Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 6634–6649.
DOI: 10.3934/math.2022370
Received: 22 September 2021
Revised: 04 January 2022
Accepted: 11 January 2022
Published: 24 January 2022

Research article

Relation-preserving generalized nonlinear contractions and related fixed point theorems

Faruk Sk[1], **Faizan Ahmad Khan**[2,*], **Qamrul Haq Khan**[1] **and Aftab Alam**[3,*]

[1] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
[2] Computational & Analytical Mathematics and Their Applications Research Group, Department of Mathematics, Faculty of Science, University of Tabuk, Tabuk, Saudi Arabia
[3] Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India

* **Correspondence:** Email: fkhan@ut.edu.sa, aafu.amu@gmail.com.

Abstract: In this paper, we present some fixed point theorems for generalized nonlinear contractions involving a new pair of auxiliary functions in a metric space endowed with a locally finitely T-transitive binary relation. Our newly proved results generalize some well-known fixed point theorems existing in the literature. We also provide an example which substantiates the utility of our results.

Keywords: (ϕ,ψ)-contractions; locally finitely T-transitive binary relations; $\mathcal{R}$-complete metric space
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Metric fixed point theory is a relatively old but still a young area of research which occupies an important place in nonlinear functional analysis. In fact, the strength of fixed point theory lies in it's wide range of applications. For recent works related to applications of metric fixed point theory, readers are referred to [7, 18, 25, 28–31]. Indeed, the most popular result of metric fixed point theory is the classical Banach contraction principle which is essentially due to S. Banach [9] (proved in 1922). Several researchers generalized this classical result by enlarging the class of underlying contraction condition, see [1, 17, 19, 20, 22] and references therein. There are few generalized contractivity mappings, which are obtained employing certain test functions (auxiliary functions) on contractivity conditions, such as:

- Nonlinear contraction or φ-contraction (Browder [13], Boyd and Wong [12])

$$d(Tx, Ty) \leq \varphi(d(x, y)).$$

- Weak nonlinear contraction or weak ϕ-contraction (Krasnosel'skiǐ et al. [23], Alber and Guerre-Delabriere [7])
$$d(Tx, Ty)) \leq d(x, y) - \phi(d(x, y)).$$
- Generalized nonlinear contraction or (ϕ, ψ)-contraction (Dutta and Choudhury [15])
$$\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)).$$

Order-theoretic aspects of metric fixed point theory were initiated by Turinici [34, 35]. Ran and Reurings [29] and Nieto and Rodríguez-López [28] extended Banach contraction principle in the setting of ordered metric spaces. In fact, the fixed point results of Ran and Reurings [29] and Nieto and Rodríguez-López [28] are consequences of the results of Turinici [34, 35]. In 2015, Alam and Imdad [3] further extended the fixed point theorem of Nieto and Rodríguez-López [28] using an amorphous binary relation instead of the partial ordering. In the recent years, fixed point theorems in the setting of ordered as well as relational metric spaces are being developed regularly, see [2, 4, 5, 8, 14, 18, 21, 31] and references therein. Harjani and Sadarangani [18] proved some fixed point theorems for generalized nonlinear contractions in context of ordered metric spaces. Very recently, Alam et al. [6] enriched the class of generalized nonlinear contractions due to Dutta and Choudhury [15] by introducing a new pair of test functions (ϕ, ψ) and utilized the same to extend the classical Banach contraction principle.

The aim of this paper is to improve the results of Harjani and Sadarangani [18] in the following respects:

(i) The underlying partial ordered relation can be replaced by a class of transitive binary relation (called, locally finitely T-transitive binary relation).

(ii) The pair (ϕ, ψ) of altering distance functions utilized in [15, 18] must be replaced by more generalized pair of auxiliary functions.

2. Relation-theoretic notions

In this section, we recall some relevant notions, which are required to prove our main results. Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and we denote the set $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$.

Definition 2.1. [26] Let X be a nonempty set. A subset $\mathcal{R}$ of X^2 is called a binary relation on X. When $(x, y) \in \mathcal{R}$, we say "x is related to y" or "x relates to y under $\mathcal{R}$" for $x, y \in X$. The subsets, X^2 and $\emptyset$ of X^2 are called the universal relation and empty relation respectively.

Definition 2.2. [3] Let $\mathcal{R}$ be a binary relation on a nonempty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

Definition 2.3. [16, 26, 27, 30, 32, 33] A binary relation $\mathcal{R}$ defined on a nonempty set X is called:

(i) Reflexive if $(x, x) \in \mathcal{R}$ for all $x \in X$,

(ii) Irreflexive if $(x, x) \notin \mathcal{R}$ for all $x \in X$,

(iii) Symmetric if $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,

(iv) Anti-symmetric if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies $x = y$,

(v) Transitive if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ implies $(x, z) \in \mathcal{R}$,

(vi) Complete (or connected or dichotomous) if $[x, y] \in \mathcal{R}$ for all $x, y \in X$,

(vii) A partial order if $\mathcal{R}$ is reflexive, anti-symmetric and transitive,

(viii) A total order (or linear order or chain) if $\mathcal{R}$ is a complete partial order.

Definition 2.4. [26] Let X be a nonempty set and $\mathcal{R}$ a binary relation on X.

(i) The inverse, transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$ is defined by, $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.

(ii) Symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$(i.e., $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$).

Remark 2.5. [3] For a binary relation $\mathcal{R}$ defined on a nonempty set X,

$$(x, y) \in \mathcal{R}^s \text{ implies } [x, y] \in \mathcal{R}.$$

Definition 2.6. [3] Let X be a nonempty set together with a self-mapping T on X. A binary relation $\mathcal{R}$ defined on X is called T-closed if for any

$$(x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}.$$

Definition 2.7. [3] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \text{for all } n \in \mathbb{N}_0.$$

Definition 2.8. [4] Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.

Remark 2.9. [4] Every complete metric space is $\mathcal{R}$-complete with respect to a binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $\mathcal{R}$-completeness coincides with usual completeness.

Definition 2.10. [4] Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation on X and $x \in X$. A mappings $T : X \to X$ is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $Tx_n \xrightarrow{d} Tx$. Moreover, T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.

Remark 2.11. [4] Every continuous mapping is $\mathcal{R}$-continuous for any binary relation $\mathcal{R}$. Particularly, under the universal relation, the notion of $\mathcal{R}$-continuity coincides with usual continuity.

Definition 2.12. [3] Let (X, d) be a metric space. A binary relation $\mathcal{R}$ defined on X is called d-self-closed if whenever $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and $x_n \xrightarrow{d} x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}_0$.

Definition 2.13. [30] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. A subset E of X is called $\mathcal{R}$-directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

Definition 2.14. [24] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. For $x, y \in X$, a path of length k in $\mathcal{R}$ from x to y is a finite sequence $\{x_0, x_1, ..., x_k\} \subset X$ satisfying the following conditions:

(i) $x_0 = x$ and $x_k = y$,

(ii) $(x_i, x_{i+1}) \in \mathcal{R}$ for each i $(0 \leq i \leq k-1)$.

Definition 2.15. [5] Let X be a nonempty set endowed with a binary relation $\mathcal{R}$. A subset Y of X is called $\mathcal{R}$-connected if for each $x, y \in Y$, there exists a path in $\mathcal{R}$ from x to y.

Definition 2.16. [10] Given $N \in \mathbb{N}_0$, $N \geq 2$, a binary relation $\mathcal{R}$ defined on a nonempty set X is called N-transitive if for any $x_0, x_1, ..., x_N \in X$ such that

$$(x_{i-1}, x_i) \in \mathcal{R} \text{ for each } i\ (1 \leq i \leq N), \text{ we have } (x_0, x_N) \in \mathcal{R}.$$

Notice that the notion of 2-transitivity coincides with transitivity. Following Turinici [36], $\mathcal{R}$ is called finitely transitive if it is N-transitive for some $N \geq 2$.

Definition 2.17. [36] A binary relation $\mathcal{R}$ defined on a nonempty set X is called locally finitely transitive if for each denumerable subset E of X, there exists $N = N(E) \geq 2$, such that $\mathcal{R}|_E$ is N-transitive.

Definition 2.18. [2] Let X be a nonempty set and T a self-mapping on X. A binary relation $\mathcal{R}$ defined on X is called locally finitely T-transitive if for each denumerable subset of E of TX, there exists $N = N(E) \geq 2$, such that $\mathcal{R}|_E$ is N-transitive.

The relation-theoretic variant of Banach contraction principle proved by Alam and Imdad [3,4] can be stated as follows.

Theorem 2.19. *[3,4] Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. Suppose that the following conditions hold:*

(a) (X, d) is $\mathcal{R}$-complete,

(b) $\mathcal{R}$ is T-closed,

(c) Either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,

(d) There exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathcal{R}$,

(e) There exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \text{ for all } x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then T has a fixed point. Moreover, if

(f) TX is $\mathcal{R}^s$-connected,

then T has a unique fixed point.

3. Test functions and auxiliary results

Let Φ be the family of test functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

Φ_1 : ϕ is right-continuous,

Φ_2 : ϕ is monotonically increasing.

Also, let Ψ be the family of of test functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

Ψ_1 : $\psi(t) > 0 \quad$ for all $t > 0$,

Ψ_2 : $\liminf\limits_{t\to r} \psi(t) > 0 \quad$ for all $r > 0$.

The above-mentioned families Φ and Ψ are introduced by Alam et al. [6].

Notice that if ϕ and ψ are altering distance functions then $(\phi, \psi) \in \Phi \times \Psi$ so that these new families define more generalized contractivity conditions as compared to those of Dutta and Choudhury [15]. Now, we indicate the following known results, which will be required in the proofs of our main results.

Proposition 3.1. *[5] Let X be a nonempty set, $\mathcal{R}$ be a binary relation on X and T a self-mapping on X. If $\mathcal{R}$ is T-closed, then, for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is also T^n-closed, where T^n denotes nth iterate of T.*

Proposition 3.2. *[6] If there exists a pair of auxiliary functions $\phi, \psi : [0, \infty) \to [0, \infty)$, wherein ϕ satisfy axiom Φ_2 while ψ satisfy axiom Ψ_1, such that for all $s \in [0, \infty)$ and $t \in (0, \infty)$,*

$$\phi(s) \leq \phi(t) - \psi(t),$$

then

$$s < t.$$

Proposition 3.3. *[2] Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. Then,*

(i) *$\mathcal{R}$ is T-transitive $\Leftrightarrow \mathcal{R}|_{TX}$ is transitive,*

(ii) *$\mathcal{R}$ is locally finitely T-transitive $\Leftrightarrow \mathcal{R}|_{TX}$ is locally finitely transitive,*

(iii) *$\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is finitely transitive $\Rightarrow \mathcal{R}$ is locally finitely transitive $\Rightarrow \mathcal{R}$ is locally finitely T-transitive,*

(iv) *$\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is T-transitive $\Rightarrow \mathcal{R}$ is locally finitely T-transitive.*

Lemma 3.4. *[11] Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*

(i) *$k \leq m_k < n_k$ for all $k \in \mathbb{N}$,*

(ii) *$d(x_{m_k}, x_{n_k}) \geq \epsilon$,*

(iii) *$d(x_{m_k}, x_{p_k}) < \epsilon$ for all $p_k \in \{m_k + 1, m_k + 2, ..., n_k - 2, n_k - 1\}$.*

In addition to this, if $\{x_n\}$ also verifies $\lim\limits_{n\to+\infty} d(x_n, x_{n+1}) = 0$, then

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k+p}) = \epsilon \text{ for all } p \in \mathbb{N}_0.$$

Lemma 3.5. *[36] Let X be a nonempty set, $\mathcal{R}$ be a binary relation on X and $\{z_n\}$ is an $\mathcal{R}$-preserving sequence in X. If $\mathcal{R}$ is a N-transitive on $Z = \{z_n : n \in \mathbb{N}_0\}$ for some natural number $N \geq 2$, then*

$$(z_n, z_{n+1+r(N-1)}) \in \mathcal{R} \text{ for all } n, r \in \mathbb{N}_0.$$

4. Main results

Now, we are equipped to prove our main result regarding the existence of fixed points under (ϕ-ψ)-contractions.

Theorem 4.1. *Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and T a self-mapping on X. Suppose that the following conditions hold:*

(a) (X, d) is $\mathcal{R}$-complete,

(b) $\mathcal{R}$ is T-closed and locally finitely T-transitive,

(c) Either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,

(d) There exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathcal{R}$,

(e) There exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)) \text{ for all } x, y \in X \text{ with } (x, y) \in \mathcal{R},$$

then T has a fixed point.

Proof. By assumption (d), there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathcal{R}$. Based at x_0, we can construct the sequence $\{x_n\}$ of Picard iteration, i.e.,

$$x_n = T^n x_0 = Tx_{n-1} \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

As $(x_0, Tx_0) \in \mathcal{R}$, by T-closedness of $\mathcal{R}$ and Proposition 3.1, we have

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}$$

which in lieu of (4.1) becomes

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \text{for all } n \in \mathbb{N}_0. \tag{4.2}$$

Thus the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. If there exists $n_0 \in \mathbb{N}_0$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then applying (4.1), we conclude x_{n_0} is a fixed point of T. Otherwise, we have

$$d_n := d(x_n, x_{n+1}) > 0 \quad \forall\, n \in \mathbb{N}_0.$$

Using (4.1) and applying contractivity condition (e), we get

$$\phi(d(x_{n+1}, x_{n+2})) = \phi(d(Tx_n, Tx_{n+1})) \leq \phi(d(x_n, x_{n+1})) - \psi(d(x_n, x_{n+1})).$$

Then,

$$\phi(d_{n+1})) \leq \phi(d_n) - \psi(d_n). \tag{4.3}$$

Using Proposition 3.2, we get

$$d_{n+1} \quad < \quad d_n \qquad \text{for all } \; n \in \mathbb{N}_0,$$

which yields that the sequence $\{d_n\} \subset (0, +\infty)$ is monotonically decreasing. As $\{d_n\}$ is also bounded below by 0, there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} d_n = r.$$

Now, we show that $r = 0$. On contrary, suppose that $r > 0$, then taking upper limit in (4.3), we get

$$\begin{aligned}\limsup_{n\to+\infty} \phi(d_{n+1}) \quad &\leq \quad \limsup_{n\to+\infty} \phi(d_n) + \limsup_{n\to+\infty}[-\psi(d_n)] \\ &\leq \quad \limsup_{n\to+\infty} \phi(d_n) - \liminf_{n\to+\infty} \psi(d_n).\end{aligned}$$

Using right-continuity of ϕ, we get

$$\phi(r) \quad \leq \quad \phi(r) - \liminf_{n\to+\infty} \psi(d_n)$$

implying thereby

$$\liminf_{d_n\to r>0} \psi(d_n) = \liminf_{n\to+\infty} \psi(d_n) \leq 0,$$

which contradicts axiom Ψ_2 so that we have

$$\lim_{n\to+\infty} d_n = \lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{4.4}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. On contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Therefore, by Lemma 3.4, there exists $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $k \leq m_k < n_k$, $d(x_{m_k}, x_{n_k}) \geq \epsilon$ and $d(x_{m_k}, x_{p_k}) < \epsilon$ where $p_k \in \{m_k + 1, m_k + 2, ..., n_k - 2, n_k - 1\}$. Further in view of (4.4), we have

$$\lim_{n\to+\infty} d(x_{m_k}, x_{n_k+p}) = \epsilon \qquad \text{for all } \; p \in \mathbb{N}_0. \tag{4.5}$$

Since $\{x_n\} \subset TX$, the range $E = \{x_n : n \in \mathbb{N}_0\}$ is a denumerable subset of TX. Hence, by locally finitely T-transitivity of $\mathcal{R}$, there exists a natural number $N = N(E) \geq 2$, such that $\mathcal{R}|_E$ is N-transitive.

As $m_k < n_k$ and $N - 1 > 0$, using Division Algorithm, we have

$$\begin{gathered} n_k - m_k = (N-1)(\mu_k - 1) + (N - \eta_k), \\ \mu_k - 1 \geq 0, 0 \leq N - \eta_k < N - 1 \end{gathered}$$

$$\Longleftrightarrow \begin{cases} n_k + \eta_k = m_k + 1 + (N-1)\mu_k, \\ \mu_k \geq 1, 1 < \eta_k \leq N. \end{cases}$$

Here μ_k and η_k are suitable natural numbers such that η_k can be taken as a finite natural number belonging in interval $(1, N]$. Hence, without loss of generality, we can choose subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ (satisfying (4.5)) such that η_k remains constant, say η, which is independent of k. Write

$$m'_k = n_k + \eta = m_k + 1 + (N-1)\mu_k, \tag{4.6}$$

where η $(1 < \eta \leq N)$ is constant.

Owing to (4.5) and (4.6), we obtain

$$\lim_{k\to+\infty} d(x_{m_k}, x_{m'_k}) = \lim_{k\to+\infty} d(x_{m_k}, x_{n_k+\eta}) = \epsilon. \tag{4.7}$$

Using triangular inequality, we have

$$d(x_{m_k+1}, x_{m'_k+1}) \leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{m'_k}) + d(x_{m'_k}, x_{m'_k+1}) \tag{4.8}$$

and

$$d(x_{m_k}, x_{m'_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{m'_k+1}) + d(x_{m'_k+1}, x_{m'_k})$$

or

$$d(x_{m_k}, x_{m'_k}) - d(x_{m_k}, x_{m_k+1}) - d(x_{m'_k+1}, x_{m'_k}) \quad \leq \quad d(x_{m_k+1}, x_{m'_k+1}). \tag{4.9}$$

Letting $k \to +\infty$ in (4.8) and (4.9) and using (4.4) and (4.7), we get

$$\lim_{k\to+\infty} d(x_{m_k+1}, x_{m'_k+1}) = \epsilon. \tag{4.10}$$

In view of (4.6) and Lemma 3.5, we have $d(x_{m_k}, x_{m'_k}) \in \mathcal{R}$. Now, using assumption (e), we get

$$\begin{aligned} \phi(d(x_{m_k+1}, x_{m'_k+1})) &= \phi(d(Tx_{m_k}, Tx_{m'_k})) \\ &\leq \phi(d(x_{m_k}, x_{m'_k})) - \psi(d(x_{m_k}, x_{m'_k})). \end{aligned}$$

Taking upper limit in the above inequality, we get

$$\limsup_{k\to+\infty} \phi(d(x_{m_k+1}, x_{m'_k+1})) \quad \leq \quad \limsup_{k\to+\infty} \phi(d(x_{m_k}, x_{m'_k})) + \limsup_{k\to+\infty}[-\psi(d(x_{m_k}, x_{m'_k}))].$$

Using the right-continuity of ϕ and (4.7), we get

$$\phi(\epsilon) \quad \leq \quad \phi(\epsilon) - \liminf_{k\to\infty} \psi(d(x_{m_k}, x_{m'_k}))$$

yielding thereby

$$\liminf_{k\to+\infty} \psi(d(x_{m_k}, x_{m'_k})) \quad \leq \quad 0,$$

which is a contradiction. It follows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is $\mathcal{R}$-complete, there exists $x \in X$ such that $x_n \stackrel{d}{\to} x$. By the $\mathcal{R}$-continuity of T, we have $Tx_n \stackrel{d}{\to} Tx$. Owing to (4.1), we obtain $Tx_n = x_{n+1} \stackrel{d}{\to} x$. Using uniqueness of limit, we get $Tx = x$, i.e., x is a fixed point of T.

Alternately, assume that $\mathcal{R}$ is d-self-closed. Since $\{x_n\}$ is $\mathcal{R}$-preserving sequence such that $x_n \xrightarrow{d} x$, d-self-closedness of $\mathcal{R}$ guarantees the existence of a subsequence of $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}$ for all $k \in \mathbb{N}$. Now, we claim that

$$\lim_{k\to+\infty} d(x_{n_k+1}, Tx) \quad = \quad 0. \tag{4.11}$$

To substantiate it, we firstly consider the case whenever $x_{n_k} = x$ for some $k \in \mathbb{N}$. Then, we have $x_{n_k+1} = Tx_{n_k} = Tx$ yielding thereby

$$\lim_{k\to+\infty} d(x_{n_k+1}, Tx) = 0.$$

Hence, in this case (4.11) holds directly. Otherwise, we have $x_{n_k} \neq x$ so that $d(x_{n_k}, x) > 0$ for all $k \in \mathbb{N}$. Using assumption (e), we get

$$\begin{aligned} \psi(d(x_{n_k+1}, Tx)) &= \psi(d(Tx_{n_k}, Tx)) \\ &\leq \psi(d(x_{n_k}, x)) - \phi(d(x_{n_k}, x)). \end{aligned}$$

Using Proposition 3.2, we obtain

$$d(x_{n_k+1}, Tx) \quad < \quad d(x_{n_k}, x). \tag{4.12}$$

Taking limit of (4.12) as $k \to \infty$ and using $x_{n_k} \xrightarrow{d} x$, we obtain

$$\lim_{k\to+\infty} d(x_{n_k+1}, Tx) = 0.$$

Hence, (4.11) holds. Now, owing to the uniqueness of limit and in view of (4.11), we obtain $Tx = x$, i.e., x is a fixed point of T. This completes the proof. □

Theorem 4.2. *In addition to the hypotheses of Theorem 4.1, if we add the following condition*

(f) TX is $\mathcal{R}^s$-connected,

then T has a unique fixed point.

Proof. In lieu of Theorem 4.1, there exists at least one fixed point of T. If x and y are two fixed points of T, then

$$T^n x = x \text{ and } T^n y = y \text{ for all } n \in \mathbb{N}_0.$$

Clearly $x, y \in TX$. By assumption (f), there exists a path $\{z_0, z_1, z_2, ..., z_k\}$ of some finite length k in $\mathcal{R}^s$ from x to y so that

$$z_0 = x, z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1). \tag{4.13}$$

As $\mathcal{R}$ is T-closed, we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1) \text{ and } n \in \mathbb{N}_0. \tag{4.14}$$

Now, for each $n \in \mathbb{N}_0$ and for each i $(0 \le i \le k-1)$, define

$$t_n^i = d(T^n z_i, T^n z_{i+1}).$$

We show that

$$\lim_{n\to+\infty} t_n^i \quad = \quad 0. \tag{4.15}$$

Fix i and distinguish two cases. Firstly, suppose that

$$t_{n_0}^i = d(T^{n_0} z_i, T^{n_0} z_{i+1}) = 0 \text{ for some } n_0 \in \mathbb{N}_0,$$

which gives rise to $T^{n_0} z_i = T^{n_0} z_{i+1}$. Now applying (4.1), we have $T^{n_0+1} z_i = T^{n_0+1} z_{i+1}$. Hence, $t_{n_0+1}^i = 0$. Thus by induction, we get $t_n^i = 0$ for all $n \ge n_0$, yielding thereby $\lim_{n\to+\infty} t_n^i = 0$.

On the other hand, suppose that $t_n^i > 0$ for all $n \in \mathbb{N}_0$. Using (4.14) along with assumption (e), we obtain

$$\begin{aligned} \phi(t_{n+1}^i) &= \phi(d(T^{n+1} z_i, T^{n+1} z_{i+1})) \\ &= \phi(d(T(T^n z_i), T(T^n z_{i+1}))) \\ &\le \phi(d(T^n z_i, T^n z_{i+1})) - \psi(d(T^n z_i, T^n z_{i+1})) \end{aligned}$$

so that

$$\phi(t_{n+1}^i) \quad \le \quad \phi(t_n^i) - \psi(t_n^i). \tag{4.16}$$

Using Proposition 3.2, we have

$$t_{n+1}^i \quad < \quad t_n^i.$$

Therefore, $\{t_n^i\}$ is a decreasing sequence of nonegative real numbers. Therefore, there exists $r \ge 0$ such that

$$\lim_{n\to+\infty} t_n^i = r.$$

We show that $r = 0$. Suppose on contrary that $r > 0$. Taking lower limit in inequality (4.16), we have

$$\liminf_{n\to+\infty} \phi(t_{n+1}^i) \quad \le \quad \liminf_{n\to+\infty} \phi(t_n^i) - \liminf_{n\to+\infty} \psi(t_n^i).$$

On using the right-continuity of ϕ, we get $\liminf_{n\to+\infty} \psi(t_n^i) \le 0$, which is a contradiction. Hence, we have $\lim_{n\to+\infty} t_n^i = 0$. Thus in both cases, (4.15) is proved for each i $(0 \le i \le k-1)$. Now, using triangle inequality, we get

$$\begin{aligned} d(x, y) &= d(T^n z_0, T^n z_k) \\ &\le t_n^0 + t_n^1 .... + t_n^{k-1} \\ &\to 0 \; as \; n \to +\infty, \end{aligned}$$

which gives rise to $x = y$. Hence, T has a unique fixed point. □

Making use of Proposition 3.3, we have the following consequence of Theorem 4.1.

Corollary 4.3. *Theorem 4.1 remains true if locally finitely T-transitivity condition is replaced by any one of the following conditions:*

(i) *$\mathcal{R}$ is transitive,*

(ii) *$\mathcal{R}$ is T-transitive,*

(iii) *$\mathcal{R}$ is finitely transitive,*

(iv) *$\mathcal{R}$ is locally finitely transitive.*

Corollary 4.4. *Theorem 4.2 remains true if we replace condition (e) by one of the following conditions:*
(f') TX is $\mathcal{R}^s$-directed,
(f'') $\mathcal{R}|_{TX}$ is complete.

Proof. Suppose condition (f') holds. Then, for each $x, y \in TX$ there exists $z \in X$ such that $[x, z] \in \mathcal{R}$ and $[y, z] \in \mathcal{R}$, which amounts to say that $\{x, z, y\}$ is a path of length 2 in $\mathcal{R}^s$ from x to y. Hence, TX is $\mathcal{R}^s$-connected and again by Theorem 4.2, we get the conclusion.

On the other hand, let the condition (f'') holds. Then, for each $x, y \in TX$, $[x, y] \in \mathcal{R}$, which yields that $\{x, y\}$ is a path of length 1 in $\mathcal{R}^s$ from x to y so that TX is $\mathcal{R}^s$-connected and then by Theorem 4.2, the conclusion is immediate. □

Remark 4.5. Under universal relation (i.e. $\mathcal{R} = X^2$), Theorems 4.1 and 4.2 reduce to the following fixed point theorem of Alam et al. [6].

Theorem 4.6. *[6] Let (X, d) be a complete metric space and T a self-mapping on X. If there exist $\phi \in \Phi$and $\psi \in \Psi$ such that*

$$\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)) \text{ for all } x, y \in X,$$

then T has a unique fixed point.

On setting $\mathcal{R} =\preceq$, the partial order in Theorem 4.1, we obtain the following result of Harjani and Sadarangani [18].

Corollary 4.7. *[18] Let (X, d) be a metric space endowed with a partial order $\preceq$ and T a self-mapping on X. Suppose that the following conditions hold:*

(a) (X, d) be complete metric space,

(b) T is increasing with respect to $\preceq$,

(c) T is continuous or $(X, d, \preceq)$ is regular,

(d) There exists $x_0 \in X$ such that $x_0 \preceq Tx_0$,

(e) There exist $\phi \in \Phi$ and $\psi \in \Psi$ such that

$$\phi(d(Tx, Ty)) \leq \phi(d(x, y)) - \psi(d(x, y)) \text{ for all } x, y \in X \text{ with } x \succeq y,$$

then T has a fixed point.

5. Illustrative examples

Now, we give some examples which show the validity and utility of Theorem 4.1.

Example 5.1. Consider $X = (-1, 1]$ with the usual metric d. On X, define a binary relation $\mathcal{R}$ by

$$\mathcal{R} = \{(x, y) \in X^2 : x > y \geq 0\},$$

then (X, d) is a $\mathcal{R}$-complete metric space.

Define the auxiliary functions $\phi, \psi : [0, +\infty) \to [0, +\infty)$ as follows:

$$\phi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1 \\ t^2, & \text{if } t > 1 \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} \frac{t^2}{2}, & \text{if } 0 \leq t \leq 1, \\ 4, & \text{if } t > 1. \end{cases}$$

Clearly, $\phi \in \Phi$ and $\psi \in \Psi$. Let $T : X \to X$ be a mapping defined by

$$Tx = \begin{cases} x + 1, & \text{if } -1 < x < 0, \\ x - \frac{x^2}{2}, & \text{if } 0 \leq x \leq 1. \end{cases}$$

Take $x, y \in X$ with $(x, y) \in \mathcal{R}$, then $x > y \geq 0$. Thus, we have

$$\begin{aligned} \phi(d(Tx, Ty)) &= (x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2) \\ &= (x - y) - \frac{1}{2}(x - y)(x + y) \\ &\leq (x - y) - \frac{1}{2}(x - y)^2 \\ &= d(x, y) - \frac{1}{2}(d(x, y))^2 \\ &= \phi(d(x, y)) - \psi(d(x, y)). \end{aligned}$$

Therefore, T satisfies condition (e) of Theorem 4.1. Notice that here $\mathcal{R}$ is locally finitely T-transitive and $\mathcal{R}$ is T-closed. It is very spontaneous that, rest of the conditions of Theorem 4.1 and Theorem 4.2 are satisfied and T has a unique fixed point, namely: $x = 0$.

Remark 5.2. Note that the above examples cannot be covered by existing results in the literature. Indeed, ψ is not continuous here. Moreover, the contraction condtion does not hold on the whole space (take $x = -\frac{1}{2}$ and $y = \frac{3}{5}$), which substantiates the effectiveness of our results.

Example 5.3. Consider $X = [0, 1] \cup \{2, 3, 4, ...\}$ with the metric d defined by

$$d(x, y) = \begin{cases} | x - y |, & \text{if } x, y \in [0, 1] \text{ and } x \neq y, \\ x + y, & \text{if at least one of } x \text{ or } y \text{ does not belong to } [0, 1] \text{ and } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

On X, define a binary relation $\mathcal{R}$ by

$$\mathcal{R} = \{(x, y) \in X^2 : x > y \text{ and } x \in \{3, 4, 5, ..\}, y \neq 2\},$$

then X is a $\mathcal{R}$-complete metric space, see [12].

Define the auxiliary functions $\phi, \psi : [0, \infty) \to [0, \infty)$ as follows:

$$\phi(t) = \begin{cases} t + 1, & \text{if } 0 \leq t < 1 \\ t^2, & \text{if } t \geq 1 \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} \frac{t^2}{4}, & \text{if } 0 \leq t \leq 1, \\ \frac{1}{5}, & \text{if } t > 1. \end{cases}$$

Clearly, $\phi \in \Phi$ and $\psi \in \Psi$. Here ϕ and ψ both are not continuous and $\phi(0) = 1$.

Let $T : X \to X$ be a mapping defined by

$$T(x) = \begin{cases} x - \frac{x^3}{4}, & \text{if } 0 \leq x \leq 1, \\ x - 1, & \text{if } x \in \{2, 3, 4, ...\}. \end{cases}$$

Notice that the (ϕ, ψ)-contractivity condition holds trivially when $x = y$.

When $x \in \{3, 4, ...\}$, then there are two possibilities of choosing y. Firstly, we take $y \in [0, 1]$, then we have

$$\begin{aligned} d(Tx, Ty) &= d\Big(x - 1, y - \frac{1}{4}y^3\Big) = x - 1 + y - \frac{y^3}{4} \\ &\leq x + y - 1. \end{aligned}$$

Otherwise, if $y \in \{3, 4, ...\}$, then we have

$$\begin{aligned} d(Tx, Ty) &= d(x - 1, y - 1) = x + y - 2 \\ &< x + y - 1. \end{aligned}$$

Therefore, in both the cases, we have

$$\begin{aligned} \phi(d(Tx, Ty)) &= (d(Tx, Ty))^2 < (x + y - 1)^2 \\ &< (x + y - 1)(x + y + 1) = (x + y)^2 - 1 \\ &< (x + y)^2 - \frac{1}{5} \\ &= \phi(d(x, y)) - \psi(d(x, y)). \end{aligned}$$

In view of all the possible cases, we conclude that all the conditions of Theorem 4.1 are satisfied. Therefore, by Theorem 4.1, T has a fixed point (namely: $x = 0$).

Remark 5.4. Notice that TX is not $\mathcal{R}^s$ connected in the above example. But still T has a unique fixed point, which attests the fact that condition (f) of Theorem 4.2 is only a necessary condition to have a unique fixed point, not a sufficient condition. Moreover, the auxiliary functions are both not continuous here, which proves the credibility of our newly proved results.

6. Conclusions

In this paper, we extended the concept of generalized nonlinear contraction mappings in a relation-theoretic sense involving some newly introduced auxiliary functions in a metric space endowed with a binary relation $\mathcal{R}$ which needs to satisfy a weak version of transitivity condition, i.e., locally finitely T-transitivity condition. We have also put out some examples where the existing theorems cannot be applied to investigate the existence of fixed points. Still, by utilizing our results, we can get the existence and uniqueness of fixed points, which guarantees the novelty of our results.

Acknowledgments

The authors would like to thank the anonymous referees for their comments that helped us improve this article. The first author would like to thank the Council of Scientific and Industrial Research (CSIR), Government of India for JRF (File No. 09/112(0604)/2018-EMR-I).

Conflict of interest

The authors declare no conflicts of interest.

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