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Research article

Neighbor full sum distinguishing total coloring of Halin graphs

Yinwan Cheng[1], **Chao Yang**[1,*], **Bing Yao**[2] **and Yaqin Luo**[1]

[1] School of Mathematics, Physics and Statistics, Center of Intelligent Computing and Applied Statistics, Shanghai University of Engineering Science, Shanghai 201620, China

[2] College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: yangchao@sues.edu.cn.

Abstract: Let $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ be a total k-coloring of G. Define a weight function on total coloring as

$$\phi(x) = f(x) + \sum_{e \ni x} f(e) + \sum_{y \in N(x)} f(y),$$

where $N(x) = \{y \in V(G) | xy \in E(G)\}$. If $\phi(x) \neq \phi(y)$ for any edge $xy \in E(G)$, then f is called a neighbor full sum distinguishing total k-coloring of G. The smallest value k for which G has such a coloring is called the neighbor full sum distinguishing total chromatic number of G and denoted by $\mathrm{fgndi}_{\sum}(G)$. Suppose that $H = T \cup C$ is a Halin graph, where T and C are called the characteristic tree and the adjoint cycle, respectively. Let $V_0 \subseteq V(H) \setminus V(C)$ and each vertex in V_0 is adjacent to some vertices on C. In this paper, we prove that the neighbor full sum distinguishing total chromatic number of two types of Halin graphs are not more than three: (i) 3-regular Halin graphs and (ii) every vertex of V_0 of a Halin graph with degree at least 4. The above results support a conjecture that $\mathrm{fgndi}_{\sum}(G) \leq 3$ for any connected graph G of order at least three (Chang et al., 2022).

Keywords: total coloring; neighbor full sum distinguishing total coloring; neighbor full sum distinguishing total chromatic number; Halin graphs
Mathematics Subject Classification: 05C15

1. Introduction and concepts

All considered graphs are finite, undirected, connected, without loops and multiple edges. Let $[1, n]$ denote the set of positive integers $\{1, 2, \ldots, n\}$. Let $d_G(v)$ and $\Delta(G)$ (or Δ) denote the degree of vertex v and the maximum degree of G, respectively. For general theoretic notations, we follow [3].

A type of distinguishing coloring on the sum of colors of vertices and edges has attracted extensive attention. Karoński et al. [9] firstly introduced and investigated neighbor sum distinguishing edge coloring of graphs. Let $f : E(G) \to [k]$ be an edge k-coloring of G. For any vertex $x \in V$, set

$$\sigma(x) = \sum_{x \in e} f(e).$$

An edge k-coloring f of G is called neighbor sum distinguishing edge coloring of G if $\sigma(x) \neq \sigma(y)$ for any edge $xy \in E(G)$. The minimum integer k for which there is a neighbor sum distinguishing edge coloring of a graph G will be denoted by $\mathrm{gndi}_{\sum}(G)$. In particular, there is a famous 1-2-3 conjecture on neighbor sum distinguishing edge coloring as follows:

Conjecture 1.1. *[9] For any connected graph G of order at least* 3, $\mathrm{gndi}_{\sum}(G) \leq 3$.

Karoński et al. [9] showed that if G is a k-colorable graph with k odd then G admits a vertex-coloring k-edge-weighting. So, for the class of 3-colorable graphs, including bipartite graphs, the answer is affirmative. Addario-Berry et al. [1] showed that every graph without isolated edges has a proper k-weighting when $k = 30$. After improvements to $k = 16$ in [2] and $k = 13$ in [16], Kalkowski et al. [8] showed that every graph without isolated edges has a proper 5-weighting. Recently, Przybylo [12] showed that every d-regular graph with $d \geq 2$ admits a vertex-coloring edge 4-weighting and every d-regular graph with $d \geq 10^8$ admits a vertex-coloring edge 3-weighting.

Przybylo and Wozniak [11] added the vertex coloring to the sum of colors of its incident edges, they gave the notation of neighbor sum distinguishing total coloring of graphs. Let $f : V(G) \cup E(G) \to [k]$ be a total k-coloring of a graph G. For every vertex x, let

$$t(x) = f(x) + \sigma(x).$$

Then f is called a neighbor sum distinguishing total coloring of G if $t(x) \neq t(y)$ for all adjacent vertices x and y in G. Similarly as above, the minimum value of k for which there exists a neighbor sum distinguishing total coloring of a graph G will be denoted by $\mathrm{tgndi}_{\sum}(G)$. Moreover, Przybylo and Wozniak [11] also put forward to a 1-2 conjecture with respect to this definition.

Conjecture 1.2. *[11] For any connected graph G,* $\mathrm{tgndi}_{\sum}(G) \leq 2$.

Note that Conjecture 1.2 is true when G is a 3-colourable, complete or 4-regular graph (see [11]). Up to now, it is known that for every graph G, $\mathrm{tgndi}_{\Sigma}(G) \leq 3$ (see [7]).

Inspired by the product versions of 1-2 conjecture (see [14]) and 1-2-3 conjecture (see [15]), and the neighbor product distinguishing total colorings of graphs were further studied in [17,18]. Flandrin et al. [6] considered that the color of the vertex is added to the sum of its neighbors and incident edges, they defined neighbor full sum distinguishing total coloring of graphs. Let $f : V(G) \cup E(G) \to [k]$ be a total k-coloring of G. Set

$$\phi(x) = t(x) + \sum_{y \in N(x)} f(y).$$

For any edge $xy \in E(G)$, if $\phi(x) \neq \phi(y)$, then f is called a neighbor full sum distinguishing (NFSD) total k-coloring of G. The smallest value k for which G has a NFSD-total coloring is called the neighbor full sum distinguishing total chromatic number of G and denoted by $\mathrm{fgndi}_{\sum}(G)$. In [5], Chang et al. obtained the neighbor full sum distinguishing total chromatic number of paths, cycles, 3-regular graphs, stars, complete graphs, trees, hypercubes, bipartite graphs and complete r-partite graphs. Meanwhile they posed a conjecture on NFSD-total coloring as follow:

Conjecture 1.3. *[5] For any connected graph G of order at least* 3*,* $\text{fgndi}_{\sum}(G) \le 3$.

A Halin graph is a plane graph H which constructed as follows. Let T be a tree having at least 4 vertices, called the characteristic tree of H. All vertices of T are either of degree 1, called leaves, or of degree at least 3. Let C be a cycle, called the adjoint cycle of H, connecting all leaves of T in such a way that C forms the boundary of the unbounded face. We usually write $H = T \cup C$ to reveal the characteristic tree and the adjoint cycle. Let $V_0 \subseteq V(H) \setminus V(C)$ and each vertex in V_0 is adjacent to some vertices on C.

This paper is organized as follows. In Section 2, by induction on the length of the adjoint cycle C, we get that $\text{fgndi}_{\sum}(H) \le 3$ for any 3-regular Halin graph H. In Section 3, for a Halin graph H with maximum degree $\Delta(H) \ge 4$, if every vertex in V_0 has degree at least four, via a coloring algorithm of tree T and by assigning colors to the edges of cycle C, we have $\text{fgndi}_{\sum}(H) \le 3$. Therefore, Conjecture 1.3 is valid for the above two types of Halin graphs.

2. 3-regular Halin graphs

A graph is said to be 3-regular if the degree of every vertex is 3. For $k \ge 1$, a 3-regular Halin graph N_{e_k}, called a necklace, was introduced in [13]. Its characteristic tree T_k consists of the path $v_0, v_1, \ldots, v_k, v_{k+1}$ and leaves $v'_1, v'_2, \ldots, v'_k$ such that the unique neighbor of v'_i in T_h is v_i for $1 \le i \le k$ and vertices $v_0, v'_1, v'_2, \ldots, v'_k, v_{k+1}$ are in order to form the adjoint cycle C_{h+2}.

Lemma 2.1. *For any necklace Ne_k,* $\text{fgndi}_{\sum}(Ne_1) = 3$ *and* $\text{fgndi}_{\sum}(Ne_k) = 2$ *for* $k \ge 2$.

Proof. Suppose that the necklace Ne_k is defined as above. Observe that Ne_k is a 3-regular Halin graph, if all vertices and edges of Ne_k are colored by 1, then $\phi(x) = 7$ for any vertex x of Ne_k, it conflicts with the definition of the NFSD-total coloring. In other words, we should use at least two colors to achieve an NFSD-total coloring of Ne_k. Observe that $Ne_1 = K_4$, it is easy to verify that $\text{fgndi}_{\sum}(K_4) = 3$. An NFSD-total 2-coloring of Ne_2 is shown in Figure 1.

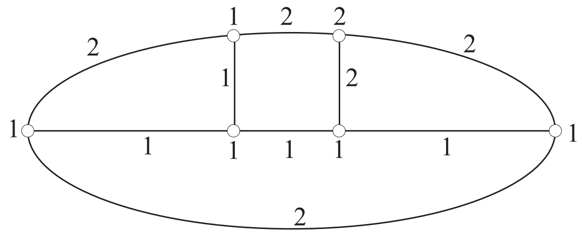

Figure 1. An NFSD-total 2-coloring of Ne_2.

For $k \ge 3$, we offer a total coloring f of Ne_k as follows:

$$f(v'_i) = \begin{cases} 1 & if\ i \equiv 1 (mod\ 2)\ and\ i \in [1, k], \\ 2 & if\ i \equiv 0 (mod\ 2)\ and\ i \in [1, k]; \end{cases}$$

$$f(v_i) = \begin{cases} 2 & if\ i = 0\ or\ i \equiv 1 (mod\ 2)\ and\ i \in [1, k], \\ 1 & if\ i \equiv 0 (mod\ 2)\ and\ i \in [1, k]. \end{cases}$$

If k is odd, set $f(v_{k+1}) = f(v_0v_1') = f(v_0v_{k+1}) = 2$ and $f(e) = 1$ for $e \in E(Ne_k) - \{v_0v_1', v_0v_{k+1}\}$, then we have $\phi(v_0) = 12$, $\phi(v_1') = \phi(v_{k+1}) = 11$, $\phi(v_{k-1}) = 10$ and $\phi(v_k) = 9$. For even k, set $f(v_{k+1}) = 1$, $f(v_0v_1') = f(v_{k-1}v_k) = 2$ and $f(e) = 1$ for $e \in E(Ne_k) - \{v_0v_1', v_{k-1}v_k\}$, then $\phi(v_0) = \phi(v_k) = 10$, $\phi(v_1') = 11$ and $\phi(v_{k-1}) = \phi(v_{k+1}) = 9$. Meanwhile,

$$\phi(v_i') = \begin{cases} 10 & if\ i \equiv 1(mod\ 2)\ and\ i \in [2, k], \\ 8 & if\ i \equiv 0(mod\ 2)\ and\ i \in [2, k]; \end{cases}$$

$$\phi(v_i) = \begin{cases} 8 & if\ i \equiv 1(mod\ 2)\ and\ i \in [2, k-2], \\ 10 & if\ i \equiv 0(mod\ 2)\ and\ i \in [2, k-2]. \end{cases}$$

Hence f is an NFSD-total 2-coloring of Ne_k.

As a fact that every vertex has degree 3 in the 3-regular Halin graph H, that is to say, the neighbor full sum distinguishing total coloring of H is actually a neighbor sum distinguishing edge coloring. Therefore, we need only to consider a neighbor sum distinguishing edge coloring of H.

Theorem 2.1. *If a 3-regular Halin graph $H = T \cup C$ is different from Ne_2 and Ne_4, then* $\text{fgndi}_{\sum}(H) \leq 3$.

Proof. We prove the theorem by induction on the length m of the adjoint cycle C. It is easy to see that the only 3-regular Halin graphs with $m = 3, 4$ and 5 are Ne_1, Ne_2 and Ne_3, respectively. They all satisfy our theorem by Lemma 2.1. Now assume that $m \geq 6$.

We firstly use 1 to color all vertices of H. In our later inductive steps, we use two basic operations to reduce a cubic Halin graph H to another cubic Halin graph H' such that the length of the adjoint cycle of H' is shorter than that of H. If H' is equal to neither Ne_2 nor Ne_4, then $\text{fgndi}_{\sum}(H') \leq 3$ by the induction hypothesis. Up to symmetry, Lih et al. [10] constructed eleven basic cubic Halin graphs based on Ne_2 and Ne_4. It is easy to give each cubic Halin graph an NSD-edge coloring by using three colors.

Let $P = u_0u_1 \dots u_h$ be a longest path in T. When $h \leq 4$, H must be a necklace $Ne_k, k \leq 3$. It is solved in Lemma 2.1. For $h \geq 5$, since P is of maximum length, all neighbors of u_1, except u_2, are leaves. We may change notation to let $w = u_3$, $u = u_2$, $v = u_1$, and v_1 and v_2, be the neighbors of v on C as depicted in Figure 2. Since $d_H(u) = 3$, there exists a path Q from u to x_1 or y_1 with $P \cap Q = u$. Without loss of generality, we may assume that Q is a path from u to y_1. Since P is a longest path in T, Q has length at most two. It follows that $uy_3 \in E(T)$ or $u = y_3$. The former implies that $uy_3 \in E(T)$ and the latter means $uy_1 \in E(T)$.

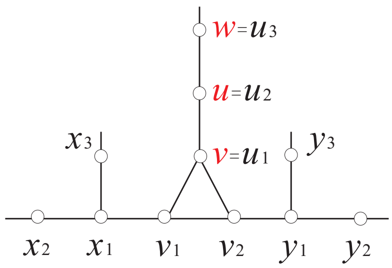

Figure 2. Around the end of a longest path in the characteristic tree.

Case 1. $uy_3 \in E(T)$. Consider Figure 3. Now let H' be the graph obtained from H by deleting six vertices $v, v_1, v_2, y_1, y_2, y_3$, and adding two new edges ux_1 and uz. By the induction hypothesis, we may assume that there exists an edge coloring f' for $E(H')$ using colors from the set $[1, 3]$. Assume that $f'(uw) = a$, $f(ux_1) = f'(uv) = b$, $f(uz) = f'(uy_3) = c$, $\{a, b, c\} \in \{1, 2, 3\}$.

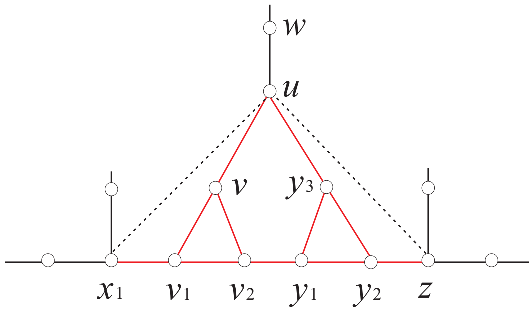

Figure 3. $uy_3 \in E(T)$.

Case 1.1. $\{a, b, c\} = \{1, 2, 3\}$.

Without loss of generality, set $a = 1,\ b = 2,\ c = 3$. Let f be an edge coloring of H, and f is defined as follows:

$$f(\theta) = \begin{cases} 1 & if\ \theta \in \{v_1v_2, v_2y_1, y_1y_2\}, \\ 2 & if\ \theta \in \{uv, x_1v_1, vv_2, y_2y_3\}, \\ 3 & if\ \theta \in \{uy_3, vv_1, y_1y_3, y_2z\}, \\ f'(\theta) & if\ otherwise. \end{cases}$$

Then $\phi(v) = 11$, $\phi(v_1) = 10$, $\phi(v_2) = 8$, $\phi(y_1) = 9$, $\phi(y_2) = 10$, $\phi(y_3) = 12$, $\phi(u) = 10$, $\phi(x_1) \neq 10$, $\phi(z) \neq 10$, and the weight of the remaining vertices keep the same as in H', which deduces that f is an NFSD-total 3-coloring of H.

Case 1.2. $a = b = c$.

For $a = b = c \neq 2$, set $d \in [1, 3] - \{a\}$ and $e \in [1, 3] - \{a, d\}$. Let f be an edge coloring of H, and f is defined as follows:

$$f(\theta) = \begin{cases} a & if\ \theta \in \{v_1v_2, y_1y_2, uv, x_1v_1, y_2y_3, uy_3, vv_1, y_2z\}, \\ d & if\ \theta \in \{v_2y_1, y_1y_3\}, \\ e & if\ \theta \in \{vv_2\}, \\ f'(\theta) & if\ otherwise. \end{cases}$$

Then $\phi(v) = 2a + e + 4$, $\phi(v_1) = 3a + 4$, $\phi(v_2) = a + d + e + 4$, $\phi(y_1) = a + 2d + 4$, $\phi(y_2) = 3a + 4$, $\phi(y_3) = 2a + d + 4$, $\phi(u) = 3a + 4$, $\phi(x_1) \neq 3a + 4$, $\phi(z) \neq 3a + 4$. Because $a \neq 2$, then $\phi(v) \neq \phi(v_2)$, $\phi(y_3) \neq \phi(u)$ and $\phi(y_3) \neq \phi(y_2)$. The weights of the remaining vertices keep the same as that in H', which implies that f is an NFSD-total 3-coloring of H.

For $a = b = c = 2$. Let f be an edge coloring of H, and f is defined as follows:

$$f(\theta) = \begin{cases} 1 & if\ \theta \in \{uy_3, y_1y_3\}, \\ 2 & if\ \theta \in \{uw, vv_1, x_1v_1, v_1v_2, y_1v_2, y_1y_2, y_2y_3, y_2z\}, \\ 3 & if\ \theta \in \{uv, vv_2\}, \\ f'(\theta) & if\ otherwise. \end{cases}$$

Then $\phi(v) = 12$, $\phi(v_1) = 10$, $\phi(v_2) = 11$, $\phi(y_1) = 9$, $\phi(y_2) = 10$, $\phi(y_3) = 8$, $\phi(u) = 10$, $\phi(x_1) \neq 10$, $\phi(z) \neq 10$, and the weights of the remaining vertices keep the same as that in H', it follows that f is an NFSD-total coloring of H.

Case 1.3. $a = b \neq c$ (or $a = c \neq b$ or $b = c \neq a$).

Set $d \in [1, 3] - \{a, c\}$. For $d \neq 2$, let f be an edge coloring of H, and f is defined as follows:

$$f(\theta) = \begin{cases} a & if\ \theta \in \{v_1v_2, y_1y_2, uv, x_1v_1, y_2y_3\}, \\ c & if\ \theta \in \{uy_3, vv_1, vv_2, y_2z\}, \\ f'(\theta) & if\ otherwise. \end{cases}$$

Then $\phi(v) = a + 2c + 4$, $\phi(v_1) = 2a + c + 4$, $\phi(v_2) = a + c + d + 4$, $\phi(y_1) = a + 2d + 4$, $\phi(y_2) = 2a + c + 4$, $\phi(y_3) = b + c + d + 4$, $\phi(u) = 2a + c + 4$, $\phi(x_1) \neq 2a + c + 4$, $\phi(z) \neq 2a + c + 4$, and the weights of the remaining vertices keep the same as that in H', it gets that f is an NFSD-total 3-coloring of H.

For $d = 2$, without loss of generality, set $a = b = 1,\ c = 3$. Let f be an edge coloring of H, and f is defined as follows:

$$f(\theta) = \begin{cases} 1 & if\ \theta \in \{uw, x_1v_1, v_1v_2, y_1y_2, y_2y_3\}, \\ 2 & if\ \theta \in \{uv, uy_3, vv_2\}, \\ 3 & if\ \theta \in \{vv_1, v_2y_1, y_1y_3, y_2z\}, \\ f'(\theta) & if\ otherwise. \end{cases}$$

Then $\phi(v) = 11$, $\phi(v_1) = 9$, $\phi(v_2) = 10$, $\phi(y_1) = 11$, $\phi(y_2) = 9$, $\phi(y_3) = 10$, $\phi(u) = 9$, $\phi(x_1) \neq 9$, $\phi(z) \neq 9$, and the weights of the remaining vertices keep the same as that in H', which deduces that f is an NFSD-total 3-coloring of H.

Case 2. $u = y_3$. Consider Figure 4. Let H' be the graph obtained from H by deleting four vertices v, v_1, v_2, y_1, and adding two new edges ux_1 and uy_2. By the induction hypothesis, we may assume that there exists an edge coloring f for $E(H')$ using colors from the set [1, 3]. Assume that $f'(uw) = a$, $f(ux_1) = f'(uv) = b$, $f(uy_2) = f'(uy_1) = c$, $\{a, b, c\} \in \{1, 2, 3\}$.

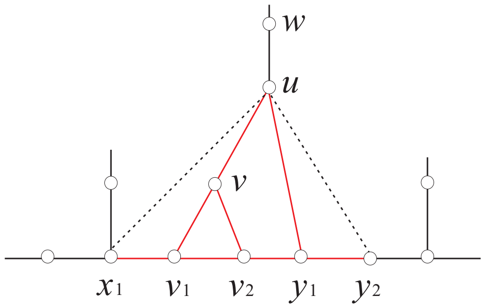

Figure 4. $u = y_3$.

Case 2.1. $a = b = c$.

Set $d \in [1, 3] - \{a\}$ and $e \in [1, 3] - \{a, d\}$. We define an edge coloring f of H as follows:

$$f(\theta) = \begin{cases} a & if\ \theta \in \{uw, v_1v_2, uv, x_1v_1, uy_1, vv_1, y_1y_2\}, \\ d & if\ \theta \in \{vv_2\}, \\ d\ or\ e & if\ \theta \in \{v_2y_1\}, \\ f'(\theta) & if\ otherwise. \end{cases}$$

Then $\phi(v) = 2a+d+4$, $\phi(v_1) = 3a+4$, $\phi(v_2) \in \{a+2d+4, a+d+e+4\}$, $\phi(y_1) \in \{e+2a+4, d+2a+4\}$, $\phi(u) = 3a + 4$, $\phi(x_1) \neq 3a + 4$. Without loss of generality, we assume that $a = b = c = 1$. Then $\phi(v) = 6+d$, $\phi(v_1) = 7$, $\phi(v_2) \in \{5+2d, 5+d+e\}$, $\phi(y_1) \in \{6+e, 6+d\}$, hence there exists a color d or e for edge v_2y_1 such that $\phi(y_1) \neq \phi(y_2)$ and $\phi(v_2) \neq \phi(y_1)$. The weights of the remaining vertices keep the same as that in H', namely, f is an NFSD-total 3-coloring of H.

Case 2.2. $\{a, b, c\} = \{1, 2, 3\}$.

We define an edge coloring f of H as follows:

$$f(\theta) = \begin{cases} a & if\ \theta \in \{uw, v_1v_2\}, \\ b & if\ \theta \in \{uv, x_1v_1\}, \\ c & if\ \theta \in \{uy_1, vv_1, vv_2, y_1y_2\}, \\ a\ or\ c & if\ \theta \in \{v_2y_1\}, \\ f'(\theta) & if\ otherwise. \end{cases}$$

Then $\phi(v) = b + 2c$, $\phi(v_1) = a + b + c + 4$, $\phi(v_2) \in \{2a + c + 4, a + 2c + 4\}$, $\phi(y_1) \in \{a + 2c + 4, 3c + 4\}$, so there exists a color a or c of edge v_2y_1 such that $\phi(y_1) \neq \phi(v_2)$ and $\phi(y_1) \neq \phi(y_2)$. The weights of the remaining vertices keep the same as that in H', hence f is an NFSD-total 3-coloring of H.

Case 2.3. $a = b \neq c$ (or $a = c \neq b$ or $b = c \neq a$).

Set $d \in [1, 3]-\{a, c\}$. Without loss of generality, assume that $2a \neq c+d$. We define an edge coloring f of H as follows:

$$f(\theta) = \begin{cases} a & if\ \theta \in \{uv, x_1v_1, vv_1, uy_1, y_1y_2\}, \\ c & if\ \theta \in \{uw, v_1v_2, v_2y_1\}, \\ f'(\theta) & if\ otherwise. \end{cases}$$

Then $\phi(v) = 2a + d + 4$, $\phi(v_1) = 2a + c + 4$, $\phi(v_2) = d + 2c + 4$, $\phi(y_1) = 2a + c + 4$ and the weights of remaining vertices keep the same as that in H', so f is an NFSD-total 3-coloring of H.

3. Every vertex in V_0 with degree at least four

Theorem 3.1. *Let $H = T \cup C$ be a Halin graph. If every vertex in V_0 has degree at least four, then* $\text{fgndi}_{\sum}(H) \leq 3$.

Proof. For any tree T, $\text{fgndi}_{\sum}(T) \leq 2$ (see [5]). Namely, there exists an NFSD-total coloring of T with two colors. Now assume that all vertices in T are colored by 2, and the edges of T are colored by 2 and 3. We propose the following algorithm and verify the feasibility of our coloring method.

Step 1. Label each vertex of T with 2.

Step 2. Select a vertex with maximum degree Δ in T and regard it as v. Initially, all incident edges of v are colored by 2.

Step 3. Let v_i be a neighboring vertex of v. If $d_T(v_i) = d_T(v)$ then color an incident edge of v_i (except for vv_i) with 3 and the other incident edges of v_i are colored by 2. If $d_T(v_i) \neq d_T(v)$ then color all incident edges (except for vv_i) of v_i with 2.

Step 4. Let v_{ij} be a neighboring vertex of v_i. Then color the incident edges of v_{ij} as follows:

Case 1. $\phi(v_i) = 4d_T(v_i) + 2$. If $d_T(v_{ij}) \neq d_T(v_i)$, then color all incident edges (except for v_iv_{ij}) of v_{ij} with 2. Color any one incident edge (except for v_iv_{ij}) of v_{ij} with 3 if $d_T(v_{ij}) = d_T(v_i)$ and the remaining incident edges of v_{ij} are colored by 2.

Case 2. $\phi(v_i) = 4d_T(v_i) + 3$. For $f(v_iv_{ij}) = 3$, if $d_T(v_{ij}) = d_T(v_i)$, then color an incident edge (except for v_iv_{ij}) of v_{ij} with 3 and the remaining incident edges of v_{ij} are colored by 2. If $d_T(v_{ij}) \neq d_T(v_i)$, then color all incident edges (except for v_iv_{ij}) of v_{ij} with 2. For $f(v_iv_{ij}) = 2$, color all incident edges (except for v_iv_{ij}) of v_{ij} with 2.

Step 5. Let v_{ijk} be a neighboring vertex of v_{ij}. Label the incident edges of v_{ijk} in the following ways.

Case 1. $\phi(v_{ij}) = 4d_T(v_{ij}) + 2$. If $d_T(v_{ijk}) \neq d_T(v_{ij})$, then color all incident edges (except for $v_{ij}v_{ijk}$) of v_{ijk} with 2. If $d_T(v_{ijk}) = d_T(v_{ij})$, then color any one incident edge (except for $v_{ij}v_{ijk}$) of v_{ijk} with 3 and the remaining incident edges of v_{ijk} are colored by 2.

Case 2. $\phi(v_{ij}) = 4d_T(v_{ij}) + 3$. For $f(v_{ij}v_{ijk}) = 3$, if $d_T(v_{ijk}) = d_T(v_{ij})$, then color one incident edge (except for $v_{ij}v_{ijk}$) of v_{ijk} with 3 and other incident edges are colored by 2. If $d_T(v_{ijk}) \neq d_T(v_{ij})$, then color all incident edges (except for $v_{ij}v_{ijk}$) of v_{ijk} with 2. For $f(v_{ij}v_{ijk}) = 2$, color all incident edges (except for $v_{ij}v_{ijk}$) of v_{ijk} with 2.

Case 3. $\phi(v_{ij}) = 4d_T(v_{ij}) + 4$. Color all incident edges (not including $v_{ij}v_{ijk}$) of v_{ijk} with 2.

Step 6. Continue this progress until all vertices of T achieving an NFSD-total coloring.

Now we use $\{1, 2, 3\}$ to color the edges on the adjoint cycle C. For an edge $xy \in E(H)$, $x \in V(C)$, $y \in V_0$, because $d_H(x) = 3$, it follows that $\phi(x) \in [12, 17]$. And for $d_H(y) \geq 4$, we have $\phi(y) \geq 18$. Hence $\phi(x) \neq \phi(y)$ is always true. The remaining task is to distinguish the weight of neighboring vertices on the adjoint circle C. Let $C = x_1x_2...x_mx_1$ and $e_i = x_ix_{i+1}$, $1 \leq i \leq m$ and the subscripts are taken modulo m.

Case A. There exist two adjacent vertices on the adjoint cycle receiving different weights in the characteristic tree T.

Without loss of generality, suppose that x_1 and x_m are the adjacent vertices such that $\phi_T(x_1) = 7$ and $\phi_T(x_m) = 6$, where $\phi_T(x)$ is the weight of vertex x in tree T. We give each edge e_i a color f in the following cases.

(1) $m \equiv 0 \ (mod\ 4)$

Set

$$f(e_i) = \begin{cases} 1 & i \equiv 1, 2 \ (mod\ 4), \\ 3 & i \equiv 0, 3 \ (mod\ 4). \end{cases}$$

Then

$$\phi(x_i) \in \begin{cases} \{12, 13\} & i \equiv 2 \ (mod\ 4), \\ \{14, 15\} & i \equiv 1 \ (mod\ 2), \\ \{16, 17\} & i \equiv 0 \ (mod\ 4). \end{cases}$$

(2) $m \equiv 1 \ (mod\ 4)$

Set

$$f(e_i) = \begin{cases} 1 & i \equiv 0, 1 \ (mod\ 4), \\ 3 & i \equiv 2, 3 \ (mod\ 4). \end{cases}$$

Then

$$\phi(x_i) \in \begin{cases} \{12\} & i = m, \\ \{13\} & i = 1, \\ \{12, 13\} & i \equiv 1 \ (mod\ 4) \ and\ i \neq 1, m, \\ \{14, 15\} & i \equiv 0 \ (mod\ 2), \\ \{16, 17\} & i \equiv 3 \ (mod\ 4). \end{cases}$$

(3) $m \equiv 2\ (mod\ 4)$

Set

$$f(e_i) = \begin{cases} 1 & i \equiv 1, 2\ (mod\ 4)\ and\ i \neq m, \\ 3 & i \equiv 0, 3\ (mod\ 4), \\ 2 & i = m. \end{cases}$$

Then

$$\phi(x_i) \in \begin{cases} \{13\} & i = m, \\ \{14\} & i = 1, \\ \{12, 13\} & i \equiv 2\ mod\ 4)\ and\ i \neq m, \\ \{14, 15\} & i \equiv 1\ (mod\ 2)\ and\ i \neq 1, \\ \{16, 17\} & i \equiv 0\ (mod\ 4). \end{cases}$$

(4) $m \equiv 3\ (mod\ 4)$

Set

$$f(e_i) = \begin{cases} 1 & i \equiv 1, 2\ (mod\ 4), \\ 3 & i \equiv 0, 3\ (mod\ 4). \end{cases}$$

Then

$$\phi(x_i) \in \begin{cases} \{14\} & i = m, \\ \{15\} & i = 1, \\ \{12, 13\} & i \equiv 2\ (mod\ 4), \\ \{14, 15\} & i \equiv 1\ (mod\ 2)\ and\ i \neq 1, m, \\ \{16, 17\} & i \equiv 0\ (mod\ 4). \end{cases}$$

Case B. All vertices on the adjoint cycle receive the same weights 6 (or 7) in the characteristic tree T. We give an edge coloring f of C as follows.

(1) $m \equiv 0\ (mod\ 3)$

Set

$$f(e_i) = \begin{cases} 1 & i \equiv 1\ (mod\ 3), \\ 2 & i \equiv 2\ (mod\ 3), \\ 3 & i \equiv 0\ (mod\ 3). \end{cases}$$

Then

$$\phi(x_i) = \begin{cases} 13 & i \equiv 2\ (mod\ 3), \\ 14 & i \equiv 1\ (mod\ 3), \\ 15 & i \equiv 0\ (mod\ 3), \end{cases}$$

or

$$\phi(x_i) = \begin{cases} 14 & i \equiv 2\ (mod\ 3), \\ 15 & i \equiv 1\ (mod\ 3), \\ 16 & i \equiv 0\ (mod\ 3). \end{cases}$$

(2) $m \equiv 1\ (mod\ 3)$

Set

$$f(e_i) = \begin{cases} 1 & i \equiv 1\ (mod\ 3), \\ 2 & i \equiv 2\ (mod\ 3), \\ 3 & i \equiv 0\ (mod\ 3). \end{cases}$$

Then

$$\phi(x_i) = \begin{cases} 12 & i = 1, \\ 13 & i \equiv 2\ (mod\ 3), \\ 14 & i \equiv 1\ (mod\ 3)\ and\ i \neq 1, \\ 15 & i \equiv 0\ (mod\ 3), \end{cases}$$

or

$$\phi(x_i) = \begin{cases} 13 & i = 1, \\ 14 & i \equiv 2\ (mod\ 3), \\ 15 & i \equiv 1\ (mod\ 3)\ and\ i \neq 1, \\ 16 & i \equiv 0\ (mod\ 3). \end{cases}$$

(3) $m \equiv 2\ (mod\ 3)$

$$f(e_i) = \begin{cases} 1 & i \equiv 1\ (mod\ 3)\ and\ i \neq m-1 or\ i = m, \\ 2 & i \equiv 2\ (mod\ 3\ and\ i \neq m, \\ 3 & i \equiv 0\ (mod\ 3)\ or\ i = m-1, \end{cases}$$

$$\phi(x_i) = \begin{cases} 12 & i = 1, \\ 13 & i \equiv 2\ (mod\ 3)\ and\ i \neq m, \\ 14 & i \equiv 1\ (mod\ 3)\ and\ i \neq 1, m-1, \\ 15 & i \equiv 0\ (mod\ 3), \\ 16 & i = m-1, \end{cases}$$

or

$$\phi(x_i) = \begin{cases} 13 & i = 1, \\ 14 & i \equiv 2\ (mod\ 3)\ and\ i \neq m, \\ 15 & i \equiv 1\ (mod\ 3)\ and\ i \neq 1, m-1, \\ 16 & i \equiv 0\ (mod\ 3), \\ 17 & i = m-1. \end{cases}$$

The above two cases implies that using three colors can achieve an NFSD-total coloring for Halin graphs where every vertex in V_0 has degree at least four.

4. Conclusions and future works

Nowadays, a large number of papers have studied graph coloring. This paper is devoted to the study of neighbor full sum distinguishing total coloring of Halin graphs. Meanwhile, we proved that the neighbor full sum distinguishing total chromatic number of two types of Halin graphs are not more than three: (i) 3-regular Halin graphs and (ii) every vertex in V_0 of a Halin graph with degree at least 4.

However, combining with Theorems 2.1 and 3.1, to confirm Conjecture 1.3 for all Halin graphs, it remains to deal with the case that vertices in V_0 have degree 3. Then it yields a problem as follow:

Problem. *Let $H = T \cup C$ be a Halin graph. If there is at least a vertex in V_0 with degree 3, then* $\mathrm{fgndi}_{\sum}(H) \leq 3$.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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