AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 5314–5327.
DOI:10.3934/math.2022296
Received: 24 November 2021
Revised: 31 December 2021
Accepted: 03 January 2022
Published: 06 January 2022

Research article

Divisibility of Fibonomial coefficients in terms of their digital representations and applications

Phakhinkon Napp Phunphayap[a] **and Prapanpong Pongsriiam***

Department of Mathematics, Faculty of Science, Silpakorn University, Nakhon Pathom, 73000, Thailand

* **Correspondence:** Email: prapanpong@gmail.com, pongsriiam_p@silpakorn.edu.

Abstract: We give a characterization for the integers $n \geq 1$ such that the Fibonomial coefficient $\binom{pn}{n}_F$ is divisible by p for any prime $p \neq 2, 5$. Then we use it to calculate asymptotic formulas for the number of positive integers $n \leq x$ such that $p \mid \binom{pn}{n}_F$. This completes the study on this problem for all primes p.

Keywords: Fibonomial coefficient; Fibonacci number; digit; sum of digits function; divisibility; asymptotic; binomial coefficient
Mathematics Subject Classification: 11B39, 11A63, 11N37, 11B65

1. Introduction

The *Fibonacci sequence* $(F_n)_{n\geq 1}$ is given by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with the initial values $F_1 = F_2 = 1$. For each $m \geq 1$ and $1 \leq k \leq m$, the *Fibonomial coefficient* $\binom{m}{k}_F$ is defined by

$$\binom{m}{k}_F = \frac{F_1F_2F_3\cdots F_m}{(F_1F_2F_3\cdots F_k)(F_1F_2F_3\cdots F_{m-k})} = \frac{F_{m-k+1}F_{m-k+2}\cdots F_m}{F_1F_2F_3\cdots F_k}.$$

As usual, if $m = k$, then the empty product $F_1F_2\cdots F_{m-k}$ is defined to be 1, and similar to the binomial coefficients, we let $\binom{m}{k}_F = 1$ if $k = 0$ and $\binom{m}{k}_F = 0$ if $k > m$. Then it is well known that $\binom{m}{k}_F$ is always an integer for every $m \geq 1$ and $k \geq 0$.

There has been some interest in the study of certain generalizations of binomial coefficients such as the Fibonomial or Lucasnomial coefficients. For instance, Marques and Trojovský [9] determined the integers $n \geq 1$ such that $\binom{3n}{n}_F$ is divisible by 3. Then Ballot [1] largely extended Marques and

[a]Napp is his nickname his parents gave him and he would like to use it as a middle name too. His first and last names read like Pa-kin-gorn Poon-pa-yap. He is the same person as Phakhinkon Phunphayap, one of the authors of the articles [13, 14].

Trojovský's results by characterizing all integers $n \geq 1$ such that $\binom{pn}{n}_U$ is divisible by p for any nondegenerate fundamental Lucas sequence U and $p = 2, 3$ and for $p = 5, 7$ in the case $U = F$. Ballot [1] also proved that $E_3(x) = O(\log x)$ and $E_7(x) = O\left((\log x)^3\right)$. Here and throughout this article, $E_p(x)$ denotes the number of positive integers n less than or equal to x such that $\binom{pn}{n}_F$ is not divisible by p. In other words,

$$E_p(x) = \sum_{\substack{1 \leq n \leq x \\ p \nmid \binom{pn}{n}_F}} 1.$$

In particular, we [13, 14] have recently provided an explicit formula for the p-adic valuation of certain Fibonomial coefficients, and have used it in the investigation of the integers $n \geq 1$ such that $\binom{p^a n}{n}_F$ is divisible by p for any prime $p \equiv \pm 2 \pmod 5$ and any integer $a \geq 1$, and also for primes $p \equiv \pm 1 \pmod 5$ and $a = 1$ in terms of the sum of digit function.

In this article, we give characterizations for the integers $n \geq 1$ such that $\binom{pn}{n}_F$ is divisible by p for any prime $p \neq 2, 5$ in terms of the digital representation of n. Then we use it in the calculation for asymptotic formulas of $E_p(x)$ for all primes p. This extends many results in the literature which focus only on small primes $p \leq 7$.

We organize this article as follows. In Section 2, we recall some definitions and useful results. In Section 3, we prove our main theorems and give some examples. For more information on Fibonacci numbers, Fibonomial coefficients, and generalizations, we refer the reader to some recent articles by Ballot [2–4], Chu and Kiliç [5], Kiliç and Akkus [7], Kiliç and Prodinger [8], Onphaeng and Pongsriiam [10–12], and Pongsriiam [15, 16].

2. Preliminaries and lemmas

Throughout this article, unless stated otherwise, x is a positive real number, p is a prime, a, b, k, m, n, q, r are integers, $m, n \geq 1$, $q \geq 2$, $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\{x\}$ is the *fractional part* of x given by $\{x\} = x - \lfloor x \rfloor$, $a \bmod m$ is the least nonnegative residue of a modulo m, and $\log x$ is the natural logarithm of x. The *p-adic valuation* of n, denoted by $\nu_p(n)$, is the exponent of p in the prime factorization of n. In addition, the *order (or the rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest positive integer m such that $n \mid F_m$. Furthermore, we define $s_q(n)$ to be the sum of digits of n when n is written in base q, that is, if

$$n = (a_k a_{k-1} \ldots a_0)_q = a_k q^k + a_{k-1} q^{k-1} + \cdots + a_0 \text{ where } 0 \leq a_i < q \text{ for every } i,$$

then $s_q(n) = a_k + a_{k-1} + \cdots + a_0$. Next, we recall some well known and useful results for the reader's convenience.

Lemma 1. *The following statements hold.*

(i) $n \mid F_m$ *if and only if* $z(n) \mid m$.
(ii) $z(p) \mid p + 1$ *if and only if* $p \equiv \pm 2 \pmod 5$.
(iii) $z(p) \mid p - 1$ *if and only if* $p \equiv \pm 1 \pmod 5$.
(iv) *If* $p \neq 5$, *then* $\gcd(z(p), p) = 1$.

Proof. These are well known. See, for example, in [13, Lemma 1] for more details. □

We will deal with a lot of calculations involving the floor function. So it is useful to recall the following results, which will be applied throughout this article without further reference.

Lemma 2. *For $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, the following statements hold.*

(i) $\lfloor k + x \rfloor = k + \lfloor x \rfloor$,
(ii) $\{k + x\} = \{x\}$,
(iii) $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} -1, & \text{if } x \notin \mathbb{Z}; \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$
(iv) $0 \le \{x\} < 1$ *and* $\{x\} = 0$ *if and only if* $x \in \mathbb{Z}$,
(v) $\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\} + \{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\} + \{y\} \ge 1, \end{cases}$
(vi) $\left\lfloor \frac{\lfloor x \rfloor}{k} \right\rfloor = \left\lfloor \frac{x}{k} \right\rfloor$ *for* $k \ge 1$.

Proof. These are well known and can be proved easily. For more details, see in [6, Chapter 3]. We also refer the reader to [11] for a nice application of these properties. □

The next three lemmas are important tools for obtaining the characterizations of the integers n such that $\binom{pn}{n}_F$ is divisible by p.

Lemma 3. [14, Corollary 13] *Suppose that $p \ne 2, 5$ and a, n are positive integers. If $n \equiv 0 \pmod{z(p)}$, then $p \mid \binom{p^a n}{n}_F$.*

Lemma 4. [14, Corollary 14] *Let $p \ne 2, 5$, $p \equiv \pm 2 \pmod 5$, $a, n \in \mathbb{N}$, $r = p^a n \bmod z(p)$, $s = n \bmod z(p)$, $A = \left\lfloor \frac{n(p^a - 1)}{p^{\nu_p(n)} z(p)} \right\rfloor$, and $n \not\equiv 0 \pmod{z(p)}$. Then the following statements hold.*

(i) *Assume that a is odd and $p \nmid n$. If $r < s$, then $p \mid \binom{p^a n}{n}_F$. If $r \ge s$, then $p \mid \binom{p^a n}{n}_F$ if and only if $s_p(A) \ge \frac{a+1}{2}(p - 1)$.*
(ii) *Assume that a is odd and $p \mid n$. If $r \ne s$, then $p \mid \binom{p^a n}{n}_F$. If $r = s$, then $p \mid \binom{p^a n}{n}_F$ if and only if $s_p(A) \ge \frac{a+1}{2}(p - 1)$.*

Lemma 5. [14, Corollary 15] *Let $p \ne 2, 5$, $p \equiv \pm 1 \pmod 5$, and $A = \frac{n(p-1)}{p^{\nu_p(n)} z(p)}$. Then $p \mid \binom{pn}{n}_F$ if and only if $s_p(A) \ge p - 1$.*

Lemma 6. *Let $k \ge 0$, $q \ge 2$, $1 \le a \le q - 1$, and $0 \le b \le q - 1$. Then*

$$s_q(a(q - 1)q^k + bq^k) \ge b.$$

Proof. When $b = 0$, the result is obvious. So we assume that $b \ge 1$. If $a = 1$, then we write $a(q - 1)q^k + bq^k = q^{k+1} + (b - 1)q^k$. If $a \ge 2$ and $b \le a - 1$, then we write $a(q - 1)q^k + bq^k = (a - 1)q^{k+1} + (q - a + b)q^k$. If $a \ge 2$ and $b \ge a$, then we write $a(q - 1)q^k + bq^k = aq^{k+1} + (b - a)q^k$. In each case, $s_q(a(q - 1)q^k + bq^k)$ is equal to, respectively, $1 + b - 1 = b$, $a - 1 + q - a + b = q + b - 1$, and $a + b - a = b$. In any case, it is at least b. □

Lemma 7. *Let $p \ge 3$, $p \equiv \pm 2 \pmod 5$, $0 \le a \le \frac{p-1}{2}$, and $1 \le k \le \frac{z(p)}{2}$. Then $az(p) + k \equiv 0 \pmod p$ if and only if $a = \frac{p-1}{2}$, $z(p)$ is even, and $k = \frac{z(p)}{2}$. In particular, if $a < \frac{p-1}{2}$, then $az(p) + k \not\equiv 0 \pmod p$.*

Proof. From the assumption, we have

$$0 < az(p) + k \leq \left(\frac{p-1}{2}\right) z(p) + \frac{z(p)}{2} = \frac{pz(p)}{2}.$$

Suppose that $az(p) + k \equiv 0 \pmod{p}$. Then $az(p) + k = np$ for some $1 \leq n \leq \frac{z(p)}{2}$. Since $p \equiv \pm 2 \pmod{5}$, we obtain by Lemma 1 that $p \equiv -1 \pmod{z(p)}$. Then $k \equiv az(p) + k \equiv np \equiv -n \pmod{z(p)}$. So there exists $m \in \mathbb{N}$ such that $k = mz(p) - n$. Therefore

$$\frac{z(p)}{2} \geq k = mz(p) - n \geq z(p) - \frac{z(p)}{2} = \frac{z(p)}{2}.$$

This implies that $k = \frac{z(p)}{2}$, $z(p)$ is even, $m = 1$, and $n = \frac{z(p)}{2}$. Since $az(p) + k = np$, we also obtain $a = \frac{p-1}{2}$. The converse can be verified easily. This completes the proof. □

We introduce the following notation for convenience.

Definition 8. Let q and i be integers such that $q \geq 2$ and $0 \leq i \leq q-1$. We define

$$H(q,i) = \{(a_m a_{m-1} \cdots a_0)_q \mid m \in \mathbb{N} \cup \{0\},\ a_k \leq a_{k-1} \text{ for all } 1 \leq k \leq m, \text{ and } a_0 = i\}.$$

In other words, $H(q,i)$ is the set of nonnegative integers n such that the q-adic representation of n is increasing (from the left to the right), and the last digit (the rightmost digit) is equal to i.

For example, if $q = 10$ and $i = 3$, then 111122233 and 11111333 are in $H(10,3)$ but 213 and 1234 are not in $H(10,3)$.

Definition 9. For positive integers k and q, we define

$$t(q,k) = \left\lfloor \frac{k(q-1)}{z(q)} \right\rfloor.$$

The next lemma is usually called stars and bars problem. Recall that if a set A has exactly n distinct elements, then the number of all possible ways in choosing m elements from A with repetitions allowed is $\binom{n+m-1}{m}$. We have the following lemma.

Lemma 10. *Let $k \geq 1$, $q \geq 2$, and $1 \leq t \leq q-1$ be integers. Then*

$$\#\{(a_k a_{k-1} \cdots a_1)_q \in \bigcup_{i=1}^{t} H(q,i) \mid a_k \neq 0\} = \binom{k+t-1}{k}.$$

Proof. This is stars and bars problem. The set A is $\{1, 2, 3, \ldots, t\}$. We would like to choose k elements from A with repetitions allowed. So the number of ways, as recalled above, is $\binom{t+k-1}{k}$, which proves this lemma. □

Lemma 11. *Let $q \geq 2$ and $1 \leq t \leq q-1$ be integers. Then*

$$\sum_{\substack{0 \leq m < q^r \\ m \in \bigcup_{i=0}^{t} H(q,i)}} 1 = \binom{r+t}{r}$$

Consequently,

$$\sum_{\substack{0\le m<q^r \\ m\in\bigcup_{i=0}^{t} H(q,i)}} 1 = \frac{r^t}{t!} + O(r^{t-1}),$$

where the implied constant depends at most on t.

Proof. The conditions $0 \le m < q^r$ and $m \in \bigcup_{i=0}^{t} H(q,i)$ mean that $m = (a_r a_{r-1} \cdots a_1)_q$ and $0 \le a_r \le a_{r-1} \le \cdots \le a_1 \le t$. So this is also stars and bars problem. The set A is $\{0, 1, 2, \dots, t\}$. We would like to choose r elements from A with repetitions allowed. Therefore the number of ways is $\binom{t+1+r-1}{r} = \binom{r+t}{r}$, which proves the first part. Next,

$$\binom{r+t}{r} = \frac{(r+t)(r+(t-1))\cdots(r+1)}{t!} = \frac{r^t}{t!} + P(r),$$

where $P(r)$ is a polynomial in r of degree $t-1$ with the coefficients depending only on t. Therefore $P(r) = O\left(r^{t-1}\right)$ and the implied constant depends at most on t. This completes the proof. □

3. Main results

In this section, we begin with a property of the sum of digit function. Then we use it in the study of the divisibility $p \mid \binom{pn}{n}_F$ in terms of the digital representation of n. After that, we determine an asymptotic formula for $E_p(x)$.

Theorem 12. *Let $m \ge 0$, $q \ge 2$, and $1 \le k \le z(q) - 1$. Then*

$$s_q\left((q-1)m + t(q,k)\right) < q-1 \text{ if and only if } m \in \bigcup_{i=0}^{t(q,k)} H(q,i).$$

Proof. Let $H = \bigcup_{i=0}^{t(q,k)} H(q,i)$ and $t = t(q,k)$. Since $k < z(q)$, we see that $t < q-1$. If $m = 0$, then $m \in H(q,0) \subseteq H$ and $s_q((q-1)m + t) = s_q(t) = t < q-1$, so we are done. From this point on, we assume that $m \ge 1$. To prove this theorem, we first show that

$$\text{if } m \notin H, \text{ then } s_q\left((q-1)m + t\right) \ge q-1. \tag{3.1}$$

We prove (3.1) by induction on r where r is the number of digits in the q-adic expansion of m. For $r = 1$, we let $m = a$, $1 \le a \le q-1$, $a \notin H$, and write

$$(q-1)m + t = (a-1)q + (q-a+t).$$

Observe that $i \in H(q,i) \subseteq H$ for each $0 \le i \le t$. Since $a \notin H$, we see that $a > t$ which implies $0 \le q-a+t \le q-1$. Therefore $s_q\left((q-1)m+t\right) = a-1+q-a+t \ge q-1$. Next, let $r \ge 1$ and suppose that (3.1) holds for any $m \in \mathbb{N}$ such that the number of digits of m in its q-adic expansion is less than or equal to r. Assume that $m = (a_{r+1} a_r \cdots a_1)_q$, $a_{r+1} \ne 0$, $0 \le a_i < q$ for all i, and $m \notin H$. Let $m_1 = (a_r a_{r-1} \cdots a_1)_q$.

Case 1. $m_1 \in H$. If $r = 1$, let $m_2 = 0$ and if $r \ge 2$, we let $m_2 = (a_{r-1} a_{r-2} \cdots a_1)_q$. Then we write $(q-1)m + t$ as $(q-1)(a_{r+1}q^r + a_r q^{r-1} + m_2) + t$, which is equal to

$$(a_{r+1}-1)q^{r+1} + (q - a_{r+1} + a_r)q^r - a_r q^{r-1} + (q-1)m_2 + t. \tag{3.2}$$

Since $m_1 \in H$, $a_r \le a_i \le t$ for all $1 \le i \le r$. So we have

$$m_2 \le t(1 + q + q^2 + \cdots + q^{r-2}) = t\left(\frac{q^{r-1} - 1}{q - 1}\right),$$

$$m_2 \ge a_r(1 + q + q^2 + \cdots + q^{r-2}) = a_r\left(\frac{q^{r-1} - 1}{q - 1}\right).$$

Therefore

$$(q-1)m_2 + t \le tq^{r-1} \text{ and } (q-1)m_2 + t \ge a_r q^{r-1} + (t - a_r) \ge a_r q^{r-1}.$$

Thus

$$0 \le -a_r q^{r-1} + (q-1)m_2 + t \le (t - a_r)q^{r-1} < q^r. \tag{3.3}$$

Since $m \notin H$ and $m_1 \in H$, $a_{r+1} > a_r$. Thus $0 \le q - a_{r+1} + a_r < q$. From this and from (3.2) and (3.3), we obtain that $s_q((q-1)m + t)$ is equal to

$$\begin{aligned}
&s_q((a_{r+1} - 1)q^{r+1} + (q - a_{r+1} + a_r)q^r) + s_q(-a_r q^{r-1} + (q-1)m_2 + t) \\
&\ge s_q((a_{r+1} - 1)q^{r+1} + (q - a_{r+1} + a_r)q^r) \\
&= (a_{r+1} - 1) + (q - a_{r+1} + a_r) = q - 1 + a_r \ge q - 1.
\end{aligned}$$

Case 2. $m_1 \notin H$. Since $(q-1)m_1 + t < (q-1)q^r + q - 1 < q^{r+1}$, we write $(q-1)m_1 + t = (b_{r+1}b_r \cdots b_1)_q$ where b_{r+1} may be zero. Since the number of digits in the q-adic representation of m_1 is less than or equal to r, we can apply the induction hypothesis on m_1 to obtain

$$q - 1 \le s_q((q-1)m_1 + t) = s_q((b_{r+1}b_r \cdots b_1)_q) = \sum_{i=1}^{r+1} b_i. \tag{3.4}$$

Next we write

$$\begin{aligned}
(q-1)m + t = (q-1)(a_{r+1}q^r + m_1) + t &= (q-1)a_{r+1}q^r + (b_{r+1}b_r \cdots b_1)_q \\
&= (q-1)a_{r+1}q^r + b_{r+1}q^r + (b_r \cdots b_1)_q.
\end{aligned}$$

By the above equation, Lemma 6, and (3.4), we obtain $s_q((q-1)m + t) =$

$$s_q((q-1)a_{r+1}q^r + b_{r+1}q^r) + s_q((b_r b_{r-1} \cdots b_1)_q) \ge b_{r+1} + \sum_{i=1}^{r} b_i \ge q - 1.$$

This proves (3.1). To prove the converse, assume that $m \in H$ and let $a = m \bmod q$ be the least nonnegative residue of m modulo q. Then a is the last digit of m in its q-adic expansion. Since $m \ge 1$ and $m \in H$, we see that $1 \le a \le t$ and $m \in H(q, a)$. So the possible digits in the q-adic representation of m with nonzero leading digit are $1, 2, 3, \ldots, a$. Therefore we can write m as

$$\sum_{i=0}^{n_a} aq^i + \sum_{i=n_a+1}^{n_a+n_{a-1}} (a-1)q^i + \sum_{i=n_a+n_{a-1}+1}^{n_a+n_{a-1}+n_{a-2}} (a-2)q^i + \cdots + \sum_{i=n_a+\cdots+n_2+1}^{n_a+n_{a-1}+\cdots+n_1} q^i, \tag{3.5}$$

where $n_1, n_2, \ldots, n_a$ are nonnegative integers and the empty sum is defined to be zero. So, for instance, if $a-1$ does not appear as a digit in the q-adic representation of m, then we let $n_{a-1} = 0$ and the second sum in (3.5) is 0. For $0 \le i \le a-1$, let $d_i = \left(\sum_{i+1 \le j \le a} n_j\right) + 1$. By (3.5), m is equal to

$$\begin{aligned}
&\sum_{j=0}^{n_a} aq^j + \sum_{j=0}^{n_{a-1}-1} (a-1)q^{d_{a-1}+j} + \sum_{j=0}^{n_{a-2}-1} (a-2)q^{d_{a-2}+j} + \cdots + \sum_{j=0}^{n_1-1} q^{d_1+j} \\
&= \sum_{j=0}^{n_a} aq^j + \sum_{i=1}^{a-1} \sum_{j=0}^{n_i-1} iq^{d_i+j} \\
&= a\sum_{j=0}^{n_a} q^j + \left(\sum_{i=1}^{a-1}\left(iq^{d_i}\sum_{j=0}^{n_i-1} q^j\right)\right) \\
&= a\left(\frac{q^{d_{a-1}}-1}{q-1}\right) + \left(\sum_{i=1}^{a-1} iq^{d_i}\left(\frac{q^{n_i}-1}{q-1}\right)\right) \\
&= \frac{1}{q-1}\left(aq^{d_{a-1}} - a + \sum_{i=1}^{a-1}\left(iq^{d_{i-1}} - iq^{d_i}\right)\right) \\
&= \frac{1}{q-1}\left(\left(\sum_{i=0}^{a-1} q^{d_i}\right) - a\right).
\end{aligned}$$

Then $(q-1)m + t = \sum_{i=0}^{a-1} q^{d_i} - a + t$. Since $d_i \ge 1$ for all i and $0 \le t - a < q - 1$, we see that $s_q((q-1)m + t)$ is equal to

$$s_q\left(\sum_{i=0}^{a-1} q^{d_i}\right) + s_q(t-a) = a + t - a < q - 1.$$

This completes the proof. □

Recall that we [14] previously gave a characterization for the divisibility $p \mid \binom{pn}{n}_F$ in terms of the sum of digits function. We are now ready to characterize it in terms of a digital representation. We first prove it for the prime $p \equiv \pm 2 \pmod 5$ in the next theorem.

Theorem 13. *Let p be an odd prime, $p \equiv \pm 2 \pmod 5$, and n a positive integer. Then $p \mid \binom{pn}{n}_F$ if and only if n is not of the form*

$$z(p)m + k \text{ where } 1 \le k \le \frac{z(p)}{2} \text{ and } m \in \bigcup_{i=0}^{t(p,k)} H(p,i). \tag{3.6}$$

Proof. We first assume that $p \nmid \binom{pn}{n}_F$. To show that n can be written as in (3.6), let $k = n \bmod z(p)$, $t = t(p,k)$, and $H = \bigcup_{i=0}^{t} H(p,i)$.

Case 1. $p \mid n$. We write $n = p^a\ell$ where $a, \ell \in \mathbb{N}$ and $p \nmid \ell$. By Lemma 3, we obtain $n \not\equiv 0 \pmod{z(p)}$. Then by Lemma 4(ii), we have

$$pn \equiv n \pmod{z(p)} \text{ and } s_p = \left(\left\lfloor \frac{n(p-1)}{p^a z(p)} \right\rfloor\right) = s_p\left(\left\lfloor \frac{\ell(p-1)}{z(p)} \right\rfloor\right) < p - 1.$$

By Lemma 1, we know that $p \equiv -1 \pmod{z(p)}$, and so $n \equiv pn \equiv -n \pmod{z(p)}$. Therefore $z(p) \mid 2n$ and $z(p) \nmid n$. This implies

$$z(p) \text{ is even and } n \bmod z(p) = \frac{z(p)}{2} = \ell \bmod z(p).$$

Then $k = \frac{z(p)}{2}$, $t = \frac{p-1}{2}$, $\ell = z(p)m_1 + k$ for some $m_1 \geq 0$, and $\left\lfloor \frac{\ell(p-1)}{z(p)} \right\rfloor = (p-1)m_1 + t$. Since $s_p\left((p-1)m_1 + t\right) < p - 1$, we obtain by Theorem 12 that $m_1 \in H$. In addition, we obtain that

$$n = \ell p^a = \left(z(p)m_1 + \frac{z(p)}{2}\right)p^a = z(p)m + k, \text{ where}$$

$$m = m_1 p^a + \frac{(p^a-1)k}{z(p)} = m_1 p^a + \frac{p^a - 1}{2} = m_1 p^a + t(p^{a-1} + p^{a-2} + \cdots + 1).$$

Since $m_1 \in H$, so is m. Hence n is of the form (3.6).

Case 2. $p \nmid n$. This case is similar to Case 1. By Lemmas 3 and 4(i), we obtain

$$1 \leq n \bmod z(p) \leq pn \bmod z(p) \text{ and } s_p\left(\left\lfloor \frac{n(p-1)}{z(p)} \right\rfloor\right) < p - 1.$$

Since $p \equiv -1 \pmod{z(p)}$, $pn \equiv -n \pmod{z(p)}$. Therefore $n \bmod z(p) \leq (-n) \bmod z(p) = z(p) - (n \bmod z(p))$. Then $n \bmod z(p) \leq \frac{z(p)}{2}$. Then

$$1 \leq k \leq \frac{z(p)}{2}, \quad n = z(p)m + k \text{ for some } m \geq 0, \text{ and } \left\lfloor \frac{n(p-1)}{z(p)} \right\rfloor = (p-1)m + t.$$

Since $s_p\left(\left\lfloor \frac{n(p-1)}{z(p)} \right\rfloor\right) < p - 1$, we obtain by Theorem 12 that $m \in H$. Therefore n is of the form (3.6). This proves the converse of this theorem.

For the other direction, assume that n is of the form (3.6). We still let $t = t(p, k)$ and $H = \bigcup_{i=0}^{t} H(p, i)$, and separate the consideration into two cases.

Case 3. $k < \frac{z(p)}{2}$. Then $0 \leq t \leq \frac{p-3}{2}$. Let $m = (a_r a_{r-1} \cdots a_1)_p$ be the p-adic expansion of m. Since $m \in H$ and $0 \leq t \leq \frac{p-3}{2}$, we see that $0 \leq a_1 \leq \frac{p-3}{2}$. So we obtain by Lemma 7 that

$$n \equiv z(p)m + k \equiv a_1 z(p) + k \not\equiv 0 \pmod{p}. \tag{3.7}$$

Applying the fact that $p \equiv -1 \pmod{z(p)}$, $n \bmod z(p) = k$, and $1 \leq k \leq \frac{z(p)}{2}$, we obtain

$$np \bmod z(p) = (-n) \bmod z(p) = (-k) \bmod z(p) = z(p) - k \geq k = n \bmod z(p). \tag{3.8}$$

Since $m \in H$ and $\left\lfloor \frac{n(p-1)}{z(p)} \right\rfloor = (p-1)m + t$, we obtain by Theorem 12 that

$$s_p\left(\left\lfloor \frac{n(p-1)}{z(p)} \right\rfloor\right) < p - 1. \tag{3.9}$$

By (3.7), (3.8), (3.9), and Lemma 4(i), we obtain $p \nmid \binom{pn}{n}_F$.

Case 4. $k = \frac{z(p)}{2}$. Similar to Case 1, we have

$$n \bmod z(p) = \frac{z(p)}{2} = np \bmod z(p) \text{ and } s_p\left(\left\lfloor \frac{n(p-1)}{z(p)} \right\rfloor\right) < p-1.$$

If $p \nmid n$, then we obtain by Lemma 4(i) that $p \nmid \binom{pn}{n}_F$. So suppose that $p \mid n$ and let $a = m \bmod p$. Since $m \in H$, we see that $a \le t = \frac{p-1}{2}$. In addition, $az(p) + k \equiv mz(p) + k \equiv n \equiv 0 \pmod p$, so we obtain by Lemma 7 that $a = \frac{p-1}{2}$. Since $m \in H(p, a)$, there are $r \ge 0$ and $m_2 \in \bigcup_{i=0}^{\frac{p-3}{2}} H(p, i)$ such that

$$m = m_2 p^{r+1} + a(p^r + p^{r-1} + \cdots + 1) = m_2 p^{r+1} + \frac{p^{r+1}-1}{2}.$$

So we have

$$n = z(p)\left(m_2 p^{r+1} + \frac{p^{r+1}-1}{2}\right) + \frac{z(p)}{2} = (z(p)m_2 + k)p^{r+1}.$$

Since $m_2 \in \bigcup_{i=0}^{\frac{p-3}{2}} H(p, i) \subseteq H$, we obtain by Theorem 12 that

$$s_p\left(\left\lfloor \frac{(z(p)m_2 + k)(p-1)}{z(p)} \right\rfloor\right) = s_p((p-1)m_2 + t) < p-1.$$

In addition, if $m_2 \bmod p = a_2$, then $0 \le a_2 \le \frac{p-3}{2}$ and we obtain by Lemma 7 that

$$z(p)m_2 + k \equiv a_2 z(p) + k \not\equiv 0 \pmod p.$$

Since $n = (z(p)m_2 + k)p^{r+1}$ and $z(p)m_2 + k \not\equiv 0 \pmod p$, we obtain $r + 1 = \nu_p(n)$. In addition,

$$np \bmod z(p) = n \bmod z(p) \text{ and } s_p\left(\left\lfloor \frac{n(p-1)}{p^{\nu_p(n)} z(p)} \right\rfloor\right) < p-1.$$

Therefore $p \nmid \binom{pn}{n}_F$, by Lemma 4(ii). This completes the proof. □

By Theorem 13, we immediately obtain the following corollary.

Corollary 14. *If $p > 2$, $p \equiv \pm 2 \pmod 5$, and $n \bmod z(p) > \frac{z(p)}{2}$, then $p \mid \binom{pn}{n}_F$.*

If $n \bmod z(p) < \frac{z(p)}{2}$, then we may still have $p \mid \binom{pn}{n}_F$ as shown in the next corollary.

Corollary 15. *Let p be an odd prime, $p \equiv \pm 2 \pmod 5$, $p \mid n$, and $n \bmod z(p) \neq \frac{z(p)}{2}$. Then $p \mid \binom{pn}{n}_F$.*

Proof. Suppose for a contradiction that $p \nmid \binom{pn}{n}_F$. Then we obtain by Theorem 13 that $n = z(p)m + k$, $1 \le k \le \frac{z(p)}{2}$, and $m \in H$ where $H = \bigcup_{i=0}^{t(p,k)} H(p, i)$. Since $n \bmod z(p) \neq \frac{z(p)}{2}$, $k < \frac{z(p)}{2}$. This implies that $t(p, k) \le \frac{p-3}{2}$. Let $m = (a_r a_{r-1} \cdots a_1)_p$ be the p-adic representation of m. Since $m \in H$ and $t(p, k) \le \frac{p-3}{2}$, we see that $0 \le a_1 \le \frac{p-3}{2}$. By Lemma 7, we obtain $a_1 z(p) + k \not\equiv 0 \pmod p$. Therefore

$$n \equiv z(p)m + k \equiv a_1 z(p) + k \not\equiv 0 \pmod p,$$

which contradicts the assumption that $p \mid n$. Hence the proof is complete. □

Next, we give a characterization for the divisibility $p \mid \binom{pn}{n}_F$ when $p \equiv \pm 1 \pmod 5$.

Theorem 16. *Let p be an odd prime such that $p \equiv \pm 1 \pmod 5$ and let n be a positive integer. Then $p \mid \binom{pn}{n}_F$ if and only if n is not of the form*

$$z(p)m + k \text{ where } 1 \le k \le z(p) - 1 \text{ and } m \in \bigcup_{i=0}^{t(p,k)} H(p,i). \tag{3.10}$$

Proof. Let $A = \frac{n(p-1)}{p^{\nu_p(n)} z(p)}$. Similar to the proof of Theorem 13, we first assume that $p \nmid \binom{pn}{n}_F$ and let $k = n \bmod z(p)$. Then $n = z(p)m + k$ for some $m \ge 0$, and by Lemma 3, $k \ne 0$. So $1 \le k \le z(p) - 1$. If remains to show that $m \in \bigcup_{i=0}^{t(p,k)} H(p,i)$. Since $p \equiv \pm 1 \pmod 5$, we obtain by Lemma 1 that $z(p) \mid p-1$. This implies $t(p,k) = \frac{k(p-1)}{z(p)}$. By Lemma 5, we have

$$p - 1 > s_p(A) = s_p\left(p^{\nu_p(n)} A\right) = s_p\left(\frac{n(p-1)}{z(p)}\right) = s_p((p-1)m + t(p,k)).$$

By Lemma 12, $m \in \bigcup_{i=0}^{t(p,k)} H(p,i)$, as required. Next, if n is of the form (3.10), then we apply Theorem 12 to obtain

$$s_p(A) = s_p\left(p^{\nu_p(n)} A\right) = s_p((p-1)m + t(p,k)) < p - 1,$$

and then use Lemma 5 to conclude that $p \nmid \binom{pn}{n}_F$. This completes the proof. □

Next we apply Theorems 13 and 16 to determine an asymptotic formula for $E_p(x)$.

Theorem 17. *Let p be an odd prime, $p \equiv \pm 2 \pmod 5$, and $t = t\left(p, \left\lfloor \frac{z(p)}{2} \right\rfloor\right)$. Then uniformly for $x \ge 2$,*

$$E_p(x) = \frac{(\log x)^t}{t!(\log p)^t} + O\left((\log x)^{t-1}\right),$$

and consequently,

$$\sum_{\substack{1 \le n \le x \\ p \mid \binom{pn}{n}_F}} 1 = x - \frac{(\log x)^t}{t!(\log p)^t} + O\left((\log x)^{t-1}\right),$$

where the implied constants depend at most on p.

Proof. In this proof, the implied constants in each estimate depend at most on p. By Theorem 13, we obtain

$$E_p(x) = \sum_{\substack{1 \le n \le x \\ p \nmid \binom{pn}{n}_F}} 1 = \sum_{1 \le k \le \frac{z(p)}{2}} \sum_{\substack{1 \le n \le x \\ n = z(p)m + k \\ m \in \bigcup_{i=0}^{t(p,k)} H(p,i)}} 1 = \sum_{1 \le k \le \frac{z(p)}{2}} \sum_{\substack{0 \le m \le \frac{x-k}{z(p)} \\ m \in \bigcup_{i=0}^{t(p,k)} H(p,i)}} 1. \tag{3.11}$$

For each $1 \le k \le \frac{z(p)}{2}$, let r_k be the number of digits in the p-adic expansion of $\left\lfloor \frac{x-k}{z(p)} \right\rfloor$ and let $r = r_{\left\lfloor \frac{z(p)}{2} \right\rfloor}$. Then

$$r_k = \left\lfloor \frac{\log \left\lfloor \frac{x-k}{z(p)} \right\rfloor}{\log p} \right\rfloor + 1 \text{ for all } 1 \le k \le \frac{z(p)}{2}.$$

By (3.11) and Lemma 11,

$$E_p(x) \leq \sum_{1\leq k\leq \frac{z(p)}{2}} \sum_{\substack{0\leq m<p^{r_k} \\ m\in\bigcup_{i=0}^{t(p,k)} H(p,i)}} 1 = \sum_{1\leq k\leq \frac{z(p)}{2}} \left(\frac{r_k^{t(p,k)}}{t(p,k)!} + O\left(r_k^{t(p,k)-1}\right)\right).$$

In addition,

$$E_p(x) \geq \sum_{1\leq k\leq \frac{z(p)}{2}} \sum_{\substack{0\leq m<p^{r_k-1} \\ m\in\bigcup_{i=0}^{t(p,k)} H(p,i)}} 1 = \sum_{1\leq k\leq \frac{z(p)}{2}} \left(\frac{(r_k-1)^{t(p,k)}}{t(p,k)!} + O\left((r_k-1)^{t(p,k)-1}\right)\right)$$
$$= \sum_{1\leq k\leq \frac{z(p)}{2}} \left(\frac{r_k^{t(p,k)}}{t(p,k)!} + O\left(r_k^{t(p,k)-1}\right)\right).$$

Therefore

$$E_p(x) = \sum_{1\leq k\leq \frac{z(p)}{2}} \left(\frac{r_k^{t(p,k)}}{t(p,k)!} + O\left(r_k^{t(p,k)-1}\right)\right). \tag{3.12}$$

Recall that $t = t\left(p, \left\lfloor \frac{z(p)}{2} \right\rfloor\right)$. Since $p \equiv \pm 2 \pmod 5$, we obtain by Lemma 1 that $z(p) = p+1$ or $z(p) \leq \frac{p+1}{2}$. If $z(p) = p+1$, then $z(p)$ is even and for $1 \leq k \leq \frac{z(p)}{2} - 1$,

$$t(p,k) \leq t\left(p, \frac{z(p)}{2} - 1\right) = \left\lfloor \frac{p-1}{2} - \frac{p-1}{z(p)} \right\rfloor = \frac{p-1}{2} - 1 = t - 1.$$

If $z(p) \leq \frac{p+1}{2}$, then for $1 \leq k \leq \frac{z(p)}{2} - 1$,

$$t(p,k) \leq t\left(p, \left\lfloor \frac{z(p)}{2} \right\rfloor - 1\right) = \frac{\left\lfloor \frac{z(p)}{2} \right\rfloor (p-1) - s}{z(p)} + \left\lfloor \frac{s-(p-1)}{z(p)} \right\rfloor$$
$$\leq \frac{\left\lfloor \frac{z(p)}{2} \right\rfloor (p-1) - s}{z(p)} + \left\lfloor \frac{(z(p)-1)-(p-1)}{z(p)} \right\rfloor \leq t-1,$$

where $s = \left\lfloor \frac{z(p)}{2} \right\rfloor (p-1) \bmod z(p)$. In any case, $t(p,k) \leq t-1$ for $1 \leq k \leq \frac{z(p)}{2} - 1$. In addition,

$$r_k = \frac{\log \left\lfloor \frac{x-k}{z(p)} \right\rfloor}{\log p} + O(1) = \frac{\log x}{\log p} + O(1) \ \text{ for } 1 \leq k \leq \frac{z(p)}{2}.$$

Therefore $r_k^{t(p,k)} \ll r^{t-1} \ll (\log x)^{t-1}$ for any $1 \leq k \leq \frac{z(p)}{2} - 1$. Therefore

$$\sum_{1\leq k\leq \frac{z(p)}{2}-1} \left(\frac{r_k^{t(p,k)}}{t(p,k)!} + O\left(r_k^{t(p,k)-1}\right)\right) = O\left((\log x)^{t-1}\right).$$

Thus (3.12) implies that

$$E_p(x) = \frac{(\log x)^t}{t!(\log p)^t} + O\left((\log x)^{t-1}\right).$$

The rest is now obvious. So the proof is completes. □

Theorem 18. *Let p be an odd prime, $p \equiv \pm 1 \pmod 5$, and $t = t(p, z(p)-1)$. Then uniformly for $x \geq 2$,*

$$E_p(x) = \frac{(\log x)^t}{t!(\log p)^t} + O\left((\log x)^{t-1}\right),$$

and consequently,

$$\sum_{\substack{1\leq n\leq x \\ p \mid \binom{pn}{n}_F}} 1 = x - \frac{(\log x)^t}{t!(\log p)^t} + O\left((\log x)^{t-1}\right),$$

where the implied constants depend at most on p.

Proof. The proof is similar to that of Theorem 17, so we omit some details, and the implied constants in the following estimates depend at most on p. We obtain by Theorem 16 that

$$E_p(x) = \sum_{1\leq k\leq z(p)-1} \sum_{\substack{1\leq n\leq x \\ n=z(p)m+k \\ m\in\bigcup_{i=0}^{t(p,k)} H(p,i)}} 1 = \sum_{1\leq k\leq z(p)-1} \sum_{\substack{0\leq m\leq \frac{x-k}{z(p)} \\ m\in\bigcup_{i=0}^{t(p,k)} H(p,i)}} 1. \tag{3.13}$$

For each $1 \leq k \leq z(p)-1$, let r_k be the number of digits in the p-adic expansion of $\left\lfloor \frac{x-k}{z(p)} \right\rfloor$ and let $r = r_{z(p)-1}$. Then $r_k = \left\lfloor \log_p \left\lfloor \frac{x-k}{z(p)} \right\rfloor \right\rfloor + 1$ for all $1 \leq k \leq z(p)-1$. Similar to the proof of Theorem 17, we apply Lemma 11 to obtain

$$E_p(x) = \sum_{1\leq k\leq z(p)-1} \left(\frac{r_k^{t(p,k)}}{t(p,k)!} + O\left(r_k^{t(p,k)-1}\right) \right). \tag{3.14}$$

Recall that

$$t = t(p, z(p)-1) = \left\lfloor \frac{(z(p)-1)(p-1)}{z(p)} \right\rfloor = p - 1 + \left\lfloor -\frac{p-1}{z(p)} \right\rfloor.$$

Since $p \equiv \pm 1 \pmod 5$, we obtain by Lemma 1 that $z(p) = p-1$ or $z(p) \leq \frac{p-1}{2}$. If $z(p) = p-1$, then for $1 \leq k \leq z(p)-2$, we have

$$t(p,k) \leq t(p, z(p)-2) = \left\lfloor \frac{(z(p)-2)(p-1)}{z(p)} \right\rfloor = p - 3 = t - 1.$$

If $z(p) \leq \frac{p-1}{2}$, then we obtain by Lemma 2(v) that for $1 \leq k \leq z(p)-2$,

$$\begin{aligned} t(p,k) \leq t(p, z(p)-2) = p - 1 + \left\lfloor -\frac{2(p-1)}{z(p)} \right\rfloor &\leq p - 1 + \left\lfloor -\frac{p-1}{z(p)} \right\rfloor + \left\lfloor -\frac{p-1}{z(p)} \right\rfloor + 1 \\ &\leq t - 1. \end{aligned}$$

In any case, $t(p,k) \leq t-1$ for all $1 \leq k \leq z(p)-2$. In addition,

$$r_k = \log_p\left(\left\lfloor \frac{x-k}{z(p)} \right\rfloor \right) + O(1) = \log_p x + O(1) \quad \text{for } 1 \leq k \leq z(p)-1.$$

Therefore $r_k^{t(p,k)} \ll r^{t-1} \ll (\log x)^{t-1}$ for any $1 \leq k \leq z(p)-2$. From (3.14) and this observation, we obtain the desired results. □

We give an example to show an application of our results. Also see [1,9,14] for the characterization of the divisibility $p \mid \binom{pn}{n}_F$ when $p = 2, 3, 5, 7$.

Example 19. Let $n \in \mathbb{N}$. Then $11 \nmid \binom{11n}{n}_F$ if and only if

$$n = 10m + k \text{ where } 1 \le k \le 9 \text{ and } m \in \bigcup_{i=0}^{k} H(11, i).$$

In addition, $13 \nmid \binom{13n}{n}_F$ if and only if

$$n = 7m + k \text{ where } 1 \le k \le 3 \; m \in \bigcup_{i=0}^{2k-1} H(13, i).$$

Furthermore,

$$E_{11}(x) = \frac{(\log x)^9}{9!(\log 11)^9} + O\left((\log x)^8\right) \text{ and } E_{13}(x) = \frac{(\log x)^5}{5!(\log 13)^5} + O\left((\log x)^4\right).$$

Proof. We have $z(11) = 10$, $z(13) = 7$, $t(11, k) = k$ for $1 \le k \le 9$, $t(13, k) = 2k - 1$ for $1 \le k \le 3$. Applying Theorems 13, 16, 17, and 18, we immediately obtain the desired results. □

4. Conclusions and a future project

We give characterizations for the integers $n \ge 1$ such that $\binom{pn}{n}_F$ is divisible by p for any prime $p \ne 2, 5$ in terms of the digital representation of n. We also obtain asymptotic formulas of $E_p(x)$ for all primes p, extending many results in the literature which focus only on small primes $p \le 7$. For a future project, we may be able to extend the results to $\binom{pn}{n}_U$ for any nondegenerate fundamental Lucas sequences U and any prime p.

Acknowledgments

We thank the referees for their comments which improve the quality of this paper. This project is funded by the National Research Council of Thailand (NRCT), grant number NRCT5-RSA63021-02.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

References

1. C. Ballot, Divisibility of Fibonomials and Lucasnomials via a general Kummer rule, *Fibonacci Quart.*, **53** (2015), 194–205.
2. C. Ballot, The congruence of Wolstenholme for generalized binomial coefficients related to Lucas sequences, *J. Integer Seq.*, **18** (2015), Article 15.5.4.

3. C. Ballot, Lucasnomial Fuss-Catalan numbers and related divisibility questions, *J. Integer Seq.*, **21** (2018), Article 18.6.5.

4. C. Ballot, Divisibility of the middle Lucasnomial coefficient, *Fibonacci Quart.*, **55** (2017), 297–308.

5. W. Chu, E. Kiliç, Quadratic sums of Gaussian q-binomial coefficients and Fibonomial coefficients, *Ramanujan J.*, **51** (2020), 229–243. https://doi.org/10.1007/s11139-018-0023-x

6. R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics : A Foundation for Computer Science*, Second Edition, Addison–Wesley, 1994

7. E. Kiliç, I. Akkus, On Fibonomial sums identities with special sign functions: analytically q-calculus approach, *Math. Slovaca*, **68** (2018), 501–512. https://doi.org/10.1515/ms-2017-0120

8. E. Kiliç, H. Prodinger, Closed form evaluation of sums containing squares of Fibonomial coefficients, *Math. Slovaca*, **66** (2016), 757–767. https://doi.org/10.3934/math.2020433

9. D. Marques, P. Trojovský, On divisibility of Fibonomial coefficients by 3, *J. Integer Seq.*, **15** (2012), Article 12.6.4.

10. K. Onphaeng, P. Pongsriiam, Jacobsthal and Jacobsthal-Lucas numbers and sums introduced by Jacobsthal and Tverberg, *J. Integer Seq.*, **20** (2017), Article 17.3.6.

11. K. Onphaeng, P. Pongsriiam, Exact divisibility by powers of the integers in the Lucas sequence of the first kind, *AIMS Math.*, **5** (2020), 6739–6748. https://doi.org/10.3934/math.2020433

12. K. Onphaeng, P. Pongsriiam, Exact divisibility by powers of the integers in the Lucas sequences of the first and second kinds, *AIMS Math.*, **6** (2021), 11733–11748. https://doi.org/10.3934/math.2021682

13. P. Phunphayap, P. Pongsriiam, Explicit formulas for the p-adic valuations of Fibonomial coefficients, *J. Integer Seq.*, **21** (2018), Article 18.3.1.

14. P. Phunphayap, P. Pongsriiam, Explicit formulas for the p-adic valuations of Fibonomial coefficients II, *AIMS Math.*, **6** (2020), 5685–5699. https://doi.org/10.3934/math.2020364

15. P. Pongsriiam, Fibonacci and Lucas numbers which have exactly three prime factors and some unique properties of F_{18} and L_{18}, *Fibonacci Quart.*, **57** (2019), 130–144.

16. P. Pongsriiam, The order of appearance of factorials in the Fibonacci sequence and certain Diophantine equations, *Period. Math. Hungar.*, **79** (2019), 141–156. https://doi.org/10.1007/s10998-018-0268-6