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Research article

Numerical solutions of space-fractional diffusion equations via the exponential decay kernel

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Abstract: The main object of this paper is to investigate the spectral collocation method for three new models of space fractional Fisher equations based on the exponential decay kernel, for which properties of Chebyshev polynomials are utilized to reduce these models to a set of differential equations. We then numerically solve these differential equations using finite differences, with the resulting algebraic equations solved using Newton 's method. The accuracy of the numerical solution is verified by computing the residual error function. Additionally, the numerical results are compared with other results obtained using the power law kernel and the Mittag-Leffler kernel. The advantage of the present work stems from the use of spectral methods, which have high accuracy and exponential convergence for problems with smooth solutions. The numerical solutions based on Chebyshev polynomials are in remarkably good agreement with numerical solutions obtained using the power law and the Mittag-Leffler kernels. Mathematica was used to obtain the numerical solutions.

Keywords: power law kernel; exponential decay kernel; Mittag-Leffler kernel; Chebyshev polynomials approximation; finite difference method; space fractional Fisher equations
Mathematics Subject Classification: 41A50, 65L12, 65N12, 65N35

1. Introduction

Recently, the importance of modeling using fractional differential equations has emerged because there are many real world problems that need to be modeled using fractional equations. There are many applications of modeling using fractional equations such as economics, biology, mechanics, geology, heat transfer, chemistry, biology, physics, signal processing, and image theory [1, 2].

As a result of the fact that these fractional models often do not have an analytical solution, many researchers have focused on developing many numerical methods to find approximate solutions [3–10]. These models incorporate several fractional differential operators like Riemann–Liouville, Liouville–Caputo, Hilfer, Riesz fractional derivative etc. [11–15].

For more clarifications and details, about the properties and definitions of the fractional derivatives, the reader is referred to the following references [1, 2]. However, the previous operators face many problems and limitations in the modeling of many real-world problems. Therefore, in a more recent time, Caputo–Fabrizio has provided a remedy to address these problems and limitations. This treatment is a fractional derivative with exponential decay and a non-singular kernel [16]. Using this operator, many researchers have provided many fractional models, as well as numerical and approximate solutions [17–29]. Before we introduce the basic definitions necessary , we describe the abbreviations that used in this work in Table 1.

Table 1. The abbreviations that used in this work.

Description	Abbreviation
EDK	Exponential Decay Kernel
PLK	Power Law Kernel
MLK	Mittag–Leffler Kernel
LC	Liouville–Caputo
CF	Caputo–Fabrizio
DEs	Differential Equations
FDM	Finite Differences Method
NIM	Newton Iteration Method
REF	Residual Error Function

The basic definitions that we will need in this paper, we will present them as follows:

Definition 1.

The LC-fractional derivative of a function $\phi(\xi)$ is given by [1, 2]:

$$^{\mathrm{PLK}}D^{\vartheta}\phi(\xi) = \frac{1}{\Gamma(1-\vartheta)} \int_0^{\xi} \frac{\phi'(\tau)}{(\xi-\tau)^{\vartheta}} d\tau, \quad \xi > 0, \quad 0 < \vartheta < 1,$$

where $\phi(\xi) \in H^1(0, b)$.

Definition 2.

The CF-fractional derivative of a function $\phi(\xi)$ is defined in the following form [16]:

$$^{\mathrm{EDK}}D^{\vartheta}\phi(\xi) = \frac{\Omega(\vartheta)}{1-\vartheta} \int_a^{\xi} e^{\frac{-\vartheta(\xi-\tau)}{1-\vartheta}} \phi'(\tau) d\tau, \quad 0 < \vartheta < 1,$$

where $\phi(\xi) \in H^1(a, b)$, $a \in (-\infty, \xi)$ and $\Omega(\vartheta)$ satisfies the condition $\Omega(0) = \Omega(1) = 1$. For any arbitrary derivative, the general definition of the CF-fractional derivative can be defined as [16].

Definition 3.

The CF-fractional derivative of a function $\phi(\xi)$ is defined in the following form:

$$^{\mathrm{EDK}}D^{\vartheta}_{a+}\phi(\xi) = {}^{\mathrm{cf}}D^{\kappa}_{a+}\left(D^n \phi(\xi)\right) = \frac{\Omega(\kappa)}{1-\kappa} \int_a^{\xi} \phi^{(n+1)}(\tau) e^{-\lambda(\xi-\tau)} d\tau,$$

$$(\lambda = \frac{\vartheta}{1-\vartheta}, \;\; n < \vartheta < n+1),$$

where $\phi(\xi) \in H^1(a,b)$, $a \in (-\infty, \xi)$, $n = \lfloor \vartheta \rfloor$=the floor($\vartheta$) (i.e. integer part) and $\kappa = \lceil \vartheta \rceil$=the ceil($\vartheta$) (i.e. the decimal part) and $\phi^{(k)}(a) = 0$, $k = 1, 2, ..., n$.
Now, we apply this definition on the function $\phi(\xi) = \xi^m$, $m > 1$, as following:

$$\begin{aligned} {}^{\mathrm{EDK}}D^{\vartheta}_{0+}\xi^m = \frac{\Omega(\kappa)\Gamma(m+1)}{1-\kappa}\Bigg[&\frac{(-1)^{m-n}e^{-\lambda\xi}}{\lambda^{m-n}} \\ &+ \sum_{i=0}^{m-n-1}\frac{(-1)^i\,\xi^{m-n-1-i}}{\Gamma(m-n-i)\lambda^{i+1}}\Bigg], \;\; m \geq \lceil \vartheta \rceil. \end{aligned} \tag{1.1}$$

In Theorem 2.3 [21], we set $a = 0$ and then prove the theorem. In [21], definitions and its new properties of CF-fractional derivatives and its application are given in details.

In our work, the focus was on finding approximate numerical solutions for different types of Fisher's equation, Space fractional generalized Fisher equation, Space fractional generalized Burger-Fisher equation, and space fractional Fisher equation with variable coefficient. These models are presented, after replacing the classical derivatives with respect to space, with the fractional derivative based on the exponential decay kernel. In many previous works, many authors, have not verify the accuracy of the approximate solutions in the case of the non-integer order, but suffice to verify this in the case of the integer order, and this is in fact not sufficient. In our work, besides introducing algorithms for the three new models, we verify the accuracy of the numerical solutions in the case of non integer orders, by calculating the residual error function. More so, the absolute error of the results presented in our work, together with previous results [27, 28], was compared.

The paper is set out as follows: In section two, the shifted Chebyshev polynomials and its properties are presented, as well as the effect of the fractional derivative based on the expansion of these polynomials. In the third section, the three models, space fractional generalized Fisher equation, Space fractional generalized Burger-Fisher equation and space fractional Fisher equation with variable coefficient will be presented with the sense of the CF–fractional derivative (i.e. with exponential decay kernel), as well as constructing the algorithm for each model. In section Four, the numerical results of the three models presented in this work will be discussed. The conclusion is presented in the fifth section.

2. The Chebyshev polynomials and approximation formula due to the exponential decay kernel

The analytic form of the $\bar{\zeta}_s(\xi)$ of degree s is given by [30]:

$$\bar{\zeta}_s(\xi) = s\sum_{k=0}^{s}(-1)^{s-k}\frac{2^{2k}\,(s+k-1)!}{(2k)!\,(s-k)!}\,\xi^k, \qquad \xi \in [0,1], \tag{2.1}$$

where $\bar{\zeta}_0(\xi) = 1$, $\bar{\zeta}_1(\xi) = 2\xi - 1$.
The relation between $\bar{\zeta}_s(\xi)$ and ζ_s is given by

$$\bar{\zeta}_s(\xi) = \zeta_s(2\xi - 1) = \zeta_{2s}(\sqrt{\xi})$$

and the set $\{\zeta_s(z)\}$ where $s = 0, 1, 2, ...$ forms a family of orthogonal Chebyshev polynomials on the interval $[-1, 1]$.

Now, we approximate the function $\Xi(\xi) \in L_2[0, 1]$, as

$$\Xi(\xi) \simeq \Xi_m(\xi) = \sum_{i=0}^{m} \Lambda_i \, \bar{\zeta}_i(\xi), \tag{2.2}$$

where the coefficients Λ_i are given by:

$$\Lambda_0 = \frac{1}{\pi} \int_0^1 \frac{\Xi(\xi)\, \bar{\zeta}_0(\xi)}{\sqrt{\xi - \xi^2}}\, d\xi, \qquad \Lambda_i = \frac{2}{\pi} \int_0^1 \frac{\Xi(\xi)\, \bar{\zeta}_i(\xi)}{\sqrt{\xi - \xi^2}}\, d\xi, \qquad i = 1, 2, \cdots$$

Theorem 1. *[20]*

Suppose that the $\Xi''(\xi) \in L_2[0, 1]$ and $|\Xi''(\xi)| \leq \beta, \beta > 0$ then the expansion (2.2) is uniformly convergent and

$$|\Lambda_\ell| < \beta/2\ell^2, \qquad \ell = 1, 2, 3, \dots . \tag{2.3}$$

Theorem 2. *[20]*

The absolute error $E_m = ||\Xi(\xi) - \Xi_m(\xi)||$, $\Xi(\xi) \in C^m[0, 1]$ can be bounded by:

$$||E_m|| \leq \frac{\hbar \Delta^{m+1}}{(m + 1)!}, \qquad \hbar = \max_{\xi \in [0,1]} \Xi^{(m+1)}(\xi), \qquad \Delta = \max[\xi_0, \xi - \xi_0]. \tag{2.4}$$

We show and obtain in the following theorem, an approximate formula of ${}^{\mathrm{cf}}D^{\vartheta}_{0+}\Xi_m(\xi)$.

Theorem 3.

From the approximation (2.2), the ${}^{\mathrm{cf}}D^{\vartheta}_{0+}(\Xi_m(\xi))$ can be defined as:

$${}^{\mathrm{EDK}}D^{\vartheta}_{0+}(\Theta_m(\xi)) = \sum_{i=\lceil\vartheta\rceil}^{m} \sum_{j=\lceil\vartheta\rceil}^{i} \Lambda_i \Pi_{i,j,\kappa} \Upsilon^{\kappa}_{i,j}(\xi) \tag{2.5}$$

where

$$\Pi_{i,j,\kappa} = \frac{i\, 2^{2j}\, (-1)^{i-j}\, \Gamma(j + 1)(i + j - 1)!}{(2j)!\, (i - j)!} \frac{\Omega(\kappa)}{1 - \kappa},$$

$$\Upsilon^{\kappa}_{i,j}(\xi) = \frac{(-1)^{j-n} e^{-\lambda \xi}}{\lambda^{j-n}} + \sum_{p=0}^{j-n-1} (-1)^p \frac{\xi^{j-n-1-p}}{\lambda^{p+1} \Gamma(j - n - p)}.$$

Proof. By affecting on (2.2) using CF–fractional differentiation and their linearity property, we obtain

$${}^{\mathrm{EDK}}D^{\vartheta}_{0+}(\Xi_m(\xi)) = \sum_{i=0}^{m} \Lambda_i \, {}^{\mathrm{cf}}D^{\vartheta}_{0+}(\bar{\zeta}_i(\xi)). \tag{2.6}$$

Back to the Eqs (1.1), (2.1) and (2.6) can be obtained the following:

$${}^{EDK}D^{\vartheta}_{0+}(\bar{\zeta}_i(\xi)) = 0, \qquad i = 0, 1, ..., \lceil\vartheta\rceil - 1, \tag{2.7}$$

$$
{}^{EDK}D^{\vartheta}_{0+}(\bar{\zeta}_i(\xi)) = \sum_{j=\lceil\vartheta\rceil}^{i} \frac{i\,2^{2j}\,(-1)^{i-j}\,(i+j-1)!\Gamma(j+1)}{(2j)!\,(i-j)!}\,\frac{\Omega(\kappa)}{1-\kappa}
$$

$$
\times\left[\frac{(-1)^{j-n}e^{-\lambda\xi}}{\lambda^{j-n}} + \sum_{p=0}^{j-n-1}\frac{(-1)^{p}\,\xi^{j-n-1-p}}{\Gamma(j-n-p)\lambda^{p+1}}\right], i=\lceil\vartheta\rceil,...,m. \tag{2.8}
$$

Now using the Eqs (2.6), (2.7) and (2.8) result can be reached, thus, we have completed the proof. □

3. Applications

In this section we will introduce the three fractional models with exponential decay kernel, as well as the algorithm for finding numerical solutions.

Model 1: Space fractional generalized Fisher equation

In this model [31], we replace the classical derivative by the fractional derivative with exponential decay kernel, we obtain

$$
\phi_t(\xi,\eta) = {}_{0}^{EDK}D^{\nu}\phi(\xi,\eta) + \phi(\xi,\eta)(1-\phi(\xi,\eta))(\phi(\xi,\eta)-\beta), \tag{3.1}
$$

$$
(0<\beta<1, \quad 0<\nu\leq 2).
$$

The exact solution of Fisher's equation in the classical case is given as follows [31]:

$$
\phi(\xi,\eta) = \frac{1}{2}(1+\beta) + \frac{1}{2}(1-\beta)\tanh\left(\sqrt{\frac{1}{8}}(1-\beta)\xi + \frac{1}{4}(1-\beta^2)\eta\right),
$$

subject to the boundary and initial conditions as:

$$
\phi(0,\eta) = g_1(\eta), \qquad \phi(1,\eta) = g_2(\eta), \tag{3.2}
$$

$$
\phi(\xi,0) = \bar{u}(\xi). \tag{3.3}
$$

In this model we explain the algorithm in detail, and then it can be directly applied it to the second and third examples without providing details. Therefore, we follow the following steps until we can reach the numerical solutions.

1. First, we approximate the function $\bar{\zeta}_i(\xi)$ as:

$$
\phi_m(\xi,\eta) = \sum_{i=0}^{m}\phi_i(\eta)\,\bar{\zeta}_i(\xi). \tag{3.4}
$$

2. Substitute from the formulae (2.5) and (3.4) in Eq (3.1) to obtain:

$$
\sum_{i=0}^{m}\frac{d\,\phi_i(\eta)}{d\eta}\,\bar{\zeta}_i(\xi) = \sum_{i=\lceil\nu\rceil}^{m}\sum_{j=\lceil\nu\rceil}^{i}\phi_i(\eta)\Pi_{i,j,\kappa}\Upsilon^{\kappa}_{i,j}(\xi) + \sum_{i=0}^{m}\phi_i(\eta)\,\bar{\zeta}_i(\xi)
$$

$$
\times\left(1-\sum_{i=0}^{m}\phi_i(\eta)\,\bar{\zeta}_i(\xi)\right)\left(\sum_{i=0}^{m}\phi_i(\eta)\,\bar{\zeta}_i(\xi)-\beta\right). \tag{3.5}
$$

3. Collocate the Eq (3.5) at $(m + 1 - \lceil\nu\rceil)$ points ξ_r to obtain the following first order system of ordinary differential equations:

$$\sum_{i=0}^{m} \frac{d\,\phi_i(\eta)}{d\eta}\,\bar{\zeta}_i(\xi_r) = \sum_{i=\lceil\nu\rceil}^{m}\sum_{j=\lceil\nu\rceil}^{i} \phi_i(\eta)\Pi_{i,j,\kappa}\Upsilon^{\kappa}_{i,j}(\xi_r) + \sum_{i=0}^{m} \phi_i(\eta)\,\bar{\zeta}_i(\xi_r)$$
$$\times\left(1 - \sum_{i=0}^{m} \phi_i(\eta)\,\bar{\zeta}_i(\xi_r)\right)\left(\sum_{i=0}^{m} \phi_i(\eta)\,\bar{\zeta}_i(\xi_r) - \beta\right). \tag{3.6}$$

4. By substituting (3.4) into (3.2), we obtain the corresponding boundary conditions of this system

$$\sum_{i=0}^{m} \bar{\zeta}_i(0)\phi_i(\eta) = g_1(\eta), \qquad \sum_{i=0}^{m} \bar{\zeta}_i(1)\phi_i(\eta) = g_2(\eta). \tag{3.7}$$

Now, using the FDM, we can solve the set of ordinary differential Eqs (3.6), (3.7) with respect to the unknowns $u_i(t), i = 0, 1, ..., m$, and obtain the following nonlinear algebraic equations

$$\sum_{i=0}^{m}\left(\frac{\phi_i^s - \phi_i^{s-1}}{\tau}\right)\bar{\zeta}_i(\xi_r) = \left(\sum_{i=\lceil\nu\rceil}^{m}\sum_{j=\lceil\nu\rceil}^{i} \phi_i(\eta)\Pi_{i,j,\kappa}\Upsilon^{\kappa}_{i,j}(\xi_r)\right)$$
$$+\left(\sum_{i=0}^{m} \phi_i(\eta)\,\bar{\zeta}_i(\xi_r)\right)\left(1 - \sum_{i=0}^{m} \phi_i(\eta)\,\bar{\zeta}_i(\xi_r)\right)$$
$$\times\left(\sum_{i=0}^{m} \phi_i(\eta)\,\bar{\zeta}_i(\xi_r) - \beta\right), \tag{3.8}$$

$$\sum_{i=0}^{m} \bar{\zeta}_i(0)\phi_i^s = g_1^s, \tag{3.9}$$

$$\sum_{i=0}^{m} \bar{\zeta}_i(1)\phi_i^s = g_2^s. \tag{3.10}$$

5. In order to be more clarified, we will explain the method in the case of $m = 4$ and using the NIM. The system (3.8)–(3.10) can be written as

$$\phi^{s+1} = \phi^s - J^{-1}(\phi^s)\,F(\phi^s), \tag{3.11}$$

where $\phi^s = (\phi_0^s, \phi_1^s, \phi_2^s, \phi_3^s, \phi_4^s)^T$, $F(\phi^s)$ is the vector which represents the nonlinear terms and $J^{-1}(u^s)$ is the inverse of the Jacobian matrix. In order to get the initial solution u^0, we set $s = 0$ and then we get the initial condition (3.3) as follows:

(a) Now, to get the

$$\phi(\xi, 0) = \bar{\phi}(\xi) \simeq \sum_{i=0}^{4} \phi_i(0)\bar{\zeta}_i(\xi), \tag{3.12}$$

we substitute (3.4) into the initial condition (3.3).

(b) Finally, to find the the components of the following initial solution ϕ^0, we solve the equation $\bar{\zeta}_5(x)$ and get the points x_r,

$$\sum_{i=0}^{4} \phi_i^0 \, \bar{\zeta}_i(\xi_r) = \bar{\phi}(\xi_r), \qquad r = 0, 1, 2, 3, 4. \tag{3.13}$$

Model 2: Space fractional generalized Burger-Fisher equation

In this model, we also replace the classical derivative by the fractional derivative with exponential decay kernel. Then in this case , the Space fractional generalized Burger-Fisher equation [31] is given by

$$\phi_\eta(\xi,\eta) = {}_0^{EDK}D^\alpha \phi(\xi,\eta) - \nu\phi(\xi,\eta)^\delta \, {}_0^{EDK}D^\beta \phi(\xi,\eta) + \gamma\phi(\xi,\eta)(1 - \phi(\xi,\eta)^\delta), \tag{3.14}$$

$$(0 < \alpha \le 2, \;\; 0 < \beta \le 1),$$

and the exact solution when $\alpha = 2, \;\; \beta = 1$ is given by,

$$\phi(\xi,\eta) = \frac{1}{2}\left(1 - \tanh\left(\frac{(\delta\nu)\left(\xi - \eta\left(\frac{\gamma(\delta+1)}{\nu} + \frac{\nu}{\delta+1}\right)\right)}{2(\delta+1)}\right)^{\frac{1}{\delta}}\right). \tag{3.15}$$

The boundary and initial conditions can be derived from this exact solution by setting $\xi = 0, \xi = 1$ and $\eta = 0$, we obtain the boundary and initial conditions respectively:

$$\phi(0,\eta) = f_1(\eta), \qquad \phi(1,\eta) = f_2(\eta), \tag{3.16}$$

$$\phi(\xi,0) = \bar{\phi}(\xi). \tag{3.17}$$

Following the same procedures as in the previous model 1 , we can solve this Model numerically using the above proposed algorithm.

Model 3: Space fractional Fisher equation with variable coefficient

In this model, we consider the nonlinear Fisher equation with variable coefficient [31] via exponential decay kernel by replacing the classical derivative with ${}_0^{EDK}D^\nu$ as

$$\phi_\eta(\xi,\eta) = -\frac{a}{6\mu^2}\coth(\frac{a}{6}\eta + c)\, {}_0^{EDK}D^\nu \phi(\xi,\eta) + a\phi(\xi,\eta)(1 - \phi(\xi,\eta)), \tag{3.18}$$

$$(0 < \nu \le 2).$$

The exact solution of (3.18) at the case $\nu = 2$ is given by

$$u(x,t) = \frac{1}{4}\coth\left(\frac{at}{6} + c\right)\operatorname{sech}^2\left(\frac{1}{12}(5a)t + \frac{\mu x}{2}\right) + \frac{1}{2}\tanh\left(\frac{1}{12}(5a)t + \frac{\mu x}{2}\right) + \frac{1}{2}. \tag{3.19}$$

By following the same procedure as in model 1 , we can get the numerical solutions.

4. Numerical results and discussion

In this section, we discuss the numerical results of the three previous new models. The fact that the exact solutions for these models are only available in the classical state, so we define the residual error function, through which we can verify the accuracy and efficiency of the algorithms that were constructed in the third section. For the first model we define REF as

$$REF(\xi,\eta) = (u_m(\xi,\eta))_\eta - {}^{EDK}D^{\alpha}_{0+}(u_m(\xi,\eta)) - u_m(\xi,\eta)\,(1 - u_m(\xi,\eta))(u_m(\xi,\eta) - \beta). \tag{4.1}$$

Figure 1(a) presents the residual error function of the first model for $m = 4, \alpha = 1.7, \beta = 0.125, \tau = 0.001$ and $T = 1$ based on the PLK, EDK and MLK, respectively. From this figure, we see that the value of the residual error function is very small and of order 10^{-3}. In all figures, including this figure, we indicate in green line the results based on the PLK, as well as the red line for the results obtained using the EDK, and in the blue line for those obtained using the MLK. Figure 1(b) shows the absolute error between the approximate solution of the first model in the presence of the EDK and MLK. The same parameters were taken as in Figure 1(a). From this figure, we can see that the error is very small and is of the order of 10^{-5}. Also in Figure 1(c), the absolute error of the first model between the approximate solution based on EDK and the approximate solution based on PLK is represented. The parameters are the same as in the Figure 1(a). Also, we observe from this figure that the value of the error is small and of order 10^{-4}.

Now for the second model, we define the residual error function as follows:

$$\begin{aligned} REF(\xi,\eta) = (u_m(\xi,\eta))_\eta - {}^{EDK}D^{\alpha}_{0+}(u_m(\xi,\eta)) - \nu\, u_m(\xi,\eta)^{\delta} \\ \times {}^{EDK}D^{\beta}_{0+}u_m(\xi,\eta) + \gamma u_m(\xi,\eta)(1 - u_m(\xi,\eta)^{\delta}). \end{aligned} \tag{4.2}$$

The numerical results for the second model will be discussed just as they were for the first model. The REF for the second model is represented by the Figure 2(a) for the parameters $m = 4, \alpha = 1.7, \beta = 0.7, \delta = 2, \nu = 0.01, \gamma = 0.01, \tau = 0.001$ and $T = 1$. In Figure 2(b), the absolute error is shown between the approximate solution in the presence of the EDK and the approximate solution in the presence of the MLK. Similarly, in Figure 2(c), but now in the presence of EDK and PLK. In all cases the errors are very small and the range of the order is $10^{-3} - 10^{-4}$. In the third model the REF is given as follows:

$$REF(\xi,\eta) = (u_m(\xi,\eta))_\eta + \frac{a}{6\mu^2} coth(\frac{a}{6}\eta + c)\, {}^{EDK}D^{\alpha}_{0+}(u_m(\xi,\eta)) - a u_m(\xi,\eta)(1 - u_m(\xi,\eta)). \tag{4.3}$$

In this model the numerical results are discussed as in the previous two models. Figure 3 shows the numerical results for the parameters $m = 5, \alpha = 1.9, \mu = 2, a = \frac{1}{5}, c = 1, \tau = 0.001$ and $T = 1$. The REF of (3.18) based on the PLK, EDK, and MLK is illustrated in Figure 3 (a) while the absolute error between the numerical solution based on MLK, PLK and EDK, are shown in Figure 3 (b)–(c), respectively. Also for this model, the error is very small and the error ranges $10^{-3} - 10^{-4}$. All these numerical results were summarized and presented clearly in the Tables 2–4. In the three proposed models, we presented the CPU time needed for the computation. In the first model, the time taken to complete the calculations was sixty seconds, while in the second model, the time taken to complete the calculations was seventeen minutes, and finally in the third model, the time was two minutes. Explanation of this in the case of the first model, we only had one fractional derivative, and then the time to complete the calculations was fast, also in the third model, the time increased slightly due to the coefficients being functions of the time variable. As for the second model, it took more time as a result of the presence of two fractional derivatives at the same time.

Table 2. The REF and the absolute error of (3.1) based on the PLK, EDK, and MLK, respectively for $m = 4, \alpha = 1.7, \mu = 0.125, \tau = 0.001$ and $T = 1$.

ξ	$REF_{PLK}(\xi,\eta)$	$REF_{EDK}(\xi,\eta)$	$REF_{MLK}(\xi,\eta)$	$\lvert MLK - EDK\rvert$	$\lvert LC - EDK\rvert$
0.1	-1.9068×10^{-4}	-8.31467×10^{-4}	-7.43203×10^{-4}	2.99439×10^{-5}	4.97441×10^{-6}
0.2	-3.63316×10^{-4}	$-0.2.02481 \times 10^{-3}$	-1.71785×10^{-3}	4.02244×10^{-5}	6.02525×10^{-5}
0.3	-2.95778×10^{-4}	-1.86699×10^{-3}	-1.53138×10^{-3}	3.64307×10^{-5}	1.29237×10^{-4}
0.4	-1.51646×10^{-4}	-1.02472×10^{-3}	-8.20059×10^{-4}	2.37457×10^{-5}	1.85228×10^{-4}
0.5	8.6528×10^{-15}	-6.11317×10^{-15}	6.94642×10^{-6}	7.43203×10^{-4}	2.1142×10^{-4}
0.6	1.19699×10^{-4}	8.44685×10^{-4}	6.51343×10^{-4}	9.59643×10^{-6}	2.00906×10^{-4}
0.7	1.81218×10^{-4}	1.26653×10^{-3}	9.6039×10^{-4}	2.19181×10^{-5}	1.56674×10^{-4}
0.8	1.65483×10^{-4}	1.12662×10^{-3}	8.39342×10^{-4}	2.64599×10^{-5}	9.16084×10^{-5}
0.9	5.78039×10^{-5}	1.12662×10^{-3}	2.75614×10^{-4}	2.00694×10^{-5}	2.84914×10^{-5}
1	-1.53568×10^{-4}	1.12662×10^{-3}	-6.74819×10^{-4}	4.81482×10^{-35}	2.77556×10^{-17}

Table 3. The REF and the absolute error of (3.14) based on the PLK, EDK, and MLK, respectively for $m = 4, \alpha = 1.7, \mu = 0.125, \tau = 0.001$ and $T = 1$.

ξ	$REF_{PLK}(\xi,\eta)$	$REF_{EDK}(\xi,\eta)$	$REF_{MLK}(\xi,\eta)$	$\lvert MLK - EDK\rvert$	$\lvert LC - EDK\rvert$
0.1	-1.95772×10^{-3}	-3.876×10^{-4}	-3.53468×10^{-4}	1.03253×10^{-4}	1.04916×10^{-4}
0.2	-1.92376×10^{-3}	-8.8975×10^{-4}	-7.05082×10^{-4}	1.21664×10^{-4}	1.23354×10^{-4}
0.3	-1.19599×10^{-3}	-7.58258×10^{-4}	-4.74322×10^{-4}	1.03267×10^{-4}	1.04363×10^{-4}
0.4	-4.36256×10^{-4}	-3.74158×10^{-4}	-1.42939×10^{-4}	8.19101×10^{-5}	8.24988×10^{-5}
0.5	-2.25313×10^{-5}	2.54306×10^{-14}	1.65293×10^{-14}	7.726×10^{-5}	7.78236×10^{-5}
0.6	-1.0766×10^{-4}	2.14128×10^{-4}	-1.4195×10^{-4}	9.47976×10^{-5}	9.59079×10^{-5}
0.7	-6.35463×10^{-4}	2.25836×10^{-4}	-4.72642×10^{-4}	01.2582×10^{-4}	1.27829×10^{-4}
0.8	-1.34712×10^{-3}	9.6654×10^{-5}	-7.03348×10^{-4}	1.47439×10^{-4}	1.50173×10^{-4}
0.9	-1.78426×10^{-3}	-1.0951×10^{-5}	-3.52772×10^{-4}	01.22585×10^{-4}	1.2503×10^{-4}
1	-1.29068×10^{-3}	1.64053×10^{-4}	1.25294×10^{-3}	0	0

Table 4. The REF and the absolute error of (3.18) based on the PLK, EDK, and MLK, respectively for $m = 4, \alpha = 1.7, \mu = 0.125, \tau = 0.001$ and $T = 1$.

ξ	$REF_{PLK}(\xi, \eta)$	$REF_{EDK}(\xi, \eta)$	$REF_{MLK}(\xi, \eta)$	$\lvert MLK - EDK \rvert$	$\lvert LC - EDK \rvert$
0.1	-2.17693×10^{-3}	$-0.00541478 \times 10^{-3}$	-5.26676×10^{-3}	5.89201×10^{-5}	1.09422×10^{-3}
0.2	-1.76375×10^{-3}	$-0.00469937 \times 10^{-3}$	-4.4629×10^{-3}	1.17245×10^{-4}	6.91422×10^{-4}
0.3	-1.22552×10^{-4}	$-0.000354133 \times 10^{-4}$	-3.26613×10^{-4}	1.80235×10^{-4}	7.80636×10^{-5}
0.4	8.13027×10^{-4}	$0.00265623 \times 10^{-3}$	2.34947×10^{-3}	2.23802×10^{-4}	6.20275×10^{-4}
0.5	7.18887×10^{-4}	$0.00295111 \times 10^{-3}$	2.43607×10^{-3}	2.14227×10^{-4}	7.45003×10^{-4}
0.6	1.83374×10^{-4}	$0.00142919 \times 10^{-3}$	1.02415×10^{-3}	1.27885×10^{-4}	5.33155×10^{-4}
0.7	1.42249×10^{-5}	$-0.0000880891 \times 10^{-5}$	-3.90126×10^{-5}	2.90368×10^{-5}	2.04127×10^{-4}
0.8	5.41745×10^{-4}	$-0.000234374 \times 10^{-4}$	3.72358×10^{-4}	2.00811×10^{-4}	1.68267×10^{-5}
0.9	9.2339×10^{-4}	$0.000519769 \times 10^{-3}$	1.18238×10^{-3}	2.62454×10^{-4}	3.12273×10^{-5}
1	-1.54685×10^{-3}	$-0.00151108 \times 10^{-3}$	-2.3627×10^{-3}	2.22045×10^{-16}	2.22045×10^{-16}

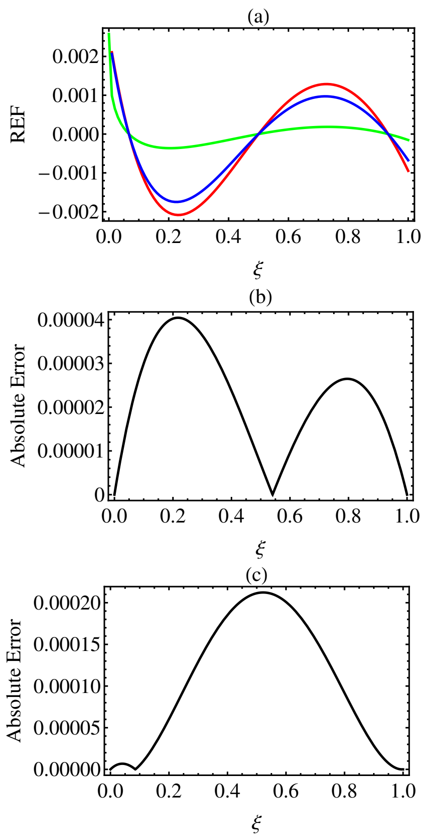

Figure 1. The REF of (3.1) based on the PLK, EDK, and MLK in (a), the absolute error between the numerical solution based on MLK and EDK in (b), and the absolute error between the numerical solution based on PLK and EDK in (c) for $m = 4, \alpha = 1.7, \mu = 0.125, \tau = 0.001$ and $T = 1$ (Green line: PLK, Red line: EDL, Blue line: EDL).

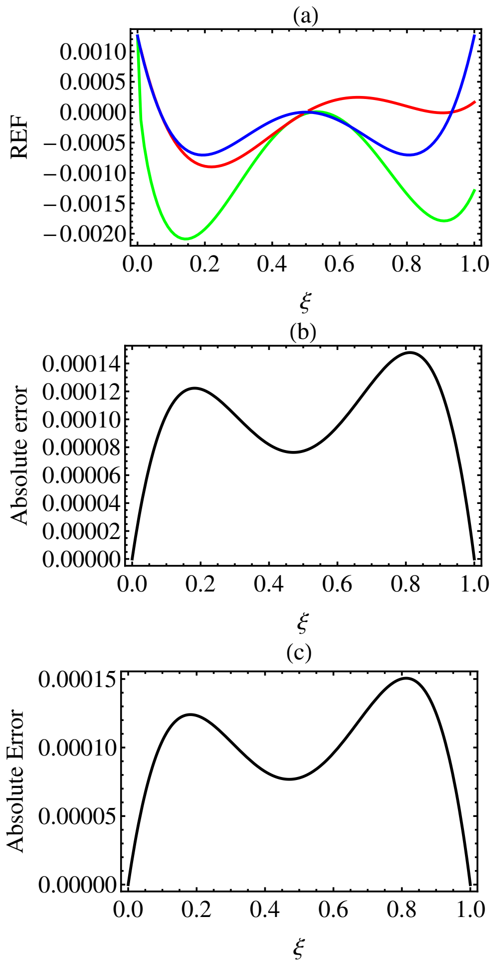

Figure 2. The REF of (3.1) based on the PLK, EDK, and MLK in (a), the absolute error between the numerical solution based on MDK and EDK in (b), and the absolute error between the numerical solution based on PLK and EDK in (c) for $m = 4, \alpha = 1.7, \beta = 0.7, \delta = 2, \nu = 0.01, \gamma = 0.01, \tau = 0.001$ and $T = 1$ (Green line: PLK, Red line: EDL, Blue line: EDL).

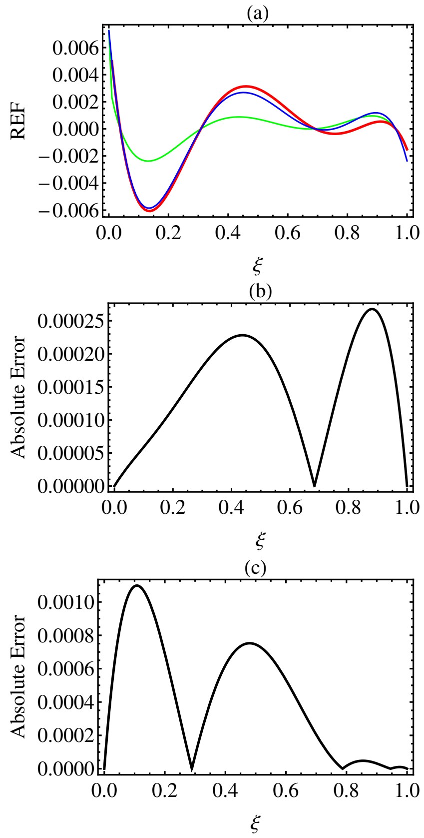

Figure 3. The REF of (3.18) based on the PLK, EDK, and MLK in (a), the absolute error between the numerical solution based on MLK and EDK in (b), and the absolute error between the numerical solution based on PLK and EDK in (c) for $m = 5, \alpha = 1.9, \mu = 2, a = \frac{1}{5}, c = 1, \tau = 0.001$ and $T = 1$ (Green line: PLK, Red line: EDL, Blue line: EDL).

5. Conclusions

In this paper, three new models, namely generalized Fisher equation, generalized Burger-Fisher equation, and Fisher equation with variable coefficient are presented based on replacing the classical derivative with respect to space with the fractional derivative based on the exponential decay kernel. Then Schema was constructed for new models using spectral methods and properties of Chebyshev polynomials, to obtain a set of differential equations and then employ the finite differences method to convert the last system into a system of algebraic equations, which was solved using the Newton iteration method. The results and the accuracy of the numerical solutions were verified, using the calculation of the absolute error between the numerical solutions to the problems presented with the previously published numerical solutions. Furthermore, the residual error function was calculated. All of these results are presented graphically and in tables. The numerical solutions based on the exponential decay kernel for the three new models resulted in good agreement with numerical solutions based on power law and Mittag-Leffler kernels. In the future, we will develop the study presented in this work for different fractal-fractional operators, with singular and non-singular kernel. Also, we can extend the study of the models for the time-space fractional derivative.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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