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Research article

Existence and multiplicity of solutions for nonlocal Schrödinger–Kirchhoff equations of convex–concave type with the external magnetic field

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Abstract: We are concerned with the following elliptic equations

$$K(|z|_{s,A}^p)(-\Delta)_{p,A}^s z + V(x)|z|^{p-2}z = a(x)|z|^{r-2}z + \lambda f(x,|z|)z \quad \text{in } \mathbb{R}^N,$$

where $(-\Delta)_{p,A}^s$ is the fractional magnetic operator, $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Kirchhoff function, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential and $V : \mathbb{R}^N \to (0,\infty)$ is continuous potential. The main purpose is to show the existence of infinitely many large- or small- energy solutions to the problem above. The strategy of the proof for these results is to approach the problem variationally by employing the variational methods, namely, the fountain and the dual fountain theorem with Cerami condition.

Keywords: Schrödinger-Kirchhoff equation; fractional magnetic operators; variational methods
Mathematics Subject Classification: 35A15, 35J60, 35R11, 47G20

1. Introduction

This paper is devoted to the study of the existence of nontrivial solutions for the following Schrödinger-Kirchhoff type problem involving the non-local fractional p-Laplacian with a magnetic potential

$$K(|z|_{s,A}^p)(-\Delta)_{p,A}^s z + V(x)|z|^{p-2}z = a(x)\,|z|^{r-2}\,z + \lambda f(x,|z|)z \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $0 < s < 1 < r < p < +\infty$

$$|z|_{s,A}^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|z(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}z(y)|^p}{|x-y|^{N+ps}}\,dxdy,$$

and the fractional magnetic operator $(-\Delta)_{p,A}^s$ is defined along all functions $z \in C_0^\infty(\mathbb{R}^N,\mathbb{C})$ as

$$(-\Delta)_{p,A}^s z(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|z(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}z(y)|^{p-2}(z(x) - e^{i(x-y)\cdot A(\frac{x+y}{2})}z(y))}{|x-y|^{N+ps}}dy$$

for $x \in \mathbb{R}^N$. Henceforward, $B_\varepsilon(x)$ denotes a ball in $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$, $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Kirchhoff function, $A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential, V and a are suitable potential functions in $(0, \infty)$ and $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ satisfies a Carathéodory condition. The operator $(-\Delta)_A^s$ in the case $p = 2$, is called a fractional magnetic operator. This nonlocal operator has been originally defined in [7] as a fractional extension of the magnetic pseudo-relativistic operator introduced in [25]. The existence and multiplicity of solutions to the fractional Schrödinger-Kirchhoff equation with an external magnetic potential have been obtained by the paper [45]; see also [35] for equations of this type involving the fractional p-Laplacian when $A \equiv 0$. The main aim of this paper is to obtain the multiplicity of solutions for the fractional magnetic Schrödinger-Kirchhoff type problem with concave-convex nonlinearities when f has a weaker condition than that of [45]. For further applications and more details on fractional magnetic operators we infer to [1,2,4,7,15,17,39,45] and to the references [24,25] for the physical background. If $A \equiv 0$, then $(-\Delta)_{p,A}^s$ is consistent with the ordinary notion of the fractional p-Laplacian. Elliptic problems involving the standard fractional Laplacian or more general integro-differential operators have been a classical topic for a long time because they are applied in various research fields, such as social sciences, fractional quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process, game theory, and Levy process, see [8, 10, 14, 22, 38, 47] and the references therein.

In order to consider the changes in the length of the strings during the vibrations, Kirchhoff in [29] initially provided a model given by the equation

$$\rho \frac{\partial^2 v}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial v}{\partial x}|^2 \, dx\Big) \frac{\partial^2 v}{\partial x^2} = 0,$$

which extends the classical D'Alembert's wave equation. In this direction, the non-local problems of Kirchhoff type have been studied in [12, 16, 19, 20, 28, 41, 44, 46, 48].

As mentioned before, this paper is concerned with the fractional magnetic equations by the case of a combined effect of concave-convex nonlinearities. From a pure mathematical point of view, many researchers have extensively studied about nonlinear elliptic equations involving the concave-convex nonlinearities (see [9, 13, 27, 43, 48]) since the celebrated paper [5] of Ambrosetti, Brezis and Cerami. In particular, the multiplicity result of solutions to the concave-convex-type elliptic problems driven by a nonlocal integro-differential operator has been proposed in [13]; see also [9, 27, 48].

It is commonly well known that the condition of Ambrosetti-Rabinowitz type in [6], that is, there exists a constant $\theta > p$ such that

$$0 < \theta F(x, \tau) \le f(x, \tau)\tau^2, \quad \text{for all } \tau \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^N, \quad \text{where} \quad F(x, \tau) = \int_0^\tau f(x, t) t \, dt,$$

is crucial to secure the boundedness of the Palais-Smale sequence of an energy functional. However, because this condition is quite restrictive and removes several nonlinearities, during the last few decades there were extensive studies which has been attempted to drop it by many researchers; see [3, 21, 23, 26, 30–32, 34]. In that sense, our main purpose is to discuss the existence of infinitely many large- or small- energy solutions to our problem for the case of a combined effect of concave–convex nonlinearities when the nonlinear growth f does not satisfy the condition of Ambrosetti-Rabinowitz type. The strategy of the proof for these results is to approach the problem using the variational methods, namely, the fountain theorem and the dual fountain theorem with Cerami condition. As far as we are aware, none have reported such multiplicity results for our problem with the external magnetic field.

This paper is organized as follows. In Section 2, we present some basic results to deal with this type equation with the fractional magnetic field and review well known facts for the fractional Sobolev space. And under certain assumption on f, we establish the existence of infinitely many large- or small- energy solutions by employing the variational methods.

2. Preliminaries and main results

In this section, we consider the existence of infinitely many solutions to problem (1,1). Firstly we assume that $V:\mathbb{R}^N\to\mathbb{R}^+$ satisfies

(V) $V\in L^1_{loc}(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N}V(x)>0$ and $\lim_{|x|\to\infty}V(x)=+\infty$.

Let $L^p_V(\mathbb{R}^N)$ denote the real valued Lebesgue space with $V(x)|z|^p\in L^1(\mathbb{R}^N)$, equipped with the norm

$$|z|^p_{p,V}=\int_{\mathbb{R}^N}V(x)|z|^p\,dx.$$

Then the fractional Sobolev space $\mathcal{H}^{s,p}_V(\mathbb{R}^N)$ is defined as for $s\in(0,1)$ and $p\in(1,+\infty)$

$$\mathcal{H}^{s,p}_V(\mathbb{R}^N)=\left\{z\in L^p_V(\mathbb{R}^N):\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(x)-z(y)|^p}{|x-y|^{N+ps}}\,dxdy<+\infty\right\}.$$

The space $\mathcal{H}^{s,p}_V(\mathbb{R}^N)$ is endowed with the norm

$$|z|^p_{\mathcal{H}^{s,p}_V(\mathbb{R}^N)}:=\left(|z|^p_{p,V}+[z]^p_s\right)\quad\text{with}\quad[z]^p_s:=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(x)-z(y)|^p}{|x-y|^{N+ps}}\,dxdy.$$

For further details on the fractional Sobolev spaces we refer the reader to [33] and the references therein. We recall the embedding theorem; see e.g., [26, 36].

Lemma 2.1. *Let* (V) *hold and let p^*_s be the fractional critical Sobolev exponent, that is $p^*_s:=\frac{Np}{N-sp}$ if $sp<N$. Then, the embedding $\mathcal{H}^{s,p}_V(\mathbb{R}^N)\to L^\gamma(\mathbb{R}^N)$ is continuous for any $\gamma\in[p,p^*_s]$ and moreover, the embedding $\mathcal{H}^{s,p}_V(\mathbb{R}^N)\hookrightarrow\hookrightarrow L^\gamma(\mathbb{R}^N)$ is compact for any $\gamma\in[p,p^*_s)$.*

Let $L^p_V(\mathbb{R}^N,\mathbb{C})$ be the Lebesgue space of functions $z:\mathbb{R}^N\to\mathbb{C}$ with $V(x)|z|^p\in L^1(\mathbb{R}^N)$. Let us define $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$ as the closure of $C^\infty_c(\mathbb{R}^N,\mathbb{C})$ with respect to the norm

$$|z|^p_{s,A}=(|z|^p_{p,V}+|z|^p_{s,A}),$$

where the magnetic Gagliardo seminorm is given by

$$|z|^p_{s,A}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}z(y)|^p}{|x-y|^{N+ps}}dxdy.$$

In fact, arguing as in [7, Proposition 2.1], we can easily show that $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$ is a reflexive and separable Banach space as the similar arguments in [35, 36, Appendix]. In the same ways as in the proof of [45, Lemma 3.4 and 3.5], the following Lemmas 2.2 and 2.3 can be proved if we consider the general exponent p instead of $p=2$.

Lemma 2.2. *Let* (V) *hold. If $r \in [p, p_s^*]$, then the embedding*

$$\mathcal{H}_{A,V}^{s,p}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^r(\mathbb{R}^N, \mathbb{C})$$

is continuous. Furthermore, for any compact subset $S \subset \mathbb{R}^N$ and $r \in [1, p_s^*)$, the embedding

$$\mathcal{H}_{A,V}^{s,p}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow \mathcal{H}_V^{s,p}(S, \mathbb{C}) \hookrightarrow\hookrightarrow L^r(S, \mathbb{C})$$

is continuous and the latter is compact, where $\mathcal{H}_V^{s,p}(S, \mathbb{C})$ is endowed with the following norm:

$$\|z\|_{s,V}^p = \Big(\int_S V(x)|z|^p \, dx + \int_S \int_S \frac{|z(x) - z(y)|^p}{|x - y|^{N+ps}} \, dxdy \Big).$$

Lemma 2.3. *Under the assumption* (V)*, for all bounded sequence $\{z_n\}$ in $\mathcal{H}_{A,V}^{s,p}(\mathbb{R}^N, \mathbb{C})$ the sequence $\{|z_n|\}$ admits a subsequence converging strongly to some z in $L^r(\mathbb{R}^N)$ for all $r \in [p, p_s^*)$.*

For our problem, we suppose that $K : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies the following conditions:

(K1) $K \in C(\mathbb{R}_0^+)$ satisfies $\inf_{\tau \in \mathbb{R}^+} K(\tau) \geq a > 0$, where $a > 0$ is a constant.
(K2) There is a positive constant $\theta \in [1, \frac{N}{N-ps})$ such that $\theta \mathcal{K}(\tau) = \theta \int_0^\tau K(\eta) d\eta \geq K(\tau)\tau$ for any $\tau \geq 0$.

A typical example for K is given by $K(\tau) = b_0 + b_1 \tau^m$ with $m > 0, b_0 > 0$, and $b_1 \geq 0$.

Let us denote $F(x, \tau) = \int_0^\tau f(x, t)t \, dt$ for all $x \in \mathbb{R}^N$ and $\tau \in \mathbb{R}^+$. We assume that for $1 < r < p \leq p\theta < q < p_s^*$ and $p \in (1, +\infty)$,

(A) $0 \leq a \in L^{\frac{p}{p-r}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\operatorname{meas}\Big\{x \in \mathbb{R}^N : a(x) \neq 0\Big\} > 0$.
(F1) $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ satisfies the Carathéodory condition.
(F2) $f \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$, and there exists nonnegative function $b \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$|f(x, \tau)| \leq b(x)\tau^{q-2}, \quad \text{for all } (x, \tau) \in \mathbb{R}^N \times \mathbb{R}^+, \quad q \in (p\theta, p_s^*).$$

(F3) There are $\nu > p\theta$ and $\mathcal{T} > 0$ such that

$$f(x, \tau)\tau^2 - \nu F(x, \tau) \geq -\varrho \tau^p - \beta(x) \quad \text{for all} \quad x \in \mathbb{R}^N \quad \text{and} \quad \tau \geq \mathcal{T},$$

where $\varrho \geq 0$ and $\beta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\beta(x) \geq 0$.

We give a simple example satisfying conditions (F3) that does not hold the condition of Ambrosetti-Rabinowitz type.

Example 2.4. *Put $\theta = 1$. If the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is defined by*

$$f(x, \tau) = \begin{cases} b(x) \Big(|\tau|^{p-2} + \frac{2}{p\tau} \sin \tau \Big) & \text{if} \quad \tau \neq 0, \\ \frac{2}{p} b(x) & \text{if} \quad \tau = 0, \end{cases}$$

where $b(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $0 < \inf_{x \in \mathbb{R}^N} b(x) \leq \sup_{x \in \mathbb{R}^N} b(x) < \infty$, then

$$F(x, \tau) = b(x) \left(\frac{1}{p} |\tau|^p - \frac{2}{p} \cos \tau + \frac{2}{p} \right).$$

If we set $\varrho := (\nu - 1)\sup_{x\in\mathbb{R}^N} b(x)$ *and* $\beta(x) := \frac{4\nu}{p} b(x)$ *with* $p < \nu$ *for all* $x \in \mathbb{R}^N$*, then*

$$\begin{aligned}
f(x,\tau)\tau^2 - \nu F(x,\tau) &= b(x)\left[|\tau|^p + \frac{2}{p}\tau\sin\tau - \frac{\nu}{p}|\tau|^p + \frac{2\nu}{p}\cos\tau - \frac{2\nu}{p}\right] \\
&\geq b(x)\left[\left(1 - \frac{\nu}{p}\right)|\tau|^p - \frac{2}{p}|\tau| - \frac{4\nu}{p}\right] \\
&= b(x)\left[(1-\nu)|\tau|^p + \frac{\nu(p-1)}{p}|\tau|^p - \frac{2}{p}|\tau|\right] - \frac{4\nu}{p} b(x) \\
&\geq b(x)(1-\nu)|\tau|^p - \frac{4\nu}{p} b(x) \\
&\geq -\varrho|\tau|^p - \beta(x)
\end{aligned}$$

for all $|\tau| \geq \mathcal{T}$*, where* $\mathcal{T} > 1$ *is chosen such that* $\nu(p-1)\mathcal{T}^p - 2\mathcal{T} \geq 0$*. Hence* (F3) *holds.*

The Euler functional $\mathcal{J}_\lambda : \mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ associated with the problem (1.1) is defined as follows:

$$\mathcal{J}_\lambda(z) = \frac{1}{p}(\mathcal{K}([z]^p_{s,A}) + \|z\|^p_{p,V}) - \frac{1}{r}\int_{\mathbb{R}^N} a(x)|z|^r\,dx - \lambda\int_{\mathbb{R}^N} F(x,|z|)\,dx.$$

Then it is obvious that the functional $\mathcal{J}_\lambda$ is Fréchet differentiable on $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C})$, and its derivative is

$$\begin{aligned}
\langle \mathcal{J}'_\lambda(z), v\rangle = \Re\Big(\mathcal{K}([z]^p_{s,A})\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & \frac{|z(x) - \mathbb{E}(x,y)z(y)|^{p-2}(z(x) - \mathbb{E}(x,y)z(y))\cdot\overline{[v(x) - \mathbb{E}(x,y)v(y)]}}{|x-y|^{N+ps}}dxdy \\
&+ \int_{\mathbb{R}^N} V(x)|z|^{p-2} z\bar{v}\,dx - \int_{\mathbb{R}^N} a(x)|z|^{r-2} z\bar{v}\,dx - \lambda\int_{\mathbb{R}^N} f(x,|z|)z\bar{v}\,dx\Big)
\end{aligned}$$

for any $z, v \in \mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C})$, where $\mathbb{E}(x,y) := e^{i(x-y)\cdot A(\frac{x+y}{2})}$ and $\bar{v}$ denotes complex conjugation of $v \in \mathbb{C}$. Hereafter, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C}))'$ and $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C})$. From [45], we observe that the critical points of $\mathcal{J}_\lambda$ are exactly the weak solutions of (1.1) and the functional $\mathcal{J}_\lambda$ is weakly lower semi-continuous in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C})$.

To begin with we introduce the Cerami condition, which was initially provided by Cerami [11].

Definition 2.5. *Let a functional* Ψ *be* C^1 *and* $c \in \mathbb{R}$*. If any sequence* $\{z_n\}$ *satisfying*

$$\Psi(z_n) \to c \quad \textit{and} \quad (1 + \|z_n\|)\|\Psi'(z_n)\| \to 0 \quad \textit{as} \quad n \to \infty,$$

possesses a convergent subsequence, then we say that Ψ *fulfils Cerami condition* ($(C)_c$*-condition in short*) *at the level c.*

Definition 2.6. A function $z \in \mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C})$ is called weak solution of problem (1.1) if z satisfies

$$\begin{aligned}
\Re\Big(\mathcal{K}([z]^p_{s,A})\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & \frac{|z(x) - \mathbb{E}(x,y)z(y)|^{p-2}(z(x) - \mathbb{E}(x,y)z(y))\cdot\overline{[\phi(x) - \mathbb{E}(x,y)\phi(y)]}}{|x-y|^{N+ps}}dxdy \\
&+ \int_{\mathbb{R}^N} V(x)|z|^{p-2} z\bar{\phi}\,dx\Big) = \Re\Big(\int_{\mathbb{R}^N} a(x)|z|^{r-2} z\bar{\phi}\,dx + \lambda\int_{\mathbb{R}^N} f(x,|z|)z\bar{\phi}\,dx\Big)
\end{aligned}$$

for all $\phi \in \mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C})$.

The following lemma plays a key role in establishing the existence of a nontrivial weak solution for the given problem.

Lemma 2.7. *Let $s \in (0,1)$, $p \in (1,+\infty)$ and $N > ps$. Suppose that* (V), (K1)–(K2), (F1)–(F3) *hold. Furthermore, assume that*

(F4) $\lim_{|\tau|\to\infty} \frac{F(x,\tau)}{|\tau|^{p\theta}} = \infty$ *uniformly for almost all $x \in \mathbb{R}^N$, where the number θ is given in* (K2).

Then the functional $\mathcal{J}_\lambda$ satisfies the $(C)_c$-condition for any $\lambda > 0$.

Proof. For $c \in \mathbb{R}$, let $\{z_n\}$ be a $(C)_c$-sequence in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$, that is,

$$\mathcal{J}_\lambda(z_n) \to c \quad \text{and} \quad \|\mathcal{J}'_\lambda(z_n)\|_{(\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C}))'}(1+\|z_n\|_{s,A}) \to 0 \qquad \text{as} \quad n\to\infty,$$

which means

$$c = \mathcal{J}_\lambda(z_n) + o(1) \quad \text{and} \quad \langle \mathcal{J}'_\lambda(z_n), z_n\rangle = o(1), \tag{2.1}$$

where $o(1) \to 0$ as $n\to\infty$. If $\{z_n\}$ is bounded sequence in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$, then the analogous argument as in the proof of Lemma 4.2 in [45] implies that $\{z_n\}$ converges strongly to z in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$. Hence, it is enough to ensure that the sequence $\{z_n\}$ is bounded in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$. We argue by contradiction. Suppose to the contrary that the sequence $\{z_n\}$ is unbounded in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$. So then we may assume that

$$\|z_n\|_{s,A} \to \infty, \quad \text{as} \quad n\to\infty.$$

Due to the condition (2.1), we have that

$$\begin{aligned} c &= \mathcal{J}_\lambda(z_n) + o(1) \\ &= \frac{1}{p}(\mathcal{K}(\|z_n\|^p_{s,A}) + \|z_n\|^p_{p,V}) - \frac{1}{r}\int_{\mathbb{R}^N} a(x)\,|z_n|^r\,dx - \lambda\int_{\mathbb{R}^N} F(x,|z_n|)\,dx + o(1). \end{aligned} \tag{2.2}$$

By Lemma 2.2, there is a constant $C_1 > 0$ such that $\|v\|_{L^\gamma(\mathbb{R}^N)} \le C_1\|v\|_{s,A}$ for any γ with $p \le \gamma < p_s^*$ and for any $v \in \mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$. Since $\|z_n\|_{s,A}\to\infty$ as $n\to\infty$, we assert by (2.2) that

$$\begin{aligned} \int_{\mathbb{R}^N} F(x,|z_n|)\,dx &\ge \frac{1}{p\lambda}(\mathcal{K}(\|z_n\|^p_{s,A}) + \|z_n\|^p_{p,V}) - \frac{1}{r\lambda}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|z_n\|^r_{L^p(\mathbb{R}^N)} - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \\ &\ge \frac{1}{p\lambda}\min\{1, a\theta^{-1}\}\|z_n\|^p_{s,A} - \frac{C_1}{r\lambda}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|z_n\|^r_{s,A} - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \to \infty \end{aligned} \tag{2.3}$$

as $n\to\infty$. Define a sequence $\{\omega_n\}$ by $\omega_n = z_n/\|z_n\|_{s,A}$. Then it is immediate that $\{\omega_n\} \subset \mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $\|\omega_n\|_{s,A} = 1$. Hence, up to a subsequence, still denoted by $\{\omega_n\}$, we obtain $\omega_n \rightharpoonup \omega$ in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$ as $n\to\infty$, and according to Lemma 2.1

$$\omega_n \to \omega \text{ a.e. in } \mathbb{R}^N \quad \text{and} \quad |\omega_n| \to |\omega| \text{ in } L^\ell(\mathbb{R}^N) \quad \text{as} \quad n\to\infty \tag{2.4}$$

for $p \le \ell < p_s^*$. Notice that $V(x)\to+\infty$ as $|x|\to\infty$, then

$$\Big(\frac{1}{p\theta} - \frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^p\,dx - C_2\int_{|z_n|\le\mathcal{T}} (|z_n|^p + b(x)\,|z_n|^q)\,dx$$

$$\geq \frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\|z_n\|_{p,V}^p-\mathcal{T}_0,$$

where C_2 and $\mathcal{T}_0$ are positive constants. Indeed we know that

$$\begin{aligned}
&\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}V(x)\,|z_n|^p\;dx-C_2\int_{|z_n|\leq\mathcal{T}}(|z_n|^p+b(x)\,|z_n|^q)\,dx\\
&\geq \frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}V(x)\,|z_n|^p\;dx+\frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{|z_n|\leq 1}V(x)\,|z_n|^p\;dx\\
&-C_2\int_{|z_n|\leq 1}(|z_n|^p+b(x)\,|z_n|^q)\,dx-C_2\int_{1<|z_n|\leq\mathcal{T}}(|z_n|^p+b(x)\,|z_n|^q)\,dx\\
&\geq \frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}V(x)\,|z_n|^p\;dx+\frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{|z_n|\leq 1}V(x)\,|z_n|^p\;dx\\
&-C_2(1+\|b\|_\infty)\int_{|z_n|\leq 1}|z_n|^p\;dx-\widetilde{C}_2,
\end{aligned}$$

where $\widetilde{C}_2>0$ is a constant. Since $|\{x\in\mathbb{R}^N:|z_n|>1\}|<\infty$, we know $\{x\in\mathbb{R}^N:|z_n|>1\}=A\cup N$ where A is bounded set and N is of measure zero. Without loss of generality, suppose that there exists $B_\tau\subseteq\mathbb{R}^N$ such that $\{x\in\mathbb{R}^N:|z_n|>1\}\subset B_\tau$. Since $V(x)\to+\infty$ as $|x|\to\infty$, there is $\tau_0>0$ such that $|x|\geq\tau_0>\tau$ implies $V(x)\geq 2C_2(1+\|b\|_\infty)\frac{p\theta\nu}{\nu-p\theta}$. Hence one has

$$\begin{aligned}
&\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}V(x)\,|z_n|^p\;dx-C_2\int_{|z_n|\leq\mathcal{T}}(|z_n|^p+b(x)\,|z_n|^q)\,dx\\
&\geq \frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}V(x)\,|z_n|^p\;dx+\frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\{|z_n|\leq 1\}\cap B_{\tau_0}^c}V(x)\,|z_n|^p\;dx\\
&+\frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\{|z_n|\leq 1\}\cap B_{\tau_0}}V(x)\,|z_n|^p\;dx-C_2(1+\|b\|_\infty)\int_{\{|z_n|\leq 1\}\cap B_{\tau_0}^c}|z_n|^p\;dx\\
&-C_2(1+\|b\|_\infty)\int_{\{|z_n|\leq 1\}\cap B_{\tau_0}}|z_n|^p\;dx-\widetilde{C}_2\\
&\geq \frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}V(x)\,|z_n|^p\;dx+\frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\{|z_n|\leq 1\}\cap B_{\tau_0}^c}V(x)\,|z_n|^p\;dx\\
&-C_2(1+\|b\|_\infty)\int_{\{|z_n|\leq 1\}\cap B_{\tau_0}^c}|z_n|^p\;dx-\mathcal{T}_0\\
&\geq \frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}V(x)\,|z_n|^p\;dx-\mathcal{T}_0,
\end{aligned}$$

as claimed. This together with (K1)–(K2) and (F2)–(F3) yields

$$\begin{aligned}
c+1&\geq\mathcal{J}_\lambda(z_n)-\frac{1}{\nu}\left\langle\mathcal{J}_\lambda'(z_n),z_n\right\rangle\\
&\geq\frac{1}{p}\mathcal{K}(\|z_n\|_{s,A}^p)-\frac{1}{\nu}K(\|z_n\|_{s,A}^p)\|z_n\|_{s,A}^p+\Big(\frac{1}{p}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N}V(x)\,|z_n|^p\;dx\\
&-\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}a(x)|z_n|^r\,dx+\lambda\int_{\mathbb{R}^N}\left(\frac{1}{\nu}f(x,|z_n|)|z_n|^2-F(x,|z_n|)\right)dx
\end{aligned}$$

$$\geq \left(\frac{1}{p\theta}-\frac{1}{\nu}\right) K(|z_n|_{s,A}^p)|z_n|_{s,A}^p + \Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^p \;dx - \left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} a(x)|z_n|^r\,dx$$
$$+\,\lambda\int_{|z_n|>\mathcal{T}}\left(\frac{1}{\nu}f(x,|z_n|)|z_n|^2 - F(x,|z_n|)\right)dx - C_2\int_{|z_n|\leq\mathcal{T}}(|z_n|^p + b(x)\,|z_n|^q)\,dx$$
$$\geq \frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\min\{1,a\}|z_n|_{s,A}^p - \left(\frac{1}{r}-\frac{1}{\nu}\right)|a|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}|z_n|^r_{L^p(\mathbb{R}^N)}$$
$$-\,\frac{\lambda}{\nu}\int_{\mathbb{R}^N}(\varrho\,|z_n|^p + \beta(x))\;dx - \mathcal{T}_0$$
$$\geq \frac{1}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\min\{1,a\}|z_n|_{s,A}^p - C_1\left(\frac{1}{r}-\frac{1}{\nu}\right)|a|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}|z_n|^r_{s.A}$$
$$-\,\frac{\lambda\varrho}{\nu}|z_n|^p_{L^p(\mathbb{R}^N)} - \frac{\lambda}{\nu}|\beta|_{L^1(\mathbb{R}^N)} - \mathcal{T}_0,$$

which implies

$$1 \leq \frac{\lambda\varrho}{\frac{\nu}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\min\{1,a\}}\limsup_{n\to\infty}|\omega_n|^p_{L^p(\mathbb{R}^N)} = \frac{\lambda\varrho}{\frac{\nu}{2}\Big(\frac{1}{p\theta}-\frac{1}{\nu}\Big)\min\{1,a\}}|\omega|^p_{L^p(\mathbb{R}^N)}. \tag{2.5}$$

Hence, it follows from (2.5) that $\omega \neq 0$. Set $\Sigma = \Big\{x\in\mathbb{R}^N : \omega(x)\neq 0\Big\}$. By (2.4), we deduce that

$$|z_n(x)| = |w_n(x)|\,|z_n|_{s,A} \to \infty \quad \text{as} \quad n\to\infty$$

for almost all $x \in \Sigma$. Then it follows from (K2) and (F4) that for all $x \in \Sigma$,

$$\begin{aligned}\lim_{n\to\infty}\frac{F(x,|z_n|)}{\mathcal{K}(|z_n|_{s,A}^p)+|z_n|_{p,V}^p} &\geq \lim_{n\to\infty}\frac{F(x,|z_n|)}{\mathcal{K}(1)(1+|z_n|_{s,A}^{p\theta})+|z_n|_{p,V}^p}\\ &\geq \lim_{n\to\infty}\frac{F(x,|z_n|)}{2\mathcal{K}(1)|z_n|_{s,A}^{p\theta}+|z_n|_{p,V}^{p\theta}}\\ &\geq \lim_{n\to\infty}\frac{F(x,|z_n|)}{(2\mathcal{K}(1)+1)\,|z_n|_{s,A}^{p\theta}}\\ &\geq \lim_{n\to\infty}\frac{F(x,|z_n|)}{(2\mathcal{K}(1)+1)\,|z_n|^{p\theta}}\,|w_n|^{p\theta}\\ &= \infty,\end{aligned} \tag{2.6}$$

where the inequality $\mathcal{K}(\xi) \leq \mathcal{K}(1)(1+\xi^\theta)$ is used for all $\xi \in \mathbb{R}_+$ because if $0 \leq \xi < 1$, then $\mathcal{K}(\xi) = \int_0^\xi K(s)\,ds \leq \mathcal{K}(1)$, and if $\xi > 1$, then $\mathcal{K}(\xi) \leq K(1)\xi^\theta$. Thus we deduce that $|\Sigma| = 0$, where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^N$. In fact, suppose that $|\Sigma| \neq 0$. By virtue of (F4) we can choose $\tau_0 > 1$ such that $F(x,\tau) > |\tau|^{p\theta}$ for all $x \in \mathbb{R}^N$ and $\tau_0 < |\tau|$. In accordance with (F1) and (F2), we derive that there is $M > 0$ such that $|F(x,\tau)| \leq M$ for all $(x,\tau) \in \mathbb{R}^N \times [-\tau_0,\tau_0]$. Hence we find a real number M_0 such that $F(x,\tau) \geq M_0$ for all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}$, and thus

$$\frac{F(x,|z_n|)-M_0}{\mathcal{K}(|z_n|_{p,A}^p)+|z_n|_{p,V}^p} \geq 0, \tag{2.7}$$

for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. In addition,

$$\begin{aligned}\mathcal{J}_\lambda(z_n) &= \frac{1}{p}\left(\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p\right) - \frac{1}{r}\int_{\mathbb{R}^N} a(x)|z_n|^r\,dx - \lambda\int_{\mathbb{R}^N} F(x,|z_n|)\,dx\\ &\le \frac{1}{p}\left(\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p\right) - \lambda\int_{\mathbb{R}^N} F(x,|z_n|)\,dx.\end{aligned}$$

Then one has

$$\frac{1}{p}\left(\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p\right) \ge \lambda\int_{\mathbb{R}^N} F(x,|z_n|)\,dx + c - o(1). \tag{2.8}$$

Since $\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p \to \infty$ as $n \to \infty$, taking (2.3), (2.6)–(2.8) and Fatou's lemma into account, we obtain that

$$\begin{aligned}\frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,|z_n|)\,dx}{\lambda\int_{\mathbb{R}^N} F(x,|z_n|)\,dx + c - o(1)}\\ &\ge \liminf_{n\to\infty}\int_{\mathbb{R}^N} \frac{pF(x,|z_n|)}{\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p}\,dx\\ &= \liminf_{n\to\infty}\int_{\Sigma} \frac{pF(x,|z_n|)}{\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p}\,dx - \limsup_{n\to\infty}\int_{\Sigma}\frac{pM_0}{\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p}\,dx\\ &= \liminf_{n\to\infty}\int_{\Sigma} \frac{p(F(x,|z_n|) - M_0)}{\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p}\,dx\\ &\ge \int_{\Sigma}\liminf_{n\to\infty} \frac{p(F(x,|z_n|) - M_0)}{\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p}\,dx\\ &= \int_{\Sigma}\liminf_{n\to\infty} \frac{pF(x,|z_n|)}{\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p}\,dx - \int_{\Sigma}\limsup_{n\to\infty}\frac{pM_0}{\mathcal{K}([z_n]_{s,A}^p) + \|z_n\|_{p,V}^p}\,dx = \infty,\end{aligned}$$

which is a contradiction. This yields $\omega(x) = 0$ for almost all $x \in \mathbb{R}^N$. Thus, we can conclude a contradiction. Therefore, $\{z_n\}$ is bounded in $\mathcal{H}_{A,V}^{s,p}(\mathbb{R}^N,\mathbb{C})$. This completes the proof. □

We are in a position to prove our main results. By making use of the fountain theorem in [42, Theorem 3.6], we demonstrate infinitely many weak solutions for problem (1.1). Let E be a real reflexive and separable Banach space, then it is known (see [18]) that there exist $\{e_n\} \subseteq E$ and $\{f_n^*\} \subseteq E^*$ such that

$$E = \overline{\operatorname{span}\{e_n : n = 1,2,\cdots\}},\quad E^* = \overline{\operatorname{span}\{f_n^* : n = 1,2,\cdots\}},$$

and

$$\left\langle f_i^*, e_j\right\rangle = \begin{cases} 1 & \text{if } i = j\\ 0 & \text{if } i \neq j.\end{cases}$$

Let us denote $\mathcal{E}_n = \operatorname{span}\{e_n\}$, $\mathcal{Y}_k = \bigoplus_{n=1}^k \mathcal{E}_n$, and $\mathcal{Z}_k = \overline{\bigoplus_{n=k}^\infty \mathcal{E}_n}$. In order to obtain the existence result, we apply the following Fountain theorem.

Lemma 2.8. ([37, 42]) *Let E be a Banach space, $\mathcal{I} \in C^1(E,\mathbb{R})$ satisfies the $(C)_c$-condition for any $c > 0$ and $\mathcal{I}$ is even. If for each sufficiently large $k \in \mathbb{N}$, there exist ϱ_k, σ_k with $\varrho_k > \sigma_k > 0$ such that the following conditions hold:*

(1) $\beta_k := \inf\{\mathcal{I}(z) : z \in \mathcal{Z}_k, \|z\|_E = \sigma_k\} \to \infty \quad \text{as} \quad k \to \infty;$
(2) $\alpha_k := \max\{\mathcal{I}(z) : z \in \mathcal{Y}_k, \|z\|_E = \varrho_k\} \leq 0.$

Then the functional $\mathcal{I}$ has an unbounded sequence of critical values, i.e., there exists a sequence $\{z_n\} \subset E$ such that $\mathcal{I}'(z_n) = 0$ and $\mathcal{I}(z_n) \to +\infty$ as $n \to +\infty$.

Theorem 2.9. *Let $s \in (0,1)$, $p \in (1,+\infty)$ and $N > ps$. Assume that* (V), (K1), (K2) *and* (F1)–(F4) *hold. Then for any $\lambda > 0$, problem* (1.1) *possesses an unbounded sequence of nontrivial weak solutions $\{z_n\}$ in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$ such that $\mathcal{J}_\lambda(z_n) \to \infty$ as $n \to \infty$.*

Proof. To apply Lemma 2.8, let us denote $E := \mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$ and $\mathcal{I} := \mathcal{J}_\lambda$. Obviously, $\mathcal{J}_\lambda$ is an even functional and ensures the $(C)_c$-condition. It is enough to prove that there exist $\varrho_k > \sigma_k > 0$ with the conditions (1) and (2) in Lemma 2.8. Firstly we prove the condition (1). Let us denote

$$\varsigma_k = \sup_{\|u\|_{s,A}=1, z\in\mathcal{Z}_k} \|z\|_{L^q(\mathbb{R}^N)}.$$

Then, it is immediate to verify that $\varsigma_k \to 0$ as $k \to \infty$. For any $z \in \mathcal{Z}_k$, suppose that $\|z\|_{s,A} > 1$. Invoking (F2), one has

$$\begin{aligned}
\mathcal{J}_\lambda(z) &= \frac{1}{p}(\mathcal{K}(\|z\|^p_{s,A}) + \|z\|^p_{p,V}) - \frac{1}{r}\int_{\mathbb{R}^N} a(x)\,|z|^r\,dx - \lambda\int_{\mathbb{R}^N} F(x,|z|)dx \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{p}\|z\|^p_{s,A} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|z\|^r_{L^p(\mathbb{R}^N)} - \lambda\int_{\mathbb{R}^N} F(x,|z|)dx \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{p}\|z\|^p_{s,A} - \frac{1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|z\|^r_{L^p(\mathbb{R}^N)} - \frac{\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\|z\|^q_{L^q(\mathbb{R}^N)} \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{p}\|z\|^p_{s,A} - \frac{C_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|z\|^r_{s,A} - \lambda\|b\|_{L^\infty(\mathbb{R}^N)}\varsigma_k^q\|z\|^q_{s,A} \\
&= \Big(\frac{\min\{1, a\theta^{-1}\}}{p} - \lambda\|b\|_{L^\infty(\mathbb{R}^N)}\varsigma_k^q\|z\|^{q-p}_{s,A}\Big)\|z\|^p_{s,A} - \frac{C_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\|z\|^r_{s,A},
\end{aligned} \tag{2.9}$$

where C_1 was given in (2.3). Choose $\sigma_k = \left[\frac{2p\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{\min\{1,a\theta^{-1}\}}\varsigma_k^q\right]^{\frac{1}{p-q}}$. Since $1 < p < q$ and $\varsigma_k \to 0$ as $k \to \infty$, we infer $\sigma_k \to \infty$ as $k \to \infty$. Hence, if $z \in \mathcal{Z}_k$ and $\|z\|_{s,A} = \sigma_k$, then we deduce by (2.9) that

$$\mathcal{J}_\lambda(z) \geq \frac{\min\{1, a\theta^{-1}\}}{2p}\sigma_k^p - \frac{C_1}{r}\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}\sigma_k^r \to \infty \quad \text{as} \quad k \to \infty,$$

which implies the condition (1).

Next we show condition (2). To do this, we claim that $\mathcal{J}_\lambda(z) \to -\infty$ as $\|z\|_{s,\mathcal{A}} \to \infty$ for all $z \in \mathcal{Y}_k$. Let us assume that this dose not hold for some k. Then we can find a positive constant M and a sequence $\{z_n\}$ in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$ such that

$$\|z_n\|_{s,A} \to \infty \text{ as } n \to \infty \quad \text{and} \quad \mathcal{J}_\lambda(z_n) \geq -M.$$

Let $\omega_n = z_n/\|z_n\|_{s,A}$. Then it is clear that $\|\omega_n\|_{s,A} = 1$. Since $\dim\mathcal{Y}_k < \infty$, there is $\omega \in \mathcal{Y}_k \setminus \{0\}$ such that up to a subsequence,

$$\|\omega_n - \omega\|_{s,A} \to 0 \quad \text{and} \quad \omega_n(x) \to \omega(x)$$

for almost all $x \in \mathbb{R}^N$ as $n \to \infty$. Thus the similar argument as in relation (2.6) implies that

$$\begin{aligned}
\frac{1}{p} + \frac{M}{\mathcal{K}(|z_n|_{s,A}^p) + |z_n|_{p,V}^p} &\geq \frac{1}{p} - \frac{\mathcal{J}_\lambda(z_n)}{\mathcal{K}(|z_n|_{s,A}^p) + |z_n|_{p,V}^p} \\
&= \lambda \int_{\mathbb{R}^N} \frac{F(x,|z_n|)}{\mathcal{K}(|z_n|_{s,A}^p) + |z_n|_{p,V}^p}\, dx + \int_{\mathbb{R}^N} \frac{a(x)|z_n|^r}{r\left(\mathcal{K}(|z_n|_{s,A}^p) + |z_n|_{p,V}^p\right)}\, dx \\
&\geq \lambda \int_{\{\omega_n(x)\neq 0\}} \frac{F(x,|z_n|)}{(2\mathcal{K}(1)+1)\,|z_n|_{s,A}^{p\theta}}\, dx. \qquad (2.10)
\end{aligned}$$

By virtue of (2.7), (2.10), (F4) and Fatou's lemma, one has

$$\begin{aligned}
\frac{1}{p\lambda} &\geq \liminf_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{F(x,|z_n|)}{(2\mathcal{K}(1)+1)\,|z_n|_{s,A}^{p\theta}}\, dx - \limsup_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{M_0}{(2\mathcal{K}(1)+1)\,|z_n|_{s,A}^{p\theta}}\, dx \\
&= \liminf_{n\to\infty} \int_{\{\omega_n(x)\neq 0\}} \frac{F(x,|z_n|)-M_0}{(2\mathcal{K}(1)+1)\,|z_n|_{s,A}^{p\theta}}\, dx \geq \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \frac{F(x,|z_n|)-M_0}{(2\mathcal{K}(1)+1)\,|z_n|_{s,A}^{p\theta}}\, dx \\
&= \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \frac{F(x,|z_n|)}{(2\mathcal{K}(1)+1)\,|z_n|_{s,A}^{p\theta}}\, dx - \int_{\{\omega_n(x)\neq 0\}} \limsup_{n\to\infty} \frac{M_0}{(2\mathcal{K}(1)+1)\,|z_n|_{s,A}^{p\theta}}\, dx \\
&\geq \frac{1}{2\mathcal{K}(1)+1} \int_{\{\omega_n(x)\neq 0\}} \liminf_{n\to\infty} \left(\frac{F(x,|z_n|)}{|z_n|^{p\theta}}|\omega_n|^{p\theta}\right) dx = \infty,
\end{aligned}$$

where M_0 was given in the proof of Lemma 2.7. This is impossible. Thus, $\mathcal{J}_\lambda(z) \to -\infty$ as $|z|_{s,A} \to \infty$ for all $z \in \mathcal{Y}_k$. Choose $\varrho_k > \sigma_k > 0$ large sufficiently and let $|z|_{s,A} = \varrho_k$, we conclude that

$$a_k = \max\{\mathcal{J}_\lambda(z) : z \in \mathcal{Y}_k, |z|_{s,A} = \varrho_k\} \leq 0,$$

and therefore the condition (2) are claimed. This completes the proof. □

Definition 2.10. Let E be a real reflexive and separable Banach space. We say that $\mathcal{I}$ satisfies the $(C)_c^*$-condition (with respect to $\mathcal{Y}_n$) if any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ for which $z_n \in \mathcal{Y}_n$, for any $n \in \mathbb{N}$,

$$\mathcal{I}(z_n) \to c \quad \text{and} \quad |(\mathcal{I}|_{\mathcal{Y}_n})'(z_n)|_{E^*}(1 + |z_n|_E) \to 0 \text{ as } n \to \infty,$$

contains a subsequence converging to a critical point of $\mathcal{I}$.

Lemma 2.11. (Dual Fountain Theorem [23, Theorem 3.11]) *Assume that E is a Banach space, $\mathcal{I} \in C^1(E,\mathbb{R})$ is an even functional. If there is $k_0 > 0$ so that, for each $k \geq k_0$, there are $\varrho_k > \sigma_k > 0$ such that*

(A1) $\inf\{\mathcal{I}(z) : z \in \mathcal{Z}_k, |z|_E = \varrho_k\} \geq 0$;
(A2) $\beta_k := \max\{\mathcal{I}(z) : z \in \mathcal{Y}_k, |z|_E = \sigma_k\} < 0$;
(A3) $\gamma_k := \inf\{\mathcal{I}(z) : z \in \mathcal{Z}_k, |z|_E \leq \varrho_k\} \to 0$ *as* $k \to \infty$;
(A4) $\mathcal{I}$ *satisfies the* $(C)_c^*$*-condition for every* $c \in [d_{k_0}, 0)$.

Then $\mathcal{I}$ has a sequence of negative critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to \infty$.

Lemma 2.12. *Let $s \in (0,1)$, $p \in (1,+\infty)$ and $N > ps$. Assume that* (V), (K1), (K2) *and* (F1)–(F4) *hold. Then the functional $\mathcal{J}_\lambda$ satisfies the $(C)_c^*$-condition.*

Proof. The proof is carried out by the analogous argument as in [40]. □

With the help of Lemmas 2.11 and 2.12 we are ready to demonstrate our second assertion.

Theorem 2.13. *Assume that all conditions of Theorem 2.9 are satisfied. In addition we assume that*

(F5) $F(x,\tau) = o(|\tau|^p)$ as $\tau \to 0$ for $x \in \mathbb{R}^N$ uniformly.

Then the problem (1.1) *has a sequence of nontrivial weak solutions $\{z_n\}$ in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N,\mathbb{C})$ such that $\mathcal{J}_\lambda(z_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$.*

Proof. Invoking Lemma 2.12 and the definition of $\mathcal{J}_\lambda$, we know that $\mathcal{J}_\lambda$ is even and satisfies the $(C)^*_c$-condition for all $c \in \mathbb{R}$. Now it remains to show that conditions (A1), (A2) and (A3) of Lemma 2.11 are satisfied.

(A1): Let us denote

$$\theta_{1,k} = \sup_{|z|_{s,A}=1, z\in\mathcal{Z}_k} |z|_{L^p(\mathbb{R}^N)}, \quad \theta_{2,k} = \sup_{|z|_{s,A}=1, z\in\mathcal{Z}_k} |z|_{L^q(\mathbb{R}^N)}.$$

Then, it is clear to ensure that $\theta_{1,k} \to 0$ and $\theta_{2,k} \to 0$ as $k \to \infty$. Set $\vartheta_k = \max\{\theta_{1,k},\ \theta_{2,k}\}$. Then we have

$$\begin{aligned}
\mathcal{J}_\lambda(z) &\geq \frac{\min\{1, a\theta^{-1}\}}{p}|z|^p_{s,A} - \frac{1}{r}|a|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}|z|^r_{L^p(\mathbb{R}^N)} - \frac{\lambda|b|_{L^\infty(\mathbb{R}^N)}}{q}|z|^q_{L^q(\mathbb{R}^N)} \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{p}|z|^p_{s,A} - \frac{|a|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\theta^r_{1,k}|z|^r_{s,A} - \frac{\lambda|b|_{L^\infty(\mathbb{R}^N)}}{q}\theta^q_{2,k}|z|^q_{s,A} \\
&\geq \frac{\min\{1, a\theta^{-1}\}}{p}|z|^p_{s,A} - \left(\frac{|a|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r} + \frac{\lambda|b|_{L^\infty(\mathbb{R}^N)}}{q}\right)\vartheta^r_k|z|^q_{s,A}
\end{aligned}$$

for k large enough and $|z|_{s,A} \geq 1$. Choose

$$\varrho_k = \left[\frac{2p}{\min\{1, a\theta^{-1}\}}\left(\frac{|a|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r} + \frac{\lambda|b|_{L^\infty(\mathbb{R}^N)}}{q}\right)\vartheta^r_k\right]^{\frac{1}{p-2q}}.$$

Let $z \in \mathcal{Z}_k$ with $|z|_{s,A} = \varrho_k > 1$ for k large enough. Then, there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mathcal{J}_\lambda(z) &\geq \frac{\min\{1, a\theta^{-1}\}}{p}|z|^p_{s,A} - \left(\frac{|a|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r} + \frac{\lambda|b|_{L^\infty(\mathbb{R}^N)}}{q}\right)\vartheta^r_k|z|^{2q}_{s,A} \\
&= \frac{\min\{1, a\theta^{-1}\}}{2p}\varrho^p_k \geq 0
\end{aligned}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$, by being

$$\lim_{k\to\infty} \frac{\min\{1, a\theta^{-1}\}}{2p}\varrho^p_k = \infty.$$

Consequently, we arrive that

$$\inf\{\mathcal{J}_\lambda(z) : z \in \mathcal{Z}_k, |z|_{s,A} = \varrho_k\} \geq 0.$$

(A2): Observe that $\|\cdot\|_{L^p(\mathbb{R}^N)}$, $\|\cdot\|_{L^{p\theta}(\mathbb{R}^N)}$ and $\|\cdot\|_{s,A}$ are equivalent on $\mathcal{Y}_k$. Then there exist positive constants $\varsigma_{1,k}$ and $\varsigma_{2,k}$ such that

$$\|z\|_{L^p(\mathbb{R}^N)} \leq \varsigma_{1,k}\|z\|_{s,A} \text{ and } \|z\|_{s,A} \leq \varsigma_{2,k}\|z\|_{L^{p\theta}(\mathbb{R}^N)}$$

for any $z \in \mathcal{Y}_k$. From (F2)–(F4), for any $\mathcal{M} > 0$ there are positive constants C_3 and $C_4(\mathcal{M})$ such that

$$F(x,\tau) \geq \mathcal{M}\varsigma_{2,k}^{p\theta}\tau^{p\theta} - C_3\tau^p - C_4(\mathcal{M})b(x)$$

for almost all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^+$. Since $\mathcal{K}(\eta) \leq \mathcal{K}(1)(1+\eta^\theta)$ for all $\eta \in \mathbb{R}_+$, it follows that

$$\begin{aligned}
\mathcal{J}_\lambda(z) &= \frac{1}{p}(\mathcal{K}(\|z\|_{s,A}^p) + \|z\|_{p,V}^p)) - \frac{1}{r}\int_{\mathbb{R}^N} a(x)\,|z|^r\,dx - \lambda\int_{\mathbb{R}^N} F(x,|z|)dx \\
&\leq \frac{1}{p}\left(\mathcal{K}(1)(1+\|z\|_{s,A}^{p\theta}) + \|z\|_{p,V}^p\right) - \lambda\mathcal{M}\varsigma_{2,k}^{p\theta}\int_{\mathbb{R}^N}|z|^{p\theta}dx + \lambda C_3\int_{\mathbb{R}^N}|z|^p dx + \lambda C_4(\mathcal{M})\int_{\mathbb{R}^N} b(x)dx \\
&\leq \frac{1}{p}\left(2\mathcal{K}(1)\|z\|_{s,A}^{p\theta} + \|z\|_{s,A}^{p\theta}\right) - \lambda\mathcal{M}\varsigma_{2,k}^{p\theta}\int_{\mathbb{R}^N}|z|^{p\theta}dx + \lambda C_3\int_{\mathbb{R}^N}|z|^p dx + C_5 \\
&\leq \frac{1}{p}\left(2\mathcal{K}(1)+1\right)\|z\|_{s,A}^{p\theta} - \lambda\mathcal{M}\|z\|_{s,A}^{p\theta} + \lambda C_3\varsigma_{1,k}^p\|z\|_{s,A}^p + C_5
\end{aligned}$$

for any $z \in \mathcal{Y}_k$ with $\|z\|_{s,A} \geq 1$ and some constant C_5. Let $h(\tau) = \frac{1}{p}(2\mathcal{K}(1)+1)\,\tau^{p\theta} - \lambda\mathcal{M}\tau^{p\theta} + \lambda C_3\varsigma_{1,k}^p\tau^p + C_5$. If $\mathcal{M}$ is large thoroughly, then $\lim_{\tau\to\infty} h(\tau) = -\infty$, and thus we look for $\tau_0 \in (1,\infty)$ such that $h(\tau) < 0$ for all $\tau \in [\tau_0,\infty)$. Hence $\mathcal{J}_\lambda(z) < 0$ for all $z \in \mathcal{Y}_k$ with $\|z\|_{s,A} = \tau_0$. Choosing $\sigma_k = \tau_0$ for all $k \in \mathbb{N}$, one has

$$\beta_k := \max\{\mathcal{J}_\lambda(z) : z \in \mathcal{Y}_k, \|z\|_{s,A} = \sigma_k\} < 0.$$

If necessary, we can change k_0 to a large value, so that $\varrho_k > \sigma_k > 0$ for all $k \geq k_0$.

(A3): Because $\mathcal{Y}_k \cap \mathcal{Z}_k \neq \phi$ and $0 < \sigma_k < \varrho_k$, we have $\gamma_k \leq \beta_k < 0$ for all $k \geq k_0$. For any $z \in \mathcal{Z}_k$ with $\|z\|_{s,A} = 1$ and $0 < \tau < \varrho_k$, one has

$$\begin{aligned}
\mathcal{J}_\lambda(\tau z) &\geq \frac{\min\{1, a\theta^{-1}\}}{p}\|\tau z\|_{s,A}^p - \frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\|\tau z\|_{L^r(\mathbb{R}^N)}^r - \frac{\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\|\tau z\|_{L^q(\mathbb{R}^N)}^q \\
&\geq -\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\tau^r\|z\|_{L^r(\mathbb{R}^N)}^r - \frac{\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\tau^q\|z\|_{L^q(\mathbb{R}^N)}^q \\
&\geq -\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\varrho_k^r\vartheta_k^r - \frac{\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\varrho_k^q\vartheta_k^q
\end{aligned}$$

for large enough k. Hence, it follows from the definition of ϱ_k that

$$\begin{aligned}
\gamma_k &\geq -\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\varrho_k^r\vartheta_k^r - \frac{\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\varrho_k^q\vartheta_k^q \\
&= -\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r}\left[\frac{2p}{\min\{1,a\theta^{-1}\}}\left(\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r} + \frac{\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\right)\right]^{\frac{r}{p-2q}}\vartheta_k^{\frac{r^2+(p-2q)r}{p-2q}} \\
&\quad - \frac{\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\left[\frac{2p}{\min\{1,a\theta^{-1}\}}\left(\frac{\|a\|_{L^{\frac{p}{p-r}}(\mathbb{R}^N)}}{r} + \frac{\lambda\|b\|_{L^\infty(\mathbb{R}^N)}}{q}\right)\right]^{\frac{q}{p-2q}}\vartheta_k^{\frac{(r+p-2q)q}{p-2q}}.
\end{aligned}$$

Because $r < p < q$ and $\vartheta_k \to 0$ as $k \to \infty$, we derive that $\lim_{k\to\infty} \gamma_k = 0$.

Hence all conditions of Lemma 2.11 are required. Therefore, we conclude that problem (1.1) has a sequence of nontrivial weak solutions $\{z_n\}$ in $\mathcal{H}^{s,p}_{A,V}(\mathbb{R}^N, \mathbb{C})$ such that $\mathcal{J}_\lambda(z_n) \to 0$ as $n \to \infty$ for any $\lambda > 0$. □

3. Conclusions

In this paper, we employ the variational methods to ensure the existence of nontrivial solutions to nonlocal Schrödinger–Kirchhoff equations of convex–concave type with the external magnetic field. As far as we can see, in these circumstances the present paper is the first attempt to study the multiplicity of nontrivial weak solutions to this non-local problems for the case of a combined effect of concave–convex nonlinearities when the nonlinear growth f does not satisfy the condition of Ambrosetti-Rabinowitz type. We point out that with an analogous analysis our main consequences still hold when $(-\Delta)^s_{p,A} z$ in (1.1) is replaced with any non-local integro-differential operator $\mathcal{L}_\Phi$ defined as follows:

$$\mathcal{L}_\Phi z(x) = 2\int_{\mathbb{R}^N} |z(x) - \mathbb{E}(x,y)z(y)|^{p-2}(z(x) - \mathbb{E}(x,y)z(y))\Phi(x-y)dy \quad \text{for all } x \in \mathbb{R}^N.$$

where $\Phi : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a kernel function satisfying properties that

(K1) $m\Phi \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^p, 1\}$;
(K2) there exists $\mu > 0$ such that $\Phi(x) \geq \mu|x|^{-(N+ps)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
(K3) $\Phi(x) = \Phi(-x)$ for all $x \in \mathbb{R}^N \setminus \{0\}$.

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Conflict of interest

The authors declare that they have no competing interests.

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