AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 6365–6380.
DOI: 10.3934/math.2022354
Received: 24 August 2021
Revised: 10 January 2022
Accepted: 12 January 2022
Published: 19 January 2022

Research article

Fekete-Szegö and Hankel inequalities for certain class of analytic functions related to the sine function

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Abstract: In this present investigation, the authors obtain Fekete-Szegö inequality for certain normalized analytic function $f(\zeta)$ defined on the open unit disk for which

$$(f'(\zeta))^{\vartheta}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^{1-\vartheta} \prec 1+\sin\zeta; \qquad (0\leq\vartheta\leq 1)$$

lies in a region starlike with respect to 1 and symmetric with respect to the real axis. As a special case of this result, the Fekete-Szegö inequality for a class of functions defined through Poisson distribution series is obtained. Further, we discuss the second Hankel inequality for functions in this new class.

Keywords: analytic function; starlike function; subordination; coefficient problem; Fekete-Szegö inequality; Hankel determinant; Poisson distribution series
Mathematics Subject Classification: 30C45, 30C50, 30C80

1. Introduction and motivation

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{D} = \{\zeta : |\zeta| < 1\}$ of the form

$$f(\zeta) = \zeta + a_2\zeta^2 + a_3\zeta^3 + \cdots \quad (\zeta \in \mathbb{D}) \tag{1.1}$$

and let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions.

Assume that f and g are two analytic functions in $\mathbb{D}$. Then, we say that the function g is subordinate to the function f, and we write

$$g(\zeta) \prec f(\zeta) \quad (\zeta \in \mathbb{D}),$$

if there exists a Schwarz function $\omega(\zeta)$ with $\omega(0) = 0$ and $|\omega(\zeta)| < 1$, such that (see [1])

$$g(\zeta) = f(\omega(\zeta)) \quad (\zeta \in \mathbb{D}).$$

The familiar coefficient conjecture for the functions $f \in \mathcal{S}$ having the series form (1.1), was given by Bieberbach in 1916 and it was later proved by de-Branges [2] in 1985. It was one of the most celebrated conjectures in classical analysis, one that has stood as a challenge to mathematician for a very long time. During this period, many mathematicians worked hard to prove this conjecture and as result they established coefficient bounds for some sub-families of the class $\mathcal{S}$ of univalent functions. Ma and Minda (see [3]) introduced two classes of analytic functions namely;

$$\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{A} : \frac{\zeta f'(\zeta)}{f(\zeta)} \prec \psi(\zeta) \quad (\zeta \in \mathbb{D}) \right\}$$

and

$$C(\psi) = \left\{ f \in \mathcal{A} : 1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} \prec \psi(\zeta) \quad (\zeta \in \mathbb{D}) \right\},$$

where the function ψ is an analytic univalent function such that $\Re\ (\psi) > 0$ in $\mathbb{D}$ with $\psi(0) = 1$, $\psi'(0) > 0$ and ψ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis and the symbol '$\prec$' denote the subordination between two analytic functions. By varying the function ψ, several familiar classes can be obtained as illustrated below:

(1) For $\psi = \frac{1+A\zeta}{1+B\zeta}$ $(-1 \leq B < A \leq 1)$, we get the class $\mathcal{S}^*(A, B)$, see [4].
(2) For different values of A and B, the class $\mathcal{S}^*(\alpha) = \mathcal{S}^*(1 - 2\alpha, -1)$ is shown in [5].
(3) For $\psi = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2$, the class was defined and studied in [6].
(4) For $\psi = \sqrt{1+\zeta}$, the class is denoted by $\mathcal{S}^*_L$, details can be seen in [7] and further studied in [8].
(5) For $\psi = \zeta + \sqrt{1+\zeta^2}$, the class is denoted by $\mathcal{S}^*_l$, see [9].
(6) If $\psi = 1 + \frac{4}{3}\zeta + \frac{2}{3}\zeta^2$, then such class denoted by $\mathcal{S}^*_C$ was introduced in [10] and further studied by [11].
(7) For $\psi = e^{\zeta}$, the class $\mathcal{S}^*_e$ was defined and studied in [12, 13].
(8) For $\psi = \cosh(\zeta)$, the class is denoted by $\mathcal{S}^*_{\cosh}$, see [14].
(9) For $\psi = 1 + \sin(\zeta)$, the class is denoted by $\mathcal{S}^*_{\sin}$, see [15] for details and further investigation, see [16].

Recently in [17–22] by choosing some particular function for ψ as above, inequalities related with coefficient bounds of some sub-classes of univalent functions have been discussed extensively .

The Fekete-Szegö inequality is one of the inequalities for the coefficients of univalent analytic functions found by Fekete and Szegö (1933), related to the Bieberbach conjecture. Another coefficient problem which is closely related with Fekete and Szegö is the Hankel determinant. Hankel determinants are very useful in the investigations of the singularities and power series with integral coefficients. For the functions $f \in \mathcal{A}$ of the form (1.1), in 1976, Noonan and Thomas [23] stated the ℓ^{th} Hankel determinant as

$$H_\ell(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\ell-1} \\ a_{n+1} & a_n & \cdots & a_{n+\ell-2} \\ \vdots & \vdots & & \vdots \\ a_{n+\ell-1} & a_{n+\ell-2} & \cdots & a_n \end{vmatrix} \qquad (a_1 = 1 \ \ \ell, n \in \mathbb{N} = \{1, 2, \cdots .\}).$$

In particular, we have

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad (a_1 = 1,\ n = 1,\ \ell = 2)$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2 \quad (n = 2,\ \ell = 2).$$

We note that $H_2(1)$ is the well-known Fekete-Szegö functional (see [24–26]).

In recent years, many papers have been devoted to finding the upper bounds for the second-order Hankel determinant $H_2(2)$, for various sub-classes of analytic functions, it is worth mentioning that [13, 19, 27–32] (also see references cited therein) and the upper bounds for the third and forth-order Hankel determinants by many researchers (see [33–38]). Recently, Cho et al. [15] introduced the following function class $\mathcal{S}_s^*$:

$$S_s^* := \left\{ f \in \mathcal{A} : \frac{\zeta f'(\zeta)}{f(\zeta)} \prec 1 + \sin\zeta \qquad (\zeta \in \mathbb{D}) \right\}, \tag{1.2}$$

which implies that the quantity $\frac{\zeta f'(\zeta)}{f(\zeta)}$ lies in an eight-shaped region in the right-half plane. Inspired by the aforementioned works, in this paper, we mainly investigate upper bounds for the second-order Hankel determinant for the new function class $\mathcal{RS}_{sin}^*$ associated with the sine function defined in Definition 1.

Definition 1. Let $0 \leq \vartheta \leq 1$. Then the class $\mathcal{RS}_{sin}^*(\vartheta)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying

$$(f'(\zeta))^{\vartheta} \left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right)^{1-\vartheta} \prec 1 + \sin\zeta = \Phi(\zeta).$$

Note that,

$$\mathcal{RS}_{sin}^*(0) = \mathcal{S}_{sin}^* = \left\{ f \in \mathcal{A} : \left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right) \prec 1 + \sin\zeta \right\}$$

and

$$\mathcal{RS}_{sin}^*(1) = \mathcal{R}_{sin} = \{ f \in \mathcal{A} : f'(\zeta) \prec 1 + \sin\zeta \}.$$

2. Auxiliary results

To prove our main result, we need the following: Let $\mathcal{P}$ represent the family of functions $h(\xi)$ that are regular with positive part in open unit disc $\mathbb{D}$ and of the form

$$p(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n \qquad (\xi \in \mathbb{D}). \tag{2.1}$$

Lemma 1. *[39] If $p(\zeta) \in \mathcal{P}$ as given in* (2.1)*, then*

$$|\, c_n \,| \leq 2 \ \ for\ all\ n \geq 1 \qquad \text{and} \qquad |c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

Lemma 2. *[40] If $p(\zeta) \in \mathcal{P}$ as given in* (2.1)*, then*

$$|c_2 - vc_1^2| \leqq 2\max\{1, |2v - 1|\}$$

and the result is sharp for the functions given by

$$p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2}, \quad p(\zeta) = \frac{1+\zeta}{1-\zeta}.$$

Lemma 3. *[41] If $p(\zeta) \in \mathcal{P}$ as given in* (2.1)*, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2, & \text{if } v \leq 0, \\ 2, & \text{if } 0 \leq v \leq 1, \\ 4v - 2, & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p(\zeta)$ is $(1+\zeta)/(1-\zeta)$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p(\zeta)$ is $(1+\zeta^2)/(1-\zeta^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p(\zeta) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+\zeta}{1-\zeta} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-\zeta}{1+\zeta} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if p is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Lemma 4. *[40] If $p(\zeta) \in \mathcal{P}$, then there exist some $x,\ \zeta$ with $|x| \leq 1,\ |\zeta| \leq 1$, such that*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - (4 - c_1^2)c_1x^2 + 2(4 - c_1^2)(1 - |x|^2)\zeta.$$

3. Coefficient bounds and Fekete-Szegö inequality for $f \in \mathcal{RS}^*_{sin}(\vartheta)$

In the first theorem,we will find the coefficient bounds for the function class $\mathcal{RS}^*_{sin}(\vartheta)$.

Theorem 5. *If the function $f(\zeta) \in \mathcal{RS}^*_{sin}(\vartheta)$ and is of the form* (1.1)*, then*

$$|a_2| \leq \frac{1}{1+\vartheta}, \tag{3.1}$$

$$|a_3| \leq \frac{1}{2+\vartheta}\max\left\{1, \left|\frac{\vartheta^2 + \vartheta - 2}{2(1+\vartheta)^2}\right|\right\}, \tag{3.2}$$

and

$$|a_3 - \mu a_2^2| \leq \frac{1}{2+\vartheta}\max\left\{1, \left|\frac{\vartheta^2 + \vartheta - 2 + 2\mu(2+\vartheta)}{2(1+\vartheta)^2}\right|\right\}, \tag{3.3}$$

where $\mu \in \mathbb{C}$.

Proof. Since $f(\zeta) \in \mathcal{RS}^*_{sin}(\vartheta)$, according to subordination relationship, thus there exists a Schwarz function $\omega(\zeta)$ with $\omega(0) = 0$ and $|\omega(\zeta)| < 1$, satisfying

$$[f'(\zeta)]^{\vartheta}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^{1-\vartheta} = 1 + \sin(\omega(\zeta)).$$

Here

$$\begin{aligned} [f'(\zeta)]^{\vartheta}\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^{1-\vartheta} &= 1 + (1+\vartheta)a_2\zeta + \frac{\zeta^2}{2}(2+\vartheta)[2a_3 - (1-\vartheta)a_2^2] \\ &+ \frac{(3+\vartheta)\zeta^3}{6}[(1-\vartheta)(2-\vartheta)a_2^3 - 6(1-\vartheta)a_2a_3 + 6a_4] + \cdots . \end{aligned} \tag{3.4}$$

Now, we define a function

$$p(\zeta) = \frac{1+\omega(\zeta)}{1-\omega(\zeta)} = 1 + c_1\zeta + c_2\zeta^2 + \cdots .$$

It is known that $p(\zeta) \in \mathcal{P}$ and

$$\omega(\zeta) = \frac{p(\zeta)-1}{1+p(\zeta)} = \frac{c_1}{2}\zeta + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)\zeta^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right)\zeta^3 + \cdots . \tag{3.5}$$

On the other hand,

$$\begin{aligned} 1 + \sin(\omega(\zeta)) &= 1 + \frac{1}{2}c_1\zeta + (\frac{c_2}{2} - \frac{c_1^2}{4})\zeta^2 \\ &+ \left(\frac{5c_1^3}{48} + \frac{c_3 - c_1c_2}{2}\right)\zeta^3 + \left(\frac{c_4 - c_1c_3}{2} + \frac{5c_1^2c_2}{16} - \frac{c_2^2}{4} - \frac{c_1^4}{32}\right)\zeta^4 + \cdots . \end{aligned} \tag{3.6}$$

Comparing the coefficients of $\zeta,\ \zeta^2,\ \zeta^3$ between the Eqs (3.4) and (3.6), we obtain

$$a_2 = \frac{c_1}{2(1+\vartheta)}, \tag{3.7}$$

$$\frac{1}{2}(2+\vartheta)[2a_3 - (1-\vartheta)a_2^2] = \frac{c_2}{2} - \frac{c_1^2}{4}, \tag{3.8}$$

$$\frac{(3+\vartheta)}{6}[(1-\vartheta)(2-\vartheta)a_2^3 - 6(1-\vartheta)a_2a_3 + 6a_4] = \frac{5c_1^3}{48} + \frac{c_3}{2} - \frac{c_1c_2}{2}. \tag{3.9}$$

Applying Lemma 1, we easily get

$$|a_2| \le \frac{1}{1+\vartheta},$$

$$a_3 = \frac{1}{2(2+\vartheta)}\left[c_2 - c_1^2\left(\frac{3\vartheta^2 + 5\vartheta}{4(1+\vartheta)^2}\right)\right], \tag{3.10}$$

$$|a_3| = \frac{1}{2(2+\vartheta)}\left|c_2 - c_1^2\left(\frac{3\vartheta^2 + 5\vartheta}{4(1+\vartheta)^2}\right)\right| = \frac{1}{2(2+\vartheta)}\left|c_2 - \nu c_1^2\right|,$$

where $\nu = \frac{3\vartheta^2+5\vartheta}{4(1+\vartheta)^2}$. Now by applying Lemma 2, we get

$$|a_3| \leq \frac{1}{2+\vartheta} \max\left\{1, \left|\frac{\vartheta^2+\vartheta-2}{2(1+\vartheta)^2}\right|\right\}.$$

From (3.7) and (3.10), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{2(2+\vartheta)}\left[c_2 - c_1^2\left(\frac{3\vartheta^2+5\vartheta}{4(1+\vartheta)^2}\right) - c_1^2 \frac{2\mu(2+\vartheta)}{4(1+\vartheta)^2}\right] \\ &= \frac{1}{2(2+\vartheta)}\left[c_2 - c_1^2\left(\frac{3\vartheta^2+5\vartheta+2\mu(2+\vartheta)}{4(1+\vartheta)^2}\right)\right] \\ &= \frac{1}{2(2+\vartheta)}\left\{c_2 - \nu c_1^2\right\}, \end{aligned} \tag{3.11}$$

where

$$\nu := \frac{3\vartheta^2+5\vartheta+2\mu(2+\vartheta)}{4(1+\vartheta)^2}.$$

Our result now follows by an application of Lemma 2 to get

$$|a_3 - \mu a_2^2| \leq \frac{1}{2+\vartheta} \max\left\{1, \left|\frac{\vartheta^2+\vartheta-2+2\mu(2+\vartheta)}{2(1+\vartheta)^2}\right|\right\}. \tag{3.12}$$

Hence the proof is complete. □

Remark 1.
By taking $\mu = 1$, we have $|a_3 - a_2^2| \leq \frac{1}{2+\vartheta} \max\left\{1, \left|\frac{\vartheta^2+3\vartheta+2}{2(1+\vartheta)^2}\right|\right\}$.
If $\vartheta = 0$ and $f \in \mathcal{S}^*_{sin}$, then we get $|a_3 - a_2^2| \leq \frac{1}{2}$ and if $\vartheta = 1$ and $f \in \mathcal{R}_{sin}$, we get $|a_3 - a_2^2| \leq \frac{1}{3}$.

Theorem 6. *If the function $f \in \mathcal{RS}^*_{sin}(\vartheta)$ is given by* (1.1)*, with $\mu \in \mathbb{R}$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{-1}{2(2+\vartheta)}\left(\dfrac{\vartheta^2+\vartheta-2}{(1+\vartheta)^2} + \dfrac{2\mu(2+\vartheta)}{(1+\vartheta)^2}\right), & \text{if} \quad \mu < \sigma_1, \\ \dfrac{1}{2+\vartheta}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{1}{2(2+\vartheta)}\left(\dfrac{\vartheta^2+\vartheta-2}{(1+\vartheta)^2} + \dfrac{2\mu(2+\vartheta)}{(1+\vartheta)^2}\right), & \text{if} \quad \mu > \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{-3\vartheta^2-5\vartheta}{2(2+\vartheta)} \qquad \text{and} \qquad \sigma_2 := \frac{\vartheta^2+3\vartheta+4}{2(2+\vartheta)}.$$

Proof. From (3.11), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{2(2+\vartheta)}\left[c_2 - \left(\frac{3\vartheta^2+5\vartheta}{4(1+\vartheta)^2} + \frac{2\mu(2+\vartheta)}{4(1+\vartheta)^2}\right)c_1^2\right] \\ &= \frac{1}{2(2+\vartheta)}\left(c_2 - \nu c_1^2\right), \end{aligned}$$

where

$$\nu := \frac{3\vartheta^2 + 5\vartheta + 2\mu(2+\vartheta)}{4(1+\vartheta)^2}. \tag{3.13}$$

The assertion of Theorem 6 now follows by an application of Lemma 3. □

4. Coefficient inequalities for the function f^{-1}

Theorem 7. *If the function $f \in \mathcal{RS}^*_{sin}(\vartheta)$ given by (1.1) and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the analytic continuation to $\mathbb{D}$ of the inverse function of f with $|w| < r_0$, where $r_0 \geq \frac{1}{4}$ the radius of the Koebe domain, then for any complex number μ, we have*

$$|d_2| \leq \frac{1}{1+\vartheta}, \tag{4.1}$$

$$|d_3| \leq \frac{1}{(2+\vartheta)} max\left\{1, \left|\frac{\vartheta^2 + 5\vartheta + 6}{2(1+\vartheta)^2}\right|\right\} \tag{4.2}$$

and

$$| d_3 - \mu d_2^2 | \leq \frac{1}{(2+\vartheta)} max\left\{1, \left|\frac{\vartheta^2 + 5\vartheta + 6}{2(1+\vartheta)^2} - \frac{\mu(2+\vartheta)}{(1+\vartheta)^2}\right|\right\}. \tag{4.3}$$

Proof. If

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{4.4}$$

is the inverse function of f, it can be seen that

$$f^{-1}(f(\zeta)) = f\left(f^{-1}(\zeta)\right) = \zeta. \tag{4.5}$$

From Eq (4.5), we have

$$f^{-1}(\zeta + \sum_{n=2}^{\infty} a_n \zeta^n) = \zeta. \tag{4.6}$$

Thus (4.5) and (4.6) yield

$$\zeta + (a_2 + d_2)\zeta^2 + (a_3 + 2a_2 d_2 + d_3)\zeta^3 + \cdots = \zeta, \tag{4.7}$$

hence by equating the corresponding coefficients of ζ, it can be seen that

$$d_2 = -a_2, \tag{4.8}$$

$$d_3 = 2a_2^2 - a_3. \tag{4.9}$$

From relations (3.7), (3.10), (4.8) and (4.9)

$$d_2 = -\frac{c_1}{2(1+\vartheta)}, \tag{4.10}$$

$$d_3 = \frac{2c_1^2}{4(1+\vartheta)^2} - \frac{1}{2(2+\vartheta)}\left[c_2 - \frac{3\vartheta^2+5\vartheta}{4(1+\vartheta)^2}c_1^2\right];$$
$$= -\frac{1}{2(2+\vartheta)}\left[c_2 - \left(\frac{3\vartheta^2+9\vartheta+8}{4(1+\vartheta)^2}\right)c_1^2\right]. \tag{4.11}$$

Taking modulus on both sides and by applying Lemma 2, we get (4.1) and (4.2). For any complex number μ, consider

$$d_3 - \mu d_2^2 = -\frac{1}{2(2+\vartheta)}\left[c_2 - \left(\frac{3\vartheta^2+9\vartheta+8}{4(1+\vartheta)^2} - \frac{\mu(2+\vartheta)}{2(1+\vartheta)^2}\right)c_1^2\right]. \tag{4.12}$$

Taking modulus on both sides and by applying Lemma 2 on the right hand side of (4.12), one can obtain the result as in (4.3). Hence this completes the proof. □

5. Functions defined by Poisson distribution

A variable $\mathcal{X}$ is said to be Poisson distributed if it takes the values $0, 1, 2, 3, \cdots$ with probabilities $e^{-\kappa}, \kappa\frac{e^{-\kappa}}{1!}, \kappa^2\frac{e^{-\kappa}}{2!}, \kappa^3\frac{e^{-\kappa}}{3!}, ...$ respectively, where κ is called the parameter. Thus

$$P(\mathcal{X} = \tau) = \frac{\kappa^r e^{-\kappa}}{\tau!},\ \tau = 0, 1, 2, 3, \cdots.$$

In [42], Porwal introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{I}(\kappa, \zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{(n-1)!} e^{-\kappa}\zeta^n, \quad \zeta \in \mathbb{D},$$

where $\kappa > 0$. We note that by the ratio test the radius of convergence of the above series is infinity. Due to the recent works in [42–45], let the linear operator

$$\mathcal{I}^\kappa(\zeta) : \mathcal{A} \to \mathcal{A}$$

be given by

$$\begin{aligned} \mathcal{I}^\kappa f(\zeta) &= \mathcal{I}(\kappa, \zeta) * f(\zeta) \\ &= \zeta + \sum_{n=2}^{\infty} \frac{\kappa^{n-1}}{(n-1)!} e^{-\kappa} a_n \zeta^n \\ &= \zeta + \sum_{n=2}^{\infty} \Upsilon_n(\kappa) a_n \zeta^n, \end{aligned}$$

where $\Upsilon_n = \Upsilon_n(\kappa) = \frac{\kappa^{n-1}}{(n-1)!}e^{-\kappa}$ and $*$ denote the convolution or the Hadamard product of two series. In particular

$$\Upsilon_2 = \kappa e^{-\kappa} \text{ and } \Upsilon_3 = \frac{\kappa^2}{2}e^{-\kappa}. \tag{5.1}$$

We define the class $\mathcal{RS}^*_{sin}(\vartheta, \Upsilon)$ in the following way:

$$\mathcal{RS}^*_{sin}(\vartheta, \Upsilon) = \{f \in \mathcal{A} : \quad \mathcal{I}^\kappa f \in \mathcal{RS}^*_{sin}(\vartheta)\},$$

where $\mathcal{RS}^{*}_{sin}(\vartheta)$ is given by Definition 1 and

$$\mathcal{I}^{\kappa} f(\zeta) = \zeta + \Upsilon_2 a_2 \zeta^2 + \Upsilon_3 a_3 \zeta^3 + \Upsilon_4 a_4 \zeta^4 \cdots .$$

Proceeding as in Theorems 5 and 6, we could obtain the coefficient estimates for functions of this class $\mathcal{RS}^{*}_{sin}(\vartheta, \Upsilon)$ from the corresponding estimates for functions of the class $\mathcal{RS}^{*}_{sin}(\vartheta)$.

Theorem 8. *Let* $0 \le \vartheta \le 1$ *and* $\mathcal{I}^{\kappa} f(\zeta) = \zeta + \Upsilon_2 a_2 \zeta^2 + \Upsilon_3 a_3 \zeta^3 + \cdots$. *If* $f \in \mathcal{RS}^{*}_{sin}(\vartheta, \Upsilon)$, *then for complex* μ, *we have*

$$|a_3 - \mu a_2^2| \le \frac{1}{(2+\vartheta)\Upsilon_3} \max\left\{1, \left|\frac{\mu(2+\vartheta)\Upsilon_3}{(1+\vartheta)^2\Upsilon_2^2} + \frac{\vartheta^2+\vartheta-2}{2(1+\vartheta)^2}\right|\right\}. \tag{5.2}$$

Proof. Since $f \in \mathcal{RS}^{*}_{sin}(\vartheta, \Upsilon)$, for $\mathcal{I}^{\kappa} f(\zeta) = \zeta + \Upsilon_2 a_2 \zeta^2 + \Upsilon_3 a_3 \zeta^3 + \cdots$ we have

$$[(\mathcal{I}^{\kappa} f(\zeta))']^{\vartheta} \left(\frac{\zeta(\mathcal{I}^{\kappa} f(\zeta))'}{\mathcal{I}^{\kappa} f(\zeta)}\right)^{1-\vartheta} = 1 + \sin(\omega(\zeta)).$$

By (3.4), we can easily get

$$\begin{aligned}
[(\mathcal{I}^{\kappa} f(\zeta))']^{\vartheta} \left(\frac{\zeta(\mathcal{I}^{\kappa} f(\zeta))'}{\mathcal{I}^{\kappa} f(\zeta)}\right)^{1-\vartheta} &= 1 + (1+\vartheta)\Upsilon_2 a_2 \zeta + (2+\vartheta)[2\Upsilon_3 a_3 - (1-\vartheta)\Upsilon_2^2 a_2^2]\frac{\zeta^2}{2} \\
&+ (3+\vartheta)\Big[(1-\vartheta)(2-\vartheta)\Upsilon_2^3 a_2^3 - 6(1-\vartheta)\Upsilon_2\Upsilon_3 a_2 a_3 \\
&+ 6\Upsilon_4 a_4\Big]\frac{\zeta^3}{6} + \cdots .
\end{aligned} \tag{5.3}$$

Thus by (5.3) and (3.6) we have

$$\begin{aligned}
&1 + (1+\vartheta)\Upsilon_2 a_2 \zeta + (2+\vartheta)[2\Upsilon_3 a_3 - (1-\vartheta)\Upsilon_2^2 a_2^2]\frac{\zeta^2}{2} \\
&\quad + (3+\vartheta)[(1-\vartheta)(2-\vartheta)\Upsilon_2^3 a_2^3 - 6(1-\vartheta)\Upsilon_2\Upsilon_3 a_2 a_3 + 6\Upsilon_4 a_4]\frac{\zeta^3}{6} + \cdots \\
&= 1 + \frac{1}{2}c_1\zeta + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)\zeta^2 \\
&\quad + \left(\frac{5c_1^3}{48} + \frac{c_3 - c_1c_2}{2}\right)\zeta^3 + \left(\frac{c_4 - c_1c_3}{2} + \frac{5c_1^2c_2}{16} - \frac{c_2^2}{4} - \frac{c_1^4}{32}\right)\zeta^4 + \cdots .
\end{aligned}$$

Now by equating corresponding coefficients of ζ, ζ^2 and proceeding as in Theorem 5,

$$a_2 = \frac{c_1}{2(1+\vartheta)\Upsilon_2}, \tag{5.4}$$

$$a_3 = \frac{1}{2(2+\vartheta)\Upsilon_3}\left[c_2 - c_1^2\left(\frac{3\vartheta^2+5\vartheta}{4(1+\vartheta)^2}\right)\right]. \tag{5.5}$$

From (5.4) and (5.5), we get

$$\begin{aligned}
a_3 - \mu a_2^2 &= \frac{1}{2(2+\vartheta)\Upsilon_3}\left[c_2 - c_1^2\left(\frac{3\vartheta^2+5\vartheta}{4(1+\vartheta)^2}\right) - c_1^2\frac{2\mu(2+\vartheta)\Upsilon_3}{4(1+\vartheta)^2\Upsilon_2^2}\right] \\
&= \frac{1}{2(2+\vartheta)}\left[c_2 - c_1^2\left(\frac{3\vartheta^2+5\vartheta}{4(1+\vartheta)^2} + \frac{2\mu(2+\vartheta)\Upsilon_3}{4(1+\vartheta)^2\Upsilon_2^2}\right)\right].
\end{aligned} \tag{5.6}$$

Now by an application of Lemma 2 we get the desired result. □

Theorem 9. *Let* $0 \leq \vartheta \leq 1$ *and* $\mathcal{I}^{\kappa} f(\zeta) = \zeta + \Upsilon_2 a_2 \zeta^2 + \Upsilon_3 a_3 \zeta^3 + \cdots$ *, with* $\mu \in \mathbb{R}$*, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{-1}{2(2+\vartheta)\Upsilon_3}\left(\dfrac{\vartheta^2+\vartheta-2}{(1+\vartheta)^2} + \dfrac{2\mu(2+\vartheta)\Upsilon_3}{(1+\vartheta)^2\Upsilon_2^2}\right), & \text{if } \mu < \sigma_1, \\ \dfrac{1}{(2+\vartheta)\Upsilon_3}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{1}{2(2+\vartheta)\Upsilon_3}\left(\dfrac{\vartheta^2+\vartheta-2}{(1+\vartheta)^2} + \dfrac{2\mu(2+\vartheta)\Upsilon_3}{(1+\vartheta)^2\Upsilon_2^2}\right), & \text{if } \mu > \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{-(3\vartheta^2+5\vartheta)}{2(2+\vartheta)}\frac{\Upsilon_2^2}{\Upsilon_3} \qquad \text{and} \qquad \sigma_2 := \frac{\vartheta^2+3\vartheta+4}{2(2+\vartheta)}\frac{\Upsilon_2^2}{\Upsilon_3}.$$

Specially, taking $\Upsilon_2 = \kappa e^{-\kappa}$ and $\Upsilon_3 = \frac{\kappa^2}{2}e^{-\kappa}$, we easily state the above results related with Poisson distribution series.

Using (5.6), and applying Lemma 3 we get desired result.

6. Second Hankel inequality for $f \in \mathcal{RS}^*_{sin}(\vartheta)$

Theorem 10. *If the function* $f \in \mathcal{RS}^*_{sin}(\vartheta)$ *and is given by* (1.1)*, then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{(2+\vartheta)^2}.$$

Proof. Using the Eqs (3.7) and (3.10) in (3.9) it follows that

$$\begin{aligned} a_4 &= \frac{1}{2(3+\vartheta)}\left[c_3 + \left(\frac{(1-\vartheta)(3+\vartheta)}{2(1+\vartheta)(2+\vartheta)} - 1\right)c_1c_2 \right. \\ &\quad \left. + \left(\frac{5}{24} - \frac{(1-\vartheta)(2-\vartheta)(3+\vartheta)}{24(1+\vartheta)^3} - \frac{(1-\vartheta)(3+\vartheta)(3\vartheta^2+5\vartheta)}{8(1+\vartheta)^2(2+\vartheta)}\right){c_1}^3\right]. \end{aligned} \tag{6.1}$$

By simple computation we get,

$$\begin{aligned} a_4 &= \frac{1}{2(3+\vartheta)}\left[c_3 - \left(\frac{3\vartheta^2+8\vartheta+1}{2(1+\vartheta)(2+\vartheta)}\right)c_1c_2 + \left(\frac{13\vartheta^4+56\vartheta^3+55\vartheta^2-2\vartheta-2}{24(1+\vartheta)^3(2+\vartheta)}\right){c_1}^3\right] \\ &= \frac{1}{2(3+\vartheta)}c_3 - \left(\frac{3\vartheta^2+8\vartheta+1}{4(\vartheta^3+6\vartheta^2+11\vartheta+6)}\right)c_1c_2 + \left(\frac{13\vartheta^4+56\vartheta^3+55\vartheta^2-2\vartheta-2}{48(1+\vartheta)^3(2+\vartheta)(3+\vartheta)}\right){c_1}^3. \end{aligned}$$

Thus we establish that the estimate of the second Hankel determinant,

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{1}{16}\left[-\left\{\frac{\vartheta^4+6\vartheta^3+5\vartheta^2+4\vartheta+8}{12(1+\vartheta)^3(2+\vartheta)^2(3+\vartheta)}\right\}{c_1}^4 \right. \\ &\quad \left. -\left\{\frac{4}{(1+\vartheta)(2+\vartheta)^2(3+\vartheta)}\right\}{c_1}^2c_2 - \frac{4}{(2+\vartheta)^2}c_2^2 + \frac{4}{(1+\vartheta)(3+\vartheta)}c_1c_3\right]. \end{aligned} \tag{6.2}$$

Since $p \in \mathcal{P}$ it follows that $p\left(e^{-i\theta}z\right) \in \mathcal{P}$; $(\theta \in \mathbb{R})$, hence we may assume without loss of generality that $c := c_1 \geq 0$. Substituting the values of c_2 and c_3 as in Lemma 4 in (6.2), we get

$$\begin{aligned}|a_2a_4 - a_3^2| &= \frac{1}{16}\Bigg| -\left(\frac{\vartheta^2+2\vartheta+5}{12(1+\vartheta)^3(3+\vartheta)}\right)c^4 \\ &- \left\{\frac{c^2}{(1+\vartheta)(3+\vartheta)} + \frac{(4-c^2)}{(2+\vartheta)^2}\right\}(4-c^2)x^2 \\ &+ \frac{2}{(1+\vartheta)(3+\vartheta)}c(4-c^2)(1-|x|^2)y\Bigg|. \qquad (6.3)\end{aligned}$$

Replacing $|x|$ by δ and by making use of the triangle inequality and the fact that $|y| \leq 1$ in the above expression, we get

$$\begin{aligned}|a_2a_4 - a_3^2| \leq \frac{1}{16}\Bigg[&\left(\frac{\vartheta^2+2\vartheta+5}{12(1+\vartheta)^3(3+\vartheta)}\right)c^4 + \frac{2c}{(1+\vartheta)(3+\vartheta)}(4-c^2) \\ &+ \left\{\frac{c^2}{(1+\vartheta)(3+\vartheta)} - \frac{2c}{(1+\vartheta)(3+\vartheta)} + \frac{(4-c^2)}{(2+\vartheta)^2}\right\}(4-c^2)\delta^2\Bigg] = \mathcal{F}(c,\delta). \qquad (6.4)\end{aligned}$$

We shall now maximize $\mathcal{F}(c,\delta)$, for $(c,\delta) \in [0,2]\times[0,1]$. Differentiating $\mathcal{F}(c,\delta)$, partially with respect to δ we get

$$\frac{\partial\mathcal{F}}{\partial\delta} = \frac{1}{8}\left\{\frac{c^2}{(1+\vartheta)(3+\vartheta)} - \frac{2c}{(1+\vartheta)(3+\vartheta)} + \frac{(4-c^2)}{(2+\vartheta)^2}\right\}(4-c^2)\delta. \qquad (6.5)$$

For $0 \leq \delta \leq 1$, and for any fixed $c \in [0,2]$, we observe that $\frac{\partial\mathcal{F}}{\partial\delta} > 0$. Thus $\mathcal{F}(c,\delta)$ is an increasing function of δ, and for $c \in [0,2]$, $\mathcal{F}(c,\delta)$ has a maximum value at $\delta = 1$. So, we have

$$\max_{0\leq\delta\leq1}\mathcal{F}(c,\delta) = \mathcal{F}(c,1) = \mathcal{G}(c).$$

On a simplification, we find that

$$\begin{aligned}\mathcal{F}(c,\delta) = \mathcal{F}(c,1) = \mathcal{G}(c) = \frac{1}{16}\Bigg[&\left(\frac{\vartheta^2+2\vartheta+5}{12(1+\vartheta)^3(3+\vartheta)}\right)c^4 \\ &+ \left\{\frac{c^2}{(1+\vartheta)(3+\vartheta)} + \frac{(4-c^2)}{(2+\vartheta)^2}\right\}(4-c^2)\Bigg]. \qquad (6.6)\end{aligned}$$

Equivalently,

$$\mathcal{F}(c,\delta) = \mathcal{F}(c,1) = \mathcal{G}(c) = \frac{1}{16}\left[\left(\frac{\vartheta^2+2\vartheta+5}{12(1+\vartheta)^3(3+\vartheta)}\right)c^4 + \frac{c^2(4-c^2)}{(1+\vartheta)(3+\vartheta)} + \frac{(4-c^2)^2}{(2+\vartheta)^2}\right].$$

Now we note that

$$\mathcal{G}'(c) = \frac{1}{16}\left[4\left(\frac{\vartheta^2+2\vartheta+5}{12(1+\vartheta)^3(3+\vartheta)}\right)c^3 + \frac{8c-4c^3}{(1+\vartheta)(3+\vartheta)} + \frac{(4c^3-16c)}{(2+\vartheta)^2}\right].$$

If $\mathcal{G}'(c) = 0$, then the root is $c = 0$. Also, we have

$$\mathcal{G}''(c) = \frac{1}{16}\left[\left(\frac{\vartheta^2+2\vartheta+5}{(1+\vartheta)^3(3+\vartheta)}\right){c_1}^3 + \left(\frac{8-12c^2}{(1+\vartheta)(3+\vartheta)} + \frac{12c^2}{(2+\vartheta)^2} - \frac{16}{(2+\vartheta)^2}\right)\right]$$
$$= \frac{1}{16}\left[\left(\frac{\vartheta^2+2\vartheta+5}{(1+\vartheta)^3(3+\vartheta)}\right){c_1}^3 - 12\left(\frac{1}{(1+\vartheta)(3+\vartheta)(2+\vartheta)^2}\right)c^2 - \frac{8(\vartheta^2+4\vartheta+2)}{(1+\vartheta)(3+\vartheta)(2+\vartheta)^2}\right]$$

is negative for $c = 0$, which means that the function $\mathcal{G}(c)$ can take the maximum value at $c = 0$, also which is

$$|a_2a_4 - a_3^2| \le \frac{1}{(2+\vartheta)^2}.$$

□

Remark 2.

When $\vartheta = 1$, then $f \in \mathcal{R}_{sin}$ and we get

$$|a_2a_4 - a_3^2| \le \frac{1}{9}.$$

Also by fixing $\vartheta = 0$, then $f \in \mathcal{S}^*_{sin}$ and we get

$$|a_2a_4 - a_3^2| \le \frac{1}{4}.$$

7. Conclusions and observations

In the present paper, we mainly get upper bounds of the second-order Hankel determinant of new class of starlike functions connected with the sine function. Also, we can discuss the related research of the coefficient problem and Fekete-Szegö inequality. Further for this function class we state the application of Poisson distribution related to Fekete-Szegö inequality. By fixing $\vartheta = 0$ and $\vartheta = 1$ we can state the above results for $f \in \mathcal{R}_{sin}$ and $f \in \mathcal{S}^*_{sin}$. For motivating further researches on the subject-matter of this, we have chosen to draw the attention of the interested readers toward a considerably large number of related recent publications (see, for example, [46–51]) and developments in the area of mathematical analysis, which are not as closely related to the subject-matter of this presentation as many of the other publications cited here. In conclusion, with an opinion mostly to encouraging and inspiring further researches on applications of the basic (or $q-$) analysis and the basic (or $q-$) calculus in geometric function theory of complex analysis along the lines (see [52]), considering our present investigation and based on recently-published works on the Fekete-Szegö and Hankel determinant problem (see, for details, [8,23,47–53], one can extend or generalize our results for $f \in \mathcal{RS}_{sin}(\vartheta)$ is left as an exercise to interested readers. In addition, we choose to reiterate an important observation, which was presented in the recently-published review-cum-expository review article by Srivastava ([52], p. 340, [54] pp. 1511–1512), who pointed out the fact that the results for the above-mentioned or new $q-$ analogues can easily (and possibly trivially) be translated into the corresponding results for the so-called $(p;q)-$ analogues (with $0 < |q| < p \le 1$) by applying some obvious parametric and argument variations with the additional parameter p being redundant.

Acknowledgements

The first-named author (Huo Tang) was partly supported by the Natural Science Foundation of the People's Republic of China under Grant 11561001, the Program for Young Talents of Science and

Technology in Universities of Inner Mongolia Autonomous Region under Grant NJYT-18-A14, the Natural Science Foundation of Inner Mongolia of the People's Republic of China under Grant 2018MS01026, and the Higher School Foundation of Inner Mongolia of the People's Republic of China under Grant NJZY20200, the Program for Key Laboratory Construction of Chifeng University (No.CFXYZD202004), the Research and Innovation Team of Complex Analysis and Nonlinear Dynamic Systems of Chifeng University (No.cfxykycxtd202005) and the Youth Science Foundation of Chifeng University (No.cfxyqn202133).

Conflict of interest

The authors declare that they have no competing interests.

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