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Research article

Positive solutions for a class of supercritical quasilinear Schrödinger equations

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Abstract: This paper deals with a class of supercritical quasilinear Schrödinger equations

$$-\Delta u + V(x)u + \kappa\Delta(\sqrt{1+u^2})\frac{u}{2\sqrt{1+u^2}} = \lambda f(u),\ x \in \mathbb{R}^N,$$

where $\kappa \geq 2,\ N \geq 3,\ \lambda > 0$. We suppose that the nonlinearity $f(t) : \mathbb{R} \to \mathbb{R}$ is continuous and only superlinear in a neighbourhood of $t = 0$. By using a change of variable and the variational methods, we obtain the existence of positive solutions for the above problem.

Keywords: quasilinear Schrödinger equations; periodic potential; asymptotically periodic potential; variational methods; L^∞-estimates
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1. Introduction

It is well-known that the generalized quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + W(x)z - \lambda l(|z|^2)z + \kappa[\Delta\rho(|z|^2)]\rho'(|z|^2)z,\ x \in \mathbb{R}^N, \tag{1.1}$$

serves as models for several physical phenomena corresponding to various forms of the given potential $W(x)$ and the given nonlinearity ρ, where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ *and* $W,\ l,\ \rho$ are real functions, κ, λ are real constants. For example, the case $\rho(s) = s$ was studied in [1] for the superfluid film equations in plasma physics. The Eq (1.1) is also related to the condensed matter theory, see [2].

In this paper, we consider the case $\rho(s) = (1 + s)^{\frac{1}{2}}$ which could be used to describe the self-channeling of a high-power ultrashort laser in matter, cf. e.g., [3, 4]. Let $z(t, x) = \exp(-iEt)u(x)$

in (1.1), where $E \in \mathbb{R}$ and u is a real function. Then we know that z satisfies (1.1) if and only if the function u solves the following equation

$$-\Delta u + V(x)u + \kappa\Delta(\sqrt{1+u^2})\frac{u}{2\sqrt{1+u^2}} = \lambda f(u),\ x \in \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x) - E$ and $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) := l(|t|^2)t$ is a new nonlinear term. We shall give the precise hypotheses on V and f latter.

In recent years, the Eq (1.2) with $\kappa < 0$ has already been investigated extensively, for example, [5–7]. But the results for $\kappa > 0$ is rarely studied, see [8–11]. In [8], when $\kappa = 2,\ N = 2$, Colin studied the existence of ground state solutions for the Eq (1.2) with $V(x) = 2w,\ f(s) = s - \frac{s}{\sqrt{1+s^2}}$, where w is a fixed positive parameter. In [9], with well defined $V(x)$ and improved (AR) condition, Shen and Wang got the better results which obtained the existence of solutions for (1.2) when $\kappa < 2$. In that paper, a change of variable was used to reduce the quasilinear problem to a semilinear one and the mountain pass theorem, the concentration compactness theorem were used to get the main existence results.

To sum up, in the past, researches of the (1.2) have mostly focused on $\kappa < 2$. Now, different from the above mentioned results, a natural question for us to pose is how about the existence of solutions for the case $\kappa \geq 2$ and $N \geq 3$. We would like to mention that the work [12] which obtained the existence of positive solutions for the supercritical quasilinear Schrödinger equations. In [12], Huang and Jia studied the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u + \Delta(u^2)u = \lambda f(u),\ x \in \mathbb{R}^N,$$

where $N \geq 3$, $f(t) : \mathbb{R} \to \mathbb{R}$ is continuous and only superlinear in a neighborhood of $t = 0$, by using the truncation methods and modifying the functional. Meanwhile, in the literatures [12–14], the authors have studied the asymptotically periodic quasilinear Schrödinger equations. So motivated by above discussions, we study the Eq (1.2) for the periodic and asymptotically periodic potentials when $\kappa \geq 2$, $\lambda > 0$ and $N \geq 3$.

Hereafter, we give the conditions of $V(x)$ and $f(t)$.

Hypothesis 1.1. Suppose that the potential $V(x)$ satisfies assumptions (v_0) *and* (v_1)

(v_0): $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$;

(v_1): $V(x) = V(x+y),\ \forall x \in \mathbb{R}^N,\ y \in \mathbb{Z}^N$.

Hypothesis 1.2. Suppose that the potential $V(x)$ satisfies the following assumption

(v_2): $V(x) = V_1(x) - m(x) \geq m_0 > 0,\ \forall x \in \mathbb{R}^N$, where $V_1(x)$ satisfies Hypothesis 1.1 and $m(x) \in \mathcal{F}$ with $m(x) \geq 0$,

$$\mathcal{F} := \{b(x) : \forall \varepsilon > 0, \lim_{|y|\to\infty} \text{meas}\{x \in B_1(y) : |b(x)| \geq \varepsilon\} = 0\}. \tag{1.3}$$

Here, we assume the inequality $m(x) > 0$ is strict on a subset of positive measures in $\mathbb{R}^N$.

We call the $V(x)$ is periodic if it satisfies Hypothesis 1.1 and is the asymptotically periodic at infinity if it satisfies the Hypothesis 1.2. In particular, if $m(x) = 0$, the asymptotically periodic problem is reduced to its corresponding periodic problem. Because the periodic potentials and the asymptotically periodic potentials are both bounded, we set $V_M = \max\{V(x)\}$.

Hypothesis 1.3. For the nonlinearity f, we suppose that it is continuous and satisfies the following conditions which give its behavior only in a neighborhood of the origin:

(f_1): $f(t) = 0$, for $t \leq 0$ and there exists $\alpha \in (2, 2^*)$ such that

$$\limsup_{t\to 0^+} \frac{f(t)}{t^{\alpha-1}} < +\infty;$$

(f_2): there exists $\beta \in (2, 2^*)$ with $\beta > \alpha$ such that

$$\liminf_{t\to 0^+} \frac{F(t)}{t^{\beta}} > 0;$$

(f_3): for $t > 0$ small, there exists $\theta \in (2, 2^*)$ such that $0 < \theta F(t) \leq t f(t)$,

where $F(t) = \int_0^t f(s)ds$.

Remark 1.1. An example of the nonlinearity satisfying Hypothesis 1.3 can be taken as

$$f(t) = \begin{cases} C_0 t^{\alpha-1} + C_1 t^{q-1}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

with $2 < \alpha < 2^* < q$, $2^* = \dfrac{2N}{N-2}$ and C_0, C_1 are positive constants.

Obviously, (1.2) is the Euler-Lagrange equation associated with the natural energy functional

$$H_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(1 - \frac{\kappa u^2}{2(1+u^2)}\right)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \lambda \int_{\mathbb{R}^N} F(u)dx, \tag{1.4}$$

which is not well defined in $H^1(\mathbb{R}^N)$. From the variational point of view, the first difficulty is to guarantee the positiveness of the principal part, that is, $\left(1 - \dfrac{\kappa u^2}{2(1+u^2)}\right) > 0$. And then, the change of variable applied in [9] loses its meaning when $\kappa \geq 2$. Besides these, since there are no conditions imposed on f at infinity, the term $\int_{\mathbb{R}^N} F(u)dx$ may not be well-defined in $H^1(\mathbb{R}^N)$. Due to these facts, we can't employ the usual variational methods directly. To overcome these difficulties, we use some variational methods to solve (1.2).

We conclude the main features of this paper as follows.

- We study the Eq (1.2) for the periodic and asymptotically periodic potentials when $\kappa \geq 2$, $\lambda > 0$ and $N \geq 3$.
- We will first establish the existence of positive solutions for a modified quasilinear Schrödinger equation which will be given more precisely in (2.3).
- Using Moser iteration we get an L^∞-estimate for the weak solutions, which depends on the parameter λ. And for λ large enough, the solutions obtained of the modified problem are solutions of the original Eq (1.2).

Now, we turn to the statement of our results.

Theorem 1.1. *Under Hypothesis 1.1 and Hypothesis 1.3, when $\kappa \geq 2$ the problem (1.2) has at least one positive solution $u \in H^1(\mathbb{R}^N)$ for λ sufficiently large.*

Theorem 1.2. *Assume that $\kappa \geq 2$, under Hypothesis 1.2 and Hypothesis 1.3, the problem (1.2) has at least one positive solution $u \in H^1(\mathbb{R}^N)$ for λ sufficiently large.*

The paper is organized as follows. In Section 2, we give a modified problem and the variational setting of the problem. In Section 3, we complete the proof of Theorem 1.1. And the proof of Theorem 1.2 is given in Section 4.

Notation

- $B_\varrho(x_0)$ denotes a ball centered at x_0 with radius $\varrho > 0$;
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$;
- the strong (respectively weak) convergence is denoted by $\to$ (respectively$\rightharpoonup$);
- $C,\ C_0, ...,$ denote suitable positive constants;
- The notation $|u|_p$ denotes the usual $L^p(\mathbb{R}^N)$ norm of the function u;
- The working space is $H^1(\mathbb{R}^N)$ endowed with the norm $||u|| = \left(\int_{\mathbb{R}^N}(u^2 + |\nabla u|^2)dx\right)^{\frac{1}{2}}$.

2. Variational setting and preliminaries results

First, we give some discussions on the nonlinearity $f(t)$. Note that from (f_1) there exist two positive constants $\delta \in (0, \frac{1}{2})$, C_2 such that

$$F(t) \leq C_2 t^\alpha,\ \text{for } 0 < t < 2\delta. \tag{2.1}$$

For the fixed $\delta > 0$ in the above, we consider a cut-off function $a(t) \in C^1(\mathbb{R}, \mathbb{R})$ satisfying

$$a(t) = \begin{cases} 1, & \text{if } t \leq \delta, \\ 0, & \text{if } t \geq 2\delta, \end{cases}$$

$|a'(t)| \leq \dfrac{2}{\delta}$ and $0 \leq a(t) \leq 1$ for $t \in \mathbb{R}$. Define

$$\tilde{F}(t) = a(t)F(t) + (1 - a(t))F_\infty(t),\ \ \tilde{f}(t) = \tilde{F}'(t), \tag{2.2}$$

where

$$F_\infty(t) = \begin{cases} C_2 t^\alpha, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

By Hypothesis 1.3 and the definition of $a(t)$, it is easy to see that $\tilde{f}(t)$ has the following properties (see [15]).

Lemma 2.1. *Let $\tilde{f}(t)$ and $\tilde{F}(t)$ be defined in (2.2). Assume that Hypothesis 1.3 hold, then we have*

(1) $\tilde{f}(t) \in C(\mathbb{R}, \mathbb{R})$, $\tilde{f}(t) = 0$ *for all* $t \leq 0$ *and* $\tilde{f}(t) \to 0$ *as* $t \to 0^+$.

(2) $\lim\limits_{t\to+\infty} \dfrac{\tilde{f}(t)}{t} = +\infty$;

(3) *there exists* $C > 0$ *such that* $\tilde{f}(t) \leq Ct^{\alpha-1}$, *for all* $t \geq 0$;

(4) $0 < \theta^* \tilde{F}(t) \leq t\tilde{f}(t)$ *for all* $t > 0$, *where* $\theta^* = \min\{\alpha, \theta\}$.

Inspired by [12], we first consider the following modified quasilinear Schrödinger equation

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \lambda\tilde{f}(u),\ x\in\mathbb{R}^N, \tag{2.3}$$

instead of the Eq (1.2). Here $g(t):[0,+\infty)\to\mathbb{R}$ in (2.3) is given by

$$g(t)=\begin{cases}\sqrt{1-\dfrac{\kappa t^2}{2(1+t^2)}}, & \text{if } 0\le t<\sqrt{\dfrac{1}{\kappa-1}},\\[2mm] \dfrac{\sqrt{\kappa-1}}{\sqrt{2}\kappa t}+\dfrac{1}{\sqrt{2}\kappa}, & \text{if } t\ge\sqrt{\dfrac{1}{\kappa-1}},\end{cases}$$

for $\kappa\ge 2$. Setting $g(t)=g(-t)$ for all $t\le 0$, we know that $g\in C^1(\mathbb{R},(\frac{1}{\sqrt{2}\kappa},1])$ and g is decreasing in $[0,\infty)$.

Now, defining a function $G(t)=\int_0^t g(s)ds$, we get that $G(t)$ is an odd function, the inverse function $G^{-1}(t)$ exists and the following properties about $G^{-1}(t)$ hold.

Lemma 2.2. *For $\kappa\ge 2$, the function $G^{-1}(t)$ satisfies the following properties:*

(1) $\lim\limits_{t\to 0^+}\dfrac{G^{-1}(t)}{t}=1$;

(2) $\lim\limits_{t\to+\infty}\dfrac{G^{-1}(t)}{t}=\sqrt{2}\kappa$;

(3) $t\le G^{-1}(t)\le\sqrt{2}\kappa t$, *for all* $t\ge 0$;

(4) $-1+\dfrac{1}{\kappa}\le\dfrac{t}{g(t)}g'(t)\le 0$, *for all* $t\ge 0$.

Proof. By the definition of $g(t)$, we get

$$\lim_{t\to 0^+}\frac{G^{-1}(t)}{t}=\lim_{t\to 0^+}\frac{1}{g(G^{-1}(t))}=1$$

and

$$\lim_{t\to+\infty}\frac{G^{-1}(t)}{t}=\lim_{t\to+\infty}\frac{1}{g(G^{-1}(t))}=\sqrt{2}\kappa,$$

which show (1) and (2).

Since g is decreasing in $[0,\infty)$, the inequality $\frac{1}{\sqrt{2}\kappa}t\le g(t)t\le G(t)\le t$ holds for all $t\ge 0$. Consequently, by replacing t with $G^{-1}(t)$ we gain the conclusion (3).

By a direct calculation, one obtains

$$\frac{t}{g(t)}g'(t)=\begin{cases}-\dfrac{\kappa t^2}{2+(4-\kappa)t^2+(2-\kappa)t^4}, & \text{if } 0\le t<\sqrt{\dfrac{1}{\kappa-1}},\\[2mm] -\dfrac{\sqrt{\kappa-1}}{\sqrt{\kappa-1}+t}, & \text{if } t\ge\sqrt{\dfrac{1}{\kappa-1}}.\end{cases}$$

Since $\frac{t}{g(t)}g'(t)$ reaches the minimum value $-1+\frac{1}{\kappa}$ at $t=\sqrt{\frac{1}{\kappa-1}}$ and $\frac{t}{g(t)}g'(t)\leq 0$, the conclusion (4) holds. □

For $\kappa\geq 2$, we observe that the Eq (2.3) is the Euler-Lagrange equation associated with the natural energy functional

$$I_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^N}g^2(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|u|^2dx-\lambda\int_{\mathbb{R}^N}\tilde{F}(u)dx.$$

In what follows, taking the change of variable

$$v=G(u), \tag{2.4}$$

we know that the functional $I_\lambda(u)$ can be reformulated in the following way

$$J_\lambda(v)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx-\lambda\int_{\mathbb{R}^N}\tilde{F}(G^{-1}(v))dx. \tag{2.5}$$

From Lemma 2.1 and Lemma 2.2, we obtain that the functional $J_\lambda(v)$ is well-defined in $H^1(\mathbb{R}^N)$ and $J_\lambda(v)\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$. Additionally, for all $\varphi\in H^1(\mathbb{R}^N)$ we have

$$\langle J'_\lambda(v),\varphi\rangle=\int_{\mathbb{R}^N}\nabla v\nabla\varphi dx+\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi dx-\lambda\int_{\mathbb{R}^N}\frac{\tilde{f}(G^{-1}(v))}{g(G^{-1}(v))}\varphi dx. \tag{2.6}$$

Lemma 2.3. *If $v\in H^1(\mathbb{R}^N)$ is a critical point of $J_\lambda(v)$, then $u=G^{-1}(v)\in H^1(\mathbb{R}^N)$ and meanwhile u is a critical point for $I_\lambda(u)$.*

Proof. Suppose that v is a critical point of J_λ. According to Lemma 2.1 and Lemma 2.2, we have $u=G^{-1}(v)\in H^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N}\nabla v\nabla\varphi dx+\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi dx-\lambda\int_{\mathbb{R}^N}\frac{\tilde{f}(G^{-1}(v))}{g(G^{-1}(v))}\varphi dx=0,\ \forall\varphi\in H^1(\mathbb{R}^N).$$

Choosing $\varphi=g(u)\psi$ with $\psi\in C_0^\infty(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N}\nabla v\nabla u g'(u)\psi dx+\int_{\mathbb{R}^N}\nabla v\nabla\psi g(u)dx+\int_{\mathbb{R}^N}V(x)u\psi dx-\lambda\int_{\mathbb{R}^N}\tilde{f}(u)\psi dx=0,$$

which can be rearranged as

$$\int_{\mathbb{R}^N}\left(-\mathrm{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2+V(x)u-\lambda\tilde{f}(u)\right)\psi dx=0.$$

Thus, we complete the proof. □

3. Proof of Theorem 1.1

In this section, we will verify the mountain pass geometry of J_λ and the boundedness of its (PS) sequences. Furthermore, we will give the proof of Theorem 1.1.

Lemma 3.1. *If Hypothesis 1.1 and Hypothesis 1.3 hold, then for $\kappa \geq 2$ there exist ρ, $\sigma > 0$ and $e \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that*
(a) $J_\lambda(v) > \sigma$, for $\|v\| = \rho$,
(b) $J_\lambda(e) < 0$, for $\|e\| > \rho$.

Proof. Combining Lemma 2.1, Lemma 2.2 and the Sobolev embedding theorem, we find

$$\begin{aligned} J_\lambda(v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \lambda\int_{\mathbb{R}^N} \tilde{F}(G^{-1}(v))dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|v|^2 dx - C\lambda\int_{\mathbb{R}^N} |G^{-1}(v)|^\alpha dx \\ &\geq \min\{1, V_0\}\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2)dx - C\lambda\int_{\mathbb{R}^N} |v|^\alpha dx \\ &\geq \min\{1, V_0\}\frac{1}{2}\|v\|^2 - C\lambda\|v\|^\alpha. \end{aligned}$$

Thus, due to the fact $2 < \alpha < 2^*$, we conclude that there exists $\sigma > 0$ such that (a) holds for $\rho = \|v\|$ sufficiently small .

In addition, Lemma 2.1 implies $\tilde{F}(t) \geq Ct^{\theta^*}$ for all $t > \varepsilon_0 > 0$. For a fixed $\omega \in C_0^\infty(\mathbb{R}^N)$, we suppose that supp $\omega = \Omega$ and $\omega \geq 1$ in $\Omega' \subset \Omega$ with $|\Omega'| > 0$. Then it turns out that

$$\begin{aligned} J_\lambda(t\omega) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla \omega|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(t\omega)|^2 dx - \lambda\int_{\mathbb{R}^N} \tilde{F}(G^{-1}(t\omega))dx \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla \omega|^2 dx + \kappa^2 t^2\int_{\mathbb{R}^N} V_M|\omega|^2 dx - C\lambda t^{\theta^*}\int_{\Omega'} |\omega|^{\theta^*} dx. \end{aligned}$$

Since $\theta^* > 2$, it follows that $J_\lambda(t\omega) \longrightarrow -\infty$ as $t \longrightarrow \infty$. Then we will prove the result (b) if we take $e = t\omega$ with t large enough. □

In consequence of Lemma 3.1, we can apply the mountain pass theorem without the (PS) condition (see [16]) to get a $(\mathrm{PS})_{d_\lambda}$ sequence $\{v_n\}$ of J_λ, where d_λ is the mountain pass level associated with J_λ, i.e.,

$$J_\lambda(v_n) \to d_\lambda,\ J'_\lambda(v_n) \to 0 \text{ as } n \to \infty.$$

Lemma 3.2. *Under the assumptions of Hypothesis 1.1 and Hypothesis 1.3, the (PS) sequence $\{v_n\}$ of J_λ is bounded.*

Proof. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a (PS) sequence of the functional J_λ. By means of (2.5) and (2.6) we know that

$$J_\lambda(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx - \lambda\int_{\mathbb{R}^N} \tilde{F}(G^{-1}(v_n))dx = d_\lambda + o_n(1) \tag{3.1}$$

and for $\varphi_n = G^{-1}(v_n)g(G^{-1}(v_n)) \in H^1(\mathbb{R}^N)$, $\langle J'_\lambda(v_n), \varphi_n\rangle = o_n(1)\|\varphi_n\|$, that is

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla v_n \nabla(G^{-1}(v_n)g(G^{-1}(v_n)))dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\ - \lambda \int_{\mathbb{R}^N} \tilde{f}(G^{-1}(v_n))G^{-1}(v_n)dx = o_n(1)\|G^{-1}(v_n)g(G^{-1}(v_n))\|.\end{aligned} \tag{3.2}$$

From Lemma 2.2 we find that

$$\left|\nabla(G^{-1}(v_n)g(G^{-1}(v_n))\right| \leq \left|1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n))\right| |\nabla v_n| \leq |\nabla v_n| \tag{3.3}$$

and

$$|G^{-1}(v_n)g(G^{-1}(v_n))| \leq \sqrt{2}\kappa|v_n|. \tag{3.4}$$

Hence, by (3.3) and (3.4), we get

$$\|G^{-1}(v_n)g(G^{-1}(v_n))\| \leq \sqrt{2}\kappa\|v_n\|.$$

Additionally, (3.3) and the fact $\langle J'_\lambda(v_n),\ G^{-1}(v_n)g(G^{-1}(v_n))\rangle = o_n(1)\|v_n\|$ imply

$$\begin{aligned} o_n(1)\|v_n\| &= \int_{\mathbb{R}^N} \left(1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n))\right) |\nabla v_n|^2 dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx - \lambda \int_{\mathbb{R}^N} \tilde{f}(G^{-1}(v_n))G^{-1}(v_n)dx \\ &\leq \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx - \lambda \int_{\mathbb{R}^N} \tilde{f}(G^{-1}(v_n))G^{-1}(v_n)dx. \end{aligned} \tag{3.5}$$

Then, from (3.1), (3.2), (3.5) and Lemma 2.1 we derive

$$\begin{aligned} \theta^* d_\lambda + o_n(1) + o_n(1)\|v_n\| &= \theta^* J_\lambda(v_n) - \langle J'_\lambda(v_n), G^{-1}(v_n)g(G^{-1}(v_n))\rangle \\ &\geq \frac{\theta^* - 2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{\theta^* - 2}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\ &\geq \frac{\theta^* - 2}{2} \min\{1, V_0\}\|v_n\|^2, \end{aligned} \tag{3.6}$$

which indicates $\|v_n\| < \infty$. □

Remark 3.1. Indeed, in Lemma 3.1 and Lemma 3.2, for the potential $V(x)$ we essentially just need it to be bounded. And there holds $m_0 \leq V(x) \leq V_M$ both in the periodic case and asymptotically periodic case. So if we replace Hypothesis 1.1 with Hypothesis 1.2, the conclusions similar to Lemma 3.1 and Lemma 3.2 still hold, which are about the asymptotically periodic case.

Lemma 3.3. *Assume that Hypothesis 1.1 and Hypothesis 1.3 hold. Then J_λ has a positive critical point.*

Proof. With the help of Lemma 3.1 and Lemma 3.2, we get that J_λ possesses a bounded (PS) sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$. Then, there exists $v \in H^1(\mathbb{R}^N)$ such that

$$v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^N),$$

$$v_n \to v \text{ in } L^p_{loc}(\mathbb{R}^N),$$
$$v_n \to v \text{ a.e. in } \mathbb{R}^N,$$

where $p \in [2, 2^*)$.

We claim that v is a critical point of J_λ, that is, $J'_\lambda(v) = 0$. To prove this claim, we only need to show that $\langle J'_\lambda(v), \varphi \rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ owing to the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$. Note that from (2.6), one has

$$\begin{aligned}
&\langle J'_\lambda(v_n) - J'_\lambda(v), \varphi \rangle \\
&= \int_{\mathbb{R}^N} (\nabla v_n - \nabla v)\nabla \varphi dx + \int_{\mathbb{R}^N} V(x)\left(\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right)\varphi dx \\
&\quad - \lambda \int_{\mathbb{R}^N} \left(\frac{\tilde{f}(G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{\tilde{f}(G^{-1}(v))}{g(G^{-1}(v))}\right)\varphi dx.
\end{aligned} \tag{3.7}$$

We will argue that the right side of (3.7) converges to zero in the following as $n \to \infty$. Considering for the (PS) sequence $\{v_n\}$, we have

$$v_n(x) \to v(x) \ a.e. \text{ in } K_\varphi := \text{supp}\, \varphi,$$

$$|v_n(x)| \le |w_p(x)| \ a.e. \text{ in } K_\varphi,$$

where $w_p \in L^p(K_\varphi)$. Hence,

$$\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi \to \frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi \ \ a.e. \text{ in } \mathbb{R}^N,$$

$$\frac{\tilde{f}(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi \to \frac{\tilde{f}(G^{-1}(v))}{g(G^{-1}(v))}\varphi \ \ a.e. \text{ in } \mathbb{R}^N.$$

From the condition (v_1), we get

$$\left| V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi \right| \le CV_M|v_n||\varphi| \le CV_M|w_p||\varphi|, \ x \in K_\varphi.$$

Then the Lebesgue Dominated Convergence theorem gives the result

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi dx = \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi dx. \tag{3.8}$$

Meanwhile, applying Lemma 2.1, we know that

$$\left| \frac{\tilde{f}(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi \right| \le C|G^{-1}(v_n)|^{\alpha-1}|\varphi| \le C|v_n|^{\alpha-1}|\varphi| \le C|w_p|^{\alpha-1}|\varphi|.$$

Making use of the Lebesgue Dominated Convergence theorem again, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{\tilde{f}(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi dx = \int_{\mathbb{R}^N} \frac{\tilde{f}(G^{-1}(v))}{g(G^{-1}(v))}\varphi dx. \tag{3.9}$$

Thus, (3.8), (3.9) and $v_n \rightharpoonup v$ yield $\langle (J'_\lambda(v_n) - J'_\lambda(v)), \varphi\rangle \to 0$ immediately. This limit together with $J'_\lambda(v_n) \to 0$ shows that $J'_\lambda(v) = 0$. Therefore, v is a critical point of J_λ.

If $v \neq 0$, we can get a nontrivial critical point of J_λ. For the case $v = 0$, similar as in [12], since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we can use a standard argument due to Lions ([16], Lemma 1.21) to prove that there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and $r, \ \sigma > 0$ such that $|y_n| \to \infty$ as $n \to \infty$ and

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |v_n|^2 dx \geq \sigma > 0. \tag{3.10}$$

Without loss of generality, we can assume that $\{y_n\} \subset \mathbb{Z}^N$. Let us consider the translation $\bar{v}_n(x) = v_n(x + y_n), \ n \in \mathbb{N}$. In this sense, $\|\bar{v}_n(x)\| = \|v_n(x)\|$ and $\{\bar{v}_n\}$ is still a bounded (PS) sequence of J_λ in view of the assumption of (v_1). Thus, taking a subsequence if necessary, we have a weak limit $\bar{v} \in H^1(\mathbb{R}^N)$ satisfying

$$\begin{aligned} &\bar{v}_n \rightharpoonup \bar{v} \ \text{ in } \ H^1(\mathbb{R}^N), \\ &\bar{v}_n \to \bar{v} \ \text{ in } \ L^2_{loc}(\mathbb{R}^N), \\ &\bar{v}_n \to \bar{v} \ \ a.e. \ \text{ in } \ \mathbb{R}^N. \end{aligned}$$

By using (3.10) we get the fact

$$0 < \sigma \leq \int_{B_r(y_n)} |v_n|^2 dx = \int_{B_r(0)} |\bar{v}_n|^2 dx \to \int_{B_r(0)} |\bar{v}|^2 dx, \tag{3.11}$$

i.e., $\bar{v} \neq 0$. Moreover, by the argument used above, we deduce a further conclusion $J'_\lambda(\bar{v})\varphi = 0$ for each $\varphi \in H^1(\mathbb{R}^N)$. Therefore, we have proved that the functional J_λ has a nontrivial critical point.

Now, assume that v is a nontrivial critical point of J_λ. Considering $\langle J'_\lambda(v), v^-\rangle = 0$, we obtain

$$\int_{\mathbb{R}^N} |\nabla v^-|^2 dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v^-)}{g(G^{-1}(v^-))} v^- dx = 0,$$

where v^-=max$\{-v, 0\}$. By using (v_0) and the definition of $g(t)$ we get $v^- = 0$, i.e., $v \geq 0$, which implies that v is positive through the strong maximum principle. Thus, J_λ has a positive critical point. □

Certainly, now we can't conclude that the origin Eq (1.2) has a positive solution. However, we note that the weak solution of (2.3) whose L^∞-norm is not bigger than $\min\{\sqrt{\frac{1}{\kappa - 1}}, \delta\}$ is also a weak solution of (1.2) for $\kappa \geq 2$. So in the following we will show the L^∞-estimates for the critical point v of J_λ.

Lemma 3.4. *If (v_0), (f_1), (f_3) hold and $v \in H^1(\mathbb{R}^N)$ is a positive critical point of J_λ, then $v \in L^\infty(\mathbb{R}^N)$. Moreover,*

$$|v|_\infty \leq C\lambda^{\frac{1}{2^*-\alpha}} \|v\|^{\frac{2^*-2}{2^*-\alpha}}, \tag{3.12}$$

where $C > 0$ only depends on α, N.

Proof. Let $v \in H^1(\mathbb{R}^N)$ be a positive critical point of J_λ. From (2.6) there holds

$$\int_{\mathbb{R}^N} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \varphi dx - \lambda \int_{\mathbb{R}^N} \frac{\tilde{f}(G^{-1}(v))}{g(G^{-1}(v))} \varphi dx = 0, \ \forall \varphi \in H^1(\mathbb{R}^N). \tag{3.13}$$

On the one hand, for $T > 0$, we define

$$v_T = \begin{cases} v, & \text{if } 0 \le v < T, \\ T, & \text{if } v \ge T. \end{cases}$$

Then there has $0 \le v_T \le v$. By taking $\varphi = v_T^{2(\gamma-1)} v$ with $\gamma > 1$ in (3.13), one obtains

$$\int_{\mathbb{R}^N} |\nabla v|^2 v_T^{2(\gamma-1)} dx + 2(\gamma - 1) \int_{\mathbb{R}^N} |\nabla v|^2 v v_T^{2(\gamma-1)-1} dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} v_T^{2(\gamma-1)} v dx$$
$$= \lambda \int_{\mathbb{R}^N} \frac{\tilde{f}(G^{-1}(v))}{g(G^{-1}(v))} v_T^{2(\gamma-1)} v dx.$$

Since the second and the third terms in the above equation are nonnegative, using Lemma 2.1 we can achieve

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v|^2 v_T^{2(\gamma-1)} dx &\le \lambda \int_{\mathbb{R}^N} \frac{\tilde{f}(G^{-1}(v))}{g(G^{-1}(v))} v_T^{2(\gamma-1)} v dx \\ &\le C\lambda \int_{\mathbb{R}^N} \frac{|G^{-1}(v)|^{\alpha-1}}{g(G^{-1}(v))} v_T^{2(\gamma-1)} v dx \\ &\le C\lambda \int_{\mathbb{R}^N} v^{\alpha} v_T^{2(\gamma-1)} dx. \end{aligned} \tag{3.14}$$

On the other hand, the Sobolev inequality implies

$$\begin{aligned} \left(\int_{\mathbb{R}^N} (v v_T^{\gamma-1})^{2^*} dx \right)^{\frac{2}{2^*}} &\le C \int_{\mathbb{R}^N} |\nabla (v v_T^{\gamma-1})|^2 dx \\ &\le C \int_{\mathbb{R}^N} |\nabla v|^2 v_T^{2(\gamma-1)} dx + C(\gamma-1)^2 \int_{\mathbb{R}^N} |\nabla v|^2 v_T^{2(\gamma-1)} dx \\ &\le C\gamma^2 \int_{\mathbb{R}^N} |\nabla v|^2 v_T^{2(\gamma-1)} dx. \end{aligned}$$

Therefore, using the above inequality, (3.14), the Hölder inequality and Sobolev embedding theorem we deduce

$$\begin{aligned} \left(\int_{\mathbb{R}^N} (v v_T^{\gamma-1})^{2^*} dx \right)^{\frac{2}{2^*}} &\le C\lambda\gamma^2 \int_{\mathbb{R}^N} v^{\alpha-2} v^2 v_T^{2(\gamma-1)} dx. \\ &\le C\lambda\gamma^2 \left(\int_{\mathbb{R}^N} v^{2^*} dx \right)^{\frac{\alpha-2}{2^*}} \left(\int_{\mathbb{R}^N} (v v_T^{\gamma-1})^{\frac{22^*}{2^*-\alpha+2}} dx \right)^{\frac{2^*-\alpha+2}{2^*}} \\ &\le C\lambda\gamma^2 \|v\|^{\alpha-2} \left(\int_{\mathbb{R}^N} v^{\frac{\gamma 22^*}{2^*-\alpha+2}} dx \right)^{\frac{2^*-\alpha+2}{2^*}}. \end{aligned}$$

From the above inequality, setting $\zeta = \frac{22^*}{2^*-\alpha+2}$, we have

$$\left(\int_{\mathbb{R}^N} (v v_T^{\gamma-1})^{2^*} dx \right)^{\frac{2}{2^*}} \le C\lambda\gamma^2 \|v\|^{\alpha-2} |v|_{\gamma\zeta}^{2\gamma}.$$

Then, by the Fatou's lemma, it follows that

$$|v|_{\gamma 2^*} \le (C\lambda\gamma^2 \|v\|^{\alpha-2})^{\frac{1}{2\gamma}} |v|_{\gamma\zeta}. \tag{3.15}$$

Define $\gamma_{n+1}\zeta = 2^*\gamma_n$ with $n = 0, 1, 2, ...,$ and $\gamma_0 = \dfrac{2^* + 2 - \alpha}{2}$. As a consequence of (3.15), we derive the following result

$$\begin{aligned}
|v|_{\gamma_1 2^*} &\le (C\lambda\gamma_1^2\|v\|^{\alpha-2})^{\frac{1}{2\gamma_1}}|v|_{2^*\gamma_0}\\
&\le (C\lambda\|v\|^{\alpha-2})^{\frac{1}{2\gamma_1}+\frac{1}{2\gamma_0}}\gamma_0^{\frac{1}{\gamma_0}}\gamma_1^{\frac{1}{\gamma_1}}|v|_{2^*}\\
&\le (C\lambda\|v\|^{\alpha-2})^{\frac{1}{2\gamma_0}(\frac{\gamma_0}{\gamma_1}+1)}\gamma_0^{(\frac{1}{\gamma_1}+\frac{1}{\gamma_0})}(\frac{\gamma_1}{\gamma_0})^{\frac{1}{\gamma_1}}|v|_{2^*}\\
&= (C\lambda\|v\|^{\alpha-2})^{\frac{1}{2\gamma_0}(\frac{\zeta}{2^*}+1)}\gamma_0^{\frac{1}{\gamma_0}(\frac{\zeta}{2^*}+1)}(\frac{2^*}{\zeta})^{\frac{1}{\gamma_1}}|v|_{2^*}.
\end{aligned}$$

Furthermore, by using the Moser iteration, we obtain

$$|v|_{\gamma_n 2^*} \le (C\lambda\|v\|^{\alpha-2})^{\frac{1}{2\gamma_0}\sum_{i=0}^{n}(\frac{\zeta}{2^*})^i}(\gamma_0)^{\frac{1}{\gamma_0}\sum_{i=0}^{n}(\frac{\zeta}{2^*})^i}(\frac{2^*}{\zeta})^{\frac{1}{\gamma_0}\sum_{i=0}^{n}i(\frac{\zeta}{2^*})^i}|v|_{2^*}.$$

Hence, from the facts that $\displaystyle\sum_{i=0}^{\infty}(\frac{\zeta}{2^*})^i = \frac{2^* + 2 - \alpha}{2^* - \alpha}$ and $\displaystyle\sum_{i=0}^{\infty} i(\frac{\zeta}{2^*})^i$ is convergent, we finally get

$$|v|_\infty \le C\lambda^{\frac{1}{2^*-\alpha}}\|v\|^{\frac{2^*-2}{2^*-\alpha}}.$$

□

Lemma 3.5. *Suppose that* (v_0), (f_1) *and* (f_3) *hold. Let* v *be a positive critical point of* J_λ *with* $J_\lambda(v) = d_\lambda$*. Then there exists* $C > 0$ *independent of* λ *such that*

$$\|v\|^2 \le Cd_\lambda.$$

Proof. By Lemma 2.2, the inequality (3.3), we get the following result

$$\begin{aligned}
\theta^* d_\lambda =&\theta^* J_\lambda(v) - \langle J_\lambda'(v), G^{-1}(v)g(G^{-1}(v))\rangle\\
=&\frac{\theta^*}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{\theta^*}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx - \lambda\theta^*\int_{\mathbb{R}^N}\tilde{F}(G^{-1}(v))dx\\
&-\int_{\mathbb{R}^N}\nabla v\nabla\left(G^{-1}(v)g(G^{-1}(v))\right)dx - \int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx\\
&+\lambda\int_{\mathbb{R}^N}\tilde{f}(G^{-1}(v))G^{-1}(v)dx\\
\ge&\frac{\theta^*-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{\theta^*-2}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx\\
\ge&\frac{\theta^*-2}{2}\min\{1, V_0\}\|v\|^2.
\end{aligned}$$

Thus, from the fact $\theta^* = \min\{\alpha, \theta\} > 2$, we get $\|v\|^2 \le Cd_\lambda$. □

Proof of Theorem 1.1 By Lemma 3.3, there exists a positive critical point v of J_λ with $J_\lambda(v) = d_\lambda$. And it follows from Lemma 2.1, Lemma 2.2 and (f_2) that

$$\begin{aligned} d_\lambda &\le \max_{t\in[0,1]} J_\lambda(te) \\ &\le \max_{t\in[0,1]}\left(\frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla e|^2 + 2\kappa^2 V_M|e|^2)dx - \lambda\int_{\mathbb{R}^N}\tilde{F}(G^{-1}(te))dx\right) \\ &\le \max_{t\in[0,1]}\left(\frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla e|^2 + 2\kappa^2 V_M|e|^2)dx - C\lambda t^{\theta^*}\int_\Omega e^{\theta^*}dx\right) \\ &\le C\lambda^{\frac{2}{2-\theta^*}}, \end{aligned} \tag{3.16}$$

where e is fixed in Lemma 3.1 and $G^{-1}(te) > \varepsilon_0$ in $\Omega \subset \mathbb{R}^N$. Then, by Lemma 3.4, Lemma 3.5 and (3.16) we have

$$|v|_\infty \le C\lambda^{\frac{2^*-\theta^*}{(2^*-\alpha)(2-\theta^*)}}.$$

Since $2 < \theta^* \le \alpha < 2^*$, from Lemma 2.2 there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$,

$$|u|_\infty = |G^{-1}(v)|_\infty \le \sqrt{2}\kappa|v|_\infty \le \min\{\sqrt{\frac{1}{\kappa-1}}, \delta\},$$

where δ is fixed in (2.1). This means that for $\lambda > \lambda_0$ the original Eq (1.2) possesses a positive solution $u = G^{-1}(v)$. □

4. Proof of Theorem 1.2

Different from the preceding section, for the case of asymptotically periodic potential we find that the inequality (3.11) is not valid. In order to overcome this difficulty, in this section, we will achieve the Lemma 4.2 which is a key point to complete the proof of Theorem 1.2. And for convenience, in this section, we give a sign $\bar{J}_\lambda(v)$ for the functional of the asymptotically periodic case, while we use $J_\lambda(v)$ to represent the functional of the corresponding periodic case. Then, there has

$$\begin{aligned} \bar{J}_\lambda(v) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx - \lambda\int_{\mathbb{R}^N}\tilde{F}(G^{-1}(v))dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}(V_1(x) - m(x))\,|G^{-1}(v)|^2dx - \lambda\int_{\mathbb{R}^N}\tilde{F}(G^{-1}(v))dx \\ &= J_\lambda(v) - \frac{1}{2}\int_{\mathbb{R}^N}m(x)|G^{-1}(v)|^2dx. \end{aligned} \tag{4.1}$$

Now, we first give the following two necessary lemmas.

Lemma 4.1. *Assume that Hypothesis 1.2 and Hypothesis 1.3 hold. If $\{v_n\}$ is bounded and $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} m(x)|G^{-1}(v_n)|^2dx = o_n(1).$$

Proof. Firstly, we claim that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\int_{\{m(x)\ge\varepsilon\}}|v|^2dx \le C_3\int_{B_{R_\varepsilon+1}(0)}|v|^2dx + C_4\varepsilon^{\frac{2}{N}}\|v\|^2,\ \forall u \in H^1(\mathbb{R}^N), \tag{4.2}$$

where $C_3,\ C_4$ are positive constants and independent on ε.

Clearly, by (1.3), for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\text{meas}\{x \in B_1(y) : |m(x)| \geq \varepsilon\} < \varepsilon,\ \forall |y| \geq R_\varepsilon.$$

Now, covering $\mathbb{R}^N$ by balls $B_1(y_i),\ i \in \mathbb{N},\ y_i \in \mathbb{R}^N$, in such a way each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls. Without loss of generality, we suppose that $|y_i| < R_\varepsilon$, for $i = 1, 2, ..., n_\varepsilon$ and $|y_i| \geq R_\varepsilon$, for $i = n_\varepsilon+1, n_\varepsilon+2, ..., +\infty$. Then we get that $|\Omega_i| < \varepsilon$, for all $|y_i| \geq R_\varepsilon$, where $\Omega_i = \{x \in B_1(y_i) : |m(x)| \geq \varepsilon\}$. Observe that from the Hölder and Sobolev inequalities one has

$$\begin{aligned}
\int_{\{m(x)\geq\varepsilon\}} |v|^2 dx &\leq \sum_{i=1}^{+\infty} \int_{\Omega_i} |v|^2 dx \\
&= \sum_{i=1}^{n_\varepsilon} \int_{\Omega_i} |v|^2 dx + \sum_{i=n_\varepsilon+1}^{+\infty} \int_{\Omega_i} |v|^2 dx \\
&\leq (N+1) \int_{B_{R_\varepsilon+1}(0)} |v|^2 dx + \sum_{i=n_\varepsilon+1}^{+\infty} |\Omega_i|^{\frac{2}{N}} \left(\int_{\Omega_i} |v|^{2^*} dx \right)^{\frac{N-2}{N}} \\
&\leq C_3 \int_{B_{R_\varepsilon+1}(0)} |v|^2 dx + C_4 \varepsilon^{\frac{2}{N}} \|v\|^2.
\end{aligned}$$

Therefore, our claim (4.2) is right.

Next, from Lemma 2.1, the boundedness of $\{v_n\}$ and $v_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$ for all $p \in [2, 2^*)$, we arrive at

$$\begin{aligned}
\int_{\mathbb{R}^N} m(x)|G^{-1}(v_n)|^2 dx &\leq C \int_{\mathbb{R}^N} m(x)|v_n|^2 dx \\
&\leq \int_{\{m(x)\geq\varepsilon\}} m(x)|v_n|^2 dx + \int_{\{m(x)<\varepsilon\}} m(x)|v_n|^2 dx \\
&\leq CV_M \left(C_3 \int_{B_{R_\varepsilon+1}(0)} |v_n|^2 dx + C_4 \varepsilon^{\frac{2}{N}} \|v_n\|^2 \right) + \varepsilon \int_{\mathbb{R}^N} |v_n|^2 dx \\
&= o_{n,\varepsilon}(1) + C_5 \varepsilon^{\frac{2}{N}} + C_6 \varepsilon \to 0,
\end{aligned} \tag{4.3}$$

as $\varepsilon \to 0$ and $n \to \infty$. Thus, we complete our proof. □

Lemma 4.2. *Assume that Hypothesis 1.2 and Hypothesis 1.3 all hold. Let $\{v_n\}$ be a bounded (PS) sequence of $\bar{J}_\lambda$ satisfying $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, as $n \to \infty$. Then $\{v_n\}$ is also a (PS) sequence for its corresponding periodic case J_λ.*

Proof. Since the above Lemma 4.1 guarantees

$$|J_\lambda(v_n) - \bar{J}_\lambda(v_n)| = \frac{1}{2} \int_{\mathbb{R}^N} m(x)|G^{-1}(v_n)|^2 dx \to 0,$$

we have $J_\lambda(v_n) \to \bar{d}_\lambda$. Taking $\varphi \in H^1(\mathbb{R}^N)$ with $\|\varphi\| \le 1$, by Hölder inequality and Lemma 4.1, we get

$$\begin{aligned}\left|\langle\left(J'_\lambda(v_n) - \bar{J}'_\lambda(v_n)\right), \varphi\rangle\right| &= \left|\int_{\mathbb{R}^N} m(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi dx\right| \\ &\le C\left(\int_{\mathbb{R}^N} m(x)|G^{-1}(v_n)|^2 dx\right)^{\frac{1}{2}} \\ &= o_n(1),\end{aligned}$$

which implies $J'_\lambda(v_n) = o_n(1)$. Hence, we know that $\{v_n\}$ is also a (PS) sequence of J_λ. □

Proof of Theorem 1.2 Firstly, notice that from the Remark 3.1 we can verify the mountain pass geometry of $\bar{J}_\lambda$ and the boundedness of its (PS) sequence $\{v_n\}$ analogously as in Lemma 3.1 and Lemma 3.2. Thus we can get a bounded $(PS)_{\bar{d}_\lambda}$ sequence $\{v_n\}$ of $\bar{J}_\lambda$, where $\bar{d}_\lambda$ is the mountain pass level of $\bar{J}_\lambda$, i.e.,

$$\bar{J}_\lambda(v_n) \to \bar{d}_\lambda,\ \bar{J}'_\lambda(v_n) \to 0 \text{ as } n \to \infty.$$

We suppose that $v \in H^1(\mathbb{R}^N)$ is the weak limit for the (PS) sequence $\{v_n\}$. Then, arguing exactly like in Lemma 3.3, we could get that v is the critical point of $\bar{J}_\lambda$. However, in the case of asymptotically periodic potential, we can't ensure that v is nontrivial directly. So, the task now is to prove that $v \ne 0$.

We suppose, by contradiction, $v \equiv 0$. From Lemma 4.2 we know that the (PS) sequence $\{v_n\}$ of $\bar{J}_\lambda$ is also a (PS) sequence of J_λ, where J_λ is the corresponding periodic case of $\bar{J}_\lambda$. Then we can define the translation $\bar{v}_n(x) = v_n(x + y_n)$ for J_λ analogously in Lemma 3.2. Furthermore, there exists a $\bar{v} \ne 0$ such that $\bar{v}_n \rightharpoonup \bar{v}$ in $H^1(\mathbb{R}^N)$ and $J'_\lambda(\bar{v}) = 0$.

Set $Q(x, v, \nabla v) = -\dfrac{G^{-1}(v)g'(G^{-1}(v))}{g(G^{-1}(v))}|\nabla v|^2$. Since $g'(v) \le 0$ for all $v \ge 0$, it is easy to see that $Q(x, v, \nabla v) \ge 0$. Moreover, we have $Q(x, v, \nabla v)$ is convex in ∇v and $\int_{\mathbb{R}^N} Q(x, v, \nabla v)dx$ is lower semi-continuous with respect to v by Theorem 1.6 in [17]. Then, from the lower semi-continuity of $\int_{\mathbb{R}^N} Q(x, v, \nabla v)dx$, Lemma 2.1, Lemma 4.1 and Fatou's Lemma we have

$$\begin{aligned}2\bar{d}_\lambda &= \lim_{n\to\infty}\left[2J_\lambda(\bar{v}_n) - \langle J'_\lambda(\bar{v}_n), G^{-1}(\bar{v}_n)g(G^{-1}(\bar{v}_n))\rangle\right] \\ &= -\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{G^{-1}(\bar{v}_n)g'(G^{-1}(\bar{v}_n))}{g(G^{-1}(\bar{v}_n))}|\nabla\bar{v}_n|^2 dx \\ &\quad -\lim_{n\to\infty}\lambda\int_{\mathbb{R}^N}\left(2\tilde{F}(G^{-1}(\bar{v}_n)) - \tilde{f}(G^{-1}(\bar{v}_n))G^{-1}(\bar{v}_n)\right)dx \\ &\ge -\int_{\mathbb{R}^N}\frac{G^{-1}(\bar{v})g'(G^{-1}(\bar{v}))}{g(G^{-1}(\bar{v}))}|\nabla\bar{v}|^2 dx \\ &\quad -\lambda\int_{\mathbb{R}^N}\left(2\tilde{F}(G^{-1}(\bar{v})) - \tilde{f}(G^{-1}(\bar{v}))G^{-1}(\bar{v})\right)dx \\ &= 2J_\lambda(\bar{v}) - \langle J'_\lambda(\bar{v}), G^{-1}(\bar{v})g(G^{-1}(\bar{v}))\rangle.\end{aligned} \tag{4.4}$$

Consequently, $\bar{v} \ne 0$ is a critical point of J_λ satisfying $J_\lambda(\bar{v}) \le \bar{d}_\lambda$. Setting

$$\Gamma := \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0,\ J_\lambda(\gamma(1)) < 0,\ \gamma(1) \ne 0\},$$

$$\bar{\Gamma} := \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0,\ \bar{J}_\lambda(\gamma(1)) \leq 0,\ \gamma(1) \neq 0\},$$

$$d_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)),$$

$$\bar{d}_\lambda = \inf_{\gamma\in\bar{\Gamma}} \max_{t\in[0,1]} \bar{J}_\lambda(\gamma(t)),$$

and using the similar arguments in [18], we get a specific path $\gamma : [0,1] \to H^1(\mathbb{R}^N)$ satisfying

$$\begin{cases} \gamma(0) = 0,\ J_\lambda(\gamma(1)) < 0,\ \bar{v} \in \gamma([0,1]), \\ \gamma(t)(x) > 0,\ \forall x \in \mathbb{R}^N,\ t \in (0,1], \\ \max\limits_{t\in[0,1]} J_\lambda(\gamma(t)) = J_\lambda(\bar{v}). \end{cases} \tag{4.5}$$

Then for the path given by (4.5), there holds $\gamma \in \Gamma \subset \bar{\Gamma}$. Since $m(x) > 0$ is strict on a subset of positive measures in $\mathbb{R}^N$ and $G^{-1}(t)$ is an odd function, we can arrive at

$$\bar{d}_\lambda \leq \max_{t\in[0,1]} \bar{J}_\lambda(\gamma(t)) = \bar{J}_\lambda(\gamma(\bar{t})) < J_\lambda(\gamma(\bar{t})) \leq \max_{t\in[0,1]} J_\lambda(\gamma(t)) = J_\lambda(\bar{v}) \leq \bar{d}_\lambda,$$

which is a contradiction. Therefore, the above arguments show that the critical point v of $\bar{J}_\lambda$ is nontrivial.

Furthermore, we repeat the same arguments used in Section 3 to verify the L^∞-estimates of v. Then, under the assumptions of Theorem 1.2 and the change of variable (2.4), we obtain a positive solution of the original Eq (1.2) for λ sufficiently large. □

5. Conclusions

In this paper, we investigated a class of quasilinear Schrödinger equations with supercritical growth on the nonlinearity $f(t)$. The nonlinearity $f(t)$ is continuous and only superlinear in a neighborhood of $t = 0$. We supposed the potentials $V(x)$ are periodic and asymptotically periodic. By using variational methods, truncation techniques and Moser iteration, we have shown that the Eq (1.2) has at least one positive solution for the periodic and asymptotically periodic potentials.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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