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Research article

A unified fixed point approach to study the existence and uniqueness of solutions to the generalized stochastic functional equation emerging in the psychological theory of learning

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Abstract: The model of decision practice reflects the evolution of moral judgment in mathematical psychology, which is concerned with determining the significance of different options and choosing one of them to utilize. Most studies on animals behavior, especially in a two-choice situation, divide such circumstances into two events. Their approach to dividing these behaviors into two events is mainly based on the movement of the animals towards a specific choice. However, such situations can generally be divided into four events depending on the chosen side and placement of the food. This article aims to fill such gaps by proposing a generic stochastic functional equation that can be used to describe several psychological and learning theory experiments. The existence, uniqueness, and stability analysis of the suggested stochastic equation are examined by utilizing the notable fixed point theory tools. Finally, we offer two examples to substantiate our key findings.

Keywords: stochastic functional equations; stability analysis; fixed points
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1. Introduction and preliminaries

Mathematical psychology is a subfield of psychology that focuses on mathematical modeling of visual, intellectual, behavioral, and physical processes and the formulation of law-like principles that link measurable functional attributes to quantitative behavior. Mathematical techniques are utilized to generate more trustworthy theories, which result in more scientifically rigorous validations. The primary difficulty with today's and most likely future applications of mathematics to psychological issues is modeling these problems.

The learning process in human beings or animals may be viewed as a chain of responses between many potential choices. Even in repeated tests conducted under well-controlled circumstances, preference sequences are often unexpected, suggesting that chance determines response selection. Thus, it is beneficial to consider the systemic changes in a series of choices that correspond to variations in response probability from trial to trial. From this perspective, all learning research is devoted to understanding the probability of trial-to-trial occurrences that define a stochastic process.

Recent studies on mathematical psychology have shown that the behavior of a basic learning experiment follows a stochastic model. It is not a novel concept (for a history of the idea, see [1]). Following 1950, two critical characteristics emerge mainly from Bush, Estes, and Mosteller's study. Firstly, one of the most critical characteristics of the proposed models is the inclusive nature of the learning process. Second, such models may be evaluated in such a manner that their statistical properties are revealed.

In psychological learning theory, the solutions to the subsequent stochastic equation have a great importance

$$\mathscr{L}(x) = x\mathscr{L}(\nu_1 + (1 - \nu_1)x) + (1 - x)\mathscr{L}((1 - \nu_2)x), \tag{1.1}$$

for all $x \in \mathscr{V} = [0, 1]$, $0 < \nu_1 \leq \nu_2 < 1$ are learning-rate parameters and $\mathscr{L} : \mathscr{V} \to \mathbb{R}$ is an unknown function. Markov transitions were used to describe such behavior and converting the states by $\mathbb{P}(x \to \nu_1 + (1 - \nu_1)x)$ and $\mathbb{P}(1 - x \to (1 - \nu_2)x)$, where $\mathbb{P}$ is the probability of that specific event.

In 1976, Istrăţescu [2] used the above stochastic equation (1.1) to inspect the involvement of predatory animals that prey two distinct types of prey.

On the other hand, Bush and Wilson [3] observed the movement of a paradise fish in a two-choice situation under the reinforcement-extinction and the habit formation behaviors. They claimed that under such behavior, there are four distinct outcomes: Left-reward, right-nonreward, right-reward, left-nonreward.

It is usually believed that being awarded for choosing one side increases the probability of selecting that specific side in the subsequent trials. However, the rationale for unrewarding experiences is less obvious. According to extinction or reinforcement theory (see Table 1), the probability of selecting an unrewarded side in the subsequent trial would decrease.

Table 1. Operators for reinforcement-extinction model with the learning rate parameters η_1 and η_2.

Fish's Responses	Outcomes (Left side)	Outcomes (Right side)	Events
Reinforcement	$\eta_1 x$	$\eta_1 x + 1 - \eta_1$	E_1^{RE}
Non-reinforcement	$\eta_2 x + 1 - \eta_2$	$\eta_2 x$	E_2^{RE}

By contrast, a model that depends on habit formation or secondary reinforcement (see Table 2) would indicate that merely picking a side increases the chances of choosing that side in subsequent trials.

Table 2. Operators for habit formation model with the learning rate parameters η_1 and η_2.

Fish's Responses	Outcomes (Left side)	Outcomes (Right side)	Events
Reinforcement	$\eta_1 x$	$\eta_1 x + 1 - \eta_1$	E_1^{HF}
Non-reinforcement	$\eta_2 x$	$\eta_2 x + +1 - \eta_2$	E_2^{HF}

In 1967, Epstein [4] proposed the following functional equation to discuss the learning process of animals in a two-choice situation

$$\mathscr{L}(x) = \left(\frac{e^x}{1-e^x}\right)\mathscr{L}(k_1x) + \left(1 - \frac{e^x}{1-e^x}\right)\mathscr{L}(k_2x), \quad \forall x \in \mathscr{V}, \tag{1.2}$$

where $\mathscr{L} : \mathscr{V} \to \mathbb{R}$ is an unknown and $k_1, k_2 : \mathscr{V} \to \mathbb{R}$ are given mappings. The analytical solution of the above equation was calculated by using the bilateral Laplace transformation.

Recently, Turab and Sintunavarat [5] utilized the above idea and suggested the functional equation stated below

$$\mathscr{L}(x) = x\mathscr{L}(\varpi_1 x + (1-\varpi_1)\Theta_1) + (1-x)\mathscr{L}(\varpi_2 x + (1-\varpi_2)\Theta_2), \quad \forall x \in \mathscr{V}, \tag{1.3}$$

where $\mathscr{L} : \mathscr{V} \to \mathbb{R}$ is an unknown, $0 < \varpi_1 \leq \varpi_2 < 1$ and $\Theta_1, \Theta_2 \in \mathscr{V}$. The aforementioned functional equation was used to study a specific kind of psychological resistance of dogs enclosed in a small box.

Several other studies on human actions in probability-learning scenarios have produced different results (see [6–11]).

The point to ponder is that most studies in the literature related to the behavior of animals in a two-choice situation just focused on the movement of the animals towards a specific choice. In contrast, by focusing on the food placement and the chosen side, Bush and Wilson [3] divided such types of responses into four events (right-reward, right-nonreward, left-reward, left-nonreward). Such events and their corresponding probabilities can be seen in Table 3 below.

Table 3. Four events and their corresponding probabilities.

Events	Responses and outcomes	Corresponding probabilities
E_1	right-reward (food side)	$\tau\nu$
E_2	right-nonreward (non-food side)	$(1-\tau)\nu$
E_3	left-reward (food side)	$\tau(1-\nu)$
E_4	left-nonreward (non-food side)	$(1-\tau)(1-\nu)$

To cover the gap discussed above, here, we propose the following general stochastic functional equation

$$\begin{aligned}\mathscr{L}(x) \;=\; & \tau\nu(x)\mathscr{L}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}(k_2(x)) \\ & +\tau(1-\nu(x))\mathscr{L}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}(k_4(x)),\end{aligned} \tag{1.4}$$

where $\mathscr{L} : \mathscr{V} \to \mathbb{R}$ is an unknown function, $0 \leq \tau \leq 1$ represents the probability of choosing the food side. Also, $\nu : \mathscr{V} \to \mathscr{V}$ and $k_1, k_2, k_3, k_4 : \mathscr{V} \to \mathscr{V}$ are given mappings that represent the four options, based on the chosen side and the reward, discussed in Table 3.

Our objective is to prove the existence and uniqueness of solutions to the above Eq (1.4) by utilizing the Banach fixed point theorem (for the details of fixed point theory, we refer [12–16]). Following that, we provide two examples to demonstrate the importance of our findings in this area. Finally, we examine the Hyers-Ulam and Hyers-Ulam-Rassias (shortly, **HU** and **HUR**) type stability of the suggested stochastic equation's solution.

The following stated outcome will be required in the advancement.

Definition 1.1 ([17]). Let $(\mathscr{V}, d)$ be a metric space. A mapping $\mathscr{L} : \mathscr{V} \to \mathscr{V}$ is called a

(1) **Banach contraction mapping** (or, **BCM**) if there is a nonnegative real number $\lambda < 1$ such that

$$d(\mathscr{L}\mu, \mathscr{L}\upsilon) \leq \lambda d(\mu, \upsilon) \tag{1.5}$$

for all $\mu, \upsilon \in \mathscr{V}$.

(2) **Contractive mapping** if

$$d(\mathscr{L}\mu, \mathscr{L}\upsilon) < d(\mu, \upsilon) \tag{1.6}$$

for all $\mu, \upsilon \in \mathscr{V}$ with $\mu \neq \upsilon$.

(3) **Non-expansive mapping** if

$$d(\mathscr{L}\mu, \mathscr{L}\upsilon) \leq d(\mu, \upsilon) \tag{1.7}$$

for all $\mu, \upsilon \in \mathscr{V}$.

Theorem 1.2 ([18]). *Let $(\mathscr{V}, d)$ be a complete metric space and $\mathscr{L}:\mathscr{V} \to \mathscr{V}$ be a **BCM** with $\lambda < 1$. Then $\mathscr{L}$ has precisely one fixed point. Furthermore, the Picard iteration $\{\mu_n\}$ in $\mathscr{V}$ which is defined as $\mu_n = \mathscr{L}\mu_{n-1}$ for all $n \in \mathbb{N}$, where $\mu_0 \in \mathscr{V}$, converges to the unique fixed point of $\mathscr{L}$.*

2. Main results

Let $\mathscr{V} = [0, 1]$. The class consisting of all continuous real-valued functions $\mathscr{L} : \mathscr{V} \to \mathbb{R}$ such that $\mathscr{L}(0) = 0$ and

$$\sup_{v_1 \neq v_2} \frac{|\mathscr{L}(v_1) - \mathscr{L}(v_2)|}{|v_1 - v_2|} < \infty$$

is denoted by $\mathscr{D}$. Here, it is straightforward that $(\mathscr{D}, \|\cdot\|)$ is a Banach space (for the detail, see [19]), where

$$\|\mathscr{L}\| = \sup_{v_1 \neq v_2} \frac{|\mathscr{L}(v_1) - \mathscr{L}(v_2)|}{|v_1 - v_2|} \tag{2.1}$$

for all $\mathscr{L} \in \mathscr{D}$.

Next, we rewrite (1.4) as

$$\begin{aligned} \mathscr{L}(x) &= \tau\nu(x)\mathscr{L}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}(k_2(x)) \\ &\quad +\tau(1-\nu(x))\mathscr{L}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}(k_4(x)), \end{aligned} \tag{2.2}$$

where $\mathscr{L} : \mathscr{V} \to \mathbb{R}$ is an unknown function and $k_1, k_2, k_3, k_4 : \mathscr{V} \to \mathscr{V}$ are given contraction mappings with contractive coefficients $\eta_1, \eta_2, \eta_3, \eta_4$, respectively and $k_3(0) = 0 = k_4(0)$. Also, $\nu : \mathscr{V} \to \mathscr{V}$ is a given non-expansive mapping with $\nu(0) = 0$ and $|\nu(x)| \leq \eta_5$ ($\eta_5 \geq 0$), for all $x \in \mathscr{V}$.

Theorem 2.1. *Consider the generalized stochastic equation (2.2). Assume that $\lambda_1 < 1$, where λ_1 is defined as*

$$\lambda_1 := \left[\tau\left(\eta_1(1+\eta_5) + 2\eta_3\right) + (1-\tau)\left(\eta_2(1+\eta_5) + 2\eta_4\right)\right], \tag{2.3}$$

and $k_1(0) = 0 = k_2(0)$. *Assume that there is a nonempty subset* $\mathscr{O}$ *of* $\mathscr{T} := \{\mathscr{L} \in \mathscr{D} | \mathscr{L}(1) \leq 1\}$ *such that* $(\mathscr{O}, \|\cdot\|)$ *is a Banach space, where* $\|\cdot\|$ *is given in (2.1). Then (2.2) has a unique solution. Furthermore, the sequence* $\{\mathscr{L}_n\}$ *in* $\mathscr{O}$ *(*$\forall n \in \mathbb{N}$*), where* $\mathscr{L}_0$ *is given in* $\mathscr{O}$*, given by*

$$\begin{aligned} \mathscr{L}_n(x) &= \tau\nu(x)\mathscr{L}_{n-1}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}_{n-1}(k_2(x)) \\ &\quad +\tau(1-\nu(x))\mathscr{L}_{n-1}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}_{n-1}(k_4(x)), \end{aligned} \tag{2.4}$$

converges to a unique solution of (2.2).

Proof. Let $d : \mathscr{O} \times \mathscr{O} \to \mathbb{R}$ be a metric induced by $\|\cdot\|$. Thus $(\mathscr{O}, d)$ is a complete metric space. We deal with the operator $\mathscr{K}$ from $\mathscr{O}$ which is defined as

$$\begin{aligned} (\mathscr{K}\mathscr{L})(x) &= \tau\nu(x)\mathscr{L}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}(k_2(x)) \\ &\quad +\tau(1-\nu(x))\mathscr{L}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}(k_4(x)), \end{aligned}$$

for all $\mathscr{L} \in \mathscr{O}$.

For each $\mathscr{L}$, we obtain $(\mathscr{K}\mathscr{L})(0) = 0$. Also, $\mathscr{K}$ is continuous and $\|\mathscr{K}\mathscr{L}\| < \infty$ for all $\mathscr{L} \in \mathscr{O}$. Thus, $\mathscr{K}$ is a self operator on $\mathscr{O}$. Moreover, the solution of (2.2) is clearly equal to $\mathscr{K}$'s fixed point. Since $\mathscr{K}$ is a linear mapping, for $\mathscr{L}_1, \mathscr{L}_2 \in \mathscr{O}$, we obtain

$$\|\mathscr{K}\mathscr{L}_1 - \mathscr{K}\mathscr{L}_2\| = \|\mathscr{K}(\mathscr{L}_1 - \mathscr{L}_2)\|.$$

Thus, to evaluate $\|\mathscr{K}\mathscr{L}_1 - \mathscr{K}\mathscr{L}_2\|$, we mark the subsequent framework

$$\Upsilon_{x_1,x_2} := \frac{\mathscr{K}(\mathscr{L}_1 - \mathscr{L}_2)(x_1) - \mathscr{K}(\mathscr{L}_1 - \mathscr{L}_2)(x_2)}{x_1 - x_2}, \quad x_1, x_2 \in \mathscr{V}, \; x_1 \neq x_2.$$

For each distinct $x_1, x_2 \in \mathscr{V}$, we get

$$\begin{aligned} \Upsilon_{x_1,x_2} &= \frac{1}{x_1 - x_2}[\tau\nu(x_1)(\mathscr{L}_1 - \mathscr{L}_2)(k_1(x_1)) + (1-\tau)\nu(x_1)(\mathscr{L}_1 - \mathscr{L}_2)(k_2(x_1)) \\ &\quad +\tau(1-\nu(x_1))(\mathscr{L}_1 - \mathscr{L}_2)(k_3(x_1)) + (1-\tau)(1-\nu(x_1))(\mathscr{L}_1 - \mathscr{L}_2)(k_4(x_1)) \\ &\quad -\tau\nu(x_2)(\mathscr{L}_1 - \mathscr{L}_2)(k_1(x_2)) - (1-\tau)\nu(x_2)(\mathscr{L}_1 - \mathscr{L}_2)(k_2(x_2)) \\ &\quad -\tau(1-\nu(x_2))(\mathscr{L}_1 - \mathscr{L}_2)(k_3(x_2)) - (1-\tau)(1-\nu(x_2))(\mathscr{L}_1 - \mathscr{L}_2)(k_4(x_2))] \\ &= \frac{1}{x_1 - x_2}[\tau\nu(x_1)(\mathscr{L}_1 - \mathscr{L}_2)(k_1(x_1)) - \tau\nu(x_1)(\mathscr{L}_1 - \mathscr{L}_2)(k_1(x_2)) \\ &\quad +(1-\tau)\nu(x_1)(\mathscr{L}_1 - \mathscr{L}_2)(k_2(x_1)) - (1-\tau)\nu(x_1)(\mathscr{L}_1 - \mathscr{L}_2)(k_2(x_2)) \\ &\quad +\tau(1-\nu(x_1))(\mathscr{L}_1 - \mathscr{L}_2)(k_3(x_1)) - \tau(1-\nu(x_1))(\mathscr{L}_1 - \mathscr{L}_2)(k_3(x_2)) \\ &\quad +(1-\tau)(1-\nu(x_1))(\mathscr{L}_1 - \mathscr{L}_2)(k_4(x_1)) - (1-\tau)(1-\nu(x_1))(\mathscr{L}_1 - \mathscr{L}_2)(k_4(x_2)) \\ &\quad +\tau\nu(x_1)(\mathscr{L}_1 - \mathscr{L}_2)(k_1(x_2)) - \tau\nu(x_2)(\mathscr{L}_1 - \mathscr{L}_2)(k_1(x_2)) \\ &\quad +(1-\tau)\nu(x_1)(\mathscr{L}_1 - \mathscr{L}_2)(k_2(x_2)) - (1-\tau)\nu(x_2)(\mathscr{L}_1 - \mathscr{L}_2)(k_2(x_2)) \\ &\quad +\tau(1-\nu(x_1))(\mathscr{L}_1 - \mathscr{L}_2)(k_3(x_2)) - \tau(1-\nu(x_2))(\mathscr{L}_1 - \mathscr{L}_2)(k_3(x_2)) \\ &\quad +(1-\tau)(1-\nu(x_1))(\mathscr{L}_1 - \mathscr{L}_2)(k_4(x_2)) - (1-\tau)(1-\nu(x_2))(\mathscr{L}_1 - \mathscr{L}_2)(k_4(x_2))]. \end{aligned}$$

Then we have

$$
\begin{aligned}
\left|\Upsilon_{x_1,x_2}\right| &= \Big|\frac{1}{x_1-x_2}\left[\tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_1))-\tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2))\right]\\
&+\frac{1}{x_1-x_2}\left[(1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_1))-(1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2))\right]\\
&+\frac{1}{x_1-x_2}\left[\tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_1))-\tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2))\right]\\
&+\frac{1}{x_1-x_2}\left[(1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_1))-(1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))\right]\\
&+\frac{1}{x_1-x_2}\left[\tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2))-\tau\nu(x_2)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2))\right]\\
&+\frac{1}{x_1-x_2}\left[(1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2))-(1-\tau)\nu(x_2)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2))\right]\\
&+\frac{1}{x_1-x_2}\left[\tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2))-\tau(1-\nu(x_2))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2))\right]\\
&+\frac{1}{x_1-x_2}\left[(1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))-(1-\tau)(1-\nu(x_2))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))\right]\Big|.
\end{aligned}
$$

As $k_1-k_4:\mathscr{V}\to\mathscr{V}$ are **BCM** with contractive coefficients $\eta_1-\eta_4$, respectively with $k_3(0)=0=k_4(0)$, and $\nu:\mathscr{V}\to\mathscr{V}$ is a non-expansive mapping with $\nu(0)=0$ and $|\nu(x)|\le\eta_5$ ($\eta_5\ge 0$), for all $x\in\mathscr{V}$. Therefore by using (2.1), we have

$$
\begin{aligned}
\left|\Upsilon_{x_1,x_2}\right| &\le \eta_1\tau|\nu(x_1)|\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_2(1-\tau)|\nu(x_1)|\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_3\tau|1-\nu(x_1)|\,\|\mathscr{L}_1-\mathscr{L}_2\|\\
&\quad+\eta_4(1-\tau)|1-\nu(x_1)|\,\|\mathscr{L}_1-\mathscr{L}_2\|+|\tau(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2))-\tau(\mathscr{L}_1-\mathscr{L}_2)(k_1(0))|\\
&\quad+|(1-\tau)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2))-(1-\tau)(\mathscr{L}_1-\mathscr{L}_2)(k_2(0))|\\
&\quad+|\tau(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2))-\tau(\mathscr{L}_1-\mathscr{L}_2)(k_3(0))|\\
&\quad+|(1-\tau)(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))-(1-\tau)(\mathscr{L}_1-\mathscr{L}_2)(k_4(0))|\\
&= \eta_1\eta_5\tau\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_2\eta_5(1-\tau)\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_3\tau\,\|\mathscr{L}_1-\mathscr{L}_2\|\\
&\quad+\eta_4(1-\tau)\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_1\tau x_2\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_2(1-\tau)x_2\,\|\mathscr{L}_1-\mathscr{L}_2\|\\
&\quad+\eta_3\tau x_2\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_4(1-\tau)x_2\,\|\mathscr{L}_1-\mathscr{L}_2\|\\
&= \lambda_1\,\|\mathscr{L}_1-\mathscr{L}_2\|,
\end{aligned}
$$

where λ_1 is given in (2.3). This gives that

$$
d(\mathscr{K}\mathscr{L}_1,\mathscr{K}\mathscr{L}_2)=\|\mathscr{K}\mathscr{L}_1-\mathscr{K}\mathscr{L}_2\|\le\lambda_1\,\|\mathscr{L}_1-\mathscr{L}_2\|=\lambda_1 d(\mathscr{L}_1,\mathscr{L}_2).
$$

As $0<\lambda_1<1$, so by Theorem 1.2, we get the unique solution of (2.2). □

Here, Theorem 2.1 leads to the conclusion stated below.

Corollary 2.2. *Consider the generalized stochastic equation (2.2). Assume that* $k_1,k_2,k_3,k_4:\mathscr{V}\to\mathscr{V}$ *are contraction mappings with contractive coefficients* $\eta_1,\eta_2,\eta_3,\eta_4$, *with* $\eta_1\le\eta_2\le\eta_3\le\eta_4$ *and* $k_1(0)=0=k_2(0)$. *Also,* $\tilde{\lambda}_1:=\eta_4(3+\eta_5)<1$, *and assume that there is a nonempty subset* $\mathscr{O}$ *of* $\mathscr{T}:=\{\mathscr{L}\in\mathscr{D}|\mathscr{L}(1)\le 1\}$ *such that* $(\mathscr{O},\|\cdot\|)$ *is a Banach space, where* $\|\cdot\|$ *is given in (2.1). Then*

(2.2) has a unique solution. Furthermore, the sequence $\{\mathscr{L}_n\}$ in $\mathscr{O}$ ($\forall n \in \mathbb{N}$), where $\mathscr{L}_0$ is given in $\mathscr{O}$, given by

$$\begin{aligned}\mathscr{L}_n(x) &= \tau\nu(x)\mathscr{L}_{n-1}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}_{n-1}(k_2(x)) \\ &\quad +\tau(1-\nu(x))\mathscr{L}_{n-1}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}_{n-1}(k_4(x)),\end{aligned} \tag{2.5}$$

converges to a unique solution of (2.2).

The conditions $k_1(0) = 0 = k_2(0)$ are sufficient but not necessary to prove the main results. Our next outcomes are independent of such conditions.

Theorem 2.3. *Consider the generalized stochastic equation (2.2). Suppose that, there exist $\eta_6, \eta_7 \geq 0$ such that*

$$|k_1(x)| \leq \eta_6 \ \text{ and } \ |k_2(x)| \leq \eta_7, \ \text{ for all } x \in \mathscr{V}, \tag{2.6}$$

and that $\lambda_2 < 1$, where λ_2 is defined as

$$\lambda_2 := \left[\tau\left(\eta_1\eta_5 + 2\eta_3 + \eta_6\right)) + (1-\tau)\left(\eta_2\eta_5 + 2\eta_4 + \eta_7\right))\right]. \tag{2.7}$$

Assume that there is a nonempty subset $\mathscr{O}$ of $\mathscr{T} := \{\mathscr{L} \in \mathscr{D} | \mathscr{L}(1) \leq 1\}$ such that $(\mathscr{O}, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is given in (2.1). Then (2.2) has a unique solution. Furthermore, the sequence $\{\mathscr{L}_n\}$ in $\mathscr{O}$ ($\forall n \in \mathbb{N}$), where $\mathscr{L}_0$ is given in $\mathscr{O}$, given by

$$\begin{aligned}\mathscr{L}_n(x) &= \tau\nu(x)\mathscr{L}_{n-1}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}_{n-1}(k_2(x)) \\ &\quad +\tau(1-\nu(x))\mathscr{L}_{n-1}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}_{n-1}(k_4(x)),\end{aligned} \tag{2.8}$$

converges to a unique solution of (2.2).

Proof. Let $d : \mathscr{O} \times \mathscr{O} \to \mathbb{R}$ be a metric induced by $\|\cdot\|$. Thus $(\mathscr{O}, d)$ is a complete metric space. We deal with the operator $\mathscr{K}$ from $\mathscr{O}$ which is defined as

$$\begin{aligned}(\mathscr{K}\mathscr{L})(x) &= \tau\nu(x)\mathscr{L}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}(k_2(x)) \\ &\quad +\tau(1-\nu(x))\mathscr{L}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}(k_4(x)),\end{aligned}$$

for all $\mathscr{L} \in \mathscr{O}$.

For each $\mathscr{L}$, we obtain $(\mathscr{K}\mathscr{L})(0) = 0$. Also, $\mathscr{K}$ is continuous and $\|\mathscr{K}\mathscr{L}\| < \infty$ for all $\mathscr{L} \in \mathscr{O}$. Thus, $\mathscr{K}$ is a self operator on $\mathscr{O}$. Moreover, the solution of (2.2) is clearly equal to $\mathscr{K}$'s fixed point. Since $\mathscr{K}$ is a linear mapping, so for $\mathscr{L}_1, \mathscr{L}_2 \in \mathscr{O}$, we get

$$\|\mathscr{K}\mathscr{L}_1 - \mathscr{K}\mathscr{L}_2\| = \|\mathscr{K}(\mathscr{L}_1 - \mathscr{L}_2)\|.$$

Thus, to evaluate $\|\mathscr{K}\mathscr{L}_1 - \mathscr{K}\mathscr{L}_2\|$, we mark the following framework

$$\Upsilon_{x_1,x_2} := \frac{\mathscr{K}(\mathscr{L}_1 - \mathscr{L}_2)(x_1) - \mathscr{K}(\mathscr{L}_1 - \mathscr{L}_2)(x_2)}{x_1 - x_2}, \ \ x_1, x_2 \in \mathscr{V}, \ x_1 \neq x_2.$$

For each distinct $x_1, x_2 \in \mathscr{V}$, we obtain

$$
\begin{aligned}
\Upsilon_{x_1,x_2} &= \frac{1}{x_1-x_2}[\tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_1)) + (1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_1)) \\
&\quad +\tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_1)) + (1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_1)) \\
&\quad -\tau\nu(x_2)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2)) - (1-\tau)\nu(x_2)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2)) \\
&\quad -\tau(1-\nu(x_2))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2)) - (1-\tau)(1-\nu(x_2))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))] \\
&= \frac{1}{x_1-x_2}[\tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_1)) - \tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2)) \\
&\quad +(1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_1)) - (1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2)) \\
&\quad +\tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_1)) - \tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2)) \\
&\quad +(1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_1)) - (1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2)) \\
&\quad +\tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2)) - \tau\nu(x_2)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2)) \\
&\quad +(1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2)) - (1-\tau)\nu(x_2)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2)) \\
&\quad +\tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2)) - \tau(1-\nu(x_2))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2)) \\
&\quad +(1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2)) - (1-\tau)(1-\nu(x_2))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))].
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\left|\Upsilon_{x_1,x_2}\right| &= \Bigg|\frac{1}{x_1-x_2}[\tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_1)) - \tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2))] \\
&\quad +\frac{1}{x_1-x_2}[(1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_1)) - (1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2))] \\
&\quad +\frac{1}{x_1-x_2}[\tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_1)) - \tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2))] \\
&\quad +\frac{1}{x_1-x_2}[(1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_1)) - (1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))] \\
&\quad +\frac{1}{x_1-x_2}[\tau\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2)) - \tau\nu(x_2)(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2))] \\
&\quad +\frac{1}{x_1-x_2}[(1-\tau)\nu(x_1)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2)) - (1-\tau)\nu(x_2)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2))] \\
&\quad +\frac{1}{x_1-x_2}[\tau(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2)) - \tau(1-\nu(x_2))(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2))] \\
&\quad +\frac{1}{x_1-x_2}[(1-\tau)(1-\nu(x_1))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2)) - (1-\tau)(1-\nu(x_2))(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))]\Bigg|.
\end{aligned}
$$

Here $k_1 - k_4 : \mathscr{V} \to \mathscr{V}$ are **BCM** with contractive coefficients $\eta_1 - \eta_4$, respectively and satisfies the condition (2.6). Also, $\nu : \mathscr{V} \to \mathscr{V}$ is a non-expansive mapping with $\nu(0) = 0$ and $|\nu(x)| \leq \eta_5$ ($\eta_5 \geq 0$), for all $x \in \mathscr{V}$. Thus by using (2.1), we have

$$
\begin{aligned}
\left|\Upsilon_{x_1,x_2}\right| &\leq \eta_1\tau|\nu(x_1)|\,\|\mathscr{L}_1-\mathscr{L}_2\| + \eta_2(1-\tau)|\nu(x_1)|\,\|\mathscr{L}_1-\mathscr{L}_2\| + \eta_3\tau|1-\nu(x_1)|\,\|\mathscr{L}_1-\mathscr{L}_2\| \\
&\quad +\eta_4(1-\tau)|1-\nu(x_1)|\,\|\mathscr{L}_1-\mathscr{L}_2\| + |\tau(\mathscr{L}_1-\mathscr{L}_2)(k_1(x_2)) - \tau(\mathscr{L}_1-\mathscr{L}_2)(0)| \\
&\quad + |(1-\tau)(\mathscr{L}_1-\mathscr{L}_2)(k_2(x_2)) - (1-\tau)(\mathscr{L}_1-\mathscr{L}_2)(0)| \\
&\quad + |\tau(\mathscr{L}_1-\mathscr{L}_2)(k_3(x_2)) - \tau(\mathscr{L}_1-\mathscr{L}_2)(k_3(0))|
\end{aligned}
$$

$$
\begin{aligned}
& +|(1-\tau)(\mathscr{L}_1-\mathscr{L}_2)(k_4(x_2))-(1-\tau)(\mathscr{L}_1-\mathscr{L}_2)(k_4(0))| \\
= \; & \eta_1\eta_5\tau\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_2\eta_5(1-\tau)\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_3\tau\,\|\mathscr{L}_1-\mathscr{L}_2\| \\
& +\eta_4(1-\tau)\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_6\tau\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_7(1-\tau)\,\|\mathscr{L}_1-\mathscr{L}_2\| \\
& +\eta_3\tau x_2\,\|\mathscr{L}_1-\mathscr{L}_2\|+\eta_4(1-\tau)x_2\,\|\mathscr{L}_1-\mathscr{L}_2\| \\
= \; & \lambda_2\,\|\mathscr{L}_1-\mathscr{L}_2\|,
\end{aligned}
$$

where λ_2 is given in (2.7). This gives that

$$d(\mathscr{K}\mathscr{L}_1,\mathscr{K}\mathscr{L}_2)=\|\mathscr{K}\mathscr{L}_1-\mathscr{K}\mathscr{L}_2\|\leq\lambda_2\,\|\mathscr{L}_1-\mathscr{L}_2\|=\lambda_2 d(\mathscr{L}_1,\mathscr{L}_2).$$

As $0<\lambda_2<1$, so by Theorem 1.2, we get the unique solution of (2.2). □

The following conclusion is derived from Theorem 2.3.

Corollary 2.4. *Consider the generalized stochastic equation (2.2). Assume that $k_1,k_2,k_3,k_4:\mathscr{V}\to\mathscr{V}$ are contraction mappings with contractive coefficients $\eta_1,\eta_2,\eta_3,\eta_4$ with $\eta_1\leq\eta_2\leq\eta_3\leq\eta_4$ and there exist $\eta_6,\eta_7\geq 0$ such that*

$$|k_1(x)|\leq\eta_6 \text{ and } |k_2(x)|\leq\eta_7, \text{ for all } x\in\mathscr{V}, \tag{2.9}$$

and that $\tilde{\lambda}_2<1$, where $\tilde{\lambda}_2$ is defined as

$$\tilde{\lambda}_2 := \left[(2+\eta_5)\,\eta_4+\eta_7+(\eta_6-\eta_7)\tau\right]. \tag{2.10}$$

Assume that there is a nonempty subset $\mathscr{O}$ of $\mathscr{T}:=\{\mathscr{L}\in\mathscr{D}|\mathscr{L}(1)\leq 1\}$ such that $(\mathscr{O},\|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is given in (2.1). Then (2.2) has a unique solution. Furthermore, the sequence $\{\mathscr{L}_n\}$ in $\mathscr{O}$ ($\forall n\in\mathbb{N}$), where $\mathscr{L}_0$ is given in $\mathscr{O}$, given as

$$
\begin{aligned}
\mathscr{L}_n(x) \;=\; & \tau\nu(x)\mathscr{L}_{n-1}(k_1(x))+(1-\tau)\nu(x)\mathscr{L}_{n-1}(k_2(x)) \\
& +\tau(1-\nu(x))\mathscr{L}_{n-1}(k_3(x))+(1-\tau)(1-\nu(x))\mathscr{L}_{n-1}(k_4(x)),
\end{aligned} \tag{2.11}
$$

converges to a unique solution of (2.2).

Remark 2.5. Our proposed generalized stochastic equation (2.2) is a generalization of many mathematical models in the particular research (including equations discussed in the introduction section). For instance

(1) If we put $\tau=0$ and define $\nu,k_2,k_4:\mathscr{V}\to\mathscr{V}$ by

$$\nu(x)=x,\,k_2(x)=\eta_1 x+1-\eta_1 \text{ and } k_4(x)=\eta_2 x,$$

where $0<\eta_1\leq\eta_2<1$, then our proposed model (2.2) is equivalent to the model examined in [20].

(2) If we put $\tau=1$. Define $\nu(x)=x$ and $k_1,k_3:\mathscr{V}\to\mathscr{V}$ as **BCM** having contractive constants η_1 and η_2 respectively with $\eta_1\leq\eta_2$, then our proposed stochastic equation (2.2) is equivalent to the functional equations examined in [21, 22].

To support our argument, we now present the subsequent examples.

Example 2.6. Consider the stochastic equation stated below

$$\begin{aligned}\mathcal{L}(x) &= \tau x\mathcal{L}\left(\frac{x}{13}\right)+(1-\tau)x\mathcal{L}\left(\frac{3x}{14}\right)+\tau(1-x)\mathcal{L}\left(\frac{x}{9}\right)\\ &\quad +(1-\tau)(1-x)\mathcal{L}\left(\frac{2x}{11}\right)\end{aligned} \tag{2.12}$$

for all $x \in \mathcal{V}$, where $\mathcal{L} : \mathcal{V} \to \mathbb{R}$ is an unknown function. If we define mappings $\nu, k_1, k_2, k_3, k_4 : \mathcal{V} \to \mathcal{V}$ by

$$\nu(x) = x, k_1(x) = \frac{x}{13},\ k_2(x) = \frac{3x}{14},\ k_3(x) = \frac{x}{9}\ \text{ and } k_4(x) = \frac{2x}{11}$$

for all $x \in \mathcal{V}$, then our generalized stochastic equation (2.2) reduces to the Eq (2.12).

Now, our aim is to use Theorem 2.1 to find the existence of a unique solution to the above problem. Here, k_1, k_2, k_3, k_4 are contraction mappings with contractive coefficients $\eta_1 = \frac{1}{13}$, $\eta_2 = \frac{3}{14}, \eta_3 = \frac{1}{9}$ and $\eta_4 = \frac{2}{11}$, respectively, and $k_1(0) = k_2(0) = k_3(0) = k_4(0) = 0$. Also, $\nu : \mathcal{V} \to \mathcal{V}$ is a non-expansive mapping with $\nu(0) = 0$ and $\eta_5 = 1$. Thus,

$$\lambda_1 := [\tau(\eta_1(1+\eta_5)+2\eta_3)+(1-\tau)(\eta_2(1+\eta_5)+2\eta_4)] = \frac{1}{13629}(10797-7409\tau) < 1,$$

for all $\tau \in \mathcal{V}$. All of the Theorem 2.1's premises are now true. As a result, there is only one solution to the functional equation (2.12).

Furthermore, if we pick $\mathcal{L}_0(x) = x$ for all $x \in \mathcal{V}$ as an initial approximation, then the next iteration will converge to a unique solution (2.12).

$$\begin{aligned}\mathcal{L}_1(x) &= \frac{1}{18018}[-1201\tau x^2+585x^2-1274\tau x+376x],\\ \mathcal{L}_2(x) &= \frac{1}{5849513501832}\begin{bmatrix} 7520845753\tau^2x^3-8677708650\tau x^3+2442459825x^3\\ +35644603904\tau^2x^2-101143007784\tau x^2+40809099240x^2\\ +29244583368\tau^2x-150400714464\tau x+193372347168x\end{bmatrix},\\ &\ \vdots\\ \mathcal{L}_n(x) &= \tau x\mathcal{L}_{n-1}\left(\frac{x}{13}\right)+(1-\tau)x\mathcal{L}_{n-1}\left(\frac{3x}{14}\right)+\tau(1-x)\mathcal{L}_{n-1}\left(\frac{x}{9}\right)\\ &\quad +(1-\tau)(1-x)\mathcal{L}_{n-1}\left(\frac{2x}{11}\right)\end{aligned}$$

for all $n \in \mathbb{N}$.

Example 2.7. Consider the stochastic equation stated below

$$\begin{aligned}\mathcal{L}(x) &= \tau x\mathcal{L}\left(\frac{ax+1-a}{2}\right)+(1-\tau)x\mathcal{L}\left(\frac{bx+1-b}{2}\right)+\tau(1-x)\mathcal{L}\left(\frac{cx}{2}\right)\\ &\quad +(1-\tau)(1-x)\mathcal{L}\left(\frac{dx}{2}\right)\end{aligned} \tag{2.13}$$

for all $x \in \mathscr{V}$ and $0 < a, b, c, d < 1$, where $\mathscr{L} : \mathscr{V} \to \mathbb{R}$ is an unknown function. Also, if we define mappings $\nu, k_1, k_2, k_3, k_4 : \mathscr{V} \to \mathscr{V}$ by

$$\nu(x) = x, k_1(x) = \frac{ax + 1 - a}{2}, \; k_2(x) = \frac{bx + 1 - b}{2}, \; k_3(x) = \frac{cx}{2} \; \text{and} \; k_4(x) = \frac{dx}{2}$$

for all $x \in \mathscr{V}$, then the generalized stochastic equation (2.2) reduces to the Eq (2.13).

We next attempt to solve the problem by using Theorem 2.3. Here, k_1, k_2, k_3, k_4 are contraction mappings with contractive coefficients $\eta_1 = \frac{a}{2}, \eta_2 = \frac{b}{2}, \eta_3 = \frac{c}{2}$ and $\eta_4 = \frac{d}{2}$, respectively, and $\nu : \mathscr{V} \to \mathscr{V}$ is a non-expansive mapping with $\nu(0) = 0$ and $\eta_5 = 1$. Also,

$$|k_1(x)| \leq \frac{1}{2} \; \text{and} \; |k_2(x)| \leq \frac{1}{2}, \; \text{for all } x \in \mathscr{V},$$

and

$$k_3(0) = k_4(0) = 0.$$

Thus,

$$\lambda_2 := [\tau(\eta_1\eta_5 + 2\eta_3 + \eta_6)) + (1 - \tau)(\eta_2\eta_5 + 2\eta_4 + \eta_7))] = \left[\frac{\tau}{2}(a + 2c + 1) + \frac{(1 - \tau)}{2}(b + 2d + 1)\right].$$

Now, all the hypotheses of Theorem 2.3 are fulfilled. Thus, (2.9) has a unique solution if $|\lambda_2| < 1$.

Furthermore, if we pick $\mathscr{L}_0(x) = x$ for all $x \in \mathscr{V}$ as an initial approximation, the next iteration will converge to a unique solution (2.13).

$$\begin{aligned}
\mathscr{L}_1(x) &= \frac{1}{2}[(b + a\tau - b\tau - c\tau + t\tau - d)x^2 + (-b - a\tau + b\tau + c\tau + 1 - d\tau + d)x],\\
\mathscr{L}_2(x) &= \tau x \mathscr{L}_1\left(\frac{ax + 1 - a}{2}\right) + (1 - \tau)x\mathscr{L}_1\left(\frac{bx + 1 - b}{2}\right) + \tau(1 - x)\mathscr{L}_1\left(\frac{cx}{2}\right)\\
&\quad +(1 - \tau)(1 - x)\mathscr{L}_1\left(\frac{dx}{2}\right),\\
&\vdots\\
\mathscr{L}_n(x) &= \tau x \mathscr{L}_{n-1}\left(\frac{ax + 1 - a}{2}\right) + (1 - \tau)x\mathscr{L}_{n-1}\left(\frac{bx + 1 - b}{2}\right) + \tau(1 - x)\mathscr{L}_{n-1}\left(\frac{cx}{2}\right)\\
&\quad +(1 - \tau)(1 - x)\mathscr{L}_{n-1}\left(\frac{dx}{2}\right)
\end{aligned}$$

for all $n \in \mathbb{N}$.

3. Stability review of the proposed generalized stochastic equation

In mathematical modeling theory, the consistency of solutions is critical. Slight changes in the data set, such as those caused by natural measurement mistakes, have no corresponding impact on the conclusion. Hence, it is essential to analyze the stability of the suggested functional equation (1.4)' solution. For the details of **HU** and **HUR** stability, we refer [23–30].

Theorem 3.1. *Under the hypothesis of Theorem 2.1, the equation $\mathscr{K}\mathscr{L} = \mathscr{L}$, where $\mathscr{K} : \mathscr{O} \to \mathscr{O}$ is given as*

$$\begin{aligned}(\mathscr{K}\mathscr{L})(x) &= \tau\nu(x)\mathscr{L}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}(k_2(x)) \\ &\quad +\tau(1-\nu(x))\mathscr{L}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}(k_4(x)),\end{aligned} \tag{3.1}$$

*for all $\mathscr{L} \in \mathscr{O}$ and $x \in \mathscr{V}$, has **HUR** stability, that is, for a fixed function $\vartheta : \mathscr{O} \to [0,\infty)$, we have that for every $\mathscr{L} \in \mathscr{O}$ with $d(\mathscr{K}\mathscr{L}, \mathscr{L}) \leq \vartheta(\mathscr{L})$, there exists a unique $\mathscr{L}^\star \in \mathscr{O}$ such that $\mathscr{K}\mathscr{L}^\star = \mathscr{L}^\star$ and $d(\mathscr{L}, \mathscr{L}^\star) \leq \ell\vartheta(\mathscr{L})$ for some $\ell > 0$.*

Proof. Let $\mathscr{L} \in \mathscr{O}$ such that $d(\mathscr{K}\mathscr{L}, \mathscr{L}) \leq \vartheta(\mathscr{L})$. By utilizing Theorem 2.1, we have a unique $\mathscr{L}^\star \in \mathscr{O}$ such that $\mathscr{K}\mathscr{L}^\star = \mathscr{L}^\star$. Therefore, we obtain

$$\begin{aligned}d(\mathscr{L}, \mathscr{L}^\star) &\leq d(\mathscr{L}, \mathscr{K}\mathscr{L}) + d(\mathscr{K}\mathscr{L}, \mathscr{L}^\star) \\ &\leq \vartheta(\mathscr{L}) + d(\mathscr{K}\mathscr{L}, \mathscr{K}\mathscr{L}^\star) \\ &\leq \vartheta(\mathscr{L}) + \lambda_1 d(\mathscr{L}, \mathscr{L}^\star),\end{aligned}$$

where λ_1 is given in (2.3), and so

$$d(\mathscr{L}, \mathscr{L}^\star) \leq \ell\vartheta(\mathscr{L}),$$

where $\ell := \dfrac{1}{1-\lambda_1}$. □

From the above analysis, we obtain the result corresponding to the **HU** stability.

Corollary 3.2. *Under the hypothesis of Theorem 2.1, the equation $\mathscr{K}\mathscr{L} = \mathscr{L}$, where $\mathscr{K} : \mathscr{O} \to \mathscr{O}$ is given as*

$$\begin{aligned}(\mathscr{K}\mathscr{L})(x) &= \tau\nu(x)\mathscr{L}(k_1(x)) + (1-\tau)\nu(x)\mathscr{L}(k_2(x)) \\ &\quad +\tau(1-\nu(x))\mathscr{L}(k_3(x)) + (1-\tau)(1-\nu(x))\mathscr{L}(k_4(x)),\end{aligned} \tag{3.2}$$

*for all $\mathscr{L} \in \mathscr{O}$ and $x \in \mathscr{V}$, has **HU** stability, that is, for a fixed $\xi > 0$, we have that for every $\mathscr{L} \in \mathscr{O}$ with $d(\mathscr{K}\mathscr{L}, \mathscr{L}) \leq \xi$, there exists a unique $\mathscr{L}^\star \in \mathscr{O}$ such that $\mathscr{K}\mathscr{L}^\star = \mathscr{L}^\star$ and $d(\mathscr{L}, \mathscr{L}^\star) \leq \ell\xi$, for some $\ell > 0$.*

4. Conclusions

Mathematical psychology is a branch of psychology that is oriented toward mathematical modeling. Simultaneously, the learning process in human beings or animals can be viewed as a set of possible reactions. From this perspective, most of the learning research focuses on determining the probability of trial-to-trial occurrences that characterizes a stochastic process. In this work, we proposed a general stochastic functional equation that can be used to discuss numerous psychological learning theory experiments on animals and humans in the existing literature. In addition, we examined the existence, uniqueness, and stability of a solution to the suggested generalized stochastic equation by utilizing the fixed-point theory tools. Two examples are also given that show the importance of our results in this area of research.

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Conflict of interest

The authors declare that they have no competing interests.

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