By (2.29), we obtain

$$\|f_o(x) - A_1(x)\| \leqslant 3\varepsilon \quad \text{and} \quad \|g_o(x) - A_2(x)\| \leqslant 3\varepsilon, \quad \|x\| \geqslant d. \tag{2.31}$$

In view of (2.24), (2.25) and (2.31), we get

$$\|h_o(x) - (A_1 + A_2)(x)\| \leqslant 7\varepsilon \quad \text{and} \quad \|k_o(x) - (A_1 - A_2)(x)\| \leqslant 7\varepsilon, \quad \|x\| \geqslant d. \tag{2.32}$$

Now we extend inequalities (2.31) and (2.32) to $\mathcal{X}$. Let $z \in \mathcal{X}$, choose $y \in \mathcal{X}$ with $\|y\| \geqslant d + \|z\|$ and let $x = z - y$. Then $\min\{\|x\|, \|x-y\|, \|y\|\} \geqslant d$. By (2.31) and (2.32), we have

$$\begin{aligned} \|g_o(x-y) - A_2(x-y)\| &\leqslant 3\varepsilon, \\ \|h_o(x) - (A_1 + A_2)(x)\| &\leqslant 7\varepsilon, \\ \|k_o(y) - (A_1 - A_2)(y)\| &\leqslant 7\varepsilon. \end{aligned}$$

Since $z = x + y$ and A_1, A_2 are additive, these inequalities and (2.7) yield

$$\|f_o(z) - A_1(z)\| \leqslant 18\varepsilon. \tag{a'}$$

Similarly, one can obtain

$$\|g_o(z) - A_2(z)\| \leqslant 18\varepsilon, \quad z \in \mathcal{X}. \tag{b'}$$

To extend (2.32), let $x \in \mathcal{X}$ and choose $y \in \mathcal{X}$ such that $\|y\| \geqslant d + \|x\|$. Then $\|x \pm y\| \geqslant d$. By (2.31), we have

$$\|f_o(x+y) - A_1(x+y)\| \leqslant 3\varepsilon \quad \text{and} \quad \|g_o(x-y) - A_2(x-y)\| \leqslant 3\varepsilon.$$

Using (2.7) and these inequalities, we get

$$\|h_o(x) + k_o(y) - A_1(x+y) - A_2(x-y)\| \leqslant 7\varepsilon. \tag{2.33}$$

Because k_o is odd and A_1, A_2 are additive, interchanging y with $-y$ in (2.33) and adding the resulting inequality to (2.33), we get

$$\|h_o(x) - (A_1 + A_2)(x)\| \leqslant 7\varepsilon. \tag{c'}$$

Similarly, one can obtain

$$\|k_o(x) - (A_1 - A_2)(x)\| \leqslant 7\varepsilon, \quad x \in \mathcal{X}. \tag{d'}$$

In view of (a) and (a'), we get (2.2). By (b) and (b'), we obtain (2.3). (2.4) follows from (c) and (c'). Finally, (d) and (d') yield (2.5). □

Corollary 2.2. *Let $\mathcal{X}, Y$ be linear normed spaces, and $f, g, h, k : \mathcal{X} \to Y$ satisfy*

$$\lim_{\|x\|+\|y\|\to\infty} \|f(x+y) + g(x-y) - h(x) - k(y)\| = 0.$$

Then (f, g, h, k)) is a solution of (1.2) *and they are given by*

$$\begin{aligned}
f(x) &= A_1(x) + Q(x) + f(0),\\
g(x) &= A_2(x) + Q(x) + g(0),\\
h(x) &= (A_1 + A_2)(x) + 2Q(x) + h(0),\\
k(x) &= (A_1 - A_2)(x) + 2Q(x) + k(0), \quad x \in \mathcal{X},
\end{aligned}$$

where $A_1, A_2 : \mathcal{X} \to Y$ are additive and $Q : \mathcal{X} \to Y$ is quadratic.

Proof. Let $\varepsilon > 0$ be an arbitrary. Then there exists $d > 0$ such that

$$\|f(x+y) + g(x-y) - h(x) - k(y)\| < \varepsilon, \quad \|x\| + \|y\| \geqslant d.$$

By Theorem 2.1 (we let $\mathcal{Y}$ be the completion of Y), we get

$$\|f(x+y) + g(x-y) - h(x) - k(y) - f(0) - g(0) + h(0) + k(0)\| \leqslant 150\varepsilon, \quad x, y \in \mathcal{X}.$$

Since $\varepsilon > 0$ was given arbitrarily, we get

$$f(x+y) + g(x-y) - h(x) - k(y) - f(0) - g(0) + h(0) + k(0) = 0, \quad x, y \in \mathcal{X}.$$

Hence the result follows from [10, Theorem 3.1]. □

Corollary 2.3. *Let $\mathcal{X}, Y$ be linear normed spaces, and $f : \mathcal{X} \to Y$ satisfy*

$$\lim_{\|x\|+\|y\|\to\infty} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| = 0.$$

Then f is given by

$$f(x) = A(x) + Q(x) + f(0), \quad x \in \mathcal{X},$$

where $A : \mathcal{X} \to Y$ is additive and $Q : \mathcal{X} \to Y$ is quadratic.

To prove the next theorem, we need the following result.

Lemma 2.4. *[24, Corollary 2.8] Let $\mathcal{X}$ be a linear normed space and $\mathcal{Y}$ be a Banach space, and let $d > 0$ and $\varepsilon \geqslant 0$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a function such that*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \varepsilon, \quad \|x\| + \|y\| \geqslant d.$$

Then there is a unique quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leqslant \frac{7}{6}\varepsilon, \quad x \in \mathcal{X}.$$

Theorem 2.5. *Let $\mathcal{X}$ be a linear normed space and $\mathcal{Y}$ be a Banach space, and let $d > 0$ and $\varepsilon \geqslant 0$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a function such that $\sup_{x\in\mathcal{X}} \|f(x)\| = +\infty$ and*

$$\|f(x+y) + f(x-y) - af(x) - bf(y)\| \leqslant \varepsilon, \tag{2.34}$$

for all $x, y \in \mathcal{X}$ with $\|x\| + \|y\| \geqslant d$, where a, b are real constants with $b \neq 0$. Then there is a unique quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leqslant \frac{7}{6}\varepsilon, \quad x \in \mathcal{X}. \tag{2.35}$$

Proof. By considering the proof of Theorem 2.1, there exists a unique quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ such that $aQ(x) = bQ(x) = 2Q(x)$ and

$$\|f(x) - Q(x) - f(0)\| \leqslant 46\varepsilon, \tag{2.36}$$

for all $x \in \mathcal{X}$. Since f is unbounded, we get $Q \neq 0$ by (2.36). So $a = b = 2$. Consequently, (2.34) implies that

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \varepsilon,$$

for all $x, y \in \mathcal{X}$ with $\|x\| + \|y\| \geqslant d$. By Lemma 2.4, we get (2.35). □

Corollary 2.6. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|f(x+y) + f(x-y) - bf(y)\| \leqslant \varepsilon,$$

for all $x, y \in \mathcal{X}$ with $\|x\| + \|y\| \geqslant d$, where b is a real constant. Then f is bounded.

Theorem 2.7. *Let $\mathcal{X}$ be a linear normed space and $\mathcal{Y}$ be a Banach space, and let $d > 0$ and $\varepsilon \geqslant 0$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a function such that $\sup_{x\in\mathcal{X}} \|f(x)\| = +\infty$ and*

$$\|f(x+y) + f(x-y) - af(x)\| \leqslant \varepsilon, \tag{2.37}$$

for all $x, y \in \mathcal{X}$ with $\|x\| + \|y\| \geqslant d$, where a is a real constant. Then there is a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x) - f(0)\| \leqslant \frac{3}{2}\varepsilon, \quad x \in \mathcal{X}. \tag{2.38}$$

Proof. By considering the proof of Theorem 2.1, there exists a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ such that $aA(x) = 2A(x)$ and

$$\|f(x) - A(x) - f(0)\| \leqslant 46\varepsilon, \tag{2.39}$$

for all $x \in \mathcal{X}$. Since f is unbounded, we get $A \neq 0$ by (2.39). So we get $a = 2$ and consequently, (2.37) implies that

$$\|f(x+y) + f(x-y) - 2f(x)\| \leqslant \varepsilon, \quad \|x\| + \|y\| \geqslant d. \tag{2.40}$$

Then

$$\|A(x) - f(x) + f(0)\| \leqslant \varepsilon, \quad \|x\| \geqslant d. \tag{2.41}$$

Now we extend (2.41) to $\mathcal{X}$. Let $x \in \mathcal{X}$ and choose $y \in \mathcal{X}$ such that $\|y\| \geqslant d + \|x\|$. It is clear that $\|x \pm y\| \geqslant d$. Then (2.41) yields that

$$\|A(x+y) - f(x+y) + f(0)\| \leqslant \varepsilon \quad \text{and} \quad \|A(x-y) - f(x-y) + f(0)\| \leqslant \varepsilon.$$

These inequalities and (2.40) imply that $\|A(x+y) + A(x-y) - 2f(x) + 2f(0)\| \leqslant 3\varepsilon$. Since A is additive, we get (2.38). □

In the following corollary, we investigate the Hyers-Ulam stability of Drygas functional equation which is a special case of Theorem 2.1. In this case we get a sharp bound.

Corollary 2.8. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \leqslant \varepsilon,$$

for all $x, y \in \mathcal{X}$ with $\|x\| + \|y\| \geqslant d$. Then there are a unique quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ and a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x) - A(x) - 2f(0)\| \leqslant \frac{8}{3}\varepsilon, \quad x \in \mathcal{X}. \tag{2.42}$$

Proof. By the assumption, we obtain

$$\|f_e(x+y) + f_e(x-y) - 2f_e(x) - 2f_e(y)\| \leqslant \varepsilon,$$
$$\|f_o(x+y) + f_o(x-y) - 2f_o(x)\| \leqslant \varepsilon,$$

for all $x, y \in \mathcal{X}$ with $\|x\| + \|y\| \geqslant d$. Then by [24, Corollary 2.8] and the argument in the proof of Theorem 2.7, there are a quadratic function $Q : \mathcal{X} \to \mathcal{Y}$ and an additive function $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f_e(x) - Q(x)\| \leqslant \frac{7}{6}\varepsilon, \quad \|f_o(x) - A(x) - f(0)\| \leqslant \frac{3}{2}\varepsilon, \quad x \in \mathcal{X}.$$

Hence we get (2.42). The uniqueness of A and Q is clear. □

3. Conclusions

We have investigated the Hyers-Ulam stability of the Pexider functional Eq (1.2) on an unbounded restricted domain. As a consequence, we have obtained asymptotic behaviors of this functional equation.

Conflict of interest

The authors declare that they have no competing interests.

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