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Research article

C^*-algebra valued $\mathcal{R}$-metric space and fixed point theorems

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Abstract: In the present manuscript, notions of C^*-algebra valued $\mathcal{R}$-metric space and C^*-algebra valued $\mathcal{R}$-contractive map are introduced along with some fixed point results which in turn generalizes and unifies certain well known results in the existing literature. Further, in support of the obtained results some illustrative examples have been provided.

Keywords: C^*-algebra; $\mathcal{R}$-metric space; C^*-algebra valued $\mathcal{R}$-metric space; C^*-algebra valued $\mathcal{R}$-contractive map; fixed point
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1. Introduction

Ever since the introduction of fixed point theorem in metric space, and in particular Banach contraction principle [2], many theories have evolved in improvising, generalizing or extending its concept in numerous ways. Some of the initial attempts were made to overcome the shortcomings of Banach contraction principle, the most prevailing one being the self map in consideration has to be a continuous map in the given complete metric space. With fairly new contraction condition, Kannan in 1968 [14] made an attempt to overcome few of the shortcomings of Banach contraction principle. The approach was followed by Meir and Keeler in 1969 [18], Reich in 1971 [23], Chatterjea in 1972 [4], Zamfirescu in 1972 [26], Hardy and Rogers in 1973 [10], Ćirić in 1974 [5] amongst others (see [3, 8, 11–13, 16, 25] and references cited therein).

Ma et al. in 2014 [20] introduced the concept of C^*-algebra valued metric spaces and proved some of the fixed point results subjected to a fairly new contractions as well as an expansion condition that, over a period of time, has been generalized by many (see [6, 9, 21, 24] and references cited therein).

Alam and Imdad in 2014 [1] came up with the idea of relational theoretic contraction principle and proved some fixed point result. Recently, Khalehoghli et al. in 2020 [16] and Prasad in 2020 [22] combined the idea of a binary relation together with some weaker contractive conditions in a metric space in order to prove some of the fixed point theorems in this setting. The idea of binary relation has opened up an exceptionally bright research framework in the field of fixed point theory.

Derived by the work done in [1, 16, 20, 22], with the help of this manuscript, we introduce the notions of C^*-algebra valued $\mathcal{R}$-metric space, C^*-algebra valued $\mathcal{R}$-contractive map and investigate some of the related fixed point theorems with the help of relatively weaker condition for which only related points in the space are subjected to hold the contractive conditions. The main results proved further generalizes and integrates results present in the literature and the outcomes are well supported by examples.

2. Preliminaries

We begin this section by discussing few of the basic concepts of C^*-algebra valued metric space followed by some definitions of relational theoretic. Readers should note that, throughout the manuscript $\mathbb{A}$ denotes a unital C^*-algebra with the unit I, θ denotes the zero element of $\mathbb{A}$, $\mathbb{A}_+ = \{\alpha \in \mathbb{A} : \theta \preceq \alpha\}$ and $\mathcal{R}$ denotes a non-empty relation on a non-empty set X. Apart from it, certain basic notations such as real numbers ($\mathbb{R}$), integers ($\mathbb{Z}$) and natural numbers ($\mathbb{N}$) are used.

Definition 2.1. *[20] For a non-empty set X, let $d : X \times X \to \mathbb{A}$ be a map satisfying:*

(1) $\theta \preceq d(\rho, \nu)$;
(2) $d(\rho, \nu) = \theta$ *if and only if* $\rho = \nu$;
(3) $d(\rho, \nu) = d(\nu, \rho)$;
(4) $d(\rho, \nu) \preceq d(\rho, \zeta) + d(\zeta, \nu)$,

for any $\rho, \nu, \zeta \in X$. Then d is said to be a C^*-algebra valued metric and $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Definition 2.2. *[17] For a non-empty set X, a binary relation $\mathcal{R}$ on X is a subset of $X \times X$.*

Definition 2.3. *[1] For a binary relation $\mathcal{R}$ defined on a non-empty set X, we say ρ and ν in X are $\mathcal{R}$-comparative (denoted by $[\rho, \nu] \in \mathcal{R}$) if either $(\rho, \nu) \in \mathcal{R}$ or $(\nu, \rho) \in \mathcal{R}$.*

Definition 2.4. *[15] For a non-empty set X and a subset Y of X, a binary relation $\mathcal{R}$ on X is said to be restricted to Y (denoted by $\mathcal{R}|_Y$) when $\mathcal{R} = \mathcal{R} \cap Y^2$.*

Definition 2.5. *[16] For a binary relation $\mathcal{R}$ defined on a non-empty set X, a sequence $\{\rho_n\}_{n\in\mathbb{N}} \in X$ is said to be a $\mathcal{R}$-sequence if $(\rho_n, \rho_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}$.*

Definition 2.6. *[16] A metric space (X, d) together with a binary relation $\mathcal{R}$ is said to be a $\mathcal{R}$-metric space. It is usually written as $(X, d, \mathcal{R})$.*

Definition 2.7. *[16] For a $\mathcal{R}$-metric space $(X, d, \mathcal{R})$, a self map $g : X \to X$ is said to be $\mathcal{R}$-continuous at $\rho \in X$ if for any $\mathcal{R}$-sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset X$ with $\lim_{n\to+\infty} \rho_n = \rho$ implies $\lim_{n\to+\infty} g\rho_n = g\rho$. Also, g is said to be $\mathcal{R}$-continuous on X if it is $\mathcal{R}$-continuous at each point of X.*

Remark 2.1. *[16] Every continuous map is $\mathcal{R}$-continuous but not conversely.*

Example 2.1. *Consider $X = (-\infty, 0]$ with usual metric d. Define a binary relation $\mathcal{R}$ on X as $(\rho, \nu) \in \mathcal{R}$ if and only if $\rho^2 = \nu^2$. Then $(X, d, \mathcal{R})$ is a $\mathcal{R}$-metric space.*

Define a self map g on X such that

$$g(\rho) = \begin{cases} 0 & for\ \rho \in X \cap \mathbb{Z}; \\ -\rho^2 & otherwise. \end{cases}$$

Then g is $\mathcal{R}$-continuous map on X but it is discontinuous at every non-integer points of X.

Definition 2.8. *[16] A $\mathcal{R}$-metric space $(X, d, \mathcal{R})$ is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-Cauchy sequence in X is convergent.*

Definition 2.9. *[16] For a $\mathcal{R}$-metric space $(X, d, \mathcal{R})$, a self map $g : X \to X$ is said to be $\mathcal{R}$-preserving if for every $(\rho, \nu) \in \mathcal{R}$ implies $(g\rho, g\nu) \in \mathcal{R}$.*

Definition 2.10. *[1] For a $\mathcal{R}$-metric space $(X, d, \mathcal{R})$, a binary relation $\mathcal{R}$ is said to be d-self-closed on X if for any $\mathcal{R}$-sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to+\infty} \rho_n = \rho$ implies there exists a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ of $\{\rho_n\}_{n\in\mathbb{N}}$ such that $[\rho_{n_k}, \rho] \in \mathcal{R}$ for all $k \in \mathbb{N}$.*

Lemma 2.1. *[7, 19] If $\mathbb{A}$ is a unital C^*-algebra with unit I:*

(1) If $a \in \mathbb{A}_+$ with $\|a\| < 1/2$, then $I - a$ is invertible and $\|a(I - a)^{-1}\| < 1$.
(2) Suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$.
(3) If $a \in \mathbb{A}'$ and $b, c \in \mathbb{A}$ where $b \succeq c \succeq \theta$ and $I - a \in \mathbb{A}'_+$ is invertible operator, then

$$(I - a)^{-1}b \succeq (I - a)^{-1}c,$$

where $\mathbb{A}_+ = \{a \in \mathbb{A} : a \succeq \theta\}$ and $\mathbb{A}' = \{a \in \mathbb{A} : ab = ba \text{ for all } b \in \mathbb{A}\}$.

3. Main results

In this section, we prove some of the fixed point results for mappings in C^*-algebra valued $\mathcal{R}$-metric space endowed with different contraction conditions. The outcomes are supported by illustrative examples.

Definition 3.1. *For a non-empty set X together with a unital C^*-algebra $\mathbb{A}$ and binary relation $\mathcal{R}$, define $d : X \times X \to \mathbb{A}$. Then $(X, \mathbb{A}, d, \mathcal{R})$ is called C^*-algebra valued $\mathcal{R}$-metric space if following are satisfied:*

(1) $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space;
(2) $\mathcal{R}$ is a binary relation on X.

Definition 3.2. *For a C^*-algebra valued $\mathcal{R}$-metric space $(X, \mathbb{A}, d, \mathcal{R})$, a self map $g : X \to X$ is said to be C^*-algebra valued $\mathcal{R}$-contractive map if for all $\rho, \nu \in X$ with $(\rho, \nu) \in \mathcal{R}$, there exists an $\alpha \in \mathbb{A}$ where $\|\alpha\| < 1$ such that $d(g\rho, g\nu) \preceq \alpha^* d(\rho, \nu)\alpha$.*

Example 3.1. *Let $X = \mathbb{R}$, $\mathbb{A} = M_2(\mathbb{R})$ with involution on $\mathbb{A}$ defined as $A^* = A^t$ for all $A \in \mathbb{A}$, where A^t denotes the transpose of matrix A and zero element $\theta = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \hat{0}$. For $A = [a_{ij}]$, let $\|A\| = \max_{1\le i,j\le 2} |a_{ij}|$. Define $d : X \times X \to \mathbb{A}$ as*

$$d(\rho, \nu) = \begin{bmatrix} |\rho - \nu| & 0 \\ 0 & |\rho - \nu| \end{bmatrix}.$$

In such case, for $A = [a_{ij}], B = [b_{ij}] \in \mathbb{A}$ *we say* $A \preceq B$ *if and only if* $a_{ij} \leq b_{ij}$ *for all* $i, j = 1, 2$. *Then* $(X, \mathbb{A}, d)$ *is* C^**-algebra valued metric space. Let* $\mathcal{R}$ *be a binary relation on* X *defined as* $(\rho, \nu) \in \mathcal{R}$ *if and only if* $\rho.\nu = 0$ *such that* $(X, \mathbb{A}, d, \mathcal{R})$ *is a* C^**-algebra valued* $\mathcal{R}$*-metric space.*

Define a self map $g : X \to X$ *as*

$$g(\rho) = \begin{cases} 3/25 & for\ \rho \in \mathbb{N}; \\ 0 & otherwise. \end{cases}$$

Now, for $(\rho, \nu) \in \mathcal{R}$*, we must have either* ρ *or* ν *or both to be zero. Consider* $\nu = 0$*, then we have following cases:*

Case(i) *If* $\rho \in X - \mathbb{N}$. *Then we have* $g\rho = 0$ *and eventually* $d(g\rho, g\nu) = d(0, 0) = \hat{0}$. *For any* $A \in \mathbb{A}$ *with* $\|A\| < 1$, *we have* $A^* d(\rho, 0)A \succeq \hat{0}$ *and thus* $d(g\rho, g\nu) \preceq A^* d(\rho, \nu)A$.

Case(ii) *If* $\rho \in \mathbb{N}$. *Then*

$$d(g\rho, g\nu) = d(3/25, 0) = \begin{bmatrix} 3/25 & 0 \\ 0 & 3/25 \end{bmatrix}, \tag{3.1}$$

and for $A = [\begin{smallmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{smallmatrix}]$, *we have*

$$A^* d(\rho, \nu)A = A^t d(\rho, \nu)A = \begin{bmatrix} \rho/5 & 0 \\ 0 & \rho/5 \end{bmatrix}. \tag{3.2}$$

Thus, from Eqs (3.1) and (3.2), we obtain $d(g\rho, g\nu) \preceq A^* d(\rho, \nu)A$ *for any* $\rho \in \mathbb{N}$. *The case when* $\rho = 0$ *can be proved in similar manner as above.*

Hence, g *is a* C^**-algebra valued* $\mathcal{R}$*-contractive map.*

Example 3.2. *Let* $X = [0, 1]$, $\mathbb{A} = M_2(\mathbb{C})$ *with involution on* $\mathbb{A}$ *defined as* $A^* = A^H$ *for all* $A \in \mathbb{A}$, *where* A^H *denotes the conjugate transpose of matrix* A *and zero element* $\theta = [\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}] = \hat{0}$. *For* $A = [a_{ij}]$, *let* $\|A\| = \max_{1 \leq i, j \leq 2} |a_{ij}|$. *Define* $d : X \times X \to \mathbb{A}$ *as*

$$d(\rho, \nu) = \begin{bmatrix} |\rho - \nu|^\zeta & 0 \\ 0 & |\rho - \nu|^\zeta \end{bmatrix}, where\ \zeta \geq 1.$$

In such case, for $A = [a_{ij}], B = [b_{ij}] \in \mathbb{A}$, *we say* $A \preceq B$ *if and only if* $|a_{ij}| \leq |b_{ij}|$ *for all* $i, j = 1, 2$. *Then* $(X, \mathbb{A}, d)$ *is* C^**-algebra valued metric space. Let* $\mathcal{R}$ *be a binary relation on* X *defined as* $(\rho, \nu) \in \mathcal{R}$ *if and only if* $\rho.\nu = 0$ *such that* $(X, \mathbb{A}, d, \mathcal{R})$ *is a* C^**-algebra valued* $\mathcal{R}$*-metric space. Define a self map* $g : X \to X$ *as*

$$g(\rho) = \begin{cases} \rho/4 & for\ \rho \in X \cap \mathbb{Q}; \\ 0 & otherwise. \end{cases}$$

Now, for $(\rho, \nu) \in \mathcal{R}$*, we must have either* ρ *or* ν *or both to be zero. Consider* $\nu = 0$*, then we have the following cases:*

Case(i) *If* $\rho \in X - \mathbb{Q}$. *Then we have* $d(g\rho, g\nu) = d(0, 0) = \hat{0}$. *For any* $A \in \mathbb{A}$ *with* $\|A\| < 1$, *we have* $\hat{0} \preceq A^* d(\rho, 0)A$ *and thus* $d(g\rho, g\nu) \preceq A^* d(\rho, \nu)A$.

Case(ii) *If $\rho \in X \cap \mathbb{Q}$. Then*

$$d(g\rho, g\nu) = d(\rho/4, 0) = \begin{bmatrix} (\rho/4)^{\zeta} & 0 \\ 0 & (\rho/4)^{\zeta} \end{bmatrix}, \tag{3.3}$$

and for $A = [\begin{smallmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{smallmatrix}]$, we have

$$A^{*}d(\rho, \nu)A = A^{H}d(\rho, \nu)A = \begin{bmatrix} \rho^{\zeta}/2 & 0 \\ 0 & \rho^{\zeta}/2 \end{bmatrix}. \tag{3.4}$$

Thus, from Eqs (3.3) and (3.4), we obtain $d(g\rho, g\nu) \preceq A^{*}d(\rho, \nu)A$ for any $\rho \in X \cap \mathbb{Q}$. The case when $\rho = 0$ can be proved in similar manner as above.

Hence, g is a C^{*}-algebra valued $\mathcal{R}$-contractive map.

Definition 3.3. *For a C^{*}-algebra valued $\mathcal{R}$-metric space $(X, \mathbb{A}, d, \mathcal{R})$, a $\mathcal{R}$-sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset X$ is said to converge to $\rho \in X$ if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|d(\rho_n, \rho)\| \leq \epsilon$ for all $n \geq n_0$.*

Definition 3.4. *For a C^{*}-algebra valued $\mathcal{R}$-metric space $(X, \mathbb{A}, d, \mathcal{R})$, a $\mathcal{R}$-sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset X$ is said to be $\mathcal{R}$-Cauchy if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|d(\rho_n, \rho_m)\| \leq \epsilon$ for all $n, m \geq n_0$.*

Definition 3.5. *A C^{*}-algebra valued $\mathcal{R}$-metric space $(X, \mathbb{A}, d, \mathcal{R})$ is said to be complete C^{*}-algebra valued $\mathcal{R}$-metric space if every $\mathcal{R}$-Cauchy sequence with respect to $\mathbb{A}$ is convergent.*

Definition 3.6. *For a C^{*}-algebra valued $\mathcal{R}$-metric space $(X, \mathbb{A}, d, \mathcal{R})$, a subset Y of X is said to be a complete C^{*}-algebra valued $\mathcal{R}$-subspace if $(Y, \mathbb{A}, d, \mathcal{R})$ is a complete C^{*}-algebra valued $\mathcal{R}$-metric space.*

Theorem 3.1. *Let $(X, \mathbb{A}, d, \mathcal{R})$ be a C^{*}-algebra valued $\mathcal{R}$-metric space and let Y be a complete C^{*}-algebra valued $\mathcal{R}$-subspace of X. If $g : X \to X$ is a self map on X such that:*

(I) $g(X) \subseteq Y$;
(II) g is $\mathcal{R}$-preserving;
(III) There exists some $\rho_0 \in X$ such that $(\rho_0, \nu) \in \mathcal{R}$ for all $\nu \in g(X)$;
(IV) g is C^{*}-algebra valued $\mathcal{R}$-contractive map;
(V) Either g is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self closed on Y.

Then g possesses a unique fixed point.

Proof. Define a sequence $\{\rho_n\}_{n\in\mathbb{N}\cup\{0\}}$ in X such that $\rho_1 = g\rho_0$, $\rho_{n+1} = g^{n}\rho_0 = g\rho_n$ for all $n \in \mathbb{N}$. By condition *(III)* for some $\rho_0 \in X$, $(\rho_0, g\rho_0) \in \mathcal{R}$, that is $(\rho_0, \rho_1) \in \mathcal{R}$.

Since g is $\mathcal{R}$-preserving, so we have $(g\rho_0, g\rho_1) = (\rho_1, \rho_2) \in \mathcal{R}$. On continuous use of $\mathcal{R}$-preserving property of g, we get $(\rho_n, \rho_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\{\rho_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\mathcal{R}$-sequence in X.

Next, by using condition *(IV)*, we obtain

$$\begin{aligned} d(\rho_{n+1}, \rho_n) &= d(g\rho_n, g\rho_{n-1}) \\ &\preceq \alpha^{*}d(\rho_n, \rho_{n-1})\alpha \\ &= \alpha^{*}d(g\rho_{n-1}, g\rho_{n-2})\alpha \\ &\preceq (\alpha^{*})^{2}d(\rho_{n-1}, \rho_{n-2})\alpha^{2} \\ &\preceq \ldots \preceq (\alpha^{*})^{n}d(\rho_1, \rho_0)\alpha^{n} = (\alpha^{*})^{n}\beta\alpha^{n}, \end{aligned} \tag{3.5}$$

where $\beta = d(\rho_1, \rho_0)$ and $\alpha \in \mathbb{A}$ with $\|\alpha\| < 1$.

Let $n > m$, for $n, m \in \mathbb{N} \cup \{0\}$, and using triangle inequality along with (3.5), we get

$$\begin{aligned} d(\rho_{n+1},\rho_m) &\preceq d(\rho_{n+1},\rho_n)+...+d(\rho_{m+2},\rho_{m+1})+d(\rho_{m+1},\rho_m)\\ &\preceq \sum_{\xi=m}^{n}(\alpha^*)^{\xi}\beta\alpha^{\xi}=\sum_{\xi=m}^{n}(\alpha^{\xi})^*\beta^{1/2}\beta^{1/2}\alpha^{\xi}\\ &= \sum_{\xi=m}^{n}(\beta^{1/2}\alpha^{\xi})^*(\beta^{1/2}\alpha^{\xi})=\sum_{\xi=m}^{n}|\beta^{1/2}\alpha^{\xi}|^2\\ &\preceq \|\sum_{\xi=m}^{n}|\beta^{1/2}\alpha^{\xi}|^2\|I\\ &\preceq \sum_{\xi=m}^{n}\|\beta^{1/2}\|^2\|\alpha^{\xi}\|^2 I\preceq \|\beta^{1/2}\|^2\sum_{\xi=m}^{+\infty}\|\alpha\|^{2\xi}I\\ &= \|\beta^{1/2}\|^2\frac{\|\alpha\|^{2m}}{(1-\|\alpha\|)}I\to\theta \quad \text{as} \quad m\to+\infty. \end{aligned}$$

Thus, $\{\rho_n = g\rho_{n-1}\}_{n\in\mathbb{N}}$ is $\mathcal{R}$-Cauchy sequence in Y and since Y is a complete C^*-algebra valued $\mathcal{R}$-subspace of X, so there exists $\rho \in Y \subset X$ such that $\lim_{n\to+\infty}\rho_n = \rho$.

Case(i) Consider g be a $\mathcal{R}$-continuous map. Since $\{\rho_n\}_{n\in\mathbb{N}\cup\{0\}}$ is $\mathcal{R}$-sequence with $\lim_{n\to+\infty}\rho_n = \rho$. Then

$$\rho = \lim_{n\to+\infty}\rho_{n+1} = \lim_{n\to+\infty} g\rho_n = g\rho.$$

Thus, g possesses a fixed point.

Case(ii) Consider $\mathcal{R}$ be d-self closed on Y. Since $\{\rho_n\}_{n\in\mathbb{N}\cup\{0\}}$ is $\mathcal{R}$-sequence such that $\rho_n \to \rho$ as $n \to +\infty$. Then there exists a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ of $\{\rho_n\}_{n\in\mathbb{N}}$ such that $[\rho_{n_k},\rho] \in \mathcal{R}|_Y$.

Now,

$$d(\rho_{n_k+1}, g\rho) = d(g\rho_{n_k}, g\rho) \preceq \alpha^* d(\rho_{n_k},\rho)\alpha \to \theta \quad \text{as} \quad k\to+\infty.$$

Therefore, $\rho_{n_k} \to g\rho$ as $k \to +\infty$ and by uniqueness of limit, we have $\rho = g\rho$. Thus, g possesses a fixed point.

For the uniqueness of the fixed point, let ν be another fixed point of g in X, that is, $g\nu = \nu$ infact $g^n\nu = \nu$. By condition *(III)*, there exists $\rho_0 \in X$ such that $(\rho_0,\nu) = (\rho_0, g\nu) \in \mathcal{R}$. Since g is $\mathcal{R}$-preserving, so $(g\rho_0, g\nu) \in \mathcal{R}$ implies $(g^n\rho_0, g^n\nu) \in \mathcal{R}$. On using contractive condition of g, we have

$$\begin{aligned} d(\rho_n,\nu) = d(g^n\rho_0, g^n\nu) &\preceq \alpha^* d(g^{n-1}\rho_0, g^{n-1}\nu)\alpha\\ &\preceq (\alpha^*)^2 d(g^{n-2}\rho_0, g^{n-2}\nu)(\alpha)^2\\ &\preceq ... \preceq (\alpha^*)^n d(\rho_0,\nu)(\alpha)^n. \end{aligned} \tag{3.6}$$

Taking limit as $n \to +\infty$ in (3.6), we get

$$d(\rho,\nu) = \theta.$$

Hence, g possesses a unique fixed point. □

Corollary 3.1. *Let $(X, \mathbb{A}, d, \mathcal{R})$ be a complete C^*-algebra valued $\mathcal{R}$-metric space. If $g : X \to X$ is a self map on X such that:*

(I) g *is* $\mathcal{R}$*-preserving;*
(II) There exists some $\rho_0 \in X$ *such that* $(\rho_0, \nu) \in \mathcal{R}$ *for all* $\nu \in g(X)$*;*
(III) g *is* C^**-algebra valued* $\mathcal{R}$*-contractive map;*
(IV) Either g *is* $\mathcal{R}$*-continuous or* $\mathcal{R}$ *is* d*-self closed.*

Then g *possesses a unique fixed point.*

Proof. In Theorem 3.1 if we take $Y = X$, then the result follows. □

Example 3.3. *Consider the* C^**-algebra valued* $\mathcal{R}$*-metric space as discussed in Example 3.2, where the defined self map* g *on* X *is a* C^**-algebra valued* $\mathcal{R}$*-contractive map and* $(X, \mathbb{A}, d, \mathcal{R})$ *is a complete* C^**-algebra valued* $\mathcal{R}$*-metric space. Also, there exists* $\rho_0 = 0 \in X$ *such that* $(\rho_0, g\rho_0) \in \mathcal{R}$*. Further,* g *is* $\mathcal{R}$*-preserving (since for any* $(\rho, \nu) \in \mathcal{R}$ *implies* $\rho = 0$ *or/and* $\nu = 0$ *implies* $(g\rho, g\nu) \in \mathcal{R}$*) and for any convergent* $\mathcal{R}$*-sequence* $\{\rho_n\}_{n\in\mathbb{N}}$ *we must have* $\lim_{n\to+\infty} \rho_n = 0$ *and so is* $\lim_{n\to+\infty} g\rho_n = 0 = g0$*. Thus,* g *is a* $\mathcal{R}$*-continuous map. Therefore, by Corollary 3.1,* g *possesses a fixed point viz.* $\rho = 0$*.*

The upcoming theorem proves an analogues result for the Kannan contractive condition [14] endowed with a binary relation $\mathcal{R}$ under similar setting.

Theorem 3.2. *Let* $(X, \mathbb{A}, d, \mathcal{R})$ *be a* C^**-algebra valued* $\mathcal{R}$*-metric space and let* Y *be a complete* C^**-algebra valued* $\mathcal{R}$*-subspace of* X*. If* $g : X \to X$ *is a self map on* X *such that:*

(I) $g(X) \subseteq Y$*;*
(II) g *is* $\mathcal{R}$*-preserving;*
(III) There exists some $\rho_0 \in X$ *such that* $(\rho_0, \nu) \in \mathcal{R}$ *for all* $\nu \in g(X)$*;*
(IV) For all $\rho, \nu \in X$ *with* $(\rho, \nu) \in \mathcal{R}$*, there exists an* $\alpha \in \mathbb{A}'_+$*, where* $\|\alpha\| < 1/2$ *such that*

$$d(g\rho, g\nu) \preceq \alpha(d(g\rho, \rho) + d(g\nu, \nu));$$

(V) Either g *is* $\mathcal{R}$*-continuous or* $\mathcal{R}$ *is* d*-self closed on* Y*.*

Then g *possesses a unique fixed point.*

Proof. Working on the lines of Theorem 3.1, we obtain a $\mathcal{R}$-sequence $\{\rho_n\}_{n\in\mathbb{N}\cup\{0\}}$ in X such that $(\rho_n, \rho_{n+1}) \in \mathcal{R}$ for $n \in \mathbb{N}$.

Using condition *(IV)*, we get

$$\begin{aligned} d(\rho_{n+1}, \rho_n) = d(g\rho_n, g\rho_{n-1}) &\preceq \alpha(d(g\rho_n, \rho_n) + d(g\rho_{n-1}, \rho_{n-1})) \\ &= \alpha(d(\rho_{n+1}, \rho_n) + d(\rho_n, \rho_{n-1})), \\ \text{therefore,} \quad (I - \alpha)d(\rho_{n+1}, \rho_n) &\preceq \alpha d(\rho_n, \rho_{n-1}). \end{aligned}$$

Now, $\alpha \in \mathbb{A}'_+$ and $\|\alpha\| < 1/2$. Thus, by Lemma 2.1, $(I - \alpha)$ and $\alpha(I - \alpha)^{-1} \in \mathbb{A}'_+$ with $\|\alpha(I - \alpha)^{-1}\| < 1$, so we have

$$\begin{aligned} d(\rho_{n+1}, \rho_n) &\preceq \alpha(I - \alpha)^{-1} d(\rho_n, \rho_{n-1}), \\ &= bd(\rho_n, \rho_{n-1}) \\ &\preceq \ldots \preceq b^n d(\rho_1, \rho_0) = b^n \beta, \end{aligned} \tag{3.7}$$

where $b = \alpha(I - \alpha)^{-1}$ and $\beta = d(\rho_1, \rho_0)$.

Let $n > m$, for $n, m \in \mathbb{N}$, and using triangle inequality along with (3.7), we get

$$\begin{aligned} d(\rho_m, \rho_{n+1}) &\leq d(\rho_m, \rho_{m+1}) + d(\rho_{m+1}, \rho_{m+2}) + ... + d(\rho_n, \rho_{n+1}) \\ &= \sum_{\xi=m}^{n} b^{\xi}\beta = \sum_{\xi=m}^{n} b^{\xi/2}b^{\xi/2}\beta^{1/2}\beta^{1/2} = \sum_{\xi=m}^{n}(b^{\xi/2}\beta^{1/2})^*(b^{\xi/2}\beta^{1/2}) \\ &= \sum_{\xi=m}^{n} |b^{\xi/2}\beta^{1/2}|^2 \leq \|\sum_{\xi=m}^{n} |b^{\xi/2}\beta^{1/2}|^2\| I \\ &\leq \|\beta^{1/2}\|^2 \sum_{\xi=m}^{+\infty} \|b\|^{\xi} I \\ &= \|\beta^{1/2}\|^2 \frac{\|b\|^m}{1 - \|b\|} I \to \theta \quad \text{as} \quad m \to +\infty. \end{aligned}$$

Thus, $\{\rho_n = g\rho_{n-1}\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\mathcal{R}$-Cauchy sequence in Y and since Y is complete C^*-algebra valued $\mathcal{R}$-subspace of X so, there exists $\rho \in Y \subset X$ such that $\lim_{n\to+\infty} \rho_n = \rho$.

Case(i) Consider g be a $\mathcal{R}$-continuous map. Then

$$\rho = \lim_{n\to+\infty} \rho_{n+1} = \lim_{n\to+\infty} g\rho_n = g\rho.$$

Thus, g possesses a fixed point.

Case(ii) Consider $\mathcal{R}$ be d-self closed on Y. Since $\{\rho_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\mathcal{R}$-sequence such that $\rho_n \to \rho$ as $n \to +\infty$. Then there exists a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ of $\{\rho_n\}_{n\in\mathbb{N}}$ such that $[\rho_{n_k}, \rho] \in \mathcal{R}|_Y$. Now,

$$d(\rho_{n_k+1}, g\rho) = d(g\rho_{n_k}, g\rho) \leq \alpha(d(g\rho_{n_k}, \rho_{n_k}) + d(g\rho, \rho)).$$

Taking norm on both sides, we get

$$\|d(\rho_{n_k+1}, g\rho)\| \leq \|\alpha\|(\|d(g\rho_{n_k}, \rho_{n_k})\| + \|d(g\rho, \rho)\|) = \|\alpha\|(\|d(\rho_{n_{k+1}}, \rho_{n_k})\| + \|d(g\rho, \rho)\|).$$

Taking limit as $k \to +\infty$ on both sides, we get

$$\|d(\rho, g\rho)\| \leq \|\alpha\|\|d(g\rho, \rho)\|.$$

For $\|\alpha\| < 1/2$, above holds only when $\|d(g\rho, \rho)\| = 0$. Thus, g possesses a fixed point.

Next, if ν is another fixed point of g in X, that is, $g\nu = \nu$ infact $g^n\nu = \nu$. By condition *(III)*, there exists $\rho_0 \in X$ such that $(\rho_0, \nu) = (\rho_0, g\nu) \in \mathcal{R}$. Since g is $\mathcal{R}$-preserving, so $(g\rho_0, g\nu) \in \mathcal{R}$ implies $(g^n\rho_0, g^n\nu) \in \mathcal{R}$ for $n \in \mathbb{N}$. On using condition *(IV)*, we have

$$\begin{aligned} d(\rho_n, \nu) = d(g^n\rho_0, g^n\nu) &\leq \alpha(d(g^n\rho_0, g^{n-1}\rho_0) + d(g^n\nu, g^{n-1}\nu)) \\ &= \alpha d(\rho_n, \rho_{n-1}) \leq ... \leq \alpha^n d(\rho_1, \rho_0). \end{aligned} \tag{3.8}$$

On taking limit as $n \to +\infty$ in (3.8), we get

$$d(\rho, \nu) = \theta.$$

Hence, g possesses a unique fixed point. □

Corollary 3.2. *Let $(X, \mathbb{A}, d, \mathcal{R})$ be a complete C^*-algebra valued $\mathcal{R}$-metric space. If $g : X \to X$ is a self map on X such that:*

(I) g is $\mathcal{R}$-preserving;
(II) There exists some $\rho_0 \in X$ such that $(\rho_0, \nu) \in \mathcal{R}$ for all $\nu \in g(X)$;
(III) For all $\rho, \nu \in X$ with $(\rho, \nu) \in \mathcal{R}$, there exists an $\alpha \in \mathbb{A}'_+$, where $\|\alpha\| < 1/2$ such that

$$d(g\rho, g\nu) \preceq \alpha(d(g\rho, \rho) + d(g\nu, \nu));$$

(IV) Either g is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self closed.

Then g possesses a unique fixed point.

Proof. In Theorem 3.2 if we take $Y = X$, then the result follows. □

In the next theorem, we establish the $\mathcal{R}$ analog of the Chatterjea contractive condition [4] for a C^*-algebra valued $\mathcal{R}$-metric space.

Theorem 3.3. *Let $(X, \mathbb{A}, d, \mathcal{R})$ be a C^*-algebra valued $\mathcal{R}$-metric space and let Y be a complete C^*-algebra valued $\mathcal{R}$-subspace of X. If $g : X \to X$ is a self map on X such that:*

(I) $g(X) \subseteq Y$;
(II) g is $\mathcal{R}$-preserving;
(III) There exists some $\rho_0 \in X$ such that $(\rho_0, \nu) \in \mathcal{R}$ for all $\nu \in g(X)$;
(IV) For all $\rho, \nu \in X$ with $(\rho, \nu) \in \mathcal{R}$, there exists an $\alpha \in \mathbb{A}'_+$, where $\|\alpha\| < 1/2$ such that

$$d(g\rho, g\nu) \preceq \alpha(d(g\rho, \nu) + d(g\nu, \rho));$$

(V) Either g is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self closed on Y.

Then g possesses a unique fixed point.

Proof. Working on the lines of Theorem 3.1, we obtain a $\mathcal{R}$-sequence $\{\rho_n\}_{n\in\mathbb{N}\cup\{0\}}$ in X such that $(\rho_n, \rho_{n+1}) \in \mathcal{R}$ for $n \in \mathbb{N}$. Using condition *(IV)*, we get

$$\begin{aligned} d(\rho_{n+1}, \rho_n) &= d(g\rho_n, g\rho_{n-1}) \\ &\preceq \alpha(d(g\rho_n, \rho_{n-1}) + d(g\rho_{n-1}, \rho_n)) = \alpha d(g\rho_n, \rho_{n-1}) \\ &= \alpha d(g\rho_n, g\rho_{n-2}) \\ &\preceq \alpha(d(g\rho_n, g\rho_{n-1}) + d(g\rho_{n-1}, g\rho_{n-2})) \\ &\preceq \alpha(d(\rho_{n+1}, \rho_n) + d(\rho_n, \rho_{n-1})), \\ \text{therefore,} \quad (I - \alpha)d(\rho_{n+1}, \rho_n) &\preceq \alpha d(\rho_n, \rho_{n-1}). \end{aligned}$$

Now, $\alpha \in \mathbb{A}'_+$ and $\|\alpha\| < 1/2$. Thus, by Lemma 2.1, $(I - \alpha)$ and $\alpha(I - \alpha)^{-1} \in \mathbb{A}'_+$ with $\|\alpha(I - \alpha)^{-1}\| < 1$, so we have

$$\begin{aligned} d(\rho_{n+1}, \rho_n) &\preceq \alpha(I - \alpha)^{-1} d(\rho_n, \rho_{n-1}) \\ &= bd(\rho_n, \rho_{n-1}) \preceq ... \preceq b^n d(\rho_1, \rho_0) = b^n \beta, \end{aligned}$$

where $b = \alpha(I-\alpha)^{-1}$ and $\beta = d(\rho_1, \rho_0)$.

By the working done in Theorem 3.2, we obtain that $\{\rho_n = g\rho_{n-1}\}_{n\in\mathbb{N}\cup\{0\}}$ is a $\mathcal{R}$-Cauchy sequence in Y, and since Y is complete C^*-algebra valued $\mathcal{R}$-subspace of X, so there exists $\rho \in Y \subset X$ such that $\lim_{n\to+\infty} \rho_n = \rho$.

Case(i) Consider g be a $\mathcal{R}$-continuous map. Then

$$\rho = \lim_{n\to+\infty} \rho_{n+1} = \lim_{n\to+\infty} g\rho_n = g\rho.$$

Thus, g possesses a fixed point.

Case(ii) Consider $\mathcal{R}$ be d-self closed on Y. Since $\{\rho_n\}_{n\in\mathbb{N}\cup\{0\}}$ is $\mathcal{R}$-sequence such that $\rho_n \to \rho$ as $n \to +\infty$. Then there exists a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ of $\{\rho_n\}_{n\in\mathbb{N}}$ such that $[\rho_{n_k}, \rho] \in \mathcal{R}|_Y$. Now,

$$d(\rho_{n_k+1}, g\rho) = d(g\rho_{n_k}, g\rho) \preceq \alpha(d(g\rho_{n_k}, \rho) + d(g\rho, \rho_{n_k})).$$

Taking norm on both sides, we get

$$\|d(\rho_{n_k+1}, g\rho)\| \leq \|\alpha\|(\|d(g\rho_{n_k}, \rho)\| + \|d(g\rho, \rho_{n_k})\|) = \|\alpha\|(\|d(\rho_{n_{k+1}}, \rho)\| + \|d(g\rho, \rho_{n_k})\|).$$

Taking limit as $k \to +\infty$ on both sides, we get

$$\|d(\rho, g\rho)\| \leq \|\alpha\|\|d(g\rho, \rho)\|.$$

For $\|\alpha\| < 1/2$, above holds only when $\|d(g\rho, \rho)\| = 0$. Thus, g possesses a fixed point.

Next, if ν is another fixed point of g in X, that is, $g\nu = \nu$ infact $g^n\nu = \nu$. By condition *(III)*, there exists $\rho_0 \in X$ such that $(\rho_0, \nu) = (\rho_0, g\nu) \in \mathcal{R}$. Since g is $\mathcal{R}$-preserving, so $(g\rho_0, g\nu) \in \mathcal{R}$ implies $(g^n\rho_0, g^n\nu) \in \mathcal{R}$ for $n \in \mathbb{N}$. On using condition *(IV)*, we have

$$\begin{aligned}
d(\rho_n, \nu) = d(g^n\rho_0, g^n\nu) &\preceq \alpha(d(g^n\rho_0, g^{n-1}\nu) + d(g^n\nu, g^{n-1}\rho_0)) \\
&= \alpha(d(\rho_n, \nu) + d(\nu, \rho_{n-1})), \\
\text{therefore,} \quad (I-\alpha)d(\rho_n, \nu) &\preceq \alpha d(\nu, \rho_{n-1}) \\
d(\rho_n, \nu) &\preceq \frac{\alpha}{(I-\alpha)} d(\nu, \rho_{n-1}) \\
&\preceq \ldots \preceq \frac{\alpha^n}{(I-\alpha)^n} d(\nu, \rho_0). \qquad (3.9)
\end{aligned}$$

Using Lemma 2.1 and taking limit as $n \to +\infty$ in (3.9), we get

$$d(\rho, \nu) = \theta.$$

Hence, g possesses a unique fixed point. □

Corollary 3.3. *Let $(X, \mathbb{A}, d, \mathcal{R})$ be a complete C^*-algebra valued $\mathcal{R}$-metric space. If $g : X \to X$ is a self map on X such that:*

(I) g is $\mathcal{R}$-preserving;
(II) There exists some $\rho_0 \in X$ such that $(\rho_0, \nu) \in \mathcal{R}$ for all $\nu \in g(X)$;

(III) For all $\rho, \nu \in X$ with $(\rho, \nu) \in \mathcal{R}$, there exists an $\alpha \in \mathbb{A}'_+$, where $\|\alpha\| < 1/2$ such that

$$d(g\rho, g\nu) \preceq \alpha(d(g\rho, \nu) + d(g\nu, \rho));$$

(IV) Either g is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self closed.

Then g possesses a unique fixed point.

Proof. In Theorem 3.3 if we take $Y = X$, then the result follows. □

Example 3.4. *Consider $X = [0, 1)$ equipped with usual metric and let the unital C^*-algebra valued metric space $\mathbb{A} = (-\infty, +\infty)$ together with $\|\alpha\| = |\alpha|$, for $\alpha, \gamma \in \mathbb{A}$ $\alpha \preceq \gamma$ if and only if $\alpha \leq \gamma$ and involution given by $\alpha^* = \alpha$. Define relation $\mathcal{R}$ on X as $(\rho, \nu) \in \mathcal{R}$ if and only if $\rho.\nu \in \{\rho, \nu\}$ and let $g : X \to X$ be defined as*

$$g(\rho) = \begin{cases} 0 & for\ \rho \in [0, 2/9]; \\ 1/9 & for\ \rho \in (2/9, 1), \end{cases}$$

then X is a complete C^*-algebra valued $\mathcal{R}$-metric space (since every $\mathcal{R}$-Cauchy sequence in X is convergent). Next, for $(\rho, \nu) \in \mathcal{R}$ then we have either $\rho = 0$ or/and $\nu = 0$. Let us consider $\nu = 0$ then following cases arise:

Case(i) *If $\rho \in [0, 2/9]$, then*

$$d(g\rho, g\nu) = d(0, 0) = 0 \tag{3.10}$$

and

$$d(\rho, \nu) = d(\rho, 0) = \rho. \tag{3.11}$$

Then for any $\alpha \in \mathbb{A}$ with $\|\alpha\| < 1$, from (3.10) and (3.11), we obtain

$$d(g\rho, g\nu) \preceq \alpha^* d(\rho, \nu)\alpha.$$

Case(ii) *If $\rho \in (2/9, 1)$, then*

$$d(g\rho, g\nu) = d(1/9, 0) = \frac{1}{9} \tag{3.12}$$

and

$$d(\rho, \nu) = d(\rho, 0) = \rho. \tag{3.13}$$

Thus, for $\alpha = \frac{1}{\sqrt{2}}$ and from (3.12) and (3.13), we obtain

$$d(g\rho, g\nu) \preceq \alpha^* d(\rho, \nu)\alpha.$$

Thus, g is C^*-algebra valued $\mathcal{R}$-contractive map. Also, g is $\mathcal{R}$-preserving and $\mathcal{R}$-continuous. Thus, by Corollary 3.1, we conclude that g possesses a unique fixed point which in this case is $\rho = 0$.

Note that the metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorem 2.1 of [20] and Banach contraction principle [2].

Example 3.5. *Let* $X = \{0,1,2,3,4\}$ *and* $\mathbb{A} = M_2(\mathbb{R})$ *with* $A^* = A$ *for all* $A \in \mathbb{A}$ *and* $\|A\| = \max_{1\leq i,j\leq 2}|a_{i,j}|$. *Let* $\theta = \hat{0} = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$. *Define* $d : X\times X \to \mathbb{A}$ *as*

$$d(\rho,\nu) = \begin{cases} \hat{0} & for\ \rho = \nu; \\ I & otherwise, \end{cases}$$

where I *denotes an identity matrix of order 2. Then* $(X,\mathbb{A},d)$ *is a* C^**-algebra valued metric space. Let* $\mathcal{R} = \{(0,0),(0,1),(1,1),(2,2),(2,4),(3,2),(3,4),(4,4)\}$, *then* $(X,\mathbb{A},d,\mathcal{R})$ *is a complete* C^**-algebra valued* $\mathcal{R}$*-metric space (since any* $\mathcal{R}$*-Cauchy sequence* $\{\rho_n\}_{n\in\mathbb{N}}$ *is convergent in* X*).*

Define a self map $g : X \to X$ *as* $g(0) = 0 = g(1)$, $g(2) = 1 = g(3) = g(4)$. *Then* g *is a* $\mathcal{R}$*-preserving map (since for any* $(\rho,\nu) \in \mathcal{R}$, $(g\rho, g\nu) = (0,0)$ *or* $(1,1)$*). Also, there is* $0 \in X$ *such that* $(0, g\nu) \in \mathcal{R}$ *for all* $\nu \in X$ *and* g *is* $\mathcal{R}$*-continuous as for any* $\mathcal{R}$*-sequence* $\{\rho_n\}_{n\in\mathbb{N}}$ *with* $\rho_n \to \rho$, *we have* $g\rho_n$ *converge to either* 0 *or* 1. *Next, for a given* $A \in \mathbb{A}$ *with* $0 < \|A\| < 1/2$, *we have*

$$d(g0, g4) = d(0,1) = I > A(d(g0,0) + d(g4,4)) = A(\hat{0} + I) = AI.$$

Here, g satisfies all the conditions of Corollary 3.2 proved in this manuscript and hence, g possesses a unique fixed point viz. $\rho = 0$.

However, the readers should note that the given self map does not satisfy Kannan contraction condition given in [14].

Example 3.6. *Consider* $X = [0,2)$ *equipped with usual metric and let the unital* C^**-algebra valued metric space* $\mathbb{A} = (-\infty,+\infty)$ *together with* $\|\alpha\| = |\alpha|$, *for* $\alpha,\gamma \in \mathbb{A}$, $\alpha \preceq \gamma$ *if and only if* $\alpha \leq \gamma$ *and an involution given by* $\alpha^* = \alpha$. *Define relation* $\mathcal{R}$ *on* X *as* $(\rho,\nu) \in \mathcal{R}$ *if and only if* $\rho.\nu \in \{0\}$ *and let* $g : X \to X$ *be defined as*

$$g(\rho) = \begin{cases} \frac{\rho^2}{6} & for\ \rho \in [0,1); \\ 0 & otherwise. \end{cases}$$

Then $(X,\mathbb{A},d,\mathcal{R})$ *is a complete* C^**-algebra valued* $\mathcal{R}$*-metric space. For* $(\rho,\nu) \in \mathcal{R}$ *we have* $\rho = 0$ *or/and* $\nu = 0$. *Let* $\nu = 0$, *then we have following cases:*

Case(i) *If* $\rho \in [0,1)$, *then*

$$d(g\rho, g\nu) = d(\frac{\rho^2}{6}, 0) = \frac{\rho^2}{6} \tag{3.14}$$

and

$$\alpha(d(g\rho,\nu) + d(g\nu,\rho)) = \alpha(d(\frac{\rho^2}{6},0) + d(0,\rho)) = \alpha(\frac{\rho^2}{6} + \rho). \tag{3.15}$$

Then for $\alpha = 1/3$ *and from (3.14) and (3.15), we obtain*

$$d(g\rho, g\nu) \preceq \alpha(d(g\rho,\nu) + d(g\nu,\rho)).$$

Case(ii) *If* $\rho \in [1,2)$, *then*

$$d(g\rho, g\nu) = d(0,0) = 0 \tag{3.16}$$

and

$$\alpha(d(g\rho,\nu) + d(g\nu,\rho)) = \alpha(d(0,0) + d(0,\rho)) = \alpha\rho. \tag{3.17}$$

Then, from (3.16) and (3.17) and for any value of $\alpha \in (0, 1/2)$, we conclude that g satisfies the contraction condition of Corollary 3.3. Moreover, g is $\mathcal{R}$-continuous and $\mathcal{R}$-preserving. Thus, by Corollary 3.3, g possesses a unique fixed point which is $\rho = 0$.

Note that the metric space (X, d) considered in above example is an incomplete metric space and thus violating the applicability of Theorem 2.3 of [20] and Chatterjea contraction theorem [4].

4. Conclusions

As discussed in the beginning, the key purpose of this manuscript is to generalize and unify some of the well known results present in the literature. Following points give a visualization of deductions made through the present manuscript.

(i) On considering binary relation $\mathcal{R}$ as a universal relation (that is, relation $\mathcal{R}$ on X such that $(\rho, \nu) \in \mathcal{R}$ for all $\rho, \nu \in X$) in Corollary 3.1 of this manuscript, we obtain the Theorem 2.1 of [20].

(ii) If we consider $\mathbb{A} = \mathbb{R}$ with $\|\alpha\| = |\alpha|$ and $\alpha^* = \alpha$ for all $\alpha \in \mathbb{A}$ in Corollary 3.1, then we obtain an analogue of Theorem 3.1 of [16].

(iii) On combining the above two conditions (that is, taking $\mathcal{R}$ as a universal relation and considering $\mathbb{A} = \mathbb{R}$ with $\|\alpha\| = |\alpha|$ and $\alpha^* = \alpha$) in Corollary 3.1, we obtain the Banach contraction principle [2].

(iv) For $\mathbb{A} = \mathbb{R}$ with $\|\alpha\| = |\alpha|$ and $\alpha^* = \alpha$ for all $\alpha \in \mathbb{A}$ in Corollary 3.2 proved herein, we obtain an analogue of Theorem 3.1 proved in [22].

(v) If we assume binary relation $\mathcal{R} = X \times X$ along with $\mathbb{A} = \mathbb{R}$, then from Corollary 3.2 we obtain the fixed point result due to Kannan given in [14].

(vi) If the binary relation $\mathcal{R}$ is considered to be a universal relation in Corollary 3.3 proved herein, we obtain the Theorem 2.3 of [20].

(vii) For $\mathbb{A} = \mathbb{R}$ with $\|\alpha\| = |\alpha|$ and $\alpha^* = \alpha$ for all $\alpha \in \mathbb{A}$ in Corollary 3.3 of this manuscript, we obtain an analogue of Theorem 3.5 proved in [22].

(viii) Under the condition $\mathbb{A} = \mathbb{R}$ together with binary relation $\mathcal{R}$ as a universal relation in Corollary 3.3, we derive the fixed point result due to Chatterjea in [4].

However, readers should note that from either of the results obtained in [2,4,14,16,20,22], we cannot obtain the results proved in this manuscript and to further substantiate the utility Examples 3.4–3.6 were considered.

Open problem

Is it possible to write the main results of this paper when the underlying C^*-algebra is replaced by a lattice?

Conflict of interest

The authors declare that they have no competing interests.

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