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*Research article*

# Modified inertial Ishikawa iterations for fixed points of nonexpansive mappings with an application

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**Abstract:** This manuscript aims to prove that the sequence $\{\nu_n\}$ created iteratively by a modified inertial Ishikawa algorithm converges strongly to a fixed point of a nonexpansive mapping $Z$ in a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Moreover, zeros of accretive mappings are obtained as an application. Our results generalize and improve many previous results in this direction. Ultimately, two numerical experiments are given to illustrate the behavior of the purposed algorithm.

**Keywords:** nonexpansive mappings; accretive mappings; uniformly Gâteaux differentiable norm; evolution equations
**Mathematics Subject Classification:** 47H09, 47J25

## 1. Introduction

Assume that $\mho$ is a non-empty, closed and convex subset of a real Banach space $\Omega$. The normalized duality mapping $J : \Omega \to 2^{\Omega^*}$ ($\Omega^*$ is the dual space of $\Omega$) is defined by

$$J(\nu) = \left\{\varkappa \in \Omega^* : \langle \nu, \varkappa \rangle = \|\nu\|^2 = \|\varkappa\|^2\right\}.$$

A mapping $Z : \mho \to \mho$ is called nonexpansive if

$$\|Z\nu - Z\wp\| \leq \|\nu - \wp\|, \ \forall \nu, \wp \in \mho.$$

Here, we denote $\nabla(Z)$ by the set of fixed points of $Z$, that is $\nabla(Z) = \{\nu \in \mho : Z\nu = \nu\}$ and we consider $\nabla(Z) \neq \emptyset$.

One of the strong contributions with important applications to fixed point theory is Banach contraction principle, which states:

**Theorem 1.1.** *[1] Every contraction mapping $Z : \hbar \to \hbar$ defined on a complete metric space $(\hbar, \sigma)$ has a unique fixed point, where $\sigma$ is the distance that describes the mapping Z, i.e.,*

$$\sigma(Z\nu, Z\wp) \leq \mu\sigma(\nu, \wp),\ \forall \nu, \wp \in \hbar,\ \mu < 1. \tag{1.1}$$

*Moreover, for arbitrary $\nu_0 \in \hbar$, the sequence $\{\nu_m\}$ created by*

$$\nu_{m+1} = Z\nu_m,\ m \geq 0, \tag{1.2}$$

*converges strongly to the unique fixed point.*

It should be noted that a mapping $Z$ verifying (1.1) is called a strict contraction and if $\mu = 1$ in (1.2), then it is called nonexpansive. The iterative sequence (1.2) is due to Picard [2]. For the iterative formula, it was observed that if the condition $\mu < 1$ on the operator $Z$ is weakened to $\mu = 1$, the sequence $\{\nu_n\}$ defined by (1.2) may fail to converge to a fixed point of $Z$. To overcome this shortcoming, Krasnoselskii [3], replaced Picard iteration formula by the following formula:

$$\nu_0 \in \hbar,\ \nu_{m+1} = \frac{1}{2}(\nu_m + Z\nu_m),\ m \geq 0.$$

He proved that the iterative sequence converges to the fixed point.

In the light of [3], the successful iterative method presented and known as Krasnoselskii-Mann iterative scheme and formulated as follows:

$$\nu_0 \in \mho,\ \nu_{m+1} = (1 - \eta_m)\nu_m + \eta_m Z\nu_m,\ m \geq 0, \tag{1.3}$$

where $\{\eta_m\}$ is a sequence of non-negative real numbers in $(0, 1)$. It was observed that via the stipulation $\nabla(Z) \neq \emptyset$ and mild assumptions forced on $\{\eta_m\}$, the sequence $\{\nu_m\}$ generated by (1.3) converges weakly to a fixed point of $Z$.

Krasnoselskii-Mann algorithm is one of many successful iteration schemes for approximating fixed points of nonexpansive mappings. It provides a unified framework for many algorithms in various disciplines, so the following approach is important.

**Theorem 1.2.** *[4] Assume that Z is a nonexpansive mapping on a real Hilbert space $\urcorner$ and $\nabla(Z) \neq \emptyset$. Then the sequence $\{\nu_n\}$ maked by (1.3) converges weakly to a fixed point of Z, provided that $\eta_m \in [0, 1]$ and $\sum\limits_{m=0}^{\infty} \eta_m = \infty$.*

It should be remarked that all previous contributions on Krasnoselskii-Mann algorithm for nonexpansive mappings have only weak convergence even in a real HS, see [4]. Further, Krasnoselskii-Mann algorithm was generalized by Yang and Zhao [5]. They introduced some important theorems about it and they called their theorems KM theorems.

Bruck [6] noted that the importance of studying nonexpansive mappings lies in two main reasons:

i) nonexpansive mappings are closely related to the monotonicity methods that were updated in the early 1960s and constitute one of the first classes of nonlinear mapping to be treated using the fixed point technique by studying the exact geometrical properties of the basic Banach spaces rather than the compactness properties.

ii) nonexpansive mappings assignments in applications appear as transitional parameters for initial value problems of differential inclusions in the form $0 \in \frac{d\mu}{d\tau} + \Upsilon(\tau)\mu = 0$, where a set-valued operators $\{\Upsilon(\tau)\}$, are accretive or minimally continuous and dissipative.

In nonlinear mapping theory and its applications, building fixed point for nonexpansive mappings assignments is a very important topic, especially, in signal processing and image recovery, see [7–9]. Study of Krasnoselskii-Mann iterative procedures to approximate fixed points of nonexpansive mappings assignments and fixed points of some of their generalizations and approximate zeros of operators of accretive-type has become more widespread and prosperous over the past thirty years or so, for further clarification, we would like to guide the reader to [10–14].

Very recently, a new form for Mann's algorithm is proposed by Bot *et al.* [15] to overcome the deficiency described before, and he described it as follows: let $\varphi_0$ be arbitrary in $\daleth$, for all $m \geq 0$,

$$\nu_{m+1} = \eta_m \nu_{m+1} + \zeta_m \left( Z(\eta_m \nu_m) - \eta_m \nu_m \right), \tag{1.4}$$

they showed that the iterative sequence $\{\nu_n\}$ generated by (1.4) is strongly convergent via suitable assumptions for $\{\eta_m\}$ and $\{\zeta_m\}$. A sequence $\{\zeta_m\}$ in (1.2) has an effective role in acceleration, it called Tikhonov regularization sequence. Many theoretical and numerical discussions to study strong convergence using Tikhonov regularization methodology have been presented by [18–20].

Recently newer types of algorithms have been developed and introduced, such as the inertia algorithm first introduced by Polyak [18]. He used an inertial extrapolation methodology for minimizing a smooth convex function. It is worth noting that, these simple changes affected positively in the performance and effectiveness of these algorithms.

After adopting this concept, researchers were able to implicate additional terms to the inertial algorithm to delve into the study of many vital applications, for example, but not limited to, inertial extragradient algorithms [19–21], inertial projection algorithms [22–26], inertial Mann algorithms [27] and inertial forward-backward splitting algorithms [28,29]. There is no doubt that these algorithms are significantly faster than the inertial algorithms.

Based on the above work, in the present manuscript, Ishikawa algorithm has been developed by adding the term of the inertial to obtain an advanced algorithm, called a modified inertial Ishikawa algorithm. A strong convergence using the proposed algorithm is also discussed in a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Moreover, as an application, we find zeros of accretive mappings. Our results generalize and extend many of the older findings in this regard. Finally, two numerical experiments are given to illustrate the behavior of the purposed algorithm.

## 2. Preliminaries

Assume that $\Omega$ is a real normed linear space and assume $\aleph = \{\nu \in \Omega : \|\nu\| = 1\}$. We say that $\Omega$ have a Gâteaux differentiable norm if the limit below exists for all $\nu, \wp \in \aleph$,

$$\lim_{\tau \to \infty} \frac{\|\nu + \tau\wp\| - \|\nu\|}{\tau},$$

and $\Omega$ is called smooth. Furthermore, we say that $\Omega$ has a uniformly Gâteaux differentiable norm, if for any $\wp \in \aleph$ the limit is attained uniformly for $\nu \in \aleph$. Also, $\Omega$ is called uniformly smooth if the limit exists uniformly for $(\nu, \wp) \in \aleph$. It is obvious that any duality mapping on $\Omega$ is a single-valued if $\Omega$ is smooth and if $\Omega$ has a uniformly Gâteaux differentiable norm then the duality mapping is norm-to-weak$^*$ uniformly continuous on bounded subsets of $\Omega$.

Suppose that $\Delta$ is a non-empty, closed, convex and bounded subset of a real Banach space $\Omega$ and let $d(\Delta) = \sup\{\|\nu - \wp\|,\ \nu, \wp \in \Delta\}$ refer to the diameter of $\Delta$, and for $w(\Delta) = \inf\{w(\nu, \wp),\ \nu \in \Delta\}$ refer to the Chebyshev radius of $\Delta$ relative to itself, where for $\nu \in \Delta$, $w(\nu, \Delta) = \sup\{\|\nu - \wp\|,\ \wp \in \Delta\}$. The normal structure coefficient $N(\Omega)$ of $\Omega$ introduced by Bynum [30] as follows: Let $\Delta$ be a non-empty, closed, convex, and bounded subset of $\Omega$, then $N(\Omega)$ is defined by $N(\Omega) = \inf\left\{\frac{d(\Delta)}{w(\Delta)} : d(\Delta) > 0\right\}$. If $N(\Omega) > 1$, then the space $\Omega$ has a uniform normal structure. It should be noted that, every space with a uniform normal structure is reflexive, this implies that all uniformly convex and uniformly smooth Banach spaces have a uniform normal structure, for example, see, [11, 31].

The lemmas below are very important in the sequel.

**Lemma 2.1.** *[32] Let $\Omega$ be a real uniformly convex Banach space. For arbitrary $u > 0$, assume that $\aleph_u(0) = \{\nu \in \Omega : \|\nu\| \leq u\}$ and $\alpha \in [0, 1]$. Then there is a continuous strictly increasing convex function $r : [0, 2u] \to \mathbb{R}$, $r(0) = 0$ so that the inequality below holds*

$$\|\alpha\nu + (1-\alpha)\wp\|^2 \leq \alpha\|\nu\|^2 + (1-\alpha)\|\wp\|^2 - \alpha(1-\alpha)r(\|\nu - \wp\|).$$

**Lemma 2.2.** *Let $\Omega$ be a real normed linear space, then for all $\nu, \wp \in \Omega$, $j(\nu + \wp) \in J(\nu + \wp)$, we have*

$$\|\nu + \wp\|^2 \leq \|\nu\|^2 + 2\langle \wp, j(\nu + \wp)\rangle.$$

**Lemma 2.3.** *[33] Let $\Omega$ be a uniformly convex Banach space, $\Delta$ be a non-empty, closed and convex, subset of $\Omega$ and $Z : \Delta \to \Delta$ be a nonexpansive mapping with a fixed point. Suppose that the sequence $\{\nu_m\}$ in $\Delta$ is so that $\nu_m \rightharpoonup \nu$ and $\nu_m - Z\nu_m \longrightarrow \wp$. Then $\nu - Z\nu = \wp$.*

**Lemma 2.4.** *[31] Assume that $\Omega$ is a Banach space with uniform normal structure, $\Delta$ is a nononexpansive mapping bounded subset of $\Omega$ and $Z : \Delta \to \Delta$ is a uniformly $L-$Lipschitzian mapping with $L < N(\Omega)^{\frac{1}{2}}$. Consider there is a non-empty bounded closed convex subset $\Re$ of $\Delta$ with the property (D) below:*

$$\nu \in \Re \Rightarrow \varpi_w(\nu) \in \Re.$$

*Then $Z$ has a fixed point in $\Delta$.*

*Note: $\varpi_w(\nu)$ here is the $\varpi-$limit set of $Z$ at $\nu$, that is, the set $\{\wp \in \Omega : y = \text{weak } \varpi - \lim Z^{n_j}\nu$, for some $n_j \to \infty\}$.*

**Lemma 2.5.** *[34] Assume that $(\nu_0, \nu_1, \nu_2, ...) \in l_\infty$, is so that $\delta_m\nu_m \leq 0$ for all Banach limits $\delta$. If $\limsup_{m\to\infty}(\nu_{m+1} - \nu_m) \leq 0$, then $\limsup_{m\to\infty} \nu_m \leq 0$.*

**Lemma 2.6.** *[35] Suppose that $\{e_n\}$ is a sequence of non-negative real numbers verifying the inequality below*

$$e_{m+1} \leq (1 - c_m)e_m + f_m\sigma_m + \pi_m,\ m \geq 1,$$

*if*

- $\{c_m\} \subset [0, 1]$, $\sum c_m = \infty$;
- $\limsup_{m\to\infty} \sigma_m \leq 0$;
- *for each $m \geq 0$, $\pi_m \geq 0$, $\sum \pi_m < \infty$.*

*Then, $\lim_{m\to\infty} e_m = 0$.*

## 3. Main results

Under mild conditions, in this section, we shall discuss the strong convergence of a modified inertial Ishikawa algorithm for nonexpansive mappings.

**Theorem 3.1.** *Let $\Omega$ be a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Suppose that $Z : \Omega \to \Omega$ is a nonexpansive mapping so that $\nabla(Z) \neq \emptyset$. Consider the hypotheses below hold:*

*(i)* $\lim_{m\to\infty} \sigma_m = 0, \sum_{m=1}^{\infty} \sigma_m = \infty, \sigma_m \in (0, 1), \rho_m \in [\ell_1, \ell_2] \subset (0, 1),$

*(ii)* $\lim_{m\to\infty} \pi_m = 0, \pi_m \in (0, 1)$ *and* $\sum_{m=0}^{\infty} \pi_m \|\nu_m - \nu_{m-1}\| < \infty.$

*Set $\nu_0, \nu_1$ arbitrary. Let the sequence $\{\nu_m\}$ created iteratively by*

$$\begin{cases} \hbar_m = \nu_m + \pi_m(\nu_m - \nu_{m-1}), \\ \wp_m = (1 - \sigma_m)\hbar_m, \\ \nu_{m+1} = (1 - \rho_m)\wp_m + \rho_m Z\wp_m, \ m \geq 1. \end{cases} \tag{3.1}$$

*Then the sequence $\{\nu_m\}$ converges strongly to a point in $\nabla(Z)$.*

*Proof.* For any $d \in \nabla(Z)$, by (3.1), we have

$$\begin{aligned} \|\nu_{m+1} - d\| &= \|(1 - \rho_m)(\wp_m - d) + \rho_m (Z\wp_m - d)\| \\ &\leq (1 - \rho_m) \|\wp_m - d\| + \rho_m \|Z\wp_m - d\| \\ &= (1 - \rho_m) \|\wp_m - d\| + \rho_m \|Z\wp_m - Zd\| \\ &\leq (1 - \rho_m) \|\wp_m - d\| + \rho_m \|\wp_m - d\| \\ &\leq \|\wp_m - d\| \\ &= \|(1 - \sigma_m)\hbar_m - d\| \\ &\leq (1 - \sigma_m) \|\hbar_m - d\| + \sigma_m \|d\| \\ &\leq (1 - \sigma_m) \|(\nu_m - d) + \pi_m(\nu_m - \nu_{m-1})\| + \sigma_m \|d\| \\ &\leq (1 - \sigma_m) \|\nu_m - d\| + (1 - \sigma_m) \pi_m \|\nu_m - \nu_{m-1}\| + \sigma_m \|d\| \\ &\leq \max\{\|\nu_m - d\|, \|\nu_m - \nu_{m-1}\|, \|d\|\}. \end{aligned}$$

By mathematical induction, it is easy to see that

$$\|\nu_m - d\| \leq \max\{\|\nu_1 - d\|, \|\nu_1 - \nu_0\|, \|d\|\}, \ \forall m \geq 0.$$

This shows that $\{\nu_m\}$ is bounded and also are $\{\hbar_m\}$ and $\{\wp_m\}$.

Based on Lemmas 2.1, 2.2 and Algorithm (3.1), one sees that

$$\begin{aligned} \|\nu_{m+1} - d\|^2 &= \|(1 - \rho_m)(\wp_m - d) + \rho_m(Z\wp_m - d)\|^2 \\ &\leq (1 - \rho_m) \|\wp_m - d\|^2 + \rho_m \|Z\wp_m - d\|^2 - \rho_m (1 - \rho_m) r (\|Z\wp_m - \wp_m\|) \\ &\leq (1 - \rho_m) \|\wp_m - d\|^2 + \rho_m \|\wp_m - d\|^2 - \rho_m (1 - \rho_m) r (\|Z\wp_m - \wp_m\|) \end{aligned}$$

$$
\begin{aligned}
&= \|\wp_m - d\|^2 - \rho_m (1-\rho_m) r(\|Z\wp_m - \wp_m\|) \\
&\leq \|\hbar_m - d\|^2 + 2\sigma_m \langle \hbar_m - d, j(\wp_m - d)\rangle - \rho_m (1-\rho_m) r(\|Z\wp_m - \wp_m\|) \\
&\leq \|\nu_m - d\|^2 + 2\pi_m \langle \nu_m - d, j(\hbar_m - d)\rangle + 2\sigma_m \langle \hbar_m - d, j(\wp_m - d)\rangle \\
&\quad -\rho_m (1-\rho_m) r(\|Z\wp_m - \wp_m\|).
\end{aligned}
$$

On the other hand, one can write

$$
\begin{aligned}
\rho_m (1-\rho_m) r(\|Z\wp_m - \wp_m\|) \leq\ & \|\nu_m - d\|^2 - \|\nu_{m+1} - d\|^2 + 2\pi_m \langle \nu_m - d, j(\hbar_m - d)\rangle \\
& + 2\sigma_m \langle \hbar_m - d, j(\wp_m - d)\rangle. \qquad (3.2)
\end{aligned}
$$

The boundedness of $\{\nu_m\}$, $\{\hbar_m\}$ and $\{\wp_m\}$ leads to there are constants $\Lambda_1, \Lambda_2 > 0$ so that

$$
\langle \nu_m - d, j(\hbar_m - d)\rangle \leq \Lambda_1 \text{ and } \langle \hbar_m - d, j(\wp_m - d)\rangle \leq \Lambda_2 \text{ for all } m \geq 1. \qquad (3.3)
$$

Applying (3.3) in (3.2), we have

$$
\rho_m (1-\rho_m) r(\|Z\wp_m - \wp_m\|) \leq \|\nu_m - d\|^2 - \|\nu_{m+1} - d\|^2 + 2\pi_m \Lambda_1 + 2\sigma_m \Lambda_2. \qquad (3.4)
$$

In order to obtain the strong convergence, we discuss the following cases:

**Case (a).** If the sequence $\{\|\nu_m - d\|\}$ is monotonically decreasing, then $\{\|\nu_m - d\|\}$ is convergent. It is easy to see that

$$
\|\nu_{m+1} - d\|^2 - \|\nu_m - d\|^2 \to 0,
$$

as $m \to \infty$, this leads to directly by (3.4),

$$
\rho_m (1-\rho_m) r(\|Z\wp_m - \wp_m\|) \to 0.
$$

By the property of $r$ and since $\rho_m \in [\ell_1, \ell_2] \subset (0, 1)$, we have

$$
\|Z\wp_m - \wp_m\| \to 0. \qquad (3.5)
$$

Combining (3.1) and (3.5), we have

$$
\|\nu_{m+1} - \wp_m\| = \rho_m (Z\wp_m - \wp_m) \to 0. \qquad (3.6)
$$

It follows from (3.1) and condition (i) that

$$
\|\wp_m - \hbar_m\| = \sigma_m \|\hbar_m\| \to 0. \qquad (3.7)
$$

By condition (ii), we get

$$
\|\hbar_m - \nu_m\| = \pi_m \|\nu_m - \nu_{m-1}\| \to 0. \qquad (3.8)
$$

Based on (3.7) and (3.8), we can write

$$
\|\wp_m - \nu_m\| \leq \|\wp_m - \hbar_m\| + \|\hbar_m - \nu_m\| \to 0. \qquad (3.9)
$$

Using (3.6) and (3.9), we have

$$
\|\nu_{m+1} - \nu_m\| \leq \|\nu_{m+1} - \wp_m\| + \|\wp_m - \nu_m\| \to 0 \text{ as } m \to \infty.
$$

From (3.5), (3.7) and (3.8), we get

$$\begin{aligned} \|Z\nu_m - \nu_m\| &\le \|Z\nu_m - Z\wp_m\| + \|Z\wp_m - \wp_m\| + \|\nu_m - \wp_m\| \\ &\le 2\,\|\nu_m - \wp_m\| + \|Z\wp_m - \wp_m\| \\ &\le 2\,(\|\wp_m - \hbar_m\| + \|\hbar_m - \nu_m\|) + \|Z\wp_m - \wp_m\| \to 0. \end{aligned}$$

Since $\{\nu_m\}$ is bounded, then there exists the subsequence $\{\nu_{m_b}\} \subset \{\nu_m\}$ so that it converges weakly to $d \in \Omega$. Furthermore, Lemma 2.3 implies that $d \in \nabla(Z)$.

Now, we shall show that

$$\limsup_{m\to\infty}\langle -d, j(\wp_m - d)\rangle \le 0.$$

For this, define a map $\chi : \Omega \to \mathbb{R}$ by

$$\chi(\nu) = \delta_m \|\wp_m - \nu\|^2, \ \forall \nu \in \Omega.$$

Then, $\chi(\nu) \to \infty$ as $\|\nu\| \to \infty$, $\chi$ is convex and continuous. As $\Omega$ is reflexive, then there is $\wp^* \in \Omega$ so that $\chi(\wp^*) = \min_{a\in\Omega}\chi(a)$. Thus, the set $\widehat{\mathfrak{R}} \neq \emptyset$, where $\widehat{\mathfrak{R}}$ is defined as

$$\widehat{\mathfrak{R}} = \left\{\nu \in \Omega : \chi(\nu) = \min_{a\in\Omega}\chi(a)\right\}.$$

Again, since $\lim_{m\to\infty}\|Z\wp_m - \wp_m\| = 0$, then by induction, we can see that $\lim_{m\to\infty}\|Z^n\wp_m - \wp_m\| = 0$ for all $n \ge 1$. Hence, from Lemma 2.4, if $\nu \in \mathfrak{R}$ and $\wp = \varpi - \lim_{j\to\infty} Z^{n_j}\nu$, then from weak lower semi-continuity of $\chi$ and $\lim_{m\to\infty}\|Z\wp_m - \wp_m\| = 0$, we get (since $\lim_{m\to\infty}\|Z\wp_m - \wp_m\| = 0$ implies $\lim_{m\to\infty}\|Z^n\wp_m - \wp_m\| = 0$, $n \ge 1$, this is easily proved by induction)

$$\begin{aligned} \chi(\wp) &\le \liminf_{j\to\infty}\chi(Z^{n_j}\nu) \\ &\le \limsup_{n\to\infty}\chi(Z^n\nu) \\ &= \limsup_{n\to\infty}\left(\delta_m\|\wp_m - Z^n\nu\|^2\right) \\ &= \limsup_{n\to\infty}\left(\delta_m\|\wp_m - Z\wp_m + Z\wp_m - Z^n\nu\|^2\right) \\ &\le \limsup_{n\to\infty}\left(\delta_m\|Z\wp_m - Z^n\nu\|^2\right) \\ &\le \limsup_{n\to\infty}\left(\delta_m\|\wp_m - \nu\|^2\right) = \chi(\nu) = \inf_{a\in\Omega}\chi(a). \end{aligned}$$

Thus, $\wp^* \in \widehat{\mathfrak{R}}$. Therefore by Lemma 2.4, $Z$ has a fixed point in $\widehat{\mathfrak{R}}$ and so $\widehat{\mathfrak{R}} \cap \nabla(Z) \neq \emptyset$. As a special case without losing the general case, suppose that $\wp^* = d \in \widehat{\mathfrak{R}} \cap \nabla(Z)$. Consider $\tau \in (0, 1)$. Then it is easy to see that $\chi(d) \le \chi(d - \tau d)$ with the helping of Lemma 2.2, one sees that

$$\|\wp_m - d + \tau d\|^2 \le \|\wp_m - d\|^2 + 2\tau\langle d, j(\wp_m - d + \tau d)\rangle,$$

by the properties of $\chi$, we can write

$$\frac{1}{\delta_m}\chi(d - \tau d) \le \frac{1}{\delta_m}\chi(d) + 2\tau\langle d, j(\wp_m - d + \tau d)\rangle,$$

By arranging the above inequality, we have

$$2\tau\delta_m\langle -d, j(\wp_m - d + \tau d)\rangle \le \chi(d) - \chi(d - \tau d) \le 0.$$

This leads to

$$\delta_m\langle -d, j(\wp_m - d + \tau d)\rangle \le 0.$$

Moreover,

$$\begin{aligned} \delta_m\langle -d, j(\wp_m - d)\rangle &\le \delta_m\langle -d, j(\wp_m - d) - j(\wp_m - d + \tau d)\rangle + \delta_m\langle -d, j(\wp_m - d + \tau d)\rangle \\ &\le \delta_m\langle -d, j(\wp_m - d) - j(\wp_m - d + \tau d)\rangle. \end{aligned} \tag{3.10}$$

Since the normalized duality mapping is norm-to-weak* uniformly continuous on bounded subsets of $\Omega$, then we have, as $\tau \to 0$ and for fixed $n$,

$$\begin{aligned} &\langle -d, j(\wp_m - d) - j(\wp_m - d + \tau d)\rangle \\ \le\ &\langle -d, j(\wp_m - d)\rangle - \langle -d, j(\wp_m - d + \tau d)\rangle \to 0. \end{aligned}$$

Thus, for each $\epsilon > 0$, there is $\varsigma_\epsilon > 0$ so that for all $\tau \in (0, \varsigma_\epsilon)$,

$$\langle -d, j(\wp_m - d)\rangle - \langle -d, j(\wp_m - d + \tau d)\rangle < \epsilon.$$

Hence,

$$\delta_m\langle -d, j(\wp_m - d)\rangle - \delta_m\langle -d, j(\wp_m - d + \tau d)\rangle \le \epsilon.$$

Because $\epsilon$ is an arbitrary, then by (3.10), one can obtain

$$\delta_m\langle -d, j(\wp_m - d)\rangle \le 0.$$

By triangle inequality, we have

$$\|\wp_{m+1} - \wp_m\| \le \|\wp_{m+1} - \nu_{m+1}\| + \|\nu_{m+1} - \wp_m\|.$$

According to (3.6) and (3.9), we get

$$\lim_{m\to\infty} \|\wp_{m+1} - \wp_m\| = 0.$$

Since the normalized duality mapping is norm-to-weak* uniformly continuous on bounded subsets of $\Omega$, then we have

$$\lim_{m\to\infty} (\langle -d, j(\wp_m - d)\rangle - \langle -d, j(\wp_{m+1} - d)\rangle) = 0.$$

It follows from Lemma 2.5 that

$$\limsup_{m\to\infty}\langle -d, j(\wp_m - d)\rangle \le 0.$$

Ultimately, from (3.1), Stipulation (ii) and Lemma 2.2, we have

$$\begin{aligned} \|\nu_{m+1} - d\|^2 &= \|(1-\rho_m)(\wp_m - d) + \rho_m(Z\wp_m - d)\|^2 \\ &\le (1-\rho_m)\|\wp_m - d\|^2 + \rho_m\|Z\wp_m - d\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|\wp_m - d\|^2 \\
&= \|(1-\sigma_m)(\hbar_m - d) - \sigma_m d\|^2 \\
&= (1-\sigma_m)\|\hbar_m - d\|^2 + 2\sigma_m\langle -d, j(\wp_m - d)\rangle \\
&\leq (1-\sigma_m)\|(\nu_m - d) + \pi_m(\nu_m - \nu_{m-1})\|^2 + 2\sigma_m\langle -d, j(\wp_m - d)\rangle \\
&\leq (1-\sigma_m)\|\nu_m - d\|^2 + 2\pi_m\langle \nu_m - \nu_{m-1}, j(\hbar_m - d)\rangle + 2\sigma_m\langle -d, j(\wp_m - d)\rangle \\
&= (1-\sigma_m)\|\nu_m - d\|^2 + 2\sigma_m\langle -d, j(\wp_m - d)\rangle. \qquad (3.11)
\end{aligned}$$

Applying Lemma 2.6, we conclude that, $\{\nu_m\} \to d \in \nabla(Z)$.

**Case (b).** If the sequence $\{\|\nu_m - d\|\}$ is not monotonically decreasing. Put $\Xi_m = \|\nu_m - d\|^2$ and assume that $\Pi : N \to N$ is a mapping defined by

$$\Pi(m) = \max\{\hbar \in N : \hbar \leq m,\ \Xi_\hbar \leq \Xi_{\hbar+1}\}.$$

Obviously, $\Pi$ is a non-decreasing sequence so that $\lim_{m\to\infty} \Pi(m) = \infty$ and $\Xi_{\Pi(m)} \leq \Xi_{\Pi(m)+1}$ for $m \geq m_0$ (for some $m_0$ large enough). Based on (3.4), one sees that

$$\begin{aligned}
&\rho_{\Pi(m)}(1-\rho_{\Pi(m)})\, r\left(\left\|Z\wp_{\Pi(m)} - \wp_{\Pi(m)}\right\|\right) \\
\leq\ & \left\|\nu_{\Pi(m)} - d\right\|^2 - \left\|\nu_{\Pi(m)+1} - d\right\|^2 + 2\pi_{\Pi(m)}\Lambda_1 + 2\sigma_{\Pi(m)}\Lambda_2 \\
=\ & \Xi_{\Pi(m)} - \Xi_{\Pi(m)+1} + 2\pi_{\Pi(m)}\Lambda_1 + 2\sigma_{\Pi(m)}\Lambda_2 \\
\leq\ & 2\pi_{\Pi(m)}\Lambda_1 + 2\sigma_{\Pi(m)}\Lambda_2 \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Furthermore, we get

$$\left\|Z\wp_{\Pi(m)} - \wp_{\Pi(m)}\right\| \to 0 \text{ as } m \to \infty.$$

By following the same scenario in Case (a) we can prove that $\{\nu_{\Pi(m)}\} \rightharpoonup d$ as $\Pi(m) \to \infty$ and $\limsup_{\Pi(m)\to\infty}\langle -d, j(\wp_{\Pi(m)} - d)\rangle \leq 0$. For all $m \geq m_0$, we obtain by (3.11) that

$$0 \leq \left\|\nu_{\Pi(m)+1} - d\right\|^2 - \left\|\nu_{\Pi(m)} - d\right\|^2 \leq \sigma_{\Pi(m)}\left[2\langle -d, j(\wp_{\Pi(m)} - d)\rangle - \left\|\nu_{\Pi(m)} - d\right\|^2\right],$$

this implies that

$$\left\|\nu_{\Pi(m)} - d\right\|^2 \leq 2\langle -d, j(\wp_{\Pi(m)} - d)\rangle.$$

Since $\limsup_{\Pi(m)\to\infty}\langle -d, j(\wp_{\Pi(m)} - d)\rangle \leq 0$, then we have after taking the limit as $m \to \infty$ in the above inequality,

$$\lim_{m\to\infty}\left\|\nu_{\Pi(m)} - d\right\|^2 = 0.$$

Hence

$$\lim_{m\to\infty}\Xi_{\Pi(m)} = \lim_{m\to\infty}\Xi_{\Pi(m)+1} = 0.$$

Moreover, for all $m \geq m_0$, it is easy to notice that $\Xi_m \leq \Xi_{\Pi(m)+1}$ if $m \neq \Pi(m)$ (that is, $\Pi(m) < m$), since $\Xi_i > \Xi_{i+1}$ for $\Pi(m) + 1 \leq i \leq m$. As a result, for all $m \geq m_0$, we get

$$0 \leq \Xi_m \leq \max\{\Xi_{\Pi(m)}, \Xi_{\Pi(m)+1}\} = \Xi_{\Pi(m)+1}.$$

Thus, $\lim_{m\to\infty}\Xi_m = 0$, this conclude that $\{\nu_m\}$ converges strongly to a point $d$. This finishes the proof. □

**Remark 3.2.** ($r_1$) Here, the results of Tan and Cho [36] are generalized from a real HS to a real uniformly convex Banach space with uniformly Gâteaux differentiable norm.

($r_2$) Because of the wide applications in most branches of mathematics and engineering for the problem of finding fixed points of nonexpansive mappings, it has attracted the attention of many researchers.

($r_3$) Since every uniformly smooth Banach space has uniformly Gâteaux differentiable norm. Then, our theorem can be stated in a uniformly convex Banach space which is also uniformly smooth. (Corollary 3.3).

**Corollary 3.3.** *Let $\Omega$ be a real uniformly convex Banach space which is also uniformly smooth. Assume that $Z : \Omega \to \Omega$ is a nonexpansive mapping so that $\nabla(Z) \neq \emptyset$. Let $\{\nu_m\}$ be a sequence created iteratively by (3.1). Then the sequence $\{\nu_m\}$ converges strongly to a point in $\nabla(Z)$.*

## 4. An application

Let $\Omega$ be a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. We say that a mapping $\Upsilon : D(\Upsilon) \to \Omega$ that has a domain $D(\Upsilon)$ is accereative if there is $j(\nu - \wp) \in J(\nu - \wp)$ such that

$$\langle j(\nu - \wp), \Upsilon\nu - \Upsilon\wp)\rangle \geq 0, \text{ for } \nu, \wp \in D(\Upsilon). \tag{4.1}$$

According to Inequality (4.1), Kato [37] introduced another definition of the accereative mapping as follows: A mapping $\Upsilon$ is called accereative if the inequality below holds

$$\|\nu - \wp\| \leq \|\nu - \wp + b\,(\Upsilon\nu - \Upsilon\wp)\| \;\; \forall s > 0 \text{ and for each } \nu, \wp \in D(\Upsilon). \tag{4.2}$$

We must recall that accerative operators are monotone if $\Omega$ is a Hilbert space. Moreover, if $\Upsilon$ is accretive and its range is $R(I + eA) = \Omega$, for all $e > 0$, then $\Upsilon$ is called $m-$accretive. Also, if $\overline{D(\Upsilon)} \subseteq R(I + e\Upsilon)$ for all $e > 0$, then $\Upsilon$ is said to satisfy the range condition, where $\overline{D(\Upsilon)}$ is the closure of the domain of $\Upsilon$. Furthermore, if $\Upsilon$ is accerative [38], then the mapping $J_\Upsilon : R(I + \Upsilon) \to D(\Upsilon)$, which defined by $J_\Upsilon = (I + \Upsilon)^{-1}$ is a single-valued nonexpansive and $\nabla(J_\Upsilon) = N(\Upsilon)$, where $N(\Upsilon) = \{\nu \in D(\Upsilon) : \Upsilon\nu = 0\}$ and $\nabla(J_\Upsilon) = \{\nu \in \Omega : J_\Upsilon\nu = \nu\}$.

In 1967, the accerative operators are presented independently by Browder [39] and Kato [37]. The study of such mappings is extremely interesting because of their firm link with the existence theory for nonlinear equations of evolution in Banach spaces.

Accerative operators are heavily involved under a suitable Banach space in many physically significant problems where these problems can be formulated as an initial boundary value problem of the form

$$\frac{d\mu}{d\tau} + \Upsilon\mu = 0, \;\mu(0) = \mu_0. \tag{4.3}$$

There are several embedded models of evolution equations such as Schrödinger, heat and wave equation [40]. Heavy work on the theory of accretive operators has been published by Browder [39] explains that if $\Upsilon$ is locally Lipschitzian and accretive on $\Omega$, then Problem (4.3) has a solution. Also, under the same conditions and the existence result of (4.3), he proved that $\Upsilon$ is $m-$accretive and there is a solution to the equation below

$$\Upsilon\mu = 0. \tag{4.4}$$

By Ray [40], Browder's results are elegantly and refined using fixed point theory of Caristi [41]. Martin [42] generalized the results of Browder by proving that in the space $\Omega$. Problem (4.3) is solvable if $\Upsilon$ is continuous and accretive. Moreover, he showed that if $\Upsilon$ is continuous and accretive, then $\Upsilon$ is $m-$accretive. For more details about theorems for zeros of accretive operators see Browder [43] and Deimling [44].

It should be noted that, if $\mu$ is independent of $\tau$ in Eq (4.3), then $\frac{d\mu}{d\tau} = 0$. Thus, Eq (4.3) reduces to (4.4) whose solution describes the stable or the equilibrium state of the problem created by (4.3). This in turn is very exciting in many elegant applications such as, to name but a few, economics, physics and ecology. As a result, strenuous efforts have been made to solve Eq (4.4) when $\Upsilon$ is accretive. Because $\Upsilon$, in general, is nonlinear, there is no known way to find a close solution to this equation, and this is what made researchers interested in studying the fixed point and approximate iterative methods for zeros of $m-$accretive mappings. So it became a thriving area for research to the present time.

In this part, we involve the results of Theorem 3.1 to describe applications of the above results to finding zeros of accretive mappings. Recall, we assume that $\Omega$ is a real uniformly convex Banach space with uniformly Gâteaux differentiable norm and consider $\Upsilon : \Omega \to \Omega$ is continuous and accretive mapping. We will find a solution to the equation:

$$\text{find } \nu \in \Omega \text{ so that } \Upsilon\nu = 0. \tag{4.5}$$

Now the statements and proof of the theorem for finding the solution to Eq (4.5) are fit for presentation.

**Theorem 4.1.** *Let $\Omega$ be a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Assume that $\Upsilon : \Omega \to \Omega$ is a continuous and accretive mapping so that $N(\Upsilon) \neq \emptyset$. Let the sequence $\{\nu_m\}$ created iteratively by $\nu_0, \nu_1 \in \Omega$,*

$$\begin{cases} \hbar_m = \nu_m + \pi_m(\nu_m - \nu_{m-1}), \\ \wp_m = (1 - \sigma_m)\hbar_m, \\ \nu_{m+1} = (1 - \rho_m)\wp_m + \rho_m J_\Upsilon \wp_m, \ m \geq 1, \end{cases}$$

*where $J_\Upsilon = (I + \Upsilon)^{-1}$. Then the sequence $\{\nu_m\}$ converges strongly to a point in $N(\Upsilon)$, provided that the assumptions below hold:*

*(i)* $\lim_{m\to\infty} \sigma_m = 0$, $\sum_{m=1}^{\infty} \sigma_m = \infty$, $\sigma_m \in (0, 1)$, $\rho_m \in [\ell_1, \ell_2] \subset (0, 1)$.

*(ii)* $\sum_{m=0}^{\infty} \pi_m \|\nu_m - \nu_{m-1}\| < \infty$.

*Proof.* Based on the results of Martin [42, 43] and Cioranescu [38], $\Upsilon$ is $m-$accretive. This shows that $J_\Upsilon = (I + \Upsilon)^{-1}$ is nonexpansive and $\nabla(J_\Upsilon) = N(\Upsilon)$. Putting $J_\Upsilon = Z$ in Theorem 3.1 and continuing with the same approach, we get the desired result. □

**Remark 4.2.**
- The problem of finding zeros of accretive mappings in a real uniformly convex Banach space with uniformly Gâteaux differentiable norm given in (4.5) above gave us the motivation to extend the result of Tan and Cho [37] from Hilbert spaces to real uniformly convex Banach spaces with uniformly Gâteaux differentiable norm.
- If we set $\pi_m = 0$ in our algorithm (3.1), then, we have Ishikawa iterative scheme [45]. So, our results extend comparable results for approximating fixed point of nonexapnsive mappings, like the results of Tan and Xu [46]. Moreover, the obtained results here complement the results of Aoyama *et al.* [47], Chapter 16 of Chidume [11] and Theorem 5.4 of Berinde [10].

## 5. Numerical experiments

Now, we study the behavior of Algorithm (3.1) for approximating the fixed point by the following two experiments:

**Example 5.1.** Assume that $\Omega = \mathbb{R}$ with the usual norm. Define a mapping $Z : \Omega \to \Omega$ by

$$Z(\nu) = (5\nu^2 - 2\nu + 48)^{\frac{1}{3}}, \ \forall \nu \in A,$$

where the set $A$ is defined by $A = \{\nu : 0 \leq \nu \leq 50\}$.

**Experiment 1:** In this experiment we have use different values of control parameter $\sigma_m = \frac{1}{(km+2)}$ for $k = 1, 2, 3, 5, 10$. Also, consider $\sigma_m = \frac{1}{(km+2)}$, $\rho_m = 0.80$, $\nu_0 = \nu_1 = 10$, $D_n = \|\nu_{n+1} - \nu_n\|$, , we have Figures 1 and 2.

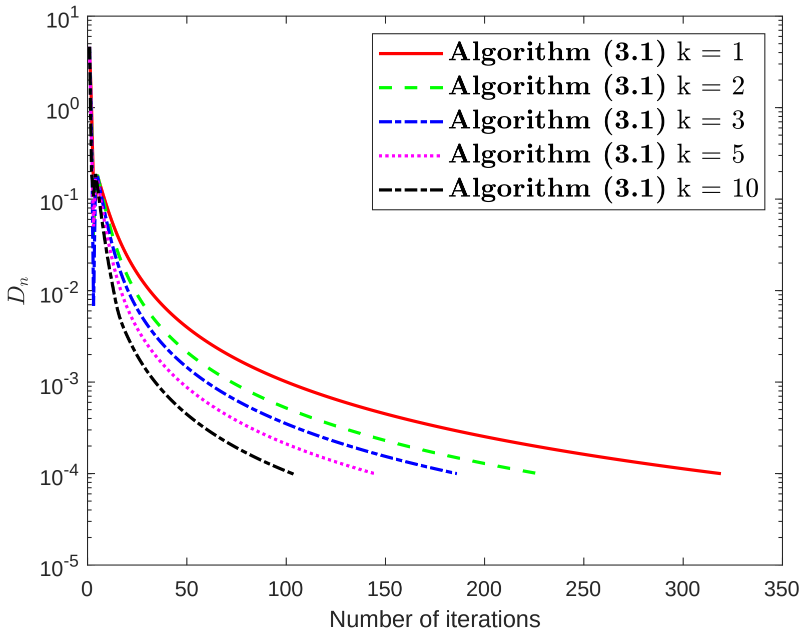


**Figure 1.** Numerical illustration of Algorithm (3.1) while $\sigma_m = \frac{1}{(k*m+2)}$ and the number of iterations are 319, 227, 186, 145, 104.

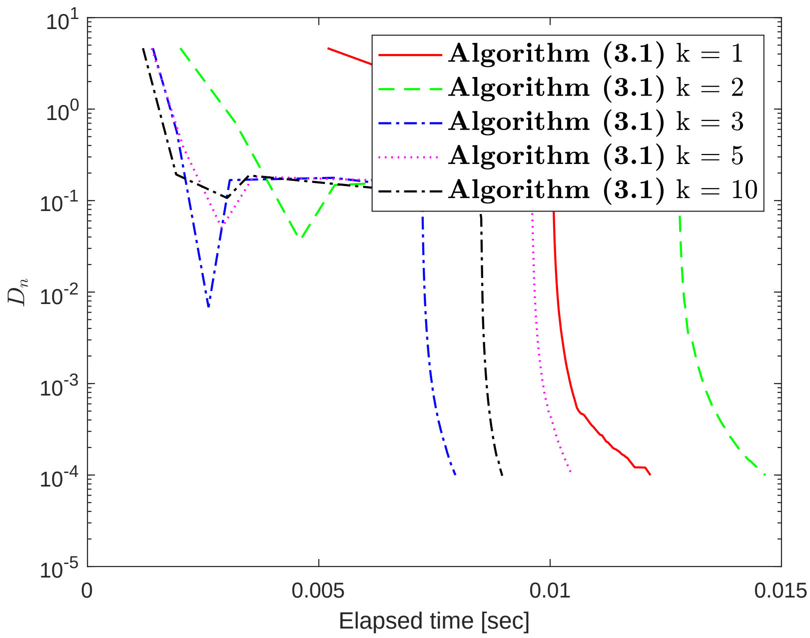


**Figure 2.** Numerical illustration of Algorithm (3.1) while $\sigma_m = \frac{1}{(km+2)}$ and elapsed time are 0.012251, 0.014747, 0.008005, 0.010552, 0.009017.

**Experiment 2:** In this experiment we have use different values of control parameter $\rho_m = k$ for $k = 0.15, 0.35, 0.55, 0.75, 0.95$.

Also, consider $\sigma_m = \frac{1}{(2m+2)}$, $\rho_m = k$, $v_0 = v_1 = 10$, $D_n = ||v_{n+1} - v_n||$, we have Figures 3 and 4.

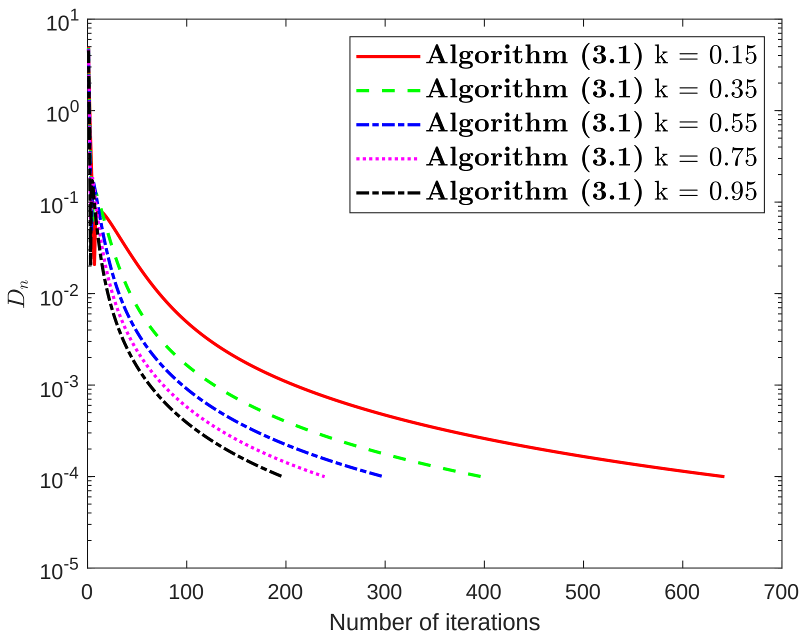


**Figure 3.** Numerical illustration of Algorithm (3.1) while $\rho_m = k$ and the number of iterations are 642, 397, 298, 239, 197.

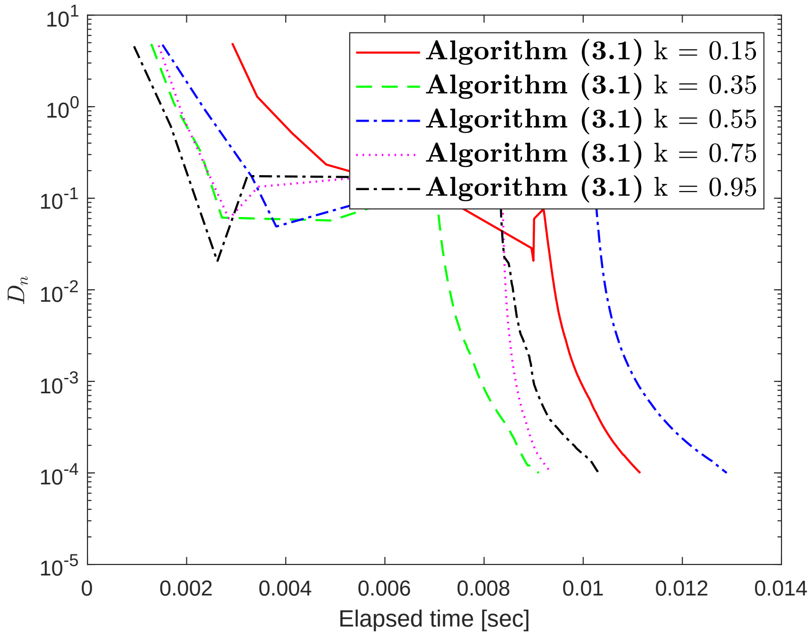


**Figure 4.** Numerical illustration of Algorithm (3.1) while $\rho_m = k$ and elapsed time are 0.011210, 0.009185, 0.012973, 0.009414, 0.010425.

## 6. Conclusions

Krasnoselskii-Mann iterative scheme is widely used in the solution of the fixed point equation which takes the shape $Zx = x$, where $Z : \mho \to \mho$ is nonexpansive mapping and $\mho$ is a non-empty, closed and convex, subset of a Banach space $\Omega$. This algorithm converges weakly to the fixed point of $Z$ provided the underlying space $\Omega$ is a Hilbert space. It is interesting to address the apparent deficiency of the previous algorithm by building an algorithm that converges strongly to the fixed point of $Z$. For this purpose, in this manuscript, we introduce an inertial Krasnoselskii-Mann Algorithm (3.1) for nonexpansive mappings in a real uniformly convex Banach space with uniformly Gâteaux differentiable norm and prove that the proposed Algorithm (3.1) has strong convergence. Moreover, it should be noted that the evidence here differs from previous literature.

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## Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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