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Research article

Existence of solutions for fractional differential equation with periodic boundary condition

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Abstract: We investigate the existence of solutions for a Caputo fractional differential equation with periodic boundary condition. Using the positivity of Green's function of the corresponding linear equation, we show the existence of positive solutions by using Krasnosel'skii fixed point theorem. Meanwhile, by using monotone iterative method and lower and upper solutions method, we also discuss the existence of extremal solutions for a special case.

Keywords: Caputo fractional differential equation; periodic boundary condition; Krasnosel'skii fixed point theorem; monotone iterative method; Green's function
Mathematics Subject Classification: 34A08, 34A12

1. Introduction

This article is devoted to the following nonlinear fractional differential equation with periodic boundary condition

$$\begin{cases} {}^cD_{0^+}^{\alpha}x(t) - \lambda x(t) = f(t, x(t)), \ 0 < t \le \omega, \\ x(0) = x(\omega), \end{cases} \tag{1.1}$$

where $\lambda \le 0$, $0 < \alpha \le 1$ and ${}^cD_{0^+}^{\alpha}$ is Caputo fractional derivative

$${}^cD_{0^+}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)ds.$$

Differential equations of fractional order occur more frequently on different research areas and engineering, such as physics, economics, chemistry, control theory, etc. In recent years, boundary value problems for fractional differential equation have become a hot research topic, see [2–7,9–13,15–21,24,25]. In [27], Zhang studied the boundary value problem for nonlinear fractional differential equation

$$\begin{cases} {}^cD^{\alpha}_{0^+}u(t) = f(t, u(t)),\ 0 < t < 1, \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0, \end{cases} \tag{1.2}$$

where $1 < \alpha \le 2$, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and ${}^cD^{\alpha}_{0^+}$ is Caputo fractional derivative

$${}^cD^{\alpha}_{0^+}u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} u''(s)ds.$$

The author obtained the existence of the positive solutions by using the properties of the Green function, Guo-Krasnosel'skill fixed point theorem and Leggett-Williams fixed point theorem.

Ahmad and Nieto [1] studied the anti-periodic boundary value problem of fractional differential equation

$$\begin{cases} {}^cD^{q}_{0^+}u(t) = f(t, u(t)),\ 0 \le t \le T,\ 1 < q \le 2, \\ u(0) = -u(T),\ {}^cD^{p}_{0^+}u(0) = -{}^cD^{p}_{0^+}u(T),\ 0 < p < 1, \end{cases} \tag{1.3}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. The authors obtained some existence and uniqueness results by applying fixed point principles. The anti-periodic boundary value condition in this article corresponds to the anti-periodic condition $u(0) = -u(T), u'(0) = -u'(T)$ in ordinary differential equation.

In [26], Zhang studied the following fractional differential equation

$$\begin{cases} D^{\delta}_{0^+}u(t) = f(t, u),\ 0 < t \le T, \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u_0, \end{cases} \tag{1.4}$$

where $0 < \delta < 1$, $T > 0$, $u_0 \in \mathbb{R}$ and $D^{\delta}_{0^+}$ is Riemann-Liouville fractional derivative

$$D^{\delta}_{0^+}u(t) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dt} \int_0^t (t-s)^{-\delta} u(s)ds.$$

The author obtained the existence and uniqueness of the solutions by the method of upper and lower solutions and monotone iterative method.

In [7], Belmekki, Nieto and Rodriguez-Lopez studied the following equation

$$\begin{cases} D^{\delta}_{0^+}u(t) - \lambda u(t) = f(t, u(t)), 0 < t \le 1, \\ \lim_{t\to 0^+} t^{1-\delta}u(t) = u(1), \end{cases} \tag{1.5}$$

where $0 < \delta < 1$, $\lambda \in \mathbb{R}$, f is continuous. The authors obtained the existence and uniqueness of the solutions by using the fixed point theorem. Cabada and Kisela [8] studied the following equation

$$\begin{cases} D^{\delta}_{0^+}u(t) - \lambda u(t) = f(t, t^{1-\alpha}u(t)), 0 < t \le 1, \\ \lim_{t\to 0^+} t^{1-\delta}u(t) = u(1), \end{cases} \tag{1.6}$$

where $0 < \delta < 1$, $\lambda \ne 0 (\lambda \in \mathbb{R})$, f is continuous. The authors studied the existence and uniqueness of periodic solutions by using Krasnosel'skii fixed point theorem and monotone iterative method. In [7, 8], the boundary condition $\lim_{t\to 0^+} t^{1-\delta}u(t) = u(1)$ was called as periodic boundary value condition of Riemann-Liouville fractional differential equation, which is different from the periodic condition for ordinary differential equation. The boundary value condition $u(0) = u(1)$ is not suitable for Riemann-Liouville fractional differential equation.

For the ordinary differential equation, the periodic boundary value problem is closely related to the periodic solution. For the Caputo fractional differential equation, the periodic boundary value condition $u(0) = u(w)$ is meaningful. As far as we know, few work involves the periodic boundary value problem for Caputo fractional. The aim of this paper is to show the existence of positive solutions of (1.1) by using Krasnosel'skii fixed point theorem. Meanwhile, we also use the monotone iterative method to study the extremal solutions problem

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t) = f(t,u(t)), 0 < t \leq \omega, \\ u(0) = u(\omega). \end{cases} \tag{1.7}$$

The paper is organized as follows. In Section 2, we recall and derive some results on Mittag-Leffler functions. In Section 3, we use the Laplace transform to obtain the solution of a linear problem and discuss some properties of Green's function. In Section 4, the existence of positive solution is studied by using the Krasnosel'skii fixed point theorem. In Section 5, the existence of extremal solutions is proved by utilizing the monotone iterative technique. Section 6 is conclusion of the paper.

2. Preliminaries

A key role in the theory of linear fractional differential equation is played by the well-known two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \Sigma_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \ z \in \mathbb{R},\ \alpha,\ \beta > 0. \tag{2.1}$$

We recall and derive some of their properties and relationships summarized in the following.

Proposition 2.1. *Let $\alpha \in (0,1], \beta > 0, \lambda \in \mathbb{R}$ and $\xi > 0$. Then it holds*

(C_1) $\lim_{t\to 0^+} E_{\alpha,\beta}(\lambda t^{\alpha}) = \frac{1}{\Gamma(\beta)}, \lim_{t\to 0^+} E_{\alpha,1}(\lambda t^{\alpha}) = 1.$
(C_2) $E_{\alpha,\alpha+1}(\lambda t^{\alpha}) = \lambda^{-1}t^{-\alpha}(E_{\alpha,1}(\lambda t^{\alpha}) - 1).$
(C_3) $E_{\alpha,\alpha}(\lambda t^{\alpha}) > 0$, $E_{\alpha,1}(\lambda t^{\alpha}) > 0$ *for all* $t \geq 0$.
(C_4) $E_{\alpha,\alpha}(\lambda t^{\alpha})$ *is decreasing in t for* $\lambda < 0$ *and increasing for* $\lambda > 0$ *for all* $t > 0$.
(C_5) $E_{\alpha,1}(\lambda t^{\alpha})$ *is decreasing in t for* $\lambda < 0$ *and increasing for* $\lambda > 0$ *for all* $t > 0$.
(C_6) $\int_0^{\xi} t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})dt = \xi^{\beta}E_{\alpha,\beta+1}(\lambda\xi^{\alpha}).$

Proof. (C_1) It is obtained by an immediate calculation from (2.1).
(C_2) By (2.1), we get

$$E_{\alpha,1}(\lambda t^{\alpha}) = \Sigma_{k=0}^{\infty}\frac{(\lambda t^{\alpha})^k}{\Gamma(\alpha k+1)} = 1 + \frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\lambda t^{\alpha})^2}{\Gamma(2\alpha+1)} + \frac{(\lambda t^{\alpha})^3}{\Gamma(3\alpha+1)} + \cdots,$$

$$E_{\alpha,\alpha+1}(\lambda t^{\alpha}) = \Sigma_{k=0}^{\infty}\frac{(\lambda t^{\alpha})^k}{\Gamma(\alpha k+\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} + \frac{\lambda t^{\alpha}}{\Gamma(2\alpha+1)} + \frac{(\lambda t^{\alpha})^2}{\Gamma(3\alpha+1)} + \cdots.$$

Hence,

$$E_{\alpha,\alpha+1}(\lambda t^{\alpha}) = \lambda^{-1}t^{-\alpha}(E_{\alpha,1}(\lambda t^{\alpha}) - 1).$$

(C_3) It follows from [23, Lemma 2.2].

(C_4) It follows from [8, Proposition 1].
(C_5) By a direct calculation, we get

$$\frac{d}{dt}E_{\alpha,1}(\lambda t^{\alpha}) = \lambda t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}),$$

since $\alpha \in (0,1]$, $t > 0$ and $E_{\alpha,\alpha}(\lambda t^{\alpha})$ is positive by Proposition (C_3), the assertion is proved.
(C_6) It follows from (1.99) of [20].

3. Linear problem

In this section, we deal with the linear case that $f(t,x) = f(t)$ is a continuous function by mean of the Laplace transform for caputo fractional derivative

$$(L^{c}D_{0^+}^{\alpha}x)(s) = s^{\alpha}X(s) - s^{\alpha-1}x(0),\ 0 < \alpha \leq 1, \tag{3.1}$$

where L denotes the Laplace transform operator, $X(s)$ denotes the Laplace transform of $x(t)$.

From Lemma 3.2 of [14], we get

$$(LE_{\alpha}(\lambda t^{\alpha}))(s) = \frac{s^{\alpha-1}}{s^{\alpha}-\lambda}, Re(s) > 0, \lambda \in \mathbb{C}, |\lambda s^{-\alpha}| < 1, \tag{3.2}$$

and

$$(Lt^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha}))(s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, Re(s) > 0, \lambda \in \mathbb{C}, |\lambda s^{-\alpha}| < 1. \tag{3.3}$$

We do Laplace transform to the equation

$${}^{c}D_{0^+}^{\alpha}x(t) - \lambda x(t) = f(t), \quad x(0) = x(\omega). \tag{3.4}$$

By (3.1), we obtain

$$s^{\alpha}X(s) - \lambda X(s) = F(s) + x(0)\cdot s^{\alpha-1},$$
$$X(s) = \frac{F(s)}{s^{\alpha}-\lambda} + \frac{s^{\alpha-1}}{s^{\alpha}-\lambda}\cdot x(0),$$

where F denotes the Laplace transform of f. By (3.2) and (3.3), we obtain that

$$x(t) = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s)ds + x(0)\cdot E_{\alpha,1}(\lambda t^{\alpha}). \tag{3.5}$$

Hence,

$$x(\omega) = x(0)E_{\alpha,1}(\lambda w^{\alpha}) + \int_0^{\omega}(\omega-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\omega-s)^{\alpha})f(s)ds = x(0),$$

which implies that

$$x(0) = \frac{\int_0^{\omega}(\omega-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\omega-s)^{\alpha})f(s)ds}{1-E_{\alpha,1}(\lambda\omega^{\alpha})}$$

if $E_{\alpha,1}(\lambda\omega^{\alpha}) \neq 1$. Therefore, if $E_{\alpha,1}(\lambda\omega^{\alpha}) \neq 1$, the solution of the problem (3.4) is

$$x(t) \quad = \quad \frac{\int_0^{\omega}(\omega-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\omega-s)^{\alpha})f(s)ds}{1-E_{\alpha,1}(\lambda\omega^{\alpha})}\cdot E_{\alpha,1}(\lambda t^{\alpha}) + \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})}{(t-s)^{1-\alpha}}f(s)ds$$

$$= \int_0^t \frac{E_{\alpha,1}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(\omega-s)^\alpha)}{(1-E_{\alpha,1}(\lambda\omega^\alpha))(\omega-s)^{1-\alpha}} f(s)ds + \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}} f(s)ds$$
$$+ \int_t^\omega \frac{E_{\alpha,1}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(\omega-s)^\alpha)}{(1-E_{\alpha,1}(\lambda\omega^\alpha))(\omega-s)^{1-\alpha}} f(s)ds.$$

Theorem 3.1. *Let $E_{\alpha,1}(\lambda\omega^\alpha) \neq 1$, the periodic boundary value problem (3.4) has a unique solution given by*

$$x(t) = \int_0^\omega G_{\alpha,\lambda}(t,s)f(s)ds,$$

where

$$G_{\alpha,\lambda}(t,s) = \begin{cases} \frac{E_{\alpha,1}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(\omega-s)^\alpha)}{(1-E_{\alpha,1}(\lambda\omega^\alpha))(\omega-s)^{1-\alpha}} + \frac{E_{\alpha,\alpha}(\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}}, & 0 \le s < t \le \omega, \\ \frac{E_{\alpha,1}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(\omega-s)^\alpha)}{(1-E_{\alpha,1}(\lambda\omega^\alpha))(\omega-s)^{1-\alpha}}, & 0 \le t \le s < \omega. \end{cases} \tag{3.6}$$

Remark 3.2. *The unique solution x of (3.4) is continuous on* $[0,\omega]$.

Lemma 3.3. *Let $0 < \alpha \le 1, \lambda \neq 0$ and sign(η) denotes the signum function. Then*

(F_1) $\lim_{t\to 0^+} G_{\alpha,\lambda}(t,s) = \frac{E_{\alpha,\alpha}(\lambda(\omega-s)^\alpha)}{(1-E_{\alpha,1}(\lambda\omega^\alpha))(\omega-s)^{1-\alpha}}$ *for any fixed $s \in [0,\omega)$,*
(F_2) $\lim_{s\to\omega^-} G_{\alpha,\lambda}(t,s) = sign(-\lambda)\cdot\infty$ *for any fixed $t \in [0,\omega]$,*
(F_3) $\lim_{t\to s^+} G_{\alpha,\lambda}(t,s) = \infty$ *for any fixed $s \in [0,\omega)$,*
(F_4) $G_{\alpha,\lambda}(t,s) > 0$ *for $\lambda < 0$ and for all $t \in [0,\omega]$ and $s \in [0,\omega)$,*
(F_5) $G_{\alpha,\lambda}(t,s)$ *changes its sign for $\lambda > 0$ for $t \in [0,\omega]$ and $s \in [0,\omega)$.*

Proof. (F_1) When $0 \le t \le s < \omega$, by Proposition 2.1 (C_1) we can get (F_1).
(F_2) When $0 \le t \le s < \omega$, it follows by Proposition 2.1 (C_5) that $1 - E_{\alpha,1}(\lambda\omega^\alpha)$ is positive for $\lambda < 0$ and negative for $\lambda > 0$. The unboundedness is implied by continuity of Mittag-Leffler function, Proposition 2.1 (C_1) and the relation $\lim_{t\to 0^+} t^{-r} = \infty$ for $r > 0$.
(F_3) When $0 \le s < t \le \omega$, the first term of (3.6) is finite due to the continuity of the involved functions. And by a similar argument as in the previous point of this proof we have the second term tends to infinity.
(F_4) It is obtained by the positivity of all involved functions (Proposition 2.1 (C_3)) and the inequation $1 - E_{\alpha,1}(\lambda\omega^\alpha) > 0$ for $\lambda < 0$.
(F_5) When $0 \le s < t \le \omega$, the second term of (3.6) is positive due to

$$\lim_{s\to t^-}(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha) = +\infty,$$

and by the positivity of all involved functions (Proposition 2.1 (C_3)) we get the proof. When $0 \le t \le s < \omega$, it is obtained by $1 - E_{\alpha,1}(\lambda\omega^\alpha) < 0$ for $\lambda > 0$ and the positivity of all involved functions (Proposition 2.1 (C_3)). □

Proposition 3.4. *Let $\alpha \in (0,1]$ and $\lambda < 0$. Then the Green's function (3.6) satisfies*

(K_1) $G_{\alpha,\lambda}(t,s) \ge m =: \frac{E_{\alpha,1}(\lambda\omega^\alpha)E_{\alpha,\alpha}(\lambda\omega^\alpha)}{|\lambda|\omega E_{\alpha,\alpha+1}(\lambda\omega^\alpha)} > 0,$
(K_2) $\int_0^\omega G_{\alpha,\lambda}(t,s)ds = M =: \frac{1}{|\lambda|}$ *for all $t \in [0,\omega]$.*

Proof. (K_1) For $0 \le t \le s < \omega$, we deduce from Proposition 2.1 $(C_4), (C_5)$ that $G_{\alpha,\lambda}$ has the minimum on the line $t = s$. Hence,

$$\begin{aligned}
G_{\alpha,\lambda}(t,s) \ge G_{\alpha,\lambda}(t,t) &= \frac{E_{\alpha,1}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(\omega-t)^\alpha)}{(1-E_{\alpha,1}(\lambda\omega^\alpha))(\omega-t)^{1-\alpha}} \\
&\ge \frac{E_{\alpha,1}(\lambda\omega^\alpha)E_{\alpha,\alpha}(\lambda\omega^\alpha)}{(1-E_{\alpha,1}(\lambda\omega^\alpha))\omega^{1-\alpha}} \\
&= \frac{E_{\alpha,1}(\lambda\omega^\alpha)E_{\alpha,\alpha}(\lambda\omega^\alpha)}{[1-(E_{\alpha,\alpha+1}(\lambda\omega^\alpha)\lambda\omega^\alpha+1)]\omega^{1-\alpha}} \\
&= \frac{E_{\alpha,1}(\lambda\omega^\alpha)E_{\alpha,\alpha}(\lambda\omega^\alpha)}{|\lambda|\omega E_{\alpha,\alpha+1}(\lambda\omega^\alpha)}.
\end{aligned}$$

For $0 \le s < t \le \omega$, we have

$$G_{\alpha,\lambda}(t,s) \quad \ge \quad \frac{E_{\alpha,1}(\lambda\omega^\alpha)E_{\alpha,\alpha}(\lambda\omega^\alpha)}{|\lambda|\omega E_{\alpha,\alpha+1}(\lambda\omega^\alpha)} + \frac{E_{\alpha,\alpha}(\lambda\omega^\alpha)}{\omega^{1-\alpha}} \ge \frac{E_{\alpha,1}(\lambda\omega^\alpha)E_{\alpha,\alpha}(\lambda\omega^\alpha)}{|\lambda|\omega E_{\alpha,\alpha+1}(\lambda\omega^\alpha)}.$$

(K_2) Employing Proposition 2.1, we get

$$\begin{aligned}
\int_0^\omega G_{\alpha,\lambda}(t,s)ds &= \int_0^\omega \frac{E_{\alpha,1}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda(\omega-s)^\alpha)}{(\omega-s)^{1-\alpha}(1-E_{\alpha,1}(\lambda\omega^\alpha))}ds + \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}}ds \\
&= \frac{E_{\alpha,1}(\lambda t^\alpha)}{1-E_{\alpha,1}(\lambda\omega^\alpha)} \int_0^\omega (\omega-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(\omega-s)^\alpha)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)ds \\
&= \frac{E_{\alpha,1}(\lambda t^\alpha)}{1-E_{\alpha,1}(\lambda\omega^\alpha)} \cdot \omega^\alpha E_{\alpha,\alpha+1}(\lambda\omega^\alpha) + t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha) \\
&= \frac{E_{\alpha,1}(\lambda t^\alpha)}{1-E_{\alpha,1}(\lambda\omega^\alpha)} \cdot \omega^\alpha \cdot \lambda^{-1}\omega^{-\alpha}(E_{\alpha,1}(\lambda\omega^\alpha)-1) \\
&\quad + t^\alpha \cdot \lambda^{-1}t^{-\alpha}(E_{\alpha,1}(\lambda t^\alpha)-1) \\
&= \frac{1}{|\lambda|},
\end{aligned}$$

which completes the proof.

4. Existence of positive solution

Let $C[0,\omega]$ be the space continuous function on $[0,\omega]$ with the norm $\|x\| = \sup\{|x(t)| : t \in [0,\omega]\}$. In this section, we always assume that $\lambda < 0$. Clearly, x is a solution of (1.1) if and only if

$$x(t) = \int_0^\omega G_{\alpha,\lambda}(t,s)f(s,x(s))ds, \tag{4.1}$$

where $G_{\alpha,\lambda}$ is Green's function defined in Theorem 3.1.

The following famous Krasnosel'skii fixed point theorem, which is main tool of this section.

Theorem 4.1. *[22] Let B be a Banach space, and let $P \subset B$ be a cone. Assume Ω_1, Ω_2 two open and bounded subsets of B with $0 \in \Omega_1, \Omega_1 \subset \Omega_2$ and let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that one of the following conditions is satisfied:*

(L_1) $\|Ax\| \leq \|x\|$, *if* $x \in P \cap \partial\Omega_1$, *and* $\|Ax\| \geq \|x\|$, *if* $x \in P \cap \partial\Omega_2$,
(L_2) $\|Ax\| \geq \|x\|$, *if* $x \in P \cap \partial\Omega_1$, *and* $\|Ax\| \leq \|x\|$, *if* $x \in P \cap \partial\Omega_2$.

Then, A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Proposition 4.2. *Assume that there exist $0 < r < R, 0 < c_1 < c_2$ such that*

$$f : [0, \omega] \times \left[\frac{mc_1\omega}{Mc_2} r, R\right] \to \mathbb{R} \ \textit{ is continuous}, \tag{4.2}$$

$$c_1 \leq f(t, u) \leq c_2, \ \forall (t, u) \in [0, \omega] \times \left[\frac{mc_1\omega}{Mc_2} r, R\right]. \tag{4.3}$$

Let $P \subset C[0, \omega]$ be the cone

$$P = \left\{ x \in C[0, \omega] : \min_{t \in [0,\omega]} x(t) \geq \frac{mc_1\omega}{Mc_2} \|x\| \right\}.$$

Then the operator $A : \overline{P}_R \setminus P_r \to P$ given by

$$Ax(t) = \int_0^\omega G_{\alpha,\lambda}(t, s) f(s, x(s)) ds \tag{4.4}$$

is completely continuous, where $P_l = \{u \in P : \|u\| < l\}$.

Proof. Let $x \in \overline{P}_R \setminus P_r$, then

$$\frac{mc_1\omega}{Mc_2} r \leq x(t) \leq R \ \ for \ \ all \ \ t \in [0, \omega]. \tag{4.5}$$

We first show that A is well-defined, i.e. that $A : \overline{P}_R \setminus P_r \to P$. Note that

$$\begin{aligned} Ax(t) &= \int_0^\omega G_{\alpha,\lambda}(t, s) f(s, x(s)) ds \\ &= \int_0^\omega \frac{E_{\alpha,1}(\lambda t^\alpha) E_{\alpha,\alpha}(\lambda(\omega - s)^\alpha)}{(\omega - s)^{1-\alpha}(1 - E_{\alpha,1}(\lambda\omega^\alpha))} f(s, x(s)) ds \\ &\quad + \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t - s)^\alpha)}{(t - s)^{1-\alpha}} f(s, x(s)) ds \\ &= kq(\omega) E_{\alpha,1}(\lambda t^\alpha) + q(t), \end{aligned} \tag{4.6}$$

where $k = \frac{1}{1 - E_{\alpha,1}(\lambda\omega^\alpha)}$ and

$$q(t) = \begin{cases} \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}} f(s, x(s)) ds, & 0 < t \leq \omega, \\ 0, & t = 0. \end{cases} \tag{4.7}$$

Clearly, for $t \in (0, \omega]$

$$0 < q(t) \le c_2 \int_0^t \frac{E_{\alpha,\alpha}(\lambda(t-s)^\alpha)}{(t-s)^{1-\alpha}} ds \le \frac{c_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds = \frac{c_2}{\Gamma(\alpha+1)} t^\alpha, \tag{4.8}$$

which implies that q is continuous at $t = 0$. On the other hand,

$$\begin{aligned} q(t) = & \sum_{k<\frac{1}{\alpha}-1} \lambda^k \int_0^t \frac{(t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} f(s, x(s)) ds \\ & + \int_0^t \sum_{k\ge\frac{1}{\alpha}-1} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} (t-s)^{\alpha k+\alpha-1} f(s, x(s)) ds \\ = & \sum_{k<\frac{1}{\alpha}-1} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} \int_0^t u^{\alpha k+\alpha-1} f(t-u, x(t-u)) du \\ & + \int_0^t \sum_{k\ge\frac{1}{\alpha}-1} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} (t-s)^{\alpha k+\alpha-1} f(s, x(s)) ds \\ =: & H_1(t) + H_2(t), \end{aligned}$$

where

$$\begin{aligned} H_1(t) &= \sum_{k<\frac{1}{\alpha}-1} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} \int_0^t u^{\alpha k+\alpha-1} f(t-u, x(t-u)) du, \\ H_2(t) &= \int_0^t \sum_{k\ge\frac{1}{\alpha}-1} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} (t-s)^{\alpha k+\alpha-1} f(s, x(s)) ds. \end{aligned}$$

Since

$$\begin{aligned} |u^{\alpha k+\alpha-1} f(t-u, x(t-u))| \le c_2 u^{\alpha k+\alpha-1}, u > 0,\ t \in (0, \omega],\ x \in \overline{P}_R/P_r, \\ \left| \frac{\lambda^k (t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} f(s, x(s)) \right| \le \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} t^{\alpha k+\alpha-1} c_2,\ 0 \le s \le t,\ x \in \overline{P}_R/P_r, \\ \int_0^t u^{\alpha k+\alpha-1} du < +\infty,\ t \in (0, \omega], \\ \sum_{k\ge\frac{1}{\alpha}-1} \frac{\lambda^k t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} < +\infty,\ t \in (0, \omega], \end{aligned}$$

we obtain that $H_1 \in C[0, \omega]$ and

$$H_2(t) = \sum_{k\ge\frac{1}{\alpha}-1} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-s)^{\alpha k+\alpha-1} f(s, x(s)) ds =: \sum_{k\ge\frac{1}{\alpha}-1} \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} u_k(t).$$

Noting that $u_k \in C(0, \omega]$

$$|u_k(t)| \le c_2 \frac{\omega^{\alpha k+\alpha}}{\alpha k+\alpha},\ t \in (0, \omega], \quad \sum \frac{\lambda^k}{\Gamma(\alpha k+\alpha)} \frac{\omega^{\alpha k+\alpha}}{\alpha k+\alpha} < +\infty,$$

we have $H_2 \in C(0, \omega]$. Hence, $q \in C[0, \omega]$.

Moreover,

$$\begin{aligned}\frac{mc_1\omega}{Mc_2}\|Ax\| =&\frac{mc_1\omega}{Mc_2}\sup\int_0^\omega G_{\alpha,\lambda}(t,s)f(s,x(s))ds\\ \leq&\frac{mc_1\omega}{M}\sup\int_0^\omega G_{\alpha,\lambda}(t,s)ds = mc_1\omega\\ \leq&\min\int_0^\omega G_{\alpha,\lambda}(t,s)f(s,x(s))ds = \min_{t\in[0,\omega]} Ax(t), \end{aligned}\tag{4.9}$$

which means that $A : \overline{P}_R \setminus P_r \to P$.

Next, we show that A is continuous on $\overline{P}_R \setminus P_r$. Let $x_n, x \in \overline{P}_R \setminus P_r$ and $\|x_n - x\| \to 0$. From (4.2), we have $\|f(t, x_n(t)) - f(t, x(t))\| \to 0$,

$$\begin{aligned}\|Ax_n - Ax\| =& \sup_{t\in[0,\omega]}\left|\int_0^\omega G_{\alpha,\lambda}(t,s)(f(s,x(s)) - f(s,y(s)))ds\right|\\ \leq& \sup_{t\in[0,\omega]}\int_0^\omega G_{\alpha,\lambda}(t,s)ds\|f(t,x_n(t)) - f(t,x(t))\|\\ \leq& M\|f(t,x_n(t)) - f(t,x(t))\| \to 0,\end{aligned}$$

which implies that A is continuous. From (4.6), we get that $Ax(t)$ is uniformly bounded. Finally, we show that $\{Ax|x \in \overline{P}_R/P_r\}$ is an equicontinuity in $C[0, \omega]$. By (4.6), we have

$$\begin{aligned}|Ax(t_1) - Ax(t_2)| =& \left|\int_0^\omega (G_{\alpha,\lambda}(t_1,s) - G_{\alpha,\lambda}(t_2,s))f(s,x(s))ds\right|\\ \leq& kq(\omega)|E_{\alpha,1}(\lambda t_1^\alpha) - E_{\alpha,1}(\lambda t_2^\alpha)| + |q(t_1) - q(t_2)|.\end{aligned}$$

Since $E_{\alpha,1}(\lambda t^\alpha) \in C[0, \omega]$, $q(t) \in C[0, \omega]$ are uniformly continuous, $|E_{\alpha,1}(\lambda t_1^\alpha) - E_{\alpha,1}(\lambda t_2^\alpha)|$ and $|q(t_1) - q(t_2)|$ tend to zero as $|t_1 - t_2| \to 0$. Hence, $\{Ax(t)|x \in \overline{P}_R \setminus P_r\}$ is equicontinuous in $C[0, \omega]$.

Finally, by Arzela-Ascoli theorem, we can obtain that A is compact. Hence, it is completely continuous. □

Theorem 4.3. *Assume that there exist $0 < r < R, 0 < c_1 < c_2$ such that (4.2) and (4.3) hold. Further suppose one of the following conditions is satisfied*

(i) $f(t,u) \geq \frac{Mc_2}{m^2\omega^2c_1}u, \ \forall (t,u) \in [0,\omega] \times \left[\frac{mc_1\omega}{Mc_2}r, r\right],$
$f(t,u) \leq |\lambda|u, \ \forall (t,u) \in [0,\omega] \times \left[\frac{mc_1\omega}{Mc_2}R, R\right],$

(ii) $f(t,u) \leq |\lambda|u, \ \forall (t,u) \in [0,\omega] \times \left[\frac{mc_1\omega}{Mc_2}r, r\right],$
$f(t,u) \geq \frac{Mc_2}{m^2\omega^2c_1}u, \ \forall (t,u) \in [0,\omega] \times \left[\frac{mc_1\omega}{Mc_2}R, R\right].$

Then (1.1) has at least a positive solution x with $r \leq \|x\| \leq R$.

Proof. Here we only consider the case (i). By Proposition 4.2, $A : \overline{P}_R \setminus P_r \to P$ is completely continuous. For $x \in \partial P_r$, we have

$$\|x\| = r, \ \ \frac{mc_1\omega}{Mc_2}r \leq x(t) \leq r, \ \forall t \in [0,\omega]$$

and

$$Ax(t) \geq m \int_0^{\omega} f(s, x(s))ds \geq \frac{Mc_2}{m\omega^2 c_1} \int_0^{\omega} x(s)ds \geq r = \|x\|.$$

Similarly, if $x \in \partial P_R$,

$$\frac{mc_1\omega}{Mc_2} R \leq x(t) \leq R, \quad t \in [0, \omega],$$

$$0 \leq Ax(t) \leq \int_0^{\omega} G_{\alpha,\lambda}(t, s)|\lambda| x(s)ds \leq |\lambda| R \int_0^{\omega} G_{\alpha,\lambda}(t, s)ds = R = \|x\|.$$

By Theorem 4.1, there exists $x \in \overline{P}_R \setminus P_r$ such that $Ax = x$ and x is a solution of (1.1). Moreover,

$$\frac{mc_1\omega}{Mc_2} r \leq x(t) \leq R.$$

Corollary 4.4. *Let $c_1 < c_2$ be positive reals and $f(t, x)$ satisfy the conditions*

(i) $c_1 \leq f(t, x) \leq c_2$ *for all* $x \geq 0$,
(ii) $f : [0, w] \times (0, +\infty) \to \mathbb{R}$ *is a continuous function.*

Then problem (1.1) has a positive solution.

Proof. Let $0 < r < \frac{c_1^2 m^2 \omega^2}{Mc_2}$, $R > \frac{c_2^2 M}{|\lambda| m_1 c_1 \omega}$, then (4.2) and (4.3) are satisfied. Clearly, for $(t, u) \in [0, \omega] \times \left[\frac{mc_1\omega}{Mc_2} r, r\right]$,

$$f(t, u) \geq c_1 \geq \frac{Mc_2}{m^2\omega^2 c_1} r \geq \frac{Mc_2}{m^2\omega^2 c_1} u$$

and for $(t, u) \in [0, \omega] \times \left[\frac{mc_1\omega}{Mc_2} R, R\right]$,

$$f(t, u) \leq c_2 \leq |\lambda| \frac{mc_1\omega}{Mc_2} R \leq |\lambda| u.$$

Hence, by Theorem 4.3 (1.1) has at least a positive solution. □

Example 4.5. *Consider the equation*

$$\begin{cases} {}^c D_{0^+}^{\alpha} x(t) - \lambda x(t) = 1 + x^{\frac{1}{\beta}}(t), \ 0 < x \leq \omega, \\ x(0) = x(\omega), \end{cases} \tag{4.10}$$

where $0 < \alpha \leq 1, \beta > 1$ *and*

$$\Lambda = \{\lambda < 0, |\lambda| E_{\alpha,1}(\lambda\omega^{\alpha}) E_{\alpha,\alpha}(\lambda\omega^{\alpha}) \geq 4E_{\alpha,\alpha+1}(\lambda\omega^{\alpha})\} \neq \emptyset.$$

Choosing $c_1 = 1, c_2 = 2, r = \frac{1}{10}\min\{1, \frac{m^2\omega^2}{2M}\}, R = 1$. It is easy to check that (4.2) and (4.3) hold. For $\lambda \in \Lambda$,

$$f(t, u) = 1 + u^{\frac{1}{\beta}} \geq 1 \geq \frac{Mc_2}{m^2\omega c_1} r \geq \frac{Mc_2}{m^2\omega c_1} u, \forall (t, u) \in [0, \omega] \times \left[\frac{mc_1\omega}{Mc_2} r, r\right],$$

$$f(t, u) \leq 2 \leq |\lambda| \cdot \frac{mc_1\omega}{Mc_2} = |\lambda| \cdot \frac{m\omega}{2M} \leq |\lambda| u, \forall (t, u) \in [0, \omega] \times \left[\frac{mc_1\omega}{Mc_2} R, R\right].$$

Hence, (4.10) has at least one positive solution for $\lambda \in \Lambda$.

5. Existence of solutions via monotone iterative techniques

In this section, by using the monotone iterative method, we discuss the existence of solutions when $\lambda = 0$ in (1.1). Firstly, we give the definition of the upper and lower solutions and get monotone iterative sequences with the help of the corresponding linear equation. Finally, we prove the limits of the monotone iterative sequences are solutions of (1.7).

Definition 5.1. *Let $h, k \in C^1[0,\omega]$. h and k are called lower solution and upper solution of problem (1.7), respectively if h and k satisfy*

$$ {}^cD^{\alpha}_{0^+}h(t) \le f(t,h(t)),\ 0 < t \le \omega,\ h(0) \le h(\omega), \tag{5.1}$$

$$ {}^cD^{\alpha}_{0^+}k(t) \ge f(t,k(t)),\ 0 < t \le \omega,\ k(0) \ge k(\omega), \tag{5.2}$$

Clearly, if g the lower solution or upper solution of (1.7), then ${}^cD^{\alpha}_{0^+}g$ is continuous on $[0,\omega]$.

Lemma 5.2. *Let $\delta \in C[0,\omega]$ with $\delta \ge 0$ and $p \in \mathbb{R}$ with $p \le 0$. Then*

$$\begin{cases} {}^cD^{\alpha}_{0^+}z(t) - \lambda z(t) = \delta(t),\ 0 < t \le \omega, \\ z(\omega) - z(0) = p, \end{cases} \tag{5.3}$$

has a unique solution $z(t) \ge 0$ for $t \in [0,\omega]$, where $0 < \alpha \le 1, \lambda < 0$.

Proof. Let z_1, z_2 are two solutions of (5.3) and $v = z_1 - z_2$, then

$$\begin{cases} {}^cD^{\alpha}_{0^+}v(t) - \lambda v(t) = 0,\ 0 < t \le \omega, \\ v(\omega) = v(0). \end{cases} \tag{5.4}$$

Using Theorem 3.1, (5.4) has trivial solution $v = 0$.

By (3.5), we can verify that problem (5.3) has a unique solution

$$z = \int_0^{\omega} G_{\alpha,\lambda}(t,s)\delta(t)ds + p \cdot \frac{E_{\alpha,1}(\lambda\omega^{\alpha})}{E_{\alpha,1}(\lambda\omega^{\alpha}) - 1}.$$

As consequence, by Proposition 2.1 (C_3), (C_5) and Lemma 3.3 (F_4), we conclude that $z(t) \ge 0$. This completes the proof. □

Theorem 5.3. *Assume that h, k are the lower and upper solutions of problem (1.7) and $h \le k$. Moreover, suppose that f satisfies the following properties:*

(M) there is $\lambda < 0$ such that for all fixed $t \in [0,\omega]$, $f(t,x) - \lambda x$ is nondecreasing in $h(t) \le x \le k(t)$,

(J) $f : [0,\omega] \times [h(t),k(t)] \to \mathbb{R}$ is a continuous function.

Then there are two monotone sequences $\{h_n\}$ and $\{k_n\}$ are nonincreasing and nondecreasing, respectively with $h_0 = h$ and $k_0 = k$ such that $\lim_{n\to\infty} h_n = \overline{h}(t)$, $\lim_{n\to\infty} k_n = \overline{k}(t)$ uniformly on $[0,\omega]$, and $\overline{h}, \overline{k}$ are the minimal and the maximal solutions of (1.7) respectively, such that

$$h_0 \le h_1 \le h_2 \le ... \le h_n \le \overline{h} \le x \le \overline{k} \le k_n \le ... \le k_2 \le k_1 \le k_0$$

on $[0,\omega]$, where x is any solution of (1.7) such that $h(t) \le x(t) \le k(t)$ on $[0,\omega]$.

Proof. Let $[h,k] = \{u \in C[0,\omega] : h(t) \leq u(t) \leq k(t), t \in [0,\omega]\}$. For any $\eta \in [h,k]$, we consider the equation

$$\begin{cases} {}^cD_{0^+}^{\alpha} x(t) - \lambda x(t) = f(t,\eta(t)) - \lambda\eta(t),\ 0 < t \leq \omega, \\ x(0) = x(\omega), \end{cases}$$

Theorem 3.1 implies the above problem has a unique solution

$$x(t) = \int_0^{\omega} G_{\alpha,\lambda}(t,s)(f(s,\eta(s)) - \lambda\eta(s))ds. \tag{5.5}$$

Define an operator B by $x = B\eta$, we shall show that
(a) $k \geq Bk, Bh \geq h$,
(b) B is nondecreasing on $[h,k]$.
To prove (a). Denote $\theta = k - Bk$, we have

$$\begin{aligned} {}^cD_{0^+}^{\alpha}\theta(t) - \lambda\theta(t) &= {}^cD_{0^+}^{\alpha}k(t) - {}^cD_{0^+}^{\alpha}Bk(t) - \lambda(k(t) - Bk(t)) \\ &\geq f(t,k(t)) - ((f(t,k(t)) - \lambda k(t)) - \lambda k(t) \\ &= 0, \end{aligned}$$

and $\theta(w) - \theta(0) \leq 0$. Since $k \in C^1[0,\omega]$,

$${}^cD_{0^+}^{\alpha}k \in C[0,\omega], \quad {}^cD_{0^+}^{\alpha}Bk \in C[0,\omega].$$

By Lemma 5.2, $\theta \geq 0$, i.e. $k \leq Bk$. In an analogous way, we can show that $Bh \geq h$.

To prove (b). We show that $B\eta_1 \leq B\eta_2$ if $h \leq \eta_1 \leq \eta_2 \leq k$. Let $z_1 = B\eta_1$, $z_2 = B\eta_2$ and $z = z_2 - z_1$, then by (M), we have

$$\begin{aligned} {}^cD_{0^+}^{\alpha}z(t) - \lambda z(t) &= {}^cD_{0^+}^{\alpha}z_2(t) - {}^cD_{0^+}^{\alpha}z_1(t) - \lambda(z_2(t) - z_1(t)) \\ &= f(t,\eta_2(t)) - \lambda\eta_2(t) - (f(t,\eta_1(t)) - \lambda\eta_1(t)) \\ &\geq 0, \end{aligned}$$

and $v(\omega) = v(0)$. By Lemma 5.2, $z(t) \geq 0$, which implies $B\eta_1 \leq B\eta_2$.

Define the sequence $\{h_n\}$, $\{k_n\}$ with $h_0 = h, k_0 = k$ such that $h_{n+1} = Bh_n, k_{n+1} = Bk_n$ for $n = 0,1,2,....$ From (a) and (b), we have

$$h_0 \leq h_1 \leq h_2 \leq ... \leq h_n \leq k_n \leq ... \leq k_2 \leq k_1 \leq k_0$$

on $t \in [0,\omega]$, and

$$\begin{aligned} h_n(t) &= \int_0^{\omega} G_{\alpha,\lambda}(t,s)(f(s,h_{n-1}(s)) - \lambda h_{n-1}(s))ds, \\ k_n(t) &= \int_0^{\omega} G_{\alpha,\lambda}(t,s)(f(s,k_{n-1}(s)) - \lambda k_{n-1}(s))ds. \end{aligned}$$

Therefore, there exist $\overline{h}, \overline{k}$ such that $\lim_{n\to\infty} h_n = \overline{h}$, $\lim_{n\to\infty} k_n = \overline{k}$.

Similar to the proof of Proposition 4.2, we can show that $B : [h,k] \to [h,k]$ is a completely continuous operator. Therefore, $\overline{h}, \overline{k}$ are solutions of (1.7).

Finally, we prove that if $x \in [h_0, k_0]$ is one solution of (1.7), then $\overline{h}(t) \leq x(t) \leq \overline{k}(t)$ on $[0, \omega]$. To this end, we assume, without loss of generality, that $h_n(t) \leq x(t) \leq k_n(t)$ for some n. From property (b), we can get that $h_{n+1}(t) \leq x(t) \leq k_{n+1}(t), t \in [0, \omega]$. Since $h_0(t) \leq x(t) \leq k_0(t)$, we can conclude that

$$h_n(t) \leq x(t) \leq k_n(t), \quad \text{for} \quad \text{all} \quad n.$$

Passing the limit as $n \to \infty$, we obtain $\overline{h}(t) \leq x(t) \leq \overline{k}(t),\ t \in [0, \omega]$. This completes the proof. □

Example 5.4. *Consider the equation*

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t) = t + 1 - x^2(t),\ 0 < x \leq 1, \\ x(0) = x(1). \end{cases} \tag{5.6}$$

It easy to check that $h = 1, k = 2$ are the low solution and upper solution of (5.6), respectively. Let $\lambda = -10$. For all $t \in [0, \omega]$,

$$f(t, u) - \lambda u = t^2 + 1 - u^2 + 10u$$

is nondecreasing on $u \in [1, 2]$ and

$$f(t, u) = t + 1 - u^2$$

is continuous on $[0, \omega] \times [1, 2]$.

Hence, there exist two monotone sequences $\{h_n\}$ and $\{k_n\}$, nonincreasing and nondecreasing respectively, that converge uniformly to the extremal solutions of (5.6) on $[h, k]$.

6. Conclusions

This paper focuses on the existence of solutions for the Caputo fractional differential equation with periodic boundary value condition. We use Green's function to transform the problem into the existence of the fixed points of some operator, and we prove the existence of positive solutions by using the Krasnosel'skii fixed point theorem. On the other hand, the existence of the extremal solutions for the special case of the problem is obtained from monotone iterative technique and lower and upper solutions method. Since the fractional differential equation is nonlocal equation, the process of verifying the compactness of operator is very tedious, and we will search for some better conditions to prove the compactness of the operator A in the follow-up research. Meanwhile, since the existence result for $0 < \alpha \leq 1$ is obtained in present paper, we will discuss the existence of solutions for the Caputo fractional differential equation when $n - 1 < \alpha \leq n$ in follow-up research.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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