AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 6971–6983.
DOI: 10.3934/math.2022387
Received: 15 September 2021
Revised: 22 January 2022
Accepted: 25 January 2022
Published: 07 February 2022

Research article

Computing vertex resolvability of benzenoid tripod structure

Maryam Salem Alatawi[1], Ali Ahmad[2], Ali N. A. Koam[3], Sadia Husain[2] and Muhammad Azeem[4]

[1] Department of Mathematics Faculty of Sciences, University of Tabuk 71491 Tabuk, Saudi Arabia
[2] College of Computer Science and Information Technology, Jazan University, Jazan, Saudi Arabia
[3] Department of Mathematics, College of Science, Jazan University, New Campus, Jazan 2097, Saudi Arabia
[4] Department of Mathematics, Riphah Institute of Computing and Applied Sciences, Riphah International University, Lahore, Pakistan

* **Correspondence:** Email: ahmadsms@gmail.com; Tel: +966595889726.

Abstract: In this paper, we determine the exact metric and fault-tolerant metric dimension of the benzenoid tripod structure. We also computed the generalized version of this parameter and proved that all the parameters are constant. Resolving set L is an ordered subset of nodes of a graph C, in which each vertex of C is distinctively determined by its distance vector to the nodes in L. The cardinality of a minimum resolving set is called the metric dimension of C. A resolving set L_f of C is fault-tolerant if $L_f \setminus b$ is also a resolving set, for every b in L_f. Resolving set allows to obtain a unique representation for chemical structures. In particular, they were used in pharmaceutical research for discovering patterns common to a variety of drugs. The above definitions are based on the hypothesis of chemical graph theory and it is a customary depiction of chemical compounds in form of graph structures, where the node and edge represents the atom and bond types, respectively.

Keywords: node-resolvability; fault-tolerant node-resolvability; benzenoid structure; benzenoid tripod
Mathematics Subject Classification: 05C12, 05C09, 05C92

1. Introduction

Mathematical chemistry has recently presented a wide range of approaches to deal with chemical structures that underpin existing chemical notions, as well as developing and exploring fresh mathematical models of chemical phenomena and applying mathematical concepts and techniques to chemistry. So far not a big range of scientists are working on this interdisciplinary research between mathematics and chemistry, so there is huge margin to use arithmetic properties to derive and anticipate new chemical properties. Mathematical approaches are widely used in several aspects of physical chemistry, notably in thermodynamics and compound energy. After physicists revealed in the first few years of the twentieth century that the key features of chemical compounds can be predicted using quantum theory approaches, a significant need for math in chemistry arose. The real discouragement to use math and its concepts into chemistry laboratories was the realization that chemistry cannot be understood without knowledge of quantum physics, including its complicated mathematical instruments. For further study about mathematical chemistry in terms of graph theory further articles like [1–5] can be studied. A novel topic of mathematical chemistry which is known as topological indices and its literature are found in [6–8].

Chemical graph theory is a branch of mathematical science that is used to characterise the structural properties of molecules, processes, crystals, polymers, clusters, and other materials. The vertex in chemical graph theory might be an electron, an atom, a molecule, a collection of atoms, intermediates, orbitals, and many other things. Intermolecular bonding, bonded and non-bonded connections, basic reactions, and other forces like as van der Waals forces, Keesom forces, Debye forces, and so on can all be used to illustrate the relationships between vertices of a structure.

Harary graph is an interested structure due to its complexity and beautiful topology, [9] find the resolvability of this structure and also discussed some other properties related to resolving vertices. Kayak paddles graph obtained by connecting two basics (path and cycle) graphs. For the resolving set of this graph we refer to see [10], and the necklace graph with these properties are discussed in [11]. The resolving set for the graph obtained from categorical product of two graphs available in [12], where a general circulant graphs are discussed in [13]. The circulant graph $C_n(1, 2, 3)$ are discussed in [14] fault-tolerant resolving sets are computed. Results related to fault-tolerant resolving set on path, cycle and some basics graph are found in [15]. Resolvability of different graphs in terms of fault-tolerant are available in [16]. Interconnection networks of fault-tolerant resolvability are discussed in [17] and some extremal structures are discussed in [18]. The general graph of convex polytopes in terms of metric, fault-tolerant and partition dimension are studied here [19–21].

The idea of resolving sets proposed by Slater [22] later discussed by Harary and Melter [23]. As described in [24,25], metric generators permit to get different representations for chemical compounds. Precisely, they were used in pharmaceutical research for determining patterns similar to a variety of drugs [38]. Metric dimension has various other applications, such as robot navigation [27], weighing problems [28], computer networks [29], combinatorial optimization [30], image processing, facility location problems, sonar and coastguard loran [22] for further detail see [31, 32]. Due to its variety of applications the concept of metric dimension is widely used to solve many difficult problems. For resolvability parameter of different chemical structure we refer to [32–34]. For the NP-harness of these topics we refer to see [36–39].

Given below are basic preliminaries for the concepts studied here.

Definition 1.1. *[40] Assume C be an associated graph of chemical structure/network, whose vertex/node set we will denote with symbol $N(C)$ or simply N, while $B(C)$ or B is the edge/bond set, the shortest distance between two bonds $b_1, b_2 \in N(C)$ denoted by S_{b_1,b_2}, and calculated by counting the number of bonds while moving through the $b_1 - b_2$ path.*

Definition 1.2. *[41] Set a subset L from $N(C)$ such that each node of $N(C)$ have unique position with respect to L. Then the chosen subset will be known as resolving set for a graph C or its node set $N(C)$.*

Mathematically, set $L = \{b_1, b_2, \ldots, b_i\}$ *in general with order or cardinality i. For a node* $b \in N(C)$*, the position is represented by* $p(b|L)$*, and defined as* $(S_{b,b_1}, S_{b,b_2}, \ldots, S_{b,b_i})$*. The position vector is actually a distance of a node from each node in the chosen subset. If all the coordinate of* $p(b|L)$ *for each node of* $N(C)$ *is unique then L is a candidate for the resolving set. While the least number of members of L, is the metric dimension of graph C and it is formally shown by* $dim(C)$*.*

Definition 1.3. *Assuming that any of the member of resolving set L is not working or any of the node from i members, is spoiled then one can not get the unique position of entire node set. To tackle this issue the definition is known as fault-tolerant resolving set which is deal with by eliminating any of the member from L and still obtain the unique position of entire node of a graph, symbolized as* L_f *and the minimum members in the set denoted as* $dim_f(C)$ *and named as fault-tolerant metric dimension.*

Definition 1.4. *The position vector* $p\left(b|L_p\right) = \{(S_b, L_{p1}),\ (S_b, L_{p2}),\ \ldots,\ (S_b, L_{pi})\}$*, of a node b with respect to* L_p*. Where* L_p *is the i-ordered proper subsets of node set and known as partition resolving set if position vector* $p\left(b|L_p\right)$ *is unique for entire node set. The notation* $pd(C)$ *is the minimum count of proper subsets of* $N(C)$ *and called as partition dimension [42, 43].*

Theorem 1.1. [22] *Let* $dim(\aleph)$*, is the metric dimension of a graph* $\aleph$*. Then* $dim(\aleph) = 1$*, iff* $\aleph$ is a path graph.

2. Construction of tripod structure $T(n, m, l)$

Benzenoid systems are the natural graph representations of benzenoid hydrocarbons that is why important in theoretical chemistry. It is established fact that hydrocarbons derived from benzenoids concerning and useful in chemical industry, food industry as we in environment [44]. The benzenoid system we discussed here can be found in [45]. In which they studied, polynomial types discussion of different catacondensed and pericondensed benzenoid structures. The benzenoid tripod structure studied here is a pericondensed system. It has $4(n + m + l) - 8$ nodes and $5(n + m + l) - 11$ bonds, with all the running parameters $n, m, l \geq 2$. Moreover, for detailed topological study of benzenoid structures are available in [46]. Following are node and bonds or vertex and edge set for the benzenoid structure $T(n, m, l)$. We use the labeling of nodes and edges defined in Figure 1 in our main results.

$$\begin{aligned}
N(T(n,m,l)) =& \{a_i : 1 \leq i \leq 2l\} \cup \{b_i : 1 \leq i \leq 2n\} \cup \{c_i,\ c_i' : 1 \leq i \leq 2m-1\} \\
& \cup \{a_i' : 1 \leq i \leq 2l-3\} \cup \{b_i' : 1 \leq i \leq 2n-3\}, \\
B(T(n,m,l)) =& \{a_i a_{i+1} : 1 \leq i \leq 2l-1\} \cup \{b_i b_{i+1} : 1 \leq i \leq 2n-1\} \\
& \cup \{c_i c_{i+1},\ c_i' c_{i+1}' : 1 \leq i \leq 2m-2\} \cup \{a_i' a_{i+1}' : 1 \leq i \leq 2l-4\} \\
& \cup \{b_i' b_{i+1}' : 1 \leq i \leq 2n-4\} \cup \{a_i a_i' : 1 \leq i\,(\text{odd}) \leq 2l-3\} \\
& \cup \{b_{i+3} b_i' : 1 \leq i\,(\text{odd}) \leq 2n-3\} \cup \{c_i c_i' : 1 \leq i\,(\text{odd}) \leq 2m-1\} \\
& \cup \{a_{2l} b_1, a_{2l-1} c_1', b_2 c_1, a_{2l-3}' c_2', b_1' c_2, \}.
\end{aligned}$$

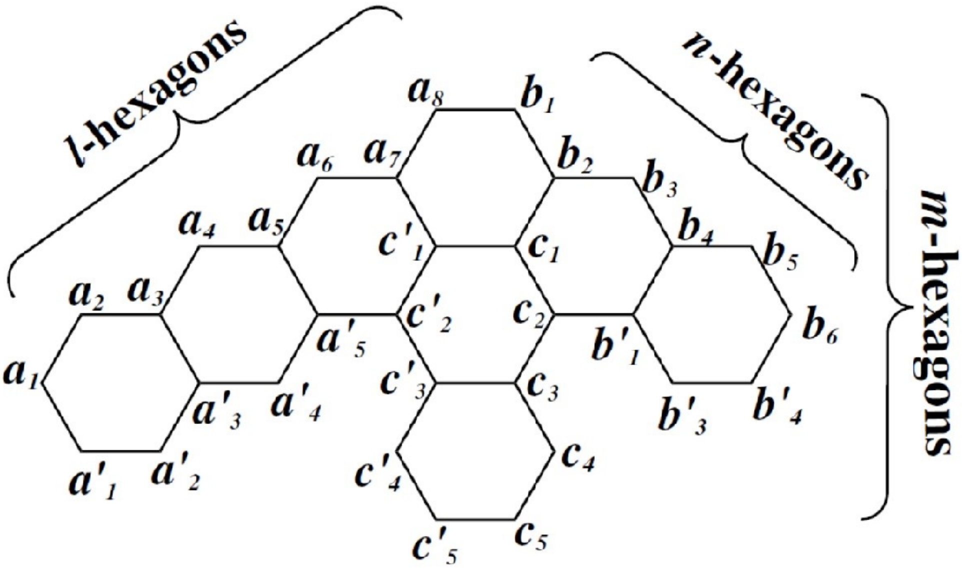

Figure 1. Benzenoid Tripod with $\{n, m, l\} = \{3, 3, 4\}$.

Lemma 2.1. *If $T(n, m, l)$ is a graph of benzenoid tripod with $n, m, l \geq 2$, then the minimum members in its resolving set are two.*

Proof. The total number of nodes in the corresponding graph of benzenoid tripod with $n, m, l \geq 2$, are $4(n + m + l) - 8$, and to check the possibilities of resolving set with cardinality two are $C(4(n + m + l) - 8, 2) = \frac{(4(n+m+l)-8)!}{2\times(4(n+m+l)-10)!}$. Here we are checking with cardinality two, because by Theorem 1.1, the resolving set with cardinality one is reserved for path graph only. Now due to NP-hardness of choosing resolving set, we can not find the exact number of resolving sets for a graph, but from $\frac{(4(n+m+l)-8)!}{2\times(4(n+m+l)-10)!}$-possibilities we choose a subset L and defined as: $L = \{a_1, b_1\}$. Now to prove this claim that L is actually one of the candidate for the resolving set of benzenoid tripod graph or $T(n, m, l)$, we will follow the Definition 1.2. To fulfill the requirements of definition, we will check the unique positions or locations of each node and the methodology is defined above in the Definition 1.2.

Positions $p(a_i|L)$ with respect to L, for the nodes a_i with $i = 1, 2, \ldots, 2l$, are given as:

$$p(a_i|L) = (i - 1,\ 2l - i + 1).$$

Positions $p(b_i|L)$ with respect to L, for the nodes b_i with $i = 1, 2, \ldots, 2n$, are given as:

$$p(b_i|L) = (2l + i - 1,\ i - 1).$$

Positions $p(c_i|L)$ with respect to L, for the nodes c_i with $i = 1, 2, \ldots, 2m - 1$, are given as:

$$p(c_i|L) = \begin{cases} (2l + i - 1,\ i + 1), & \text{if } i = 1, 2; \\ (2l + i - 3,\ i + 1), & \text{if } i = 3, 4, \ldots, 2m - 1. \end{cases}$$

Positions $p\left(a_i'|L\right)$ with respect to L, for the nodes a_i' with $i = 1, 2, \ldots, 2l - 3$, are given as:

$$p(a_i'|L) = (i,\ 2(l + 1) - i).$$

Positions $p\left(b'_i|L\right)$ with respect to L, for the nodes b'_i with $i = 1, 2, \ldots, 2n-3$, are given as:

$$p\left(b'_i|L\right) = (2l + i + 1,\ i + 3).$$

Positions $p\left(c'_i|L\right)$ with respect to L, for the nodes c'_i with $i = 1, 2, \ldots, 2m-1$, are given as:

$$p\left(c'_i|L\right) = \begin{cases} (2l - 1,\ i + 2), & \text{if } i = 1; \\ (2(l-2) + i,\ i + 2), & \text{if } i = 2, 3, \ldots, 2m-1. \end{cases}$$

By the given positions $p(.|L)$ of all $4(n+m+l) - 8$-nodes of $T(n,m,l)$ graph of benzenoid tripod with $n, m, l \geq 2$, with respect to L, are unique and no two nodes have same position p. So we can conclude that we resolve the nodes of $T(n,m,l)$ with two nodes. It is implied that the minimum members in the resolving set of $T(n,m,l)$ are two. □

Remark 2.1. *If $T(n,m,l)$ is a graph of benzenoid tripod with $n, m, l \geq 2$, then*

$$dim(T(n,m,l)) = 2.$$

Proof. From the definition of metric dimension, the concept is solemnly based on the selected subset (L) chooses in such a way that the entire vertex set have unique position with respect to the selected nodes or subset. In the Lemma 2.1, we already discussed the possibility of selected subset (resolving set) and according to the definition, its minimum possible cardinality. In that lemma we choose $L = \{a_1, b_1\}$ as a resolving set for the graph of benzenoid tripod or $T(n,m,l)$ for all the possible combinatorial values of $n, m, l \geq 2$. We also proved in such lemma that $|L| = 2$ is the least possible cardinality of resolving set for the benzenoid tripod $T(n,m,l)$. It is enough for the prove of what we claim in the statement that metric dimension of benzenoid tripod is two, which completes the prove. □

Lemma 2.2. *If $T(n,m,l)$ is a graph of benzenoid tripod with $n, m, l \geq 2$, then the minimum members in its fault-tolerant resolving set are four.*

Proof. The total number of nodes in the corresponding graph of benzenoid tripod with $n, m, l \geq 2$, are $4(n+m+l) - 8$, and to check the possibilities of fault-tolerant resolving set with cardinality four are $C(4(n+m+l) - 8, 4) = \frac{(4(n+m+l)-8)!}{2\times(4(n+m+l)-12)!}$. Here we are checking with cardinality four, later we will also check cardinality three. Now due to NP-hardness of choosing fault-tolerant resolving set, we can not find the exact number of fault-tolerant resolving sets for a graph, but from $\frac{(4(n+m+l)-8)!}{2\times(4(n+m+l)-12)!}$-possibilities we choose a subset L_f and defined as: $L_f = \{a_1, b_1, a_{2l}, b_{2n}\}$. Now to prove this claim that L_f is actually one of the candidate for the fault-tolerant resolving set of benzenoid tripod graph or $T(n,m,l)$, we will follow the Definition 1.3. To fulfill the requirements of definition, we will check the unique positions or locations of each node and the methodology is defined above in the Definition 1.3.
Positions $p\left(a_i|L_f\right)$ with respect to L_f, for the nodes a_i with $i = 1, 2, \ldots, 2l$, are given as:

$$p\left(a_i|L_f\right) = (i - 1,\ 2l - i + 1,\ 2l - i,\ 2(l+n) - i).$$

Positions $p\left(b_i|L_f\right)$ with respect to L_f, for the nodes b_i with $i = 1, 2, \ldots, 2n$, are given as:

$$p\left(b_i|L_f\right) = (2l + i - 1,\ i - 1,\ i,\ 2n - i).$$

Positions $p\left(c_i|L_f\right)$ with respect to L_f, for the nodes c_i with $i = 1, 2, \ldots, 2m-1$, are given as:

$$p\left(c_i|L_f\right) = \begin{cases} (2l+i-1,\ i+1,\ i+2,\ 2n-1), & \text{if } i = 1; \\ (2l+i-1,\ i+1,\ i+2,\ 2(n-2)+i), & \text{if } i = 2; \\ (2l+i-3,\ i+1,\ i+2,\ 2(n-2)+i), & \text{if } i = 3, 4, \ldots, 2m-1. \end{cases}$$

Positions $p\left(a'_i|L_f\right)$ with respect to L_f, for the nodes a'_i with $i = 1, 2, \ldots, 2l-3$, are given as:

$$p\left(a'_i|L_f\right) = (i,\ 2(l+1)-i,\ 2l+1-i,\ 2(l+n)-1-i).$$

Positions $p\left(b'_i|L_f\right)$ with respect to L_f, for the nodes b'_i with $i = 1, 2, \ldots, 2n-3$, are given as:

$$p\left(b'_i|L_f\right) = (2l+i+1,\ i+3,\ i+4,\ 2(n-1)-i).$$

Positions $p\left(c'_i|L_f\right)$ with respect to L_f, for the nodes c'_i with $i = 1, 2, \ldots, 2m-1$, are given as:

$$p\left(c'_i|L_f\right) = \begin{cases} (2l-1,\ i+2,\ i+1,\ 2n-1+i), & \text{if } i = 1; \\ (2(l-2)+i,\ i+2,\ i+1,\ 2n-1+i), & \text{if } i = 2; \\ (2(l-2)+i,\ i+2,\ i+1,\ 2n-3+i), & \text{if } i = 3, 4, \ldots, 2m-1. \end{cases}$$

On the behalf of given fact for the fulfillment of definition of fault-tolerant resolving set, we can say that L_f with cardinality four is possible, but when it comes to optimized value of $\left|L_f\right|$, we still need to investigate about the minimize value of $\left|L_f\right|$. Following are some possible cases to check that whether $\left|L_f\right| = 3$ is possible or not. Though we find the fault-tolerant resolving set with the help of algorithm and satisfied that $\left|L_f\right| \neq 3$, but for the proving purpose we build some general cases and try to conclude that only $\left|L_f\right| > 3$ is possible.

Case 1: Assume that $L'_f \subset \{a_i : i = 1, 2, \ldots, 2l\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(a'_s|L'_f\right)$, where $1 \leq r \leq 2l$ and $1 \leq s \leq 2l-3$.

Case 2: Assume that $L'_f \subset \{b_i : i = 1, 2, \ldots, 2n\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(a'_s|L'_f\right)$, where $1 \leq r \leq 2l$ and $1 \leq s \leq 2l-3$.

Case 3: Assume that $L'_f \subset \{c_i : i = 1, 2, \ldots, 2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(b_s|L'_f\right)$, where $1 \leq r \leq 2l$ and $1 \leq s \leq 2n$.

Case 4: Assume that $L'_f \subset \{a_{i'} : i = 1, 2, \ldots, 2l-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(c_s|L'_f\right)$, where $1 \leq r \leq 2l$ and $1 \leq s \leq 2m-1$.

Case 5: Assume that $L'_f \subset \{b_{i'} : i = 1, 2, \ldots, 2n-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(c'_s|L'_f\right)$, where $1 \leq r \leq 2l$ and $1 \leq s \leq 2m-1$.
Case 6: Assume that $L'_f \subset \{c_{i'} : i = 1, 2, \ldots, 2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a'_r|L'_f\right) = p\left(c_s|L'_f\right)$, where $1 \leq r \leq 2l-3$ and $1 \leq s \leq 2m-1$.
Case 7: Assume that $L'_f \subset \{a_i, b_j : i = 1, 2, \ldots, 2l, \ j = 1, 2, \ldots, 2n\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(a'_s|L'_f\right)$, where $1 \leq r \leq 2l$ and $1 \leq s \leq 2l-3$.
Case 8: Assume that $L'_f \subset \{a_i, \ c_j : i = 1, 2, \ldots, 2l, \ j = 1, 2, \ldots, 2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(b_r|L'_f\right) = p\left(b'_s|L'_f\right)$, where $1 \leq r \leq 2n$ and $1 \leq s \leq 2n-3$.
Case 9: Assume that $L'_f \subset \{a_i, \ a'_j : i = 1, 2, \ldots, 2l, \ j = 1, 2, \ldots, 2l-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(b_r|L'_f\right) = p\left(c_s|L'_f\right)$, where $1 \leq r \leq 2n$ and $1 \leq s \leq 2m-1$.
Case 10: Assume that $L'_f \subset \{a_i, \ b'_j : i = 1, 2, \ldots, 2l, \ j = 1, 2, \ldots, 2n-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(c_r|L'_f\right) = p\left(c_s|L'_f\right)$, where $1 \leq r, s \leq 2m-1$.
Case 11: Assume that $L'_f \subset \{a_i, \ c'_j : i = 1, 2, \ldots, 2l, \ j = 1, 2, \ldots, 2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(a'_s|L'_f\right)$, where $1 \leq r \leq 2l$ and $1 \leq s \leq 2l-3$.
Case 12: Assume that $L'_f \subset \{b_i, \ c_j : i = 1, 2, \ldots, 2n, \ j = 1, 2, \ldots, 2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(a'_s|L'_f\right)$, or $p\left(a_r|L'_f\right) = p\left(b'_k|L'_f\right)$, where $1 \leq r \leq 2l$, $1 \leq s \leq 2l-3$ and $1 \leq k \leq 2n-3$.
Case 13: Assume that $L'_f \subset \{b_i, \ a'_j : i = 1, 2, \ldots, 2n, \ j = 1, 2, \ldots, 2l-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(c'_s|L'_f\right)$, where $1 \leq r \leq 2l$, $1 \leq s \leq 2m-1$.
Case 14: Assume that $L'_f \subset \{b_i, \ b'_j : i = 1, 2, \ldots, 2n, \ j = 1, 2, \ldots, 2l-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(c_r|L'_f\right) = p\left(c'_s|L'_f\right)$, where $1 \leq r, s \leq 2m-1$.

Case 15: Assume that $L'_f \subset \{b_i,\ c'_j\ :\ i = 1,2,\dots,2n,\ j = 1,2,\dots,2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(b_r|L'_f\right) = p\left(b_s|L'_f\right)$, where $1 \leq r, s \leq 2n$.
Case 16: Assume that $L'_f \subset \{c_i,\ a'_j\ :\ i = 1,2,\dots,2m-1,\ j = 1,2,\dots,2l-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(b_r|L'_f\right) = p\left(b_s|L'_f\right)$, where $1 \leq r, s \leq 2n$.
Case 17: Assume that $L'_f \subset \{c_i,\ b'_j\ :\ i = 1,2,\dots,2m-1,\ j = 1,2,\dots,2n-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(a_s|L'_f\right)$, where $1 \leq r, s \leq 2l$.
Case 18: Assume that $L'_f \subset \{c_i,\ c'_j\ :\ i, j = 1,2,\dots,2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(a_s|L'_f\right)$, where $1 \leq r, s \leq 2l$.
Case 19: Assume that $L'_f \subset \{a'_i,\ b'_j\ :\ i = 1,2,\dots,2l-3,\ j = 1,2,\dots,2n-3\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(a_r|L'_f\right) = p\left(c_s|L'_f\right)$, where $1 \leq r \leq 2l$, $1 \leq s \leq 2m-1$.
Case 20: Assume that $L'_f \subset \{a'_i,\ c'_j\ :\ i = 1,2,\dots,2l-3,\ j = 1,2,\dots,2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(b_r|L'_f\right) = p\left(b'_s|L'_f\right)$, where $1 \leq r \leq 2n$, $1 \leq s \leq 2n-3$.
Case 21: Assume that $L'_f \subset \{b'_i,\ c'_j\ :\ i = 1,2,\dots,2n-3,\ j = 1,2,\dots,2m-1\}$, with a condition according to our requirement of theorem that $|L'_f| = 3$, and removal of any vertex from L'_f to fulfill the definition. The result is implied in same vertex's position and contradict our assumption with the fact that $p\left(b_r|L'_f\right) = p\left(b_s|L'_f\right)$, where $1 \leq r, s \leq 2n$.
By the given positions $p\left(.|L_f\right)$ of all $4(n+m+l)-8$-nodes of $T(n,m,l)$ graph of benzenoid tripod with $n, m, l \geq 2$, with respect to L_f, having $\left|L_f\right| = 4$ are unique and no two nodes have same position p. It is also can be accessed that by eliminating any of arbitrary nodes from L_f will not effect the definition of resolving set. We also checked that the fault-tolerant resolving set L_f with $\left|L_f\right| = 3$ are resulted in with two nodes have same position p. So we can conclude that we resolve the nodes of $T(n,m,l)$ with four nodes. It is implied that the minimum members in the fault-tolerant resolving set of $T(n,m,l)$ are four. □

Remark 2.2. *If $T(n,m,l)$ is a graph of benzenoid tripod with $n, m, l \geq 2$, then*

$$dim_f(T(n,m,l)) = 4.$$

Proof. From the definition of fault-tolerant metric dimension (same as in parent concept), the concept is solemnly based on the selected subset (L_f) chooses in such a way that the entire vertex set have unique position with respect to the selected nodes or subset, with addition is by removal of any arbitrary single member of L_f, does not effect the resolvability of vertices or position of entire nodes of graph remains unique. In the Lemma 2.2, we already discussed the possibility of selected subset (fault-tolerant resolving set) and according to the definition, its minimum possible cardinality. In that lemma we choose $L_f = \{a_1, b_1, a_{2l}, b_{2n}\}$ as a fault-tolerant resolving set for the graph of benzenoid tripod or $T(n,m,l)$ for all the possible combinatorial values of $n,m,l \geq 2$. We also proved in such lemma that $\left|L_f\right| = 4$ is the least possible cardinality of fault-tolerant resolving set for the benzenoid tripod $T(n,m,l)$. It is enough for the prove of what we claim in the statement that fault-tolerant metric dimension of benzenoid tripod is four, which completes the prove. □

Lemma 2.3. *If $T(n,m,l)$ is a graph of benzenoid tripod with $n,m,l \geq 2$, then the minimum numbers of subsets of its partition resolving set are three.*

Proof. The total number of nodes in the corresponding graph of benzenoid tripod with $n,m,l \geq 2$, are $4(n+m+l)-8$, and to check the possible combinations given by Bell number which is $Bell(4(n+m+l)-8) = \sum_{\alpha=0}^{4(n+m+l)-8} S(4(n+m+l)-8, \alpha)$, where $S(4(n+m+l)-8, \alpha)$ is the Stirling number of second kind [?]. $Bell(4(n+m+l)-8)$ is the possible number of choosing partition resolving set for $T(n,m,l)$, but the best and suited one are presented here and defined as $L_p = \{L_{p_1},\ L_{p_2},\ L_{p_3}\}$, where $L_{p_1} = \{a_1\}$, $L_{p_2} = \{b_1\}$, $L_{p_3} = N(T(n,m,l)) \setminus \{a_1,\ b_1\}$. Here we are checking with cardinality three, because by Theorem 1.1, the partition resolving set with cardinality two is reserved for path graph only. Now due to NP-hardness of choosing partition resolving set, we can not find the exact number of resolving sets for a graph, but from $Bell(4(n+m+l)-8) = \sum_{\alpha=0}^{4(n+m+l)-8} S(4(n+m+l)-8, \alpha)$,-possibilities we choose a subset L_p. Now to prove this claim that L_p is actually one of the candidate for the partition resolving set of benzenoid tripod graph or $T(n,m,l)$, we will follow the Definition 1.4. To fulfill the requirements of definition, we will check the unique positions or locations of each node and the methodology is defined above in the Definition 1.4.
Positions $p\left(a_i|L_p\right)$ with respect to L_p, for the nodes a_i with $i = 1,2,\ldots,2l$, are given as:

$$p\left(a_i|L_p\right) = (i-1,\ 2l-i+1,\ z_1).$$

where $z_1 = \begin{cases} 1, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$

Positions $p\left(b_i|L_p\right)$ with respect to L_p, for the nodes b_i with $i = 1,2,\ldots,2n$, are given as:

$$p\left(b_i|L_p\right) = (2l+i-1,\ i-1,\ z_1).$$

Positions $p\left(c_i|L_p\right)$ with respect to L_p, for the nodes c_i with $i = 1,2,\ldots,2m-1$, are given as:

$$p\left(c_i|L_p\right) = \begin{cases} (2l+i-1,\ i+1,\ 0), & \text{if } i = 1,2; \\ (2l+i-3,\ i+1,\ 0), & \text{if } i = 3,4,\ldots,2m-1. \end{cases}$$

Positions $p\left(a'_i|L_p\right)$ with respect to L_p, for the nodes a'_i with $i = 1,2,\ldots,2l-3$, are given as:

$$p\left(a'_i|L_p\right) = (i,\ 2(l+1)-i,\ 0).$$

Positions $p\left(b'_i|L_p\right)$ with respect to L_p, for the nodes b'_i with $i = 1, 2, \ldots, 2n-3$, are given as:

$$p\left(b'_i|L_p\right) = (2l + i + 1,\ i + 3,\ 0).$$

Positions $p\left(c'_i|L_p\right)$ with respect to L_p, for the nodes c'_i with $i = 1, 2, \ldots, 2m-1$, are given as:

$$p\left(c'_i|L_p\right) = \begin{cases} (2l-1,\ i+2,\ 0), & \text{if } i = 1; \\ (2(l-2)+i,\ i+2,\ 0), & \text{if } i = 2, 3, \ldots, 2m-1. \end{cases}$$

By the given positions $p\left(.|L_p\right)$ of all $4(n+m+l)-8$-nodes of $T(n,m,l)$ graph of benzenoid tripod with $n, m, l \geq 2$, with respect to L_p, are unique and no two nodes have same position p. So we can conclude that we resolve the nodes of $T(n,m,l)$ by making three subsets of nodes of graph. It is implied that the minimum members in the partition resolving set of $T(n,m,l)$ are three.

□

Remark 2.3. *If $T(n,m,l)$ is a graph of benzenoid tripod with $n, m, l \geq 2$, then*

$$pd(T(n,m,l)) = 3.$$

Proof. From the definition of partition dimension (same as in parent concept), the concept is solemnly based on the selected way of doing subsets (L_p) in such a way that the entire vertex set have unique position with respect to the selected way of making subsets of entire node set. In the Lemma 2.3, we already discussed the possibility of making the subset (partition resolving set) and according to the definition, its minimum possible cardinality. In that lemma we choose $L_p = \{L_{p_1},\ L_{p_2},\ L_{p_3}\}$, where $L_{p_1} = \{a_1\}$, $L_{p_2} = \{b_1\}$, $L_{p_3} = N(T(n,m,l)) \backslash \{a_1,\ b_1\}$, as a partition resolving set for the graph of benzenoid tripod or $T(n,m,l)$ for all the possible combinatorial values of $n, m, l \geq 2$. We also proved in such lemma that $\left|L_p\right| = 3$ is the least possible cardinality of partition resolving set for the benzenoid tripod $T(n,m,l)$. It is enough for the prove of what we claim in the statement that partition dimension of benzenoid tripod is three, which completes the prove. □

3. Conclusions

It is established fact that mathematical chemistry especially graphical chemistry, made easier way to study complex networks and chemical structures in their most easiest forms. Similarly resolvability is a parameter in which entire node or edge set and sometimes both components reshaped into specific arrangement to call or accessed them. Metric dimension is also a parameter with this property to gain each node of a structure into unique form. In this work, we consider benzenoid tripod structure to achieve its resolvability and found its minimum node-resolving set. We concluded that metric, fault-tolerant-metric and partition resolving set are with constant and exact number of members for this structure. Similar to this work, many more chemical structures can be discussed in terms of resolvability parameters. Moreover, this structure can be discussed in terms of their edge based resolvability and also edge-vertex-based resolvability. These are motivation and future direction on this topic.

References

1. B. Yang, M. Rafiullah, H. M. A. Siddiqui, S. Ahmad, On Resolvability parameters of some wheel-related graphs, *J. Chem.*, **2019** (2019), 1–9. http://dx.doi/10.1155/2019/9259032
2. S. Imran, M. K. Siddiqui, M. Hussain, Computing the upper bounds for the metric dimension of cellulose network, *Appl. Math. E-Notes*, **19** (2019), 585–605. https://www.math.nthu.edu.tw/ amen/2019/AMEN-181121.pdf
3. M. K. Siddiqui, M. Imran, Computing the metric and partition dimension of H-Naphtalenic and VC5C7 nanotubes, *J. Optoelectron. Adv. M.*, **17** (2015), 790–794. https://www.semanticscholar.org/paper/Computing-the-metric-and-partition-dimension-of-and-Siddiqui-Imran/b79571b2517ff2f37c731687cd5ea1787d9d05b6
4. S. Manzoor, M. K. Siddiqui, S. Ahmad, On entropy measures of polycyclic hydroxychloroquine used for novel Coronavirus (COVID-19) treatment, *Polycycl. Aromat. Comp.*, (2020). http://dx.doi/10.1080/10406638.2020.1852289
5. M. K. Siddiqui, M. Naeem, N. A. Rahman, M. Imran, Computing topological indices of certain networks, *J. Optoelectron. Adv. M.*, **18** (2016), 9–10. http://dx.doi/10.1016/j.amc.2014.04.091
6. D. Maji, G. Ghorai, A novel graph invariant: The third leap Zagreb index under several graph operations, *Discret. Math. Algorit.*, **11** (2019), 1950054. http://dx.doi/10.1142/s179383091950054x
7. D. Maji, G. Ghorai, M. K. Mahmood, M. A. Alam, On the inverse problem for some topological indices, *J. Math.*, **2021** (2021), 1–8. http://dx.doi/10.1155/2021/9411696
8. D. Maji, G. Ghorai, Computing F-index, coindex and Zagreb polynomials of the kth generalized transformation graphs, *Heliyon*, **6** (2020), e05781. http://dx.doi/10.1016/j.heliyon.2020.e05781
9. A. Ahmad, M. Bača, S. Sultan, On metric dimension and minimal doubly resolving sets of Harary graph, *Acta Math. Univ. Comen.*, **89** (2020), 123–129. http://www.iam.fmph.uniba.sk/amuc/ojs/index.php/amuc/article/view/1032/795
10. A. Ahmad, M. Bača, S. Sultan, Computing the metric dimension of Kayak Paddles graph and Cycles with chord, *Proyecciones J. Math.*, **39** (2020), 287–300. http://dx.doi/10.22199/issn.0717-6279-2020-02-0018
11. A. Ahmad, M. Baă, S. Sultan, Minimal doubly resolving sets of Necklace graph, *Math. Rep.*, **20** (2018), 123–129. http://imar.ro/journals/Mathematical-Reports/Pdfs/2018/2/2.pdf
12. T. Vetrik, A. Ahmad, Computing the metric dimension of the categorial product of graphs, *Int. J. Comput. Math.*, **94** (2017), 363–371. http://dx.doi/10.1080/00207160.2015.1109081
13. A. Ahmad, S. Sultan, On minimal doubly resolving sets of circulant graphs, *Acta Mechanica Slovaca*, **20** (2017), 6–11. http://dx.doi/10.21496/ams.2017.002
14. M. Basak, L. Saha, G. K. Das, K. Tiwary, Fault-tolerant metric dimension of circulant graphs $C_n(1;2;3)$, *Theor. Comput. Sci.*, **817** (200), 66–79. http://dx.doi/10.1016/j.tcs.2019.01.011
15. M. A. Chaudhry, I. Javaid, M. Salman, Fault-Tolerant metric and partition dimension of graphs, *Utliltas Mathematica*, **83** (2010), 187–199. https://www.researchgate.net/publication/259574039-Fault-Tolerant-Metric-and-Partition-Dimension-of-Graphs

16. I. Javaid, M. Salman, M. A. Chaudhry, S. Shokat, Fault-tolerance in resolvability, *Utliltas Mathematica*, **80** (2009), 263–275. https://www.researchgate.net/publication/259573732-Fault-tolerance-in-resolvability
17. H. Raza, S. Hayat, X. F. Pan, On the fault-tolerant metric dimension of certain interconnection networks, *J. Appl. Math. Comput.*, **60** (2019), 517–535. http://dx.doi/10.1109/ACCESS.2020.3014883
18. H. Raza, S. Hayat, M. Imran, X. F. Pan, Fault-tolerant resolvability and extremal structures of graphs, *Mathematics*, **7** (2019), 78–97. http://dx.doi/10.3390/math7010078
19. H. Raza, S. Hayat, X. F. Pan, On the fault-tolerant metric dimension of convex polytopes, *Appl. Math. Comput.*, **339** (2018), 172–185. http://dx.doi/10.1016/j.amc.2018.07.010
20. Y. M. Chu, M. F. Nadeem, M. Azeem, M. K. Siddiqui, On sharp bounds on partition dimension of convex polytopes, *IEEE Access*, **8** (2020), 224781–224790. http://dx.doi/10.1109/ACCESS.2020.3044498
21. J. B. Liu, M. F. Nadeem, M. Azeem, Bounds on the partition dimension of convex polytopes, *Comb. Chem. Hight. T. Scr.*, (2020). http://dx.doi/10.2174/1386207323666201204144422
22. P. J. Slater, Leaves of trees, Proceeding of the 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congressus Numerantium, **14** (1975), 549–559.
23. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria*, **2** (1976), 191–195.
24. G. Chartrand, E. Salehi, P. Zhang, The partition dimension of graph, *Aequationes Math.*, **59** (2000), 45–54. http://dx.doi/10.1007/PL00000127
25. G. Chartrand, L. Eroh, M. A. O. Johnson, R. Ortrud, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), 99–113. http://dx.doi/10.1016/S0166-218X(00)00198-0
26. M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *J. Biopharm. Stat.*, **3** (1993), 203–236. http://dx.doi/10.1080/10543409308835060
27. S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.*, **70** (1996), 217–229. http://dx.doi/10.1016/0166-218X(95)00106-2
28. S. Söderberg, H. S. Shapiro, A combinatory detection problem, *Am. Math. Mon.*, **70** (1963), 1066–1070. http://dx.doi/10.2307/2312835
29. P. Manuel, R. Bharati, I. Rajasingh, M. C. Monica, On minimum metric dimension of honeycomb networks, *J. Discrete Algorit.*, **6** (2008), 20–27. http://dx.doi/10.1016/j.jda.2006.09.002
30. A. Sebö, E. Tannier, On metric generators of graphs, *Math. Operat. Res.*, **29** (2004), 383–393.
31. M. Perc, J. Gómez-Gardeñes, A. Szolnoki, L. M. Floría, Y. Moreno, Evolutionary dynamics of group interactions on structured populations: A review, *J. R. Soc. Interface*, **10** (2013). http://dx.doi/10.1098/rsif.2012.0997
32. M. Perc, A. Szolnoki, Coevolutionary games-A mini review, *Biosystems*, **99** (2010), 109–125. http://dx.doi/10.1016/j.biosystems.2009.10.003
33. Z. Hussain, M. Munir, M. Choudhary, S. M. Kang, Computing metric dimension and metric basis of 2*D* lattice of alpha-boron nanotubes, *Symmetry*, **10** (2018). http://dx.doi/10.3390/sym10080300

34. S. Krishnan, B. Rajan, Fault-tolerant resolvability of certain crystal structures, *Appl. Math.*, **7** (2016), 599–604. http://dx.doi/10.4236/am.2016.77055

35. H. M. A. Siddiqui, M. Imran, Computing metric and partition dimension of 2-Dimensional lattices of certain nanotubes, *J. Comput. Theor. Nanos.*, **11** (2014), 2419–2423.

36. M. Hauptmann, R. Schmied, C. Viehmann, Approximation complexity of metric dimension problem, *J. Discrete Algorit.*, **14** (2012), 214–222. http://dx.doi/10.1016/j.jda.2011.12.010

37. H. R. Lewis, M. R. Garey, D. S. Johnson, Computers and intractability, A guide to the theory of NP-completeness, *J. Symbolic Logic*, **48** (1983), 498–500. https://dl.acm.org/doi/book/10.5555/578533

38. M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *J. Biopharm. Stat.*, **3** (1993), 203–236. http://dx.doi/10.1080/10543409308835060

39. M. A. Johnson, Browsable structure-activity datasets, Advances in molecular similarity, *JAI Press Connecticut*, (1998), 153–170.

40. M. F. Nadeem, M. Azeem, A. Khalil, The locating number of hexagonal Möbius ladder network *J. Appl. Math. Comput.*, (2020). http://dx.doi/10.1007/s12190-020-01430-8

41. A. N. A. Koam, A. Ahmad, M. E. Abdelhag, M. Azeem, Metric and fault-tolerant metric dimension of hollow coronoid, *IEEE Access*, **9** (2021), 81527–81534. http://dx.doi/10.1109/ACCESS.2021.3085584

42. A. Shabbir, M. Azeem, On the partition dimension of tri-hexagonal alpha-boron nanotube, *IEEE Access*, **9** (2021), 55644–55653. http://dx.doi/10.1109/ACCESS.2021.3071716

43. N. Mehreen, R. Farooq, S. Akhter, On partition dimension of fullerene graphs, *AIMS Math.*, **3** (2018), 343–352. http://dx.doi/10.3934/Math.2018.3.343

44. A. Ali, W. Nazeer, M. Munir, S. M. Kang, M-Polynomials and topological indices Of Zigzag and rhombic benzenoid systems, *Open Chem.*, **16** (2018), 122–135. http://dx.doi/10.1515/chem-2018-0010

45. C. P. Chou, Y. Li, H. A. Witek, Zhang–Zhang polynomials of various classes of benzenoid systems, *MATCH-Commun. Math. Co.*, **68** (2012), 31–64. vvhttps://waseda.pure.elsevier.com/en/publications/determination-of-zhang-zhang-polynomials-for-various-classes-of-b

46. M. K. Jamil, M. Imran, K. A. Sattar, Novel face index for benzenoid hydrocarbons, *Mathematics*, **8** (2020). http://dx.doi/10.3390/math8030312