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Research article

Infinite growth of solutions of second order complex differential equations with meromorphic coefficients

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Abstract: This paper is devoted to studying the growth of solutions of $f'' + A(z)f' + B(z)f = 0$, where $A(z)$ and $B(z)$ are meromorphic functions. With some additional conditions, we show that every non-trivial solution f of the above equation has infinite order. In addition, we also obtain the lower bound of measure of the angular domain, in which the radial order of f is infinite.

Keywords: meromorphic function; infinite growth; complex differential equation; radial order
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1. Introduction and main results

Throughout this paper, we assume that the reader is familar with the fundamental results and the standard notations of Nevanlinna's value distribution theory (see [9, 14, 28]). In addition, we use $\rho(f)$ and $\mu(f)$ to denote the order and lower order of a meromorphic function $f(z)$ respectively, which are defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

The second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

where $A(z)$ and $B(z)$ are meromorphic functions, is the focus of this paper. To begin, we look for the conditions of coefficients that guarantee that every non-trivial meromorphic solution of Eq (1.1) has infinite order. Every non-trivial solution of Eq (1.1) must be an entire function, if $A(z)$ and $B(z)$ are entire functions, as is widely known. When $A(z)$ and $B(z)$ are entire functions, a lot of progress has been made, see Gundersen [5], Hellerstein, Miles and Rossi [10], and Ozawa [19]. The following is a summary of their work.

Theorem A. Suppose that $A(z)$ and $B(z)$ are entire functions satisfying any one of the following additional hypotheses:
(1) $\rho(A) < \rho(B)$, see [5];
(2) $A(z)$ is a polynomial and $B(z)$ is transcendental, see [10];
(3) $\rho(B) < \rho(A) \leq \frac{1}{2}$, see [19].
Then, every non-trivial solution f of Eq (1.1) is of infinite order.

One may ask a question based on Theorem A.

Question 1: If $\rho(A) = \rho(B)$, or if $\rho(A) > \rho(B)$ and $\rho(A) > \frac{1}{2}$, is every non-trivial solution f of Eq (1.1) is of infinite order ?

In general, the answer to Question 1 is negative.

Example 1.1. *Let $Q(z)$ be any non-constant polynomial, let $B(z) \not\equiv 0$ be any entire function with $\rho(B) < \deg(Q)$, let f be any antiderivative of $e^{Q(z)}$ that satisfies $\rho(f) = \deg(Q)$, and set $A(z) = -Q' - B(z)fe^{-Q}$. Then $\rho(B) < \rho(A) = \deg(Q) = \rho(f)$, and $f'' + A(z)f' + B(z)f = 0$. This shows that it is possible to have a finite order non-trivial solution f of Eq (1.1) where $\rho(B) < \rho(A)$ and $\rho(A)$ may be any positive integer.*

Example 1.2. *Let $Q(z)$ be any non-constant polynomial, let $A(z) \not\equiv 0$ be any entire function, and set $B(z) = -Q'' - (Q')^2 - A(z)Q'$. Then $\rho(A) = \rho(B)$ and it can be verifies that $f(z) = e^{Q(z)}$ satisfies the equation $f'' + A(z)f' + B(z)f = 0$. This shows that it is possible to have a finite order non-trivial solution f of Eq (1.1) where $\rho(A) = \rho(B)$.*

In some special cases, however, an entire solution of Eq (1.1) can have infinite order, see, for example, [3, 13, 15, 16, 22, 23]. Gundersen [8] took into account a special case in which the coefficient $A(z)$ of Eq (1.1) is an exponential function.

Theorem B. [8] Let $A(z) = e^{-z}$ and $B(z)$ is a transcendental entire function with order $\rho(B) \neq 1$. Then every non-trivial solution f of Eq (1.1) has infinite order.

When $\rho(B) = 1$, the entire solution of Eq (1.1) may be finite order, according to Theorem B. What conditions can guarantee that every non-trivial solution f of Eq (1.1) has infinite order if $\rho(B) = 1$? Chen [3] considered this question and proved the following result.

Theorem C. [3] Suppose that $A_j(z)(j = 0, 1)$ are entire functions with $\rho(A_j) < \infty$, a and b are complex constants with $ab \neq 0$ and $a = cb(c > 1)$. Let $A(z) = A_1(z)e^{az}$, $B(z) = A_2(z)e^{bz}$. Then every non-trivial solution f of Eq (1.1) has infinite order.

Remark 1.1. *Because $\rho(A) = 1$ and $\rho(B) \neq 1$ in Theorem B, $\rho(A) > \rho(B)$ and $\rho(A) > \frac{1}{2}$ can occur, whereas $\rho(A) = \rho(B) = 1$ in Theorem C. As a result, Theorems B and C partially answer Question 1 as well.*

Question 1 was recently studied by several scholars who assumed that $A(z)$ is a nontrivial solution of a second order differential equation. We have the following collection theorem.

Theorem D. Let $A(z)$ be a nontrivial solution of $w'' + P(z)w = 0$, where $P(z)$ is a nonconstant polynomials with $\deg(P) = n$, and satisfying any one of the following additional hypotheses:
(1) $\rho(B) < \frac{1}{2}$, see [24];
(2) $B(z)$ is an entire function with Fabry gaps, see [16]; Here, an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is said

to have Fabry gaps if $\frac{\lambda_n}{n} \to \infty$ as $n \to \infty$.
(3) $T(r, B) \sim \log M(r, B)$ as $r \to \infty$ outside a set of finite logarithmic measure such that $\rho(A) \neq \rho(B)$, see [29]. Then every nontrivial solution of Eq (1.1) is of infinite order.

Remark 1.2. *We know that $\rho(A) = \frac{n+2}{2}$ based on Theorem D's assumptions. Clearly, Theorem D partially answers Question 1 because $\rho(A) = \rho(B)$ or $\rho(A) > \rho(B)$ with $\rho(A) > \frac{1}{2}$ can occur if $B(z)$ meets one of the conditions (1)-(3) in Theorem D. We also find that the proof of three cases of Theorem D is based on the observation that if $A(z)$ is a nontrivial solution of $w'' + P(z)w = 0$, then the whole plane may be divided into $n + 2$ sectors $S_j (j = 0, 1, \cdots, n + 1)$ in which $A(z)$ either blows up or decays to zero exponentially.*

We will continue to study Question 1 in this paper. We consider a more general case in which $A(z)$ in some angular domains either blows up or decays to zero rapidly. The coefficients of Eq (1.1) in particular, are meromorphic rather than entire.

Theorem 1.1. *Let $\{\phi_k\}$ be a finite set of real numbers satisfying $\phi_1 < \phi_2 < ... < \phi_{2n} < \phi_{2n+1}$ with $\phi_{2n+1} = \phi_1 + 2\pi$, and set*

$$\nu = \max_{1\leq k\leq 2n} (\phi_{k+1} - \phi_k).$$

Suppose that $A(z)$ and $B(z)$ are meromorphic functions such that for some constant $\alpha \geq 0$ and a set $H \subset [0, 2\pi)$ of linear measure zero,

$$|A(z)| = O(|z|^{\alpha})$$

as $z \to \infty$ in $\arg z \in (\phi_{2k-1}, \phi_{2k})\backslash H$ for $k = 1, ..., n$, and where $B(z)$ is transcendental with a deficient value ∞ and

$$\mu(B) < \frac{4 \arcsin \sqrt{\frac{\delta(\infty,B)}{2}}}{\nu}.$$

Then every non-trivial meromorphic solution f of Eq (1.1) has infinite order.

Remark 1.3. *We can use a specific example to illustrate our point. If $A(z) = e^{P(z)}$, where $P(z)$ is a polynomial with $\deg(P) = n$, and $B(z)$ is an entire function with $\mu(B) < n$, then the example meets Theorem 1.1's criteria.*

Corollary 1.1. *Suppose that $A(z) = h_1(z)e^{p_1(z)} + h_2(z)e^{p_2(z)} + p_3(z)$, where $p_3(z)$ is a polynomial, $p_i(z) = a_i z^n + ...(i = 1, 2)$ are two non-constant polynomials of degree n with $\arg a_1 - \arg a_2 \neq \pm\pi$, and $h_i(z)(i = 1, 2)$ are meromorphic functions of order less than n. Let $B(z)$ be given as Theorem 1.1. Then every non-trivial solution f of Eq (1.1) has infinite order.*

Remark 1.4. *For $p_1(z)$ and $p_2(z)$, from remark 2.1, there exist $\Omega_k(p_1)$ and $\Omega_k(p_2)$ such that when k is odd, $\delta(p_1, \theta) < 0$ if $\theta \in \Omega_k(p_1)$ and $\delta(p_2, \theta) < 0$ if $\theta \in \Omega_k(p_2)$. Since $\arg a_1 - \arg a_2 \neq \pm\pi$, $\Omega_k(p_1) \cap \Omega_k(p_2) \neq \emptyset$. Then we can redivide the plane into $2n$ open angles $S_j (j = 0, 1, ..., 2n - 1)$ such that for $\theta \in S_j$, $\delta(p_1, \theta) < 0$ and $\delta(p_2, \theta) < 0$ if j is odd, while $\delta(p_1, \theta) \geq 0$ or $\delta(p_2, \theta) \geq 0$ if j is even. Suppose that $\deg(p_3) = m$. By Lemma 2.3, we know $|A(re^{i\theta})| = O(r^m)$ in S_j outside a set of linear measure zero when j is odd. By Theorem 1.1, Corollary 1.1 holds.*

Corollary 1.2. *Suppose $A(z) = h_1(z)e^{p_1(z)} + h_2(z)e^{p_2(z)} + p_3(z)$, where p_3 is a polynomial, $p_i(i = 1, 2)$ are two non-constant polynomials with $\deg(p_1) \neq \deg(p_2)$, and $h_i(z)(i = 1, 2)$ are meromorphic functions of order less than $\deg(p_i)$. Let $B(z)$ be given as Theorem 1.1. Then every non-trivial solution f of Eq (1.1) has infinite order.*

Remark 1.5. *Let* $p_1(z) = a_n z^n + ... + a_0$, $p_2(z) = b_t z^t + ... + b_0$. *Similarly as remark 1.4, we also can redivide the plane into* $2s$ *open angles* $S_j (j = 0, 1, ..., 2s-1)$ *where* s *may be different from* n *and* t, *and depends on* $\deg(p_i)(i = 1, 2)$, a_n *and* b_t, *such that for* $\theta \in S_j$, $\delta(p_1, \theta) < 0$ *and* $\delta(p_2, \theta) < 0$ *if* j *is odd; while* j *is even,* $\delta(p_1, \theta) \geq 0$ *or* $\delta(p_2, \theta) \geq 0$. *Then Corollary 1.2 holds.*

Next, we consider the lower bound estimate of the measure of the angular domain, in which the radial order of every non-trivial solution of (1.1) is infinite. The following notations and notions are provided to further illustrate our concerns.

Assuming $0 \leq \alpha < \beta \leq 2\pi$, we denote

$$\Omega(\alpha, \beta) = \{z \in \mathbb{C} : \arg z \in (\alpha, \beta)\},$$

and use $\overline{\Omega}(\alpha, \beta)$ to denote the closure of $\Omega(\alpha, \beta)$. Similarly, we denote

$$\Omega(r, \alpha, \beta) = \{z \in \mathbb{C} : \arg z \in (\alpha, \beta), |z| > r\}.$$

Suppose that $f(z)$ is an analytic function in $\overline{\Omega}(\alpha, \beta)$, we use $\rho_{\alpha,\beta}$ to denote the order of growth of f on $\Omega(\alpha, \beta)$, that is

$$\rho_{\alpha,\beta}(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, \Omega(\alpha, \beta), f)}{\log r},$$

where $M(r, \Omega(\alpha, \beta), f) = \sup_{\alpha \leq \theta \leq \beta} |f(re^{i\theta})|$. Furthermore, we denote the radial order of $f(z)$ by

$$\rho_\theta(f) = \lim_{\varepsilon\to 0} \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), f)}{\log r}.$$

In recent years, some progress in the angular distribution of entire solutions of linear differential equations have been obtained by several researchers, see [12, 20, 21, 24–27]. In particular, Huang and Wang [11] considered the following question. We know that for a transcendental entire function $f(z)$ of infinite order, at least one ray $\arg z = \theta$ exists with the radial order $\rho_\theta(f) = \infty$. However, for any angular domain $\Omega(\alpha, \beta)$, $\rho(f) = \infty$ cannot ensure $\rho_{\alpha,\beta}(f) = \infty$, $f(z) = e^{e^z}$, for example, satisfies $\rho(f) = \infty$, whereas $\rho_{\pi/2, 3\pi/2}(f) = 0$. Motivated by this fact, we raise an interesting question: how wide are such $\Omega(\alpha, \beta)$ with $\rho_{\alpha,\beta}(f) = \infty$? Let

$$I(f) = \{\theta \in [0, 2\pi) : \rho_\theta(f) = \infty\}.$$

Clearly, $I(f)$ is closed and measurable. In 2015, Huang and Wang [11] proved the following result.

Theorem E. Suppose that $A(z)$, $B(z)$ are entire function with $\rho(A) < \mu(B)$. If $f(z)$ is a non-trivial solution of Eq (1.1), then $mesI(f) \geq \min\{2\pi, \pi/\mu(B)\}$.

Inspired by Theorem E, we consider the lower bound of measure of the set of infinite radial order solutions of Eq (1.1). As a result, we have the following.

Theorem 1.2. *Let* $A(z)$ *be an entire function of finite order having a finite Borel exceptional value, and* $B(z)$ *be a transcendental entire function with* $\mu(B) < \rho(A)$. *Then, every non-trivial solution* f *of Eq (1.1) is of infinite order and satisfies* $mes(I(f)) \geq \min\{k_1, k_2\}$, *where* $k_1 := \pi$, $k_2 :=$
$$\begin{cases} \frac{[\rho(A)/\mu(B)]}{2} \frac{\pi}{\rho(A)}, & \text{if } [\frac{\rho(A)}{\mu(B)}] \text{ is even} \\ \frac{\pi}{\mu(B)} - \frac{[\rho(A)/\mu(B)]+1}{2} \frac{\pi}{\rho(A)}, & \text{if } [\frac{\rho(A)}{\mu(B)}] \text{ is odd.} \end{cases}$$

Remark 1.6. *In Theorem 1.2, we use* $[x]$ *to denote the largest integer not exceeding the real number* x*, for example,* $[1.2] = 1$.

Remark 1.7. *Under the assumptions of Corollary 1.1, we know every non-trivial solution* f *of Eq (1.1) is an entire function with infinite order. Hence, from the proof of Theorem 1.2, we obtain that* $mes(I(f)) \geq k_1$ *if* $\mu(B) \in [0, \frac{1}{2})$*, while* $mes(I(f)) \geq k_2$ *if* $\mu(B) \in [\frac{1}{2}, \rho(A))$.

2. Preliminary lemmas

Lemma 2.1. *[17] Let* f *be an entire function of finite order having a finite Borel exceptional value* c*. Then*

$$f(z) = h(z)e^{Q(z)} + c,$$

where $h(z)$ *is an entire function with* $\rho(h) < \rho(f)$*,* $Q(z)$ *is a polynomial of degree* $deg(Q) = \rho(f)$.

Lemma 2.2. *[6] Let* $g(z)$ *be a transcendental meromorphic function of order* $\rho(g) = \rho < \infty$*. Then there exists a set* $E \subset [0, 2\pi)$ *of linear measure zero such that, for any* $\theta \in [0, 2\pi)\backslash E$*, there exists a positive constant* $R_0 = R_0(\theta) > 1$ *such that, for all* z *satisfying* $\arg z = \theta$ *and* $|z| > R_0$,

$$\left|\frac{g^{(k)}(z)}{g(z)}\right| \leq |z|^{k(\rho(g)-1+\varepsilon)},$$

where $k = 1, 2$ *and* ε *is given positive real constant.*

Remark 2.1. *For the polynomial* $p(z)$ *with degree* n*, set* $p(z) = (\alpha + i\beta)z^n + p_{n-1}(z)$ *with* α, β *real, and denote* $\delta(p, \theta) := \alpha \cos n\theta - \beta \cos n\theta$*. The rays*

$$\arg z = \theta_k = \frac{\arctan \frac{\alpha}{\beta} + k\pi}{n}, \quad k = 0, 1, 2, ..., 2n - 1$$

satisfying $\delta(p, \theta_k) = 0$ *can split the complex domain into* $2n$ *equal angles. We denote these angle domains as*

$$\Omega_k(p) = \{\theta : -\frac{\arctan \frac{\alpha}{\beta}}{n} + (2k-1)\frac{\pi}{2n} < \theta < -\frac{\arctan \frac{\alpha}{\beta}}{n} + (2k+1)\frac{\pi}{2n}\}, k = 0, 1, ..., 2n - 1.$$

And we know that for $\theta \in \Omega_k(p)$*,* $\delta(p, \theta) > 0$ *if* k *is even, while* $\delta(p, \theta) < 0$ *if* k *is odd.*

Lemma 2.3. *[4, Lemma 1.20] Let* $p(z)$ *be a polynomial of degree* $n \geq 1$*. Suppose that* $h(z) \not\equiv 0$ *is a meromorphic function and* $\rho(h) < n$*. Consider the function* $g(z) := h(z)e^{p(z)}$*, there exists a set* $H_1 \subset [0, 2\pi)$ *of linear measure zero, for every* $\theta \in [0, 2\pi)\backslash(H_1 \cup H_2)$ *and* $r > r_0(\theta) > 0$*, we have*
(1) If $\delta(p, \theta) > 0$,

$$|g(re^{i\theta})| \geq \exp\{\frac{1}{2}\delta(p.\theta)r^n\}$$

holds here;
(1) If $\delta(p, \theta) < 0$,

$$|g(re^{i\theta})| \leq \exp\{\frac{1}{2}\delta(p.\theta)r^n\}$$

holds here, where $H_2 = \{\theta : \delta(p, \theta) = 0, 0 \leq \theta < 2\pi\}$ *is a finite set.*

The following lemma due to Markushevich [18].

Lemma 2.4. *[18] Suppose $f(z) = e^{P(z)}$, where $P(z)$ is a polynomial of degree n:*

$$P(z) = b_n z^n + b_{n-1} z^{n-1} + ... + b_0 \quad (b_n \neq 0, n \geq 1).$$

We know that $f(z)$ is a function of order n. Let $b_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$ and $z = re^{i\theta}$, we now divide the plane into $2n$ equal open angles

$$S_j = \left\{\theta : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n}\right\},$$

where $j = 0, 1, 2, ..., 2n-1$. We also introduce $2n$ closed angles

$$S_j(\varepsilon) = \left\{\theta : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \frac{\varepsilon}{n} \leq \theta \leq -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \frac{\varepsilon}{n}\right\}, \quad j = 0, 1, 2, ..., 2n-1,$$

where $0 < \varepsilon < \pi/2$ and $S_j(\varepsilon) \subset S_j$. Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$,

$$|f(z)| > \exp\{\alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n\}$$

if $\arg z \in S_j(\varepsilon)$ when j is even; While

$$|f(z)| < \exp\{-\alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n\}$$

if $\arg z \in S_j(\varepsilon)$ when j is odd.

Remark 2.2. *Clearly, for any $\arg z \in S_j$, we always find an ε, $\varepsilon \in (0, \pi/2)$ and a positive number $R = R(\varepsilon)$ such that $\arg z \in S_j(\varepsilon)$, and for $|z| = r > R$,*

$$|f(z)| > \exp\{\alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n\}$$

if j is even; While

$$|f(z)| < \exp\{-\alpha_n(1-\varepsilon)\sin(n\varepsilon)r^n\}$$

if j is odd.

Lemma 2.5. *[1] Suppose that $g(z)$ is an entire function with $\mu(g) \in [0, 1)$. Then, for every $\alpha \in (\mu(g), 1)$, there exists a set $E \subset [0, \infty)$ such that $\overline{\log dens} E \geq 1 - \frac{\mu(g)}{\alpha}$, where $E = \{r \in [0, \infty) : m(r) > M(r)\cos\pi\alpha\}$, $m(r) = \inf_{|z|=r} \log|g(z)|$ and $M(r) = \sup_{|z|=r} \log|g(z)|$.*

Lemma 2.6. *[22] Suppose that f is an entire function and lower order $\mu(f) \in [\frac{1}{2}, +\infty)$. Then, there exists an domain $\Omega(\alpha, \beta)$, where α, β satisfy $\beta - \alpha \geq \frac{\pi}{\mu(f)}$ and $0 \leq \alpha \leq \beta \leq 2\pi$, such that*

$$\limsup_{r\to\infty} \frac{\log\log|f(re^{i\psi})|}{\log r} \geq \mu(f)$$

for all $\psi \in (\alpha, \beta)$.

Lemma 2.7. *[7] Let $z = r\exp(i\psi), r_0 + 1 < r$ and $\alpha \leq \psi \leq \beta$, where $0 < \beta - \alpha \leq 2\pi$. Suppose that $n(\geq 2)$ is an integer, and that $g(z)$ is analytic in $\Omega(r_0, \alpha, \beta)$ with $\rho_{\alpha,\beta} < \infty$. Choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon \in (0, \frac{\beta_j - \alpha_j}{2})(j = 1, 2, ..., n-1)$ outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s \quad and \quad \beta_j = \beta + \sum_{s=1}^{j-1} \varepsilon_s, j = 2, 3, ..., n-1,$$

there exist $K > 0$ and $M > 0$ only depending g, $\varepsilon_1, ..., \varepsilon_{n-1}$ and $\Omega(\alpha_{n-1}, \beta_{n-1})$, and not depend on z such that

$$\left|\frac{g'(z)}{g(z)}\right| \leq Kr^M (\sin k(\psi - \alpha))^{-2}$$

and

$$\left|\frac{g^{(n)}(z)}{g(z)}\right| \leq Kr^M \left(\sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_j \left(\psi - \alpha_j\right)\right)^{-2}$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R-set D, where $k = \pi/(\beta - \alpha)$ and $k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j(j = 1, 2, ..., n-1))$.

Lemma 2.8. *[2] Let $f(z)$ be a transcendental meromorphic function of finite lower order μ, and have one deficient value a. Let $\Lambda(r)$ be a positive function with $\Lambda(r) = o(T(r, f))$ as $r \to \infty$. Then for any fixed sequence of Pólya peaks $\{r_n\}$ of order μ, we have*

$$\liminf_{n\to\infty} \operatorname{mes} D_\Lambda(r_n, a) \geq \min\left\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}}\right\},$$

where $D_\Lambda(r, a)$ is defined by

$$D_\Lambda(r, \infty) = \left\{\theta \in [0, 2\pi) : \left|f\left(re^{i\theta}\right)\right| > e^{\Lambda(r)}\right\},$$

and for finite a,

$$D_\Lambda(r, a) = \left\{\theta \in [0, 2\pi) : \left|f\left(re^{i\theta}\right) - a\right| < e^{-\Lambda(r)}\right\}.$$

3. Proof of Theorem 1.1

Suppose on the contrary to the assertion that there exists a non-trivial solution f with $\rho(f) < \infty$. We aim for a contradiction. Let $\{\phi_k\}$ be a finite collection of real numbers satisfying $\phi_1 < \phi_2 < ... < \phi_{2n} < \phi_{2n+1}$ with $\phi_{2n+1} = \phi_1 + 2\pi$, and set

$$v = \max_{1\leq k\leq 2n} (\phi_{k+1} - \phi_k).$$

Suppose that $A(z)$ is a meromorphic function such that for some constant $\alpha \geq 0$ and a set $H \subset [0, 2\pi)$ of linear measure zero,

$$|A(z)| = O(|z|^\alpha) \tag{3.1}$$

as $z \to \infty$ in $\arg z \in (\phi_{2k-1}, \phi_{2k})\backslash H$ for $k = 1, ..., n$. From Eq (1.1), we get the following inequality

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + \left|\frac{f'(z)}{f(z)}\right| |A(z)|. \tag{3.2}$$

From Lemma 2.2, we can obtain that there exists a set $E^* \subset [0, 2\pi)$ of linear measure zero and a positive constant $R_0 > 1$ such that

$$\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \leq r^{\rho(f)}, \left|\frac{f''(re^{i\theta})}{f(re^{i\theta})}\right| \leq r^{2\rho(f)}, \tag{3.3}$$

for any $\theta \in [0, 2\pi)\backslash E^*$ and $r > R_0$.

Let $B(z)$ be a transcendental meromorphic function having a deficient value ∞ and $\mu(B) < 4 \arcsin \sqrt{\frac{\delta(\infty,B)}{2}}/\nu$. Then we have

$$\frac{4}{\mu(B)} \arcsin \sqrt{\frac{\delta(\infty, B)}{2}} > \nu. \tag{3.4}$$

Next, we define

$$\Lambda(r) = \sqrt{T(r, B) \log r}.$$

It is clear that $\Lambda(r) = o(T(r, B))$ and $\Lambda(r)/\log r \to \infty$ as $r \to \infty$ since $B(z)$ is a transcendental meromorphic function. Applying Lemma 2.8 to $B(z)$ gives the existence of the Pólya peaks $\{r_j\}$ of order $\mu(B)$ such that for sufficiently large j,

$$mesD_\Lambda\left(r_j, \infty\right) \geq \min\left\{2\pi, \frac{4}{\mu(B)} \arcsin \sqrt{\frac{\delta(\infty, B)}{2}}\right\},$$

where $D_\Lambda(r_j, \infty)$ is defined by

$$D_\Lambda(r_j, \infty) = \left\{\theta \in [0, 2\pi) : \left|B\left(r_j e^{i\theta}\right)\right| > e^{\Lambda(r_j)}\right\}.$$

From (3.4), there exists at least one sector (ϕ_{2k-1}, ϕ_{2k}) such that

$$mes((\phi_{2k-1}, \phi_{2k}) \cap D_\Lambda\left(r_j, \infty\right)) > 0,$$

where $k = 1, ..., n$.

Let $F_j := (\phi_{2k-1}, \phi_{2k}) \cap D_\Lambda\left(r_j, \infty\right)$. On the one hand, for any $\theta \in F_j$ and sufficiently large j, we have

$$|B(r_j e^{i\theta})| > \exp\{\Lambda(r_j)\}.$$

Since $\Lambda(r)/\log r \to \infty$ as $r \to \infty$,

$$\lim_{j\to\infty} \frac{\log |B(r_j e^{i\theta})|}{\log r_j} = \infty. \tag{3.5}$$

Substituting (3.1) and (3.3) into (3.2), for any $\theta \in F_j\backslash(E^* \cup H)$ and sufficiently large j, we have

$$|B(r_j e^{i\theta})| < O(r_j^{2\rho(f)+\alpha}), \tag{3.6}$$

where $\alpha \geq 0$ is a constant. Coupling (3.5)and (3.6) yields a contradiction. Thus, every non-trivial solution f of Eq (1.1) is of infinite order.

4. Proof of Theorem 1.2

We assume the contrary to the assertion that $m(I(f)) < k$, where $k = k_1$ or k_2. Then $t := k - \text{mes}\, I(f) > 0$. Our goal is to obtain a contradiction. Since $I(f)$ is closed, $H := (0, 2\pi)\backslash I(f)$ is open. So it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)(i = 1, 2, ..., m)$ in H such that

$$mes(H\backslash \bigcup_{i=1}^{m} I_i) < \frac{t}{4}. \tag{4.1}$$

It easy to see that $I_i \cap I(f) = \emptyset$, and hence $\rho_{\alpha_i,\beta_i}(f) < \infty$ for each $i = 1, 2, ..., m$. Apply Lemma 2.7 to f, for sufficiently small $\xi > 0$, there exist two constants $M > 0$ and $K > 0$ such that

$$|\frac{f^{(s)}(z)}{f(z)}| \leq Kr^M, \quad (s = 1, 2) \tag{4.2}$$

for all $z \in \cup_{i=1}^{m} \Omega(r_j, \alpha_i + \xi, \beta_i - \xi)$ outside an R-set D.

Suppose that $A(z)$ is an entire function of finite order having a finite Borel exceptional value a. By Lemma 2.1, we have

$$A(z) = h(z)e^{Q(z)} + a,$$

where $h(z)$ is an entire function with $\rho(h) < \rho(A)$, $Q(z)$ is a polynomial of degree $deg(Q) = \rho(A)$. Let $Q(z) = b_d z^d + b_{d-1} z^{d-1} + ... + b_0$, where d is a positive integer and $b_d = \alpha_d e^{i\theta_d}$, $\alpha_d > 0$ and $\theta_d \in [0, 2\pi)$. We now divide the plane into $2d$ equal open angles

$$S_j = \left\{\theta : -\frac{\theta_d}{d} + (2j - 1)\frac{\pi}{2d} < \theta < -\frac{\theta_d}{d} + (2j + 1)\frac{\pi}{2d}\right\},$$

where $j = 0, 1, 2, ..., 2d - 1$. We also introduce $2d$ closed angles

$$S_j(\varepsilon) = \left\{\theta : -\frac{\theta_d}{d} + (2j - 1)\frac{\pi}{2d} + \frac{\varepsilon}{d} \leq \theta \leq -\frac{\theta_d}{d} + (2j + 1)\frac{\pi}{2d} - \frac{\varepsilon}{d}\right\}, \quad j = 0, 1, 2, ..., 2d - 1,$$

where $0 < \varepsilon < \pi/4$ and $S_j(\varepsilon) \subset S_j$. According to Lemma 2.4, for $z = re^{i\theta}$, if $\theta \in S_j(\varepsilon)$ and j is odd, there exists a positive number $R(\varepsilon)$, such that

$$|e^{Q(z)}| < \exp\{-\alpha_d(1 - \varepsilon)\sin(d\varepsilon)r^d\}$$

for $r > R(\varepsilon)$. Then for any given $\varepsilon \in (0, \pi/4)$, $\rho(h) < d = \rho(A)$ and $\theta \in S_j(\varepsilon)$($j$ is odd), we have

$$|A(re^{i\theta}) - a| < \exp\{-C(\varepsilon)r^d\},$$

for all sufficiently large r, where $C(\varepsilon)$ is a positive only constant depending on α_d, ε and $\rho(A)$. Therefore, for any $\theta \in S_j$(j is odd), we can find an ε and a positive constant $C(\varepsilon)$ such that $\theta \in S_j(\varepsilon)$ and

$$|A(re^{i\theta})| \leq |A(re^{i\theta}) - a| + |a| < \exp\{-C(\varepsilon)r^d\} + |a| \tag{4.3}$$

for all sufficiently large r. From Eq (1.1), we have

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + \left|\frac{f'(z)}{f(z)}\right| |A(z)|. \tag{4.4}$$

In the following, we consider three cases.

Case 1. Suppose $\mu(B) = 0$. Then $k = k_1$. Define

$$S := \bigcup_{i=1}^{d} S_{2i-1}.$$

Clearly, mes $S = \pi$. Thus,

$$mes(S \cap H) = mes(S \backslash (I(f) \cap S)) \geq mes(S) - mes(I(f)) > \frac{3t}{4}.$$

Hence

$$mes((\bigcup_{i=1}^{m} I_i) \cap S) = mes(S \cap H) - mes((H \backslash \bigcup_{i=1}^{m} I_i) \cap S) > \frac{3t}{4} - \frac{t}{4} = \frac{t}{2}.$$

Then we can conclude that exists at least one open interval $I_{i0} = (\alpha_0, \beta_0)$ such that

$$mes(S \cap (\alpha_0, \beta_0)) > \frac{t}{2m} > 0,$$

and there exists at least one sector S_j(j is odd) such that

$$mes(S_j \cap (\alpha_0, \beta_0)) > \frac{t}{2md} > 0.$$

We can find an $\varepsilon > 0$ such that $mes(S_j(\varepsilon) \cap (\alpha_0, \beta_0)) > 0$. So $D(\varepsilon) := S_j(\varepsilon) \cap (\alpha_0 + \xi, \beta_0 - \xi) \neq \emptyset$, where $0 < \xi < mes(S_j(\varepsilon) \cap (\alpha_0, \beta_0))/4$. Applying Lemma 2.5 to $B(z)$, we can choose $\alpha = \frac{1}{4}$ and there is a set $E \subset [0, \infty)$ such that $\overline{\log dens} E = 1$ such that, for all $r \in E$,

$$\log |B(re^{i\theta})| > \frac{\sqrt{2}}{2} \log M(r, B), \theta \in [0, 2\pi) \tag{4.5}$$

where $M(r, B) = \sup_{|z|=r} |B(z)|$. By substituting (4.5), (4.3) and (4.2) into (4.4), for any $\theta \in D(\varepsilon)$ and $r \in E$ outside an R-set D, we have

$$M(r, B)^{\frac{\sqrt{2}}{2}} < Kr^M (1 + \exp\{-C(\varepsilon) r^d\} + |a|) \tag{4.6}$$

for all sufficiently large r. Since $B(z)$ is a transcendental entire function, we know that

$$\liminf_{r \to \infty} \frac{\log M(r, B)}{\log r} = +\infty. \tag{4.7}$$

From (4.6) and (4.7), we can deduce a contradiction. Therefore, $mes(I(f)) \geq k_1$.

Case 2. Suppose $0 < \mu(B) < \frac{1}{2}$. So $k = k_1$. Similarly as in Case 1, we have a sector $D(\varepsilon) := S_j(\varepsilon) \cap (\alpha_0 + \xi, \beta_0 - \xi) \neq \emptyset$. According to Lemma 2.5, for any given $\alpha \in (\mu(B), \frac{1}{2})$, there exists a set $E \subset [0, \infty)$ such that $\overline{\log dens} E \geq 1 - \frac{\mu(B)}{\alpha}$, where $E = \{r \in [0, \infty) : m(r) > M(r) \cos \pi\alpha\}$, $m(r) = \inf_{|z|=r} \log |B(z)|$ and $M(r) = \sup_{|z|=r} \log |B(z)|$. Thus, there exists a constant $R_1 > 0$ such that, for arbitrarily small $\eta > 0$ and all $r \in E \backslash [0, R_1]$,

$$\left| B\left(re^{i\theta}\right) \right| \geq \exp\{r^{\mu(B)-\eta}\}, \theta \in [0, 2\pi). \tag{4.8}$$

Taking (4.8), (4.3) and (4.2) into (4.4), for any $\theta \in D(\varepsilon)$ and $r \in E\backslash[0, R_1]$ outside an R-set D, we deduce

$$\exp\{r^{\mu(B)-\eta}\} < Kr^M(1 + \exp\{-Cr^d\} + |a|) \tag{4.9}$$

for all sufficiently large r. This is a contradiction. Therefore, $mes(I(f)) \geq k_1$.

Case 3. Suppose $\frac{1}{2} \leq \mu(B) < \rho(A)$. So $k = k_2$. By using Lemma 2.6 to $B(z)$, we can get a sector $\Omega(\alpha,\beta)$ with $\beta - \alpha \geq \frac{\pi}{\mu(B)} > \frac{\pi}{\rho(A)} = \frac{\pi}{d}$, which satisfies

$$\limsup_{r\to\infty} \frac{\log\log|B(re^{i\theta})|}{\log r} \geq \mu(B)$$

for all $\theta \in (\alpha,\beta)$. That is, there exist a sequence $\{r_n\}$ such that, for arbitrarily small $\eta > 0$,

$$\left|B\left(r_n e^{i\theta}\right)\right| \geq \exp\{r_n^{\mu(B)-\eta}\}, \theta \in (\alpha,\beta). \tag{4.10}$$

Let

$$G := S \cap (\alpha,\beta).$$

Clearly, mes$(G) \geq k_2$. By using the similar method in Case 1, we have at least one open interval $I_{i1} = (\alpha_1,\beta_1)$ such that

$$\text{mes}(G \cap (\alpha_1,\beta_1)) > \frac{t}{2m} > 0.$$

Then there exists at least one sector S_j(j is odd) such that

$$\text{mes}(S_j \cap (\alpha_0,\beta_0)) > \frac{t}{2md} > 0.$$

We can find an $\varepsilon > 0$ such that mes $(S_j(\varepsilon) \cap (\alpha_1,\beta_1)) > 0$. So $F(\varepsilon) := S_j(\varepsilon) \cap (\alpha_1 + \xi, \beta_1 - \xi) \neq \emptyset$. Substituting (4.10), (4.3) and (4.2) into (4.4), for any $\arg z = \theta \in F(\varepsilon)$ and $\{r_n\}$ outside an R-set D, we have

$$\exp\{r_n^{\mu(B)-\eta}\} < Kr_n^M(1 + \exp\{-Cr_n^d\} + |a|) \tag{4.11}$$

for all sufficiently large n. A contradiction. Therefore, $mes(I(f)) \geq k_2$.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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