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Research article

A posteriori error estimates of hp spectral element method for parabolic optimal control problems

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Abstract: In this paper, we investigate the spectral element approximation for the optimal control problem of parabolic equation, and present a hp spectral element approximation scheme for the parabolic optimal control problem. For improve the accuracy of the algorithm and construct an adaptive finite element approximation. Under the Scott-Zhang type quasi-interpolation operator, a $L^2(H^1) - L^2(L^2)$ posteriori error estimates of the hp spectral element approximated solutions for both the state variables and the control variable are obtained. Adopting two auxiliary equations and stability results, a $L^2(L^2)-L^2(L^2)$ posteriori error estimates are derived for the hp spectral element approximation of optimal parabolic control problem.

Keywords: parabolic optimal control problems; hp spectral element method; a posteriori error estimates
Mathematics Subject Classification: 49J20, 65N30

1. Introduction

Optimal control problems are frequently used in practical problems of physical, social, economic processes, and other fields, and the numerical solution of optimal control problems is of great significance for better performance in these fields [30]. Consequently, it is particularly important to need some effective numerical methods to approximate the solution of the optimal control problem. As we all know, finite element method is one of the most commonly used numerical methods to solve optimal control problems. Applying finite element methods, the emergence of errors has captured the attention of scholars. One of the main sources of errors is the error caused by the discretisation of the model, so a large number of researchers have analyzed it in all aspects by using the finite element method. Bonifacius and Pieper, Lu and huang have studied the prior error estimates of the nonlinear optimal control problem [29,30]. Also, Boulaaras has analysed the posteriori error estimates of the finite element method for nonlinear optimal control problems [25, 26]. Boulaaras, Touati Brahim, Bouzenada and et all used the Euler time scheme combined with Galerkin spatial method, a posteriori error estimates for the generalized Schwartz method with Dirichlet boundary conditions on the interfaces for advection-diffusion equation with second order boundary value problems are proved [27]. And Boulaaras and Haiour dealed with the semi-implicit scheme with respect to the t-variable combined with a finite element spatial approximation of evolutionary Hamilton-Jacobi-Bellman equations with nonlinear source terms [28]. Simultaneously, the spectral method, the finite volume method, the mixed finite element method and other numerical methods have also been applied to the approximate solution of the optimal control problem [1, 5, 6, 8, 10, 13, 18, 19] and there are references.

It is common knowledge that the hp spectral element method, which combines the advantages of the spectral method and the hp finite element method, emphasizes the use of the hp-version adaptive by simply applying the spectral method for each element, because the spectral accuracy provides very accurate approximations when smoothing the solution, with relatively few unknowns. And the spectral element method can solve complex problems, for example, a posteriori error estimates for parameter identification problem, complex nonlinear optimal control problems and etc. A lot of literatures dealt with the optimal control problem and many solutions are proposed, such as the finite element method, mixed finite element method, spectral method and so on. For a brief introduction, there has been an amount of work on constrained optimal control problems for numerically solving via the finite element methods [14–17]. Also, the mixed finite element method for the optimal control problems [2–4, 7, 23, 24]. The hp spectral element method for optimal control problems seems to have not been much studied. Therefore, it is of great significance to solve the parabolic optimal control problem by using the hp spectral element method to solve the parabolic optimal control problem is of great significance.

Let us to introduce the hp spectral element method into the parabolic optimal control problem, which is due to the adaptation of hp-version, it can choose to segment an element (h-refinement) or increase its approximate order (p-refinement). For instance, some authors have studied the hp spectral element method for the optimal control problem controlled by elliptic equations. They have derived the a posteriori error estimation of the hp spectral element approximation of the optimal control problem, in which they used $L^2(\Omega)$-norm to estimate the control approximation error and $H^1(\Omega)$-norm of the state and common state approximation error [8]. In order to emphasize the hp spectral element method and its high precision, we study the hp spectral element method for optimal control problems governed by parabolic equations comparing with [8]. First, we propose a fully discrete scheme, which uses the backward Euler scheme in time, and then uses the hp spectral approximation in space. By using the Scott-Zhang type quasi-interpolation operator, we obtain a posteriori error estimate for the approximate solution of hp spectral elements of both the state and the co-state in $L^2(0,T;H^1(\Omega))$-norm or $L^2(0,T;L^2(\Omega))$-norm and the control in $L^2(0,T;L^2(\Omega))$-norm.

The remainder of this paper is organized as follows. In Section 2, We will use the spectral approximation in space and the inverse Euler scheme in time to construct the spectral approximation scheme for parabolic optimal control problems. In Section 3, a $L^2(H^1)-L^2(L^2)$ posteriori error estimate is derived for the parabolic optimal control problem. In Section 4, by using two auxiliary equations, we derive a $L^2(L^2)-L^2(L^2)$ posteriori error estimates for parabolic optimal control problems. In the last section, the conclusions and some possible future work are briefly given.

In our paper, the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and the semi-norm $|\cdot|_{W^{m,q}(\Omega)}$ are adopted. We set $W_0^{m,q}(\Omega)\equiv\{w\in W^{m,q}(\Omega):\ w|_{\partial\Omega}=0\}$. We denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$). We denote by $L^s(0,T;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,q}(\Omega))}=(\int_0^T\|v\|^s_{W^{m,q}(\Omega)}dt)^{\frac{1}{s}}$ for $s\in[0,\infty)$ and the standard modification for $s=\infty$. Similarly, one define the spaces $H^1(0,T;W^{m,q}(\Omega))$ and $C^l(0,T;W^{m,q}(\Omega))$. The details can be found in [13].

2. hp spectral element approximation

In this section, the hp spectral element method and the backward Euler discretisation approximation for distributed convex optimal control problems governed by parabolic equations is investigated as follows:

$$\min_{u(t)\in K}\left\{\frac{1}{2}\int_0^T(\|\,y-y_d\,\|^2_{L^2(\Omega)}+\|\,u\,\|^2_{L^2(\Omega)})dt\right\},\tag{2.1}$$

$$y_t-\operatorname{div}(A\nabla y)=f+Bu,\qquad x\in\Omega,\quad t\in(0,T],\tag{2.2}$$

$$y|_{\partial\Omega}=0,\qquad t\in[0,T],\tag{2.3}$$

$$y(x,0)=y_0(x),\qquad x\in\Omega,\tag{2.4}$$

where Ω is bounded open subset in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, and B is a linear continuous operator from X to $L^2(0,T;Y')$. Now K is a set defined by

$$K=\left\{v\in X:\int_0^T\int_\Omega vdxdt\geq 0\right\}.$$

Obviously $f,y_d\in L^2(0,T;H)$, $y_0\in H_0^1(\Omega)$ and $A(\cdot)=(a_{i,j}(\cdot))_{n\times n}\in(C^\infty(\bar{\Omega}))^{n\times n}$, such that there exists a constant $c>0$ satisfying

$$\xi^tA\xi\geq c\|\xi\|^2,\qquad \xi\in\mathbb{R}^2.$$

We shall take the state space $W=L^2(0,T;Y)$ with $Y=H_0^1(\Omega)$, the control space $X=L^2(0,T;U)$ with $U=L^2(\Omega)$ to fixed the idea. Then there holds

$$a(y,\omega)=\int_\Omega(A\nabla y)\cdot\nabla\omega dx,\qquad\forall\ y,\ \omega\in Y,$$

$$(f_1,f_2)=\int_\Omega f_1f_2dx,\qquad\forall\ f_1,\ f_2\in U,$$

$$(u,v)=\int_\Omega uvdx, \qquad \forall\, u,\ v\in U.$$

On the basis of the assumptions on A, there exist constants $c>0$ and $C>0$ such that

$$a(\upsilon,\upsilon)\geq c\|\upsilon\|^2_{1,\Omega}, \qquad |a(\upsilon,\omega)|\leq C|\upsilon|_{1,\Omega}|\omega|_{1,\Omega}, \qquad \forall\, \upsilon,\omega\in Y.$$

Then a weak formula of the convex optimal control problem reads:

$$\min_{u(t)\in K}\left\{\frac{1}{2}\int_0^T(\|y-y_d\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)})dt\right\}, \tag{2.5}$$

where $y\in W,\ u\in X,\ u(t)\in K$ subject to

$$\left(\frac{\partial y}{\partial t},\omega\right)+a(y,\omega)=(f+Bu,\omega), \qquad \forall\, \omega\in Y,\ t\in(0,T], \tag{2.6}$$

$$y(x,0)=y_0(x), \qquad x\in\Omega. \tag{2.7}$$

Apparently, the optimal control problem (2.5)–(2.7) has a unique solution (y,u), and a pair (y,u) is the solution of (2.5)–(2.7) if and only if there is a co-state $p\in W$ such that the triplet (y,p,u) satisfies the following optimality conditions [12]:

$$\left(\frac{\partial y}{\partial t},\omega\right)+a(y,\omega)=(f+Bu,\omega), \qquad \forall\, \omega\in Y, \quad y(0)=y_0, \tag{2.8}$$

$$-\left(\frac{\partial p}{\partial t},q\right)+a(q,p)=(y-y_d,q), \qquad \forall\, q\in Y, \quad p(T)=0, \tag{2.9}$$

$$\int_0^T(u+B^*p,\upsilon-u)dt\geq 0, \qquad \forall\, \upsilon(t)\in K,\ \upsilon\in X=L^2(0,T;U), \tag{2.10}$$

where B^* is the adjoint operators of B.

Now, let's consider the hp spectral element approximation of the parabolic optimal control problem (2.5)–(2.7). As we all know, the spectral element method proposed by Patera combines the advantages of Galerkin spectral method and finite element method by a simple application of the spectral method per element [21]. Also, it is similar to the finite element method that the domain is divided into N_τ non-overlapping subdomains elements τ_i, $1\leq i\leq N_\tau$:

$$\overline{\Omega}=\bigcup_{i=1}^{N_\tau}\overline{\tau_i}, \quad \tau_i\bigcap\tau_j=\emptyset, \quad i\neq j, \quad 1\leq i,j\leq N_\tau.$$

Considering the hp spectral element approximation of (2.5)–(2.7), we set $\mathcal{T}=\{\tau\}$ be a local quasi-uniform partitioning of Ω into non-overlapping regular element τ. We denote by the $\hat{\tau}=(-1,1)^2$ the reference element, and let $\mathcal{E}(\mathcal{T})$ denote all edges, and $\mathcal{E}_0(\mathcal{T})$ denote all edges which do not lie on the boundary $\partial\Omega$. Each element τ can be the image of the reference element $\hat{\tau}$ under an affine map $F_\tau:\hat{\tau}\to\tau$. We write $h_\tau:=\text{diam}\ \tau$. If we assume that the triangulation is γ-shape regular, we have

$$h_\tau^{-1}\|F'_\tau\|+h_\tau\|(F'_\tau)^{-1}\|\leq\gamma. \tag{2.11}$$

For γ-shape regular meshes $\mathcal{T}$ on the domain Ω, we associate with each element $\tau \in \mathcal{T}$ a polynomial degree $p_\tau \in \mathbb{N}_0$. Moreover, these polynomial degrees $\{p_\tau\}$ are collected into the polynomial degree vector $\boldsymbol{p} = \{p_\tau\}$. Therefore, we can define the spaces of hp spectral element approximation $U^{\boldsymbol{p}}(\mathcal{T}, \Omega)$, $S^{\boldsymbol{p}}(\mathcal{T}, \Omega)$, $S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega)$ as described below:

$$U^{\boldsymbol{p}}(\mathcal{T}, \Omega) := \{u \in L^2(\Omega) : u|_e \circ F_\tau \in \mathcal{P}_{p_\tau}(\hat{\tau})\},$$

$$S^{\boldsymbol{p}}(\mathcal{T}, \Omega) := \{v \in H^1(\Omega) : v|_e \circ F_\tau \in \mathcal{P}_{p_\tau}(\hat{\tau})\},$$

$$S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega) := S^{\boldsymbol{p}}(\mathcal{T}, \Omega) \bigcap H_0^1(\Omega),$$

where $\mathcal{P}_{p_\tau}(\hat{\tau})$ denotes the spaces of polynomials in $\hat{\tau}$ of degree $\leq p_\tau$ in each variable, respectively. As to polynomial degree distribution $\boldsymbol{p}$, similar to (2.11), we assume that the polynomial degrees of neighboring elements are comparable. As a result, there exists a constant $\gamma > 0$ such that

$$\gamma^{-1}(p_\tau + 1) \leq p_{\tau'} + 1 \leq \gamma(p_\tau + 1), \quad \forall \tau, \tau' \in \mathcal{T}, \quad \bar{\tau} \bigcap \bar{\tau}' \neq \emptyset. \tag{2.12}$$

Let $K^{h,\boldsymbol{p}}(\mathcal{T}, \Omega) := K \bigcap U^{\boldsymbol{p}}(\mathcal{T}, \Omega)$ be the space of hp spectral element approximation for the control, and $S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega)$ be the space of hp spectral element approximation for the state and co-state. Then the semi-discrete hp spectral element approximation of (2.5)–(2.7) is as follows:

$$\min_{u_{hp}(t) \in K^{h,\boldsymbol{p}}} \left\{ \frac{1}{2} \int_0^T (\|y_{hp} - y_d\|^2_{L^2(\Omega)} + \|u_{hp}\|^2_{L^2(\Omega)}) dt \right\}, \tag{2.13}$$

$$\left(\frac{\partial y_{hp}}{\partial t}, w_{hp} \right) + a(y_{hp}, w_{hp}) = (f + Bu_{hp}, w_{hp}), \quad \forall\, w_{hp} \in S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega),\ t \in (0, T], \tag{2.14}$$

$$y_{hp}(x, 0) = y_0^{hp}(x), \qquad x \in \Omega, \tag{2.15}$$

where $y_{hp} \in H^1(0, T; S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega))$ and $y_0^{h,\boldsymbol{p}} \in S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega)$ is a hp spectral element approximation of y_0.

It follows that the optimal control problem (2.13)–(2.15) has a unique solution (y_{hp}, u_{hp}) and that a pair (y_{hp}, u_{hp}) is the solution of (2.13)–(2.15) if and only if there is a co-state p_{hp} such that the triplet (y_{hp}, p_{hp}, u_{hp}) satisfies the following optimality conditions:

$$\left(\frac{\partial y_{hp}}{\partial t}, w_{hp} \right) + a(y_{hp}, w_{hp}) = (f + Bu_{hp}, w_{hp}), \qquad \forall\, w_{hp} \in S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega), \tag{2.16}$$

$$y_{hp}(x, 0) = y_0^{hp}(x), \qquad x \in \Omega, \tag{2.17}$$

$$-\left(\frac{\partial p_{hp}}{\partial t}, q_{hp} \right) + a(q_{hp}, p_{hp}) = (y_{hp} - y_d, q_{hp}), \qquad \forall\, q_{hp} \in S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega), \tag{2.18}$$

$$p_{hp}(x, T) = 0, \qquad x \in \Omega, \tag{2.19}$$

$$(u_{hp} + B^* p_{hp}, \upsilon_{hp} - u_{hp})_U \geq 0, \qquad \forall\, \upsilon_{hp} \in K^{h,\boldsymbol{p}}(\mathcal{T}, \Omega). \tag{2.20}$$

Now, we shall consider the fully discrete hp spectral element approximation for above semi-discrete problem by using the backward Euler scheme in time. Let $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T,\ k_i = t_i - t_{i-1}, i = 1, 2, \cdots, M,\ k = \max_{1 \leq i \leq N}\{k_i\}$. For $i = 1, 2, \cdots, M$, we construct the hp spectral element approximation spaces $S_{i0}^{\boldsymbol{p}}(\mathcal{T}, \Omega) \subset H_0^1(\Omega)$ (similar as $S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega)$) on the i-th time step. Similarly, we construct the hp spectral element approximation spaces $K_i^{h,\boldsymbol{p}}(\mathcal{T},\Omega)\subset K$ (similar as $K^{h,\boldsymbol{p}}(\mathcal{T},\Omega)$) on the i-th time step. Then the fully discrete hp spectral element approximation scheme (2.21)–(2.23) is to find $(y_{hp}^i, u_{hp}^i)\in S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)\times K_i^{h,\boldsymbol{p}}(\mathcal{T},\Omega), i=1,2,\cdots,M$, such that

$$\min_{u_{hp}^i\in K_i^{h,\boldsymbol{p}}(\mathcal{T},\Omega)}\left\{\frac{1}{2}\sum_{i=1}^{N}k_i\left(\| y_{hp}^i - y_d(x,t_i)\|_{L^2(\Omega)}^2 + \| u_{hp}^i\|_{L^2(\Omega_U)}^2\right)\right\}, \tag{2.21}$$

$$\left(\frac{y_{hp}^i - y_{hp}^{i-1}}{k_i}, w_{hp}\right) + a(y_{hp}^i, w_{hp}) = (f(x,t_i) + Bu_{hp}^i, w_{hp}), \tag{2.22}$$

$$\forall\, w_{hp}\in S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)\subset H_0^1(\Omega),\quad i=1,2,\cdots,M,$$

$$y_{hp}^0(x) = y_0^{hp}(x),\quad x\in\Omega. \tag{2.23}$$

It follows that the optimal control problem (2.21)–(2.23) has a unique solution (Y_{hp}^i, U_{hp}^i), $i = 1,2,\cdots,M$, and that a pair $(Y_{hp}^i, U_{hp}^i)\in S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)\times K_i^{h,\boldsymbol{p}}(\mathcal{T},\Omega)$, $i=1,2,\cdots,M$, is the solution of (2.21)–(2.23) if and only if there is a co-state $P_{hp}^{i-1}\in S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)$, $i=1,2,\cdots,M$, such that the triplet $(Y_{hp}^i, P_{hp}^{i-1}, U_{hp}^i)\in S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)\times S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)\times K_i^{h,\boldsymbol{p}}(\mathcal{T},\Omega)$, $i=1,2,\cdots,M$, satisfies the following optimality conditions:

$$\left(\frac{Y_{hp}^i - Y_{hp}^{i-1}}{k_i}, w_{hp}\right) + a(Y_{hp}^i, w_{hp}) = (f(x,t_i) + BU_{hp}^i, w_{hp}), \tag{2.24}$$

$$\forall\, w_{hp}\in S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)\subset H_0^1(\Omega),\quad i=1,2,\cdots,M,$$

$$Y_{hp}^0(x) = y_0^{hp}(x),\quad x\in\Omega, \tag{2.25}$$

$$\left(\frac{P_{hp}^{i-1} - P_{hp}^{i}}{k_i}, q_{hp}\right) + a(q_{hp}, P_{hp}^{i-1}) = (Y_{hp}^i - y_d(x,t_i), q_{hp}), \tag{2.26}$$

$$\forall\, q_{hp}\in S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)\subset H_0^1(\Omega),\quad i=1,2,\cdots,M,$$

$$P_{hp}^M(x) = 0,\quad x\in\Omega, \tag{2.27}$$

$$(U_{hp}^i + B^*P_{hp}^{i-1}, \upsilon_{hp} - U_{hp}^i)\geq 0,\quad \forall\, \upsilon_{hp}\in K_i^{h,\boldsymbol{p}}(\mathcal{T},\Omega)\subset K,\quad i=1,2,\cdots,M. \tag{2.28}$$

For $i=1,2,\cdots,M$, let

$$\begin{aligned} Y_{hp}|_{(t_{i-1},t_i]} &= ((t_i - t)Y_{hp}^{i-1} + (t-t_{i-1})Y_{hp}^i)/k_i,\\ P_{hp}|_{(t_{i-1},t_i]} &= ((t_i - t)P_{hp}^{i-1} + (t-t_{i-1})P_{hp}^i)/k_i,\\ U_{hp}|_{(t_{i-1},t_i]} &= U_{hp}^i. \end{aligned}$$

For any function $w\in C(0,T;L_\Omega^2)$, let $\hat{w}(x,t)|_{t\in(t_{i-1},t_i]} = w(x,t_i)$, $\tilde{w}(x,t)|_{t\in(t_{i-1},t_i]} = w(x,t_{i-1})$. Then the optimality conditions (2.24)–(2.28) can be restated as :

$$\left(\frac{\partial Y_{hp}}{\partial t}, w_{hp}\right) + a(\hat{Y}_{hp}, w_{hp}) = (\hat{f} + BU_{hp}, w_{hp}), \tag{2.29}$$

$$\forall\, w_{hp}\in S_{i0}^{\boldsymbol{p}}(\mathcal{T},\Omega)\subset H_0^1(\Omega),\ t\in(t_{i-1},t_i],\ i=1,2,\cdots,M,$$

$$Y_{hp}(x,0) = y_0^{hp}(x),\qquad x\in\Omega, \tag{2.30}$$

$$-\left(\frac{\partial P_{hp}}{\partial t}, q_{hp}\right) + a(q_{hp}, \tilde{P}_{hp}) = (\hat{Y}_{hp} - \hat{y}_d, q_{hp}), \tag{2.31}$$

$$\forall\, q_{hp} \in S^{\boldsymbol{p}}_{i0}(\mathcal{T}, \Omega) \subset H^1_0(\Omega),\ t \in (t_{i-1}, t_i],\ i = M, \cdots, 2, 1,$$

$$P_{hp}(x, T) = 0, \qquad x \in \Omega, \tag{2.32}$$

$$(U_{hp} + B^* \tilde{P}_{hp}, \upsilon_{hp} - U_{hp}) \geq 0, \quad U_{hp} \in K^{h,\boldsymbol{p}}_i(\mathcal{T}, \Omega) \subset K, \tag{2.33}$$

$$\forall\, v_{hp} \in K^{h,\boldsymbol{p}}_i(\mathcal{T}, \Omega), \quad t \in (t_{i-1}, t_i],\ i = 1, 2, \cdots, M.$$

In the follows, we introduce a lemma which generalize the well-known Clément-type interpolation operators of [22] to the hp context.

Lemma 2.1. *(Scott-Zhang type quasi-interpolation). Let $\mathcal{T}$ be a γ-shape regular triangulation (see (2.11)) of a domain $\Omega \in \mathbb{R}^2$ and $\boldsymbol{p}$ be a polynomial degree distribution which is comparable (see (2.12)). Then there exists a linear operator $\hat{\Pi} : H^1_0(\Omega) \to S^{\boldsymbol{p}}_0(\mathcal{T}, \Omega)$, and there exists a constant $C > 0$, which depends only on γ, such that for every $u \in H^1_0(\Omega)$ and all elements $\tau \in \mathcal{T}$ and all edges $e \in \mathcal{E}(\tau)$,*

$$\|u - \hat{\Pi}u\|_{L^2(\tau)} + \frac{h_\tau}{p_\tau}\|\nabla(u - \hat{\Pi}u)\|_{L^2(\tau)} \leq C\frac{h_\tau}{p_\tau}\|\nabla u\|_{L^2(\omega_\tau)}, \tag{2.34}$$

$$\|u - \hat{\Pi}u\|_{L^2(e)} \leq C\sqrt{\frac{h_e}{p_e}}\|\nabla u\|_{L^2(\omega_e)}, \tag{2.35}$$

where h_e is the length of the edge e and $p_e = \max(p_\tau, p_{\tau'})$, where τ, τ' are elements sharing the edge e, ω_τ, ω_e are patches covering τ and e with a few layers, respectively. See [20] for more details on ω_τ and ω_e.

3. A $L^2(H^1) - L^2(L^2)$ posteriori error estimates

In this section, we shall derive a $L^2(H^1) - L^2(L^2)$ posteriori error estimates for the hp spectral approximation of the optimal control problem governed by parabolic equations. Set

$$J(u) = \frac{1}{2}\int_0^T \left(\|y - y_d\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right) dt,$$

$$J_{hp}(U_{hp}) = \frac{1}{2}\int_0^T \left(\|Y_{hp} - y_d\|^2_{L^2(\Omega)} + \|U_{hp}\|^2_{L^2(\Omega)}\right) dt.$$

According to [11], it can be shown that

$$(J'(u), v) = (u + B^* p, v), \tag{3.1}$$

$$(J'_{hp}(U_{hp}), v) = (U_{hp} + B^* \tilde{P}_{hp}, v), \tag{3.2}$$

$$(J'(U_{hp}), v) = (U_{hp} + B^* p(U_{hp}), v), \tag{3.3}$$

where $p(U_{hp})$ is the solution of the auxiliary equations:

$$\left(\frac{\partial y(U_{hp})}{\partial t}, w\right) + a(y(U_{hp}), w) = (f + BU_{hp}, w), \quad \forall\, w \in H^1_0(\Omega), \tag{3.4}$$

$$y(U_{hp})(x,0) = y_0(x), \qquad x \in \Omega, \tag{3.5}$$

$$-\left(\frac{\partial p(U_{hp})}{\partial t}, q\right) + a(q, p(U_{hp})) = (y(U_{hp}) - y_d, q), \quad \forall\, q \in H_0^1(\Omega), \tag{3.6}$$

$$p(U_{hp})(x,T) = 0, \qquad x \in \Omega. \tag{3.7}$$

Theorem 3.1. *Let (y, p, u) and (Y_{hp}, P_{hp}, U_{hp}) be the solutions of (2.8)–(2.10) and (2.29)–(2.33), respectively. Then we have*

$$\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \leq C\eta_1^2 + C\|p(U_{hp}) - \tilde{P}_{hp}\|^2_{L^2(0,T;L^2(\Omega))}, \tag{3.8}$$

where $p(U_{hp})$ is defined by (3.4)–(3.7) and

$$\eta_1^2 = \sum_{\tau \in \mathcal{T}} \left(\sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} \|U_{hp} + B^* \tilde{P}_{hp}\|^2_{L^2(\tau)} dt \right).$$

Proof. According to the definition of norm $\|\cdot\|_{L^2(0,T;L^2(\Omega))}$, there are

$$\begin{aligned} c\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T (u - U_{hp}, u - U_{hp}) dt \\ = \int_0^T (u + B^* p, u - U_{hp}) dt &+ \int_0^T (U_{hp} + B^* \tilde{P}_{hp}, U_{hp} - u) dt \\ + \int_0^T (B^*(\tilde{P}_{hp} - p(U_{hp})), u - U_{hp}) dt &+ \int_0^T (B^*(p(U_{hp}) - p), u - U_{hp}) dt. \end{aligned} \tag{3.9}$$

From (2.8), (2.9) and (3.4)–(3.7), we obtain

$$\begin{aligned} &\int_0^T (B^*(p(U_{hp}) - p), u - U_{hp}) dt = \int_0^T (p(U_{hp}) - p, B(u - U_{hp})) dt \\ &= \int_0^T \left(\left(\frac{\partial}{\partial t}(y - y(U_{hp})), p(U_{hp}) - p \right) + a(y - y(U_{hp}), p(U_{hp}) - p) \right) dt \\ &= \int_0^T \left(-\left(y - y(U_{hp}), \frac{\partial}{\partial t}(p(U_{hp}) - p) \right) + a(y - y(U_{hp}), p(U_{hp}) - p) \right) dt \\ &= \int_0^T (y - y(U_{hp}), y(U_{hp}) - y) dt \leq 0. \end{aligned} \tag{3.10}$$

Moreover, note that $U_{hp} \in K^{h,\boldsymbol{p}}(\mathcal{T}, \Omega) \subset K$. It follows from (2.10) that

$$\int_0^T (u + B^* p, u - U_{hp}) dt \leq 0. \tag{3.11}$$

Combining (3.10) with (3.11) from (3.9), we obtain

$$\begin{aligned} c\|u^* - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \leq& \int_0^T (U_{hp} + B^* \tilde{P}_{hp}, U_{hp} - u) dt \\ &+ \int_0^T (B^*(\tilde{P}_{hp} - p(U_{hp})), u - U_{hp}) dt \\ :=& I_1 + I_2. \end{aligned} \tag{3.12}$$

We first estimate I_1 here. It is clear that

$$\begin{aligned} I_1 =& \int_0^T (U_{hp} + B^*\tilde{P}_{hp}, U_{hp} - u)dt = \sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} (U_{hp} + B^*\tilde{P}_{hp}, U_{hp} - u)dt \\ \le& C(\delta) \sum_{\tau\in\mathcal{T}} \left(\sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} \|U_{hp} + B^*\tilde{P}_{hp}\|^2_{L^2(\tau)} dt \right) + \delta \sum_{\tau\in\mathcal{T}} \left(\sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} \|U_{hp} - u\|^2_{L^2(\tau)} dt \right) \\ \le& C(\delta)\eta_1^2 + \delta\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))}, \end{aligned} \tag{3.13}$$

for any sufficiently small positive number δ. Then for I_2 form (3.12), we obtain

$$\begin{aligned} I_2 =& \int_0^T (B^*(\tilde{P}_{hp} - p(U_{hp})), u - U_{hp})dt \\ \le& C(\delta) \sum_{\tau\in\mathcal{T}} \left(\int_0^T \|B^*(\tilde{P}_{hp} - p(U_{hp})\|^2_{L^2(\tau)} dt \right) + \delta \sum_{\tau\in\mathcal{T}} \left(\int_0^T \|U_{hp} - u\|^2_{L^2(\tau)} dt \right) \\ \le& C\|\tilde{P}_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} + \delta\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))}, \end{aligned} \tag{3.14}$$

for any sufficiently small positive number δ. Thus, applying Eqs (3.12) and (3.14) gives the estimate

$$\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \le C\eta_1^2 + C\|p(U_{hp}) - \tilde{P}_{hp}\|^2_{L^2(0,T;L^2(\Omega))}.$$

This proves (3.8). $\square$

Theorem 3.2. *Let (Y_{hp}, P_{hp}, U_{hp}) be the solution of (2.13)–(2.15) and $(y(U_{hp}), p(U_{hp}))$ be defined by (3.4)–(3.7). Then*

$$\|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;H^1(\Omega))} + \|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;H^1(\Omega))} \le C \sum_{i=2}^{8} \eta_i^2, \tag{3.15}$$

where

$$\begin{aligned} \eta_2^2 =& \sum_{\tau\in\mathcal{T}} \int_0^T \frac{h_\tau^2}{p_\tau^2} \int_\tau \left(\hat{Y}_{hp} - \hat{y}_d + \mathrm{div}(A^*\nabla\tilde{P}_{hp}) + \frac{\partial P_{hp}}{\partial t} \right)^2 dxdt, \\ \eta_3^2 =& \sum_{\tau\in\mathcal{T}} \int_0^T \int_\tau |A^*\nabla(\tilde{P}_{hp} - P_{hp})|^2 dxdt, \\ \eta_4^2 =& \|\hat{y}_d - y_d\|^2_{L^2(0,T;L^2(\Omega))}, \\ \eta_5^2 =& \|Y_{hp} - \hat{Y}_{hp}\|^2_{L^2(0,T;L^2(\Omega))}, \\ \eta_6^2 =& \sum_{\tau\in\mathcal{T}} \int_0^T \frac{h_\tau^2}{p_\tau^2} \int_\tau \left(\hat{f} + BU_{hp} + \mathrm{div}(A\nabla\hat{Y}_{hp}) - \frac{\partial Y_{hp}}{\partial t} \right)^2 dxdt, \\ \eta_7^2 =& \|f - \hat{f}\|^2_{L^2(0,T;L^2(\Omega))}, \\ \eta_8^2 =& \sum_{\tau\in\mathcal{T}} \int_0^T \int_\tau |A\nabla(\hat{Y}_{hp} - Y_{hp})|^2 dxdt, \\ \eta_9^2 =& \|y_0(x) - Y_{hp}(x,0)\|^2_{L^2(\Omega)}. \end{aligned}$$

Proof. Let $e^p = p(U_{hp}) - P_{hp}$ and $e_I^p = \hat{\Pi} e^p$, where $\hat{\Pi}$ be the Scott-Zhang type quasi-interpolator defined as in Lemma 2.1. Note that $(p(U_{hp}) - P_{hp})(x,T) = 0$, hence

$$\int_0^T -\left(\frac{\partial (p(U_{hp}) - P_{hp})}{\partial t}, e^p\right) dt \geq 0.$$

Then there holds the estimate:

$$\begin{aligned}
c\|e^p\|^2_{L^2(0,T;H^1(\Omega))} \leq& \int_0^T a(e^p, p(U_{hp}) - P_{hp}) dt \\
\leq& \int_0^T (\nabla e^p, A^* \nabla (p(U_{hp}) - P_{hp})) dt - \int_0^T \left(\frac{\partial (p(U_{hp}) - P_{hp})}{\partial t}, e^p\right) dt \\
=& \int_0^T (\nabla e^p, A^* \nabla (p(U_{hp}) - \tilde{P}_{hp})) dt - \int_0^T \left(\frac{\partial (p(U_{hp}) - P_{hp})}{\partial t}, e^p\right) dt \\
&+ \int_0^T (\nabla e^p, A^* \nabla (\tilde{P}_{hp} - P_{hp}) dt \\
=& \int_0^T (\nabla (e^p - e_I^p), A^* \nabla (p(U_{hp}) - \tilde{P}_{hp})) dt - \int_0^T \left(\frac{\partial (p(U_{hp}) - P_{hp})}{\partial t}, e^p - e_I^p\right) dt \\
&- \int_0^T \left(\frac{\partial (p(U_{hp}) - P_{hp})}{\partial t}, e_I^p\right) dt \\
&+ \int_0^T (\nabla e_I^p, A^* \nabla (p(U_{hp}) - \tilde{P}_{hp})) dt + \int_0^T (\nabla e^p, A^* \nabla (\tilde{P}_{hp} - P_{hp})) dt.
\end{aligned} \tag{3.16}$$

By using the Eqs (2.16)–(2.20) and (3.4)–(3.7), note that $e_I^p = \hat{\pi} e^p \in S_0^{\boldsymbol{p}}(\mathcal{T}, \Omega)$, then the above formula (3.16) can be written as

$$\begin{aligned}
c\|e^p\|^2_{L^2(0,T;H^1(\Omega))} \leq& \int_0^T \left(y(U_{hp}) - y_d + \mathrm{div}(A^* \nabla \tilde{P}_{hp}) + \frac{\partial P_{hp}}{\partial t}, e^p - e_I^p\right) dt \\
&+ \int_0^T (y(U_{hp}) - \hat{Y}_{hp}, e_I^p) dt + \int_0^T (\hat{y}_d - y_d, e_I^p) dt \\
&+ \int_0^T (\nabla e^p, A^* \nabla (\tilde{P}_{hp} - P_{hp})) dt \\
=& \int_0^T \left(\hat{Y}_{hp} - \hat{y}_d + \mathrm{div}(A^* \nabla \tilde{P}_{hp}) + \frac{\partial P_{hp}}{\partial t}, e^p - e_I^p\right) dt \\
&+ \int_0^T (y(U_{hp}) - \hat{Y}_{hp}, e^p) dt + \int_0^T (\hat{y}_d - y_d, e^p) dt \\
&+ \int_0^T (\nabla e^p, A^* \nabla (\tilde{P}_{hp} - P_{hp})) dt \\
:=& J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.17}$$

Employing Lemma 2.1, the first estimate J_1 becomes as

$$\begin{aligned} J_1 =& \int_0^T \left(\hat{Y}_{hp} - \hat{y}_d + div(A^*\nabla\tilde{P}_{hp}) + \frac{\partial P_{hp}}{\partial t}, e^p - e_I^p\right) dt \\ \leq& C(\delta) \sum_{\tau\in\mathcal{T}} \int_0^T \frac{h_\tau^2}{p_\tau^2} \int_\tau \left(\hat{Y}_{hp} - \hat{y}_d + \mathrm{div}(A^*\nabla\tilde{P}_{hp}) + \frac{\partial P_{hp}}{\partial t}\right)^2 dxdt \\ &+ \delta \sum_{\tau\in\mathcal{T}} \int_0^T \|e^p\|_{H^1(\tau)}^2 dt \\ \leq& C(\delta)\eta_2^2 + \delta\|p(U_{hp}) - P_{hp}\|_{L^2(0,T;H^1(\Omega))}^2, \end{aligned} \tag{3.18}$$

where δ is an arbitrary positive number, $C(\delta)$ is a constant dependent on δ. Similarly,

$$\begin{aligned} J_2 =& \int_0^T (y(U_{hp}) - \hat{Y}_{hp}, e^p) dt \\ \leq& C(\delta) \sum_{\tau\in\mathcal{T}} \int_0^T \int_\tau |y(U_{hp}) - \hat{Y}_{hp}|^2 dxdt + \delta\|p(U_{hp}) - P_{hp}\|_{L^2(0,T;H^1(\Omega))}^2 \\ \leq& C(\delta)\|y(U_{hp}) - Y_{hp}\|_{L^2(0,T;H^1(\Omega))}^2 + C(\delta)\|Y_{hp} - \hat{Y}_{hp}\|_{L^2(0,T;L^2(\Omega))}^2 \\ &+ \delta\|p(U_{hp}) - P_{hp}\|_{L^2(0,T;H^1(\Omega))}^2. \end{aligned} \tag{3.19}$$

And for J_3 and J_4, we obtain

$$\begin{aligned} J_3 =& \int_0^T (y_d - \hat{y}_d, e^p) dt \\ \leq& C(\delta)\|y_d - \hat{y}_d\|_{L^2(0,T;L^2(\Omega))}^2 + \delta\|p(U_{hp}) - P_{hp}\|_{L^2(0,T;H^1(\Omega))}^2, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} J_4 =& \int_0^T (\nabla e^p, A^*\nabla(\tilde{P}_{hp} - P_{hp})) dt \\ \leq& C(\delta) \sum_{\tau\in\mathcal{T}} \int_0^T \int_\tau |A^*\nabla(\tilde{P}_{hp} - P_{hp})|^2 dxdt + \delta \sum_{\tau\in\mathcal{T}} \int_0^T \int_\tau |\nabla e^p|^2 dxdt \\ \leq& C(\delta)\eta_3^2 + \delta\|p(U_{hp}) - P_{hp}\|_{L^2(0,T;H^1(\Omega))}^2. \end{aligned} \tag{3.21}$$

Then, let δ be small enough, from (3.16)–(3.21), we obtain

$$\|p(U_{hp}) - P_{hp}\|_{L^2(0,T;H^1(\Omega))}^2 \leq C(\delta) \sum_{i=2}^5 \eta_i^2 + C(\delta)\|y(U_{hp}) - Y_{hp}\|_{L^2(0,T;L^2(\Omega))}^2. \tag{3.22}$$

Similarly, let $e^y = y(U_{hp}) - Y_{hp}$, $e_I^y = \hat{\Pi}e^y$, where $\hat{\Pi}$ be the Scott-Zhang type quasi-interpolator defined as in Lemma 2.1. Note that

$$\int_0^T \left(\frac{\partial(y(U_{hp}) - Y_{hp})}{\partial t}, e^y\right) dt = \sum_{\tau\in\mathcal{T}} \int_\tau \int_0^T e^y \frac{\partial(y(U_{hp}) - Y_{hp})}{\partial t} dtdx$$

$$
\begin{aligned}
&=\sum_{\tau\in\mathcal{T}}\int_{\tau}\int_0^T e^y d(y(U_{hp})-Y_{hp})dx \\
&=\frac{1}{2}\sum_{\tau\in\mathcal{T}}\int_{\tau}((y(U_{hp})-Y_{hp})(x,T))^2dx \\
&\quad-\frac{1}{2}\sum_{\tau\in\mathcal{T}}\int_{\tau}((y(U_{hp})-Y_{hp})(x,0))^2dx \\
&=\frac{1}{2}\sum_{\tau\in\mathcal{T}}\int_{\tau}((y(U_{hp})-Y_{hp})(x,T))^2dx \\
&\quad-\frac{1}{2}\|y_0(x)-Y_{hp}(x,0)\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.23}
$$

Thus

$$
\int_0^T\left(\frac{\partial(y(U_{hp})-Y_{hp})}{\partial t},e^y\right)dt+\frac{1}{2}\|y_0(x)-Y_{hp}(x,0)\|^2_{L^2(\Omega)}\geq 0.
$$

And then we can derive

$$
\begin{aligned}
c\|e^y\|^2_{L^2(0,T;H^1(\Omega))}\leq&\int_0^T a(y(U_{hp})-Y_{hp},e^y)dt+\int_0^T\left(\frac{\partial(y(U_{hp})-Y_{hp})}{\partial t},e^y\right)dt \\
&+\frac{1}{2}\|y_0(x)-Y_{hp}(x,0)\|^2_{L^2(\Omega)} \\
=&\int_0^T(A\nabla(y(U_{hp})-\hat{Y}_{hp}),\nabla e^y)dt+\int_0^T\left(\frac{\partial(y(U_{hp})-Y_{hp})}{\partial t},e^y\right)dt \\
&+\int_0^T(A\nabla(\hat{Y}_{hp}-Y_{hp}),\nabla e^y)dt+\frac{1}{2}\|y_0(x)-Y_{hp}(x,0)\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.24}
$$

Similar as (3.17), by using the Eqs (3.4)–(3.7) and (2.16)–(2.20), for $e^y_I=\hat{\pi}e^y\in S^{\boldsymbol{p}}_0(\mathcal{T},\Omega)$ and from (3.24), we obtain

$$
\begin{aligned}
c\|e^y\|^2_{L^2(0,T;H^1(\Omega))}\leq&\int_0^T(\hat{f}+BU_{hp}+\mathrm{div}(A\nabla\hat{Y}_{hp})-\frac{\partial Y_{hp}}{\partial t},e^y-e^y_I)dt \\
&+\frac{1}{2}\|y_0(x)-Y_{hp}(x,0)\|^2_{L^2(\Omega)}+\int_0^T(f-\hat{f},e^y)dt \\
&+\int_0^T(A\nabla(\hat{Y}_{hp}-Y_{hp}),\nabla e^y)dt \\
\leq& C(\delta)\sum_{\tau\in\mathcal{T}}\int_0^T\frac{h_\tau^2}{p_\tau^2}\int_\tau\left(\hat{f}+BU_{hp}+\mathrm{div}(A\nabla\hat{Y}_{hp})-\frac{\partial Y_{hp}}{\partial t}\right)^2dxdt \\
&+C(\delta)\|f-\hat{f}\|^2_{L^2(0,T;L^2(\Omega))}+C(\delta)\sum_{\tau\in\mathcal{T}}\int_0^T\int_\tau|A\nabla(\hat{Y}_{hp}-Y_{hp})|^2dxdt \\
&+\frac{1}{2}\|y_0(x)-Y_{hp}(x,0)\|^2_{L^2(\Omega)}+\delta\|y(U_{hp})-Y_{hp}\|^2_{L^2(0,T;H^1(\Omega))} \\
=& C(\delta)\sum_{i=5}^{9}\eta_i^2+\delta\|y(U_{hp})-Y_{hp}\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
\tag{3.25}
$$

Hence, there is

$$\|y(U_{hp}) - Y_{hp}\|^2_{L^2(0,T;H^1(\Omega))} \leq C(\delta) \sum_{i=5}^{9} \eta_i^2. \tag{3.26}$$

Finally, we can obtain (3.15) from (3.22) and (3.26). □

Theorem 3.3. *Let* (y, p, u) *and* (Y_{hp}, P_{hp}, U_{hp}) *be the solutions of (2.8)–(2.10) and (2.29)–(2.33), respectively. Assume that all the conditions in Theorem 3.1 are valid. Then*

$$\|Y_{hp} - y\|^2_{L^2(0,T;H^1(\Omega))} + \|P_{hp} - p\|^2_{L^2(0,T;H^1(\Omega))} + \|U_{hp} - u\|^2_{L^2(0,T;L^2(\Omega))} \leq C \sum_{i=1}^{9} \eta_i^2, \tag{3.27}$$

where η_i^2, $i = 1, \cdots, 9$ *are defined in Theorems 3.1 and 3.2.*

Proof. It follows from Theorem 3.1 and Theorem 3.2, we have

$$\begin{aligned}
\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \leq& C\eta_1^2 + C\|\tilde{P}_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} \\
\leq& C\eta_1^2 + C\|\tilde{P}_{hp} - P_{hp}\|^2_{L^2(0,T;L^2(\Omega))} + C\|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} \\
\leq& C \sum_{i=1}^{9} \eta_i^2 + C\|\tilde{P}_{hp} - P_{hp}\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned} \tag{3.28}$$

Note that A is positive definite and it follows from the Poincaré inequality that

$$\|\tilde{P}_{hp} - P_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \leq C \sum_{\tau\in\mathcal{T}} \int_0^T \int_\tau |A^*\nabla(\tilde{P}_{hp} - P_{hp})|^2 dxdt = C\eta_3^2. \tag{3.29}$$

Then, it follows from (3.28) and (3.29) that

$$\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \leq C \sum_{i=1}^{9} \eta_i^2. \tag{3.30}$$

Note that

$$\|Y_{hp} - y\|^2_{L^2(0,T;H^1(\Omega))} \leq \|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;H^1(\Omega))} + \|y(U_{hp}) - y\|^2_{L^2(0,T;H^1(\Omega))}, \tag{3.31}$$

$$\|P_{hp} - p\|^2_{L^2(0,T;H^1(\Omega))} \leq \|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;H^1(\Omega))} + \|p(U_{hp}) - p\|^2_{L^2(0,T;H^1(\Omega))}, \tag{3.32}$$

and

$$\|y(U_{hp}) - y\|^2_{L^2(0,T;H^1(\Omega))} \leq C\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))}, \tag{3.33}$$

$$\|p(U_{hp}) - p\|^2_{L^2(0,T;H^1(\Omega))} \leq \|y(U_{hp}) - y\|^2_{L^2(0,T;L^2(\Omega))} \leq C\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))}. \tag{3.34}$$

From (3.30), (3.31), (3.33), and Theorem 3.2, we derive

$$\begin{aligned}
\|Y_{hp} - y\|^2_{L^2(0,T;H^1(\Omega))} \leq& \|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;H^1(\Omega))} + C\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \\
\leq& C \sum_{i=1}^{9} \eta_i^2.
\end{aligned} \tag{3.35}$$

Similarly, from (3.30), (3.32), (3.34), and Theorem 3.2, there is

$$
\begin{aligned}
\|P_{hp}-p\|^2_{L^2(0,T;H^1(\Omega))} \leq & \|P_{hp}-p(U_{hp})\|^2_{L^2(0,T;H^1(\Omega))}+C\|u-U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \\
\leq & C\sum_{i=1}^{9}\eta_i^2.
\end{aligned}
\tag{3.36}
$$

Therefore, we obtain (3.27) follows from (3.30), (3.35) and (3.36). □

4. A $L^2(L^2)-L^2(L^2)$ posteriori error estimates

In this section, we shall derive a $L^2(L^2)-L^2(L^2)$ posteriori error estimate for the hp spectral element approximation of the optimal control problem governed by parabolic equations. In order to estimate the errors $\|Y_{hp}-y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}$ and $\|\tilde{P}_{hp}-p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}$, we shall use two auxiliary equations.

We set the following dual auxiliary equations:

$$
\begin{cases}
\frac{\partial \xi}{\partial t}-\operatorname{div}(A\nabla\xi)=F, & x\in\Omega, t\in(0,T],\\
\xi|_{\partial\Omega}=0, & t\in[0,T],\\
\xi(x,0)=0, & x\in\Omega.
\end{cases}
\tag{4.1}
$$

$$
\begin{cases}
-\frac{\partial \zeta}{\partial t}-\operatorname{div}(A^*\nabla\zeta)=F, & x\in\Omega, t\in(0,T],\\
\zeta|_{\partial\Omega}=0, & t\in[0,T],\\
\zeta(x,T)=0, & x\in\Omega.
\end{cases}
\tag{4.2}
$$

The following well known stability results are presented in [13].

Lemma 4.1. *Assume that Ω is a convex domain. Let ξ and ζ be the solutions of (3.28) and (3.29), respectively. Then, for $\upsilon=\xi$ or $\upsilon=\zeta$,*

$$\|\upsilon\|_{L^\infty(0,T;L^2(\Omega))}\leq C\|F\|_{L^2(0,T;L^2(\Omega))}, \tag{4.3}$$

$$\|\nabla\upsilon\|_{L^2(0,T;L^2(\Omega))}\leq C\|F\|_{L^2(0,T;L^2(\Omega))}, \tag{4.4}$$

$$\|D^2\upsilon\|_{L^2(0,T;L^2(\Omega))}\leq C\|F\|_{L^2(0,T;L^2(\Omega))}, \tag{4.5}$$

$$\left\|\frac{\partial\upsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}\leq C\|F\|_{L^2(0,T;L^2(\Omega))}, \tag{4.6}$$

where $D^2\upsilon=\frac{\partial^2\upsilon}{\partial x_i\partial x_j}, 1\leq i,j\leq n$.

Theorem 4.1. *Let (Y_{hp},P_{hp},U_{hp}) be the solution of (2.13)–(2.15) and let $(y(U_{hp}),p(U_{hp}))$ be defined by (3.4)–(3.7). Then*

$$\|Y_{hp}-y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}+\|P_{hp}-p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}\leq C\sum_{i=2}^{9}\eta_i^2, \tag{4.7}$$

where

$$\hat{\eta}_2^2=\sum_{\tau\in\mathcal{T}}\int_0^T\frac{h_\tau^4}{p_\tau^4}\int_\tau\left(\frac{\partial}{\partial t}P_{hp}+\operatorname{div}(A^*\nabla\tilde{P}_{hp})+\hat{Y}_{hp}-\hat{y}_d\right)^2dxdt,$$

$$\hat{\eta}_3^2 = \sum_{\tau\in\mathcal{T}} \int_0^T \int_\tau |A^*\nabla(P_{hp} - \tilde{P}_{hp})|^2 dxdt,$$
$$\hat{\eta}_4^2 = \|y_d - \hat{y}_d\|^2_{L^2(0,T;L^2(\Omega))},$$
$$\hat{\eta}_5^2 = \|\hat{Y}_{hp} - Y_{hp}\|^2_{L^2(0,T;L^2(\Omega))},$$
$$\hat{\eta}_6^2 = \sum_{\tau\in\mathcal{T}} \int_0^T \frac{h_\tau^4}{p_\tau^4} \int_\tau \left(\frac{\partial Y_{hp}}{\partial t} - \mathrm{div}(A\nabla\hat{Y}_{hp}) - \hat{f} - BU_{hp}\right)^2 dxdt,$$
$$\hat{\eta}_7^2 = |\hat{f} - f\|^2_{L^2(0,T;L^2(\Omega))},$$
$$\hat{\eta}_8^2 = \sum_{\tau\in\mathcal{T}} \int_0^T \int_\tau |A\nabla(Y_{hp} - \hat{Y}_{hp})|^2 dxdt,$$
$$\hat{\eta}_9^2 = \|Y_{hp}(x;0) - y_0(x)\|^2_{L^2(\Omega)}.$$

Proof. Let ξ be the solution of (4.1) with $F = P_{hp} - p(U_{hp})$. Let $\xi_I = \hat{\Pi}\xi$, where $\hat{\Pi}$ be the Scott-Zhang type quasi-interpolator defined as in Lemma 2.1. Then it follows from (3.4)–(3.7) and (2.14) that

$$\begin{aligned}
&\|(P_{hp} - p(U_{hp}))\|^2_{L^2(0,T;L^2(\Omega))}\\
=&\int_0^T (P_{hp} - p(U_{hp}), F)dt\\
=&\int_0^T \left(-\left(\frac{\partial}{\partial t}(P_{hp} - p(U_{hp})), \xi\right) + a(\xi, P_{hp} - p(U_{hp}))\right) dt\\
=&\int_0^T \left(-\left(\frac{\partial}{\partial t}(P_{hp} - p(U_{hp})), \xi - \xi_I\right) + a(\xi - \xi_I, \tilde{P}_{hp} - p(U_{hp}))\right) dt\\
&+ \int_0^T \left(-\left(\frac{\partial}{\partial t}(P_{hp} - p(U_{hp})), \xi_I\right) + a(\xi_I, \tilde{P}_{hp} - p(U_{hp}))\right) dt + \int_0^T a(\xi, P_{hp} - \tilde{P}_{hp})dt \qquad (4.8)\\
=&\int_0^T \left(-\frac{\partial}{\partial t}P_{hp} - div(A^*\nabla\tilde{P}_{hp}) - (\hat{Y}_{hp} - \hat{y}_d), \xi - \xi_I\right) dt + \int_0^T (\hat{Y}_{hp} - Y_{hp}, \xi)dt\\
&+ \int_0^T (Y_{hp} - y(U_{hp}), \xi)dt + \int_0^T (y_d - \hat{y}_d, \xi)dt + \int_0^T a(\xi, P_{hp} - \tilde{P}_{hp})dt\\
=&K_1 + K_2 + K_3 + K_4 + K_5.
\end{aligned}$$

It follows from Lemma 2.1 and Lemma 4.1 that

$$\begin{aligned}
K_1 =& \int_0^T \left(-\frac{\partial}{\partial t}P_{hp} - \mathrm{div}(A^*\nabla\tilde{P}_{hp}) - (\hat{Y}_{hp} - \hat{y}_d), \xi - \xi_I\right) dt\\
\le& C(\delta) \sum_{\tau\in\mathcal{T}} \int_0^T \frac{h_\tau^4}{p_\tau^4} \int_\tau \left(\frac{\partial}{\partial t}P_{hp} + \mathrm{div}(A^*\nabla\tilde{P}_{hp}) + \hat{Y}_{hp} - \hat{y}_d\right)^2 dxdt\\
&+ \delta \int_0^T \|\xi\|^2_{H^2(\Omega)}dt \qquad (4.9)\\
\le& C(\delta)\hat{\eta}_2^2 + \delta\|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

Similarly, here is

$$\begin{aligned} K_2 =& \int_0^T (\hat{Y}_{hp} - Y_{hp}, \xi)dt \\ \leq& C(\delta) \sum_{\tau \in \mathcal{T}} \int_0^T \int_\tau |\hat{Y}_{hp} - Y_{hp})|^2 dxdt + \delta \sum_{\tau \in \mathcal{T}} \int_0^T \|\xi\|^2_{L^2(\tau)} dt \\ \leq& C(\delta)\hat{\eta}_5^2 + \delta \|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{4.10}$$

And for K_3, K_4, and K_5, we derive

$$\begin{aligned} K_3 =& \int_0^T (Y_{hp} - y(U_{hp}), \xi)dt \\ \leq& C(\delta)\|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} + \delta\|\xi\|^2_{L^2(0,T;L^2(\Omega))} \\ \leq& C(\delta)\|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} + \delta\|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} K_4 =& \int_0^T (y_d - \hat{y}_d, \xi)dt \\ \leq& C(\delta)\|y_d - \hat{y}_d\|^2_{L^2(0,T;L^2(\Omega))} + \delta\|\xi\|^2_{L^2(0,T;L^2(\Omega))} \\ \leq& C(\delta)\hat{\eta}_4^2 + \delta\|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} K_5 =& \int_0^T a(\xi, P_{hp} - \tilde{P}_{hp})dt \\ \leq& C(\delta) \sum_{\tau \in \mathcal{T}} \int_0^T \int_\tau |A^* \nabla (P_{hp} - \tilde{P}_{hp})|^2 dxdt + \delta \sum_{\tau \in \mathcal{T}} \int_0^T \int_\tau |\nabla \xi|^2 dxdt \\ \leq& C(\delta)\eta_3^2 + \delta\|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{4.13}$$

Then, let δ be small enough, from (4.9)–(4.13), we obtain

$$\|p(U_{hp}) - P_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \leq C(\delta) \sum_{i=2}^{5} \hat{\eta}_i^2 + C(\delta)\|y(U_{hp}) - Y_{hp}\|^2_{L^2(0,T;L^2(\Omega))}. \tag{4.14}$$

Similarly, let ζ be the solution of (4.2) with $F = Y_{hp} - y(U_{hp})$, there is

$$\begin{aligned} & \|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} \\ =& \int_0^T (Y_{hp} - y(U_{hp}), F) \\ =& \int_0^T \left(\left(\frac{\partial}{\partial t}(Y_{hp} - y(U_{hp})), \zeta \right) + a(Y_{hp} - y(U_{hp}), \zeta) \right) dt \\ &+ ((Y_{hp} - y(U_{hp}))(x,0), \zeta(x,0)) \end{aligned}$$

$$
\begin{aligned}
&\le C(\delta)\sum_{\tau\in\mathcal{T}}\int_0^T \frac{h_\tau^4}{p_\tau^4}\int_\tau\left(\frac{\partial Y_{hp}}{\partial t} - div(A\nabla\hat{Y}_{hp} - \hat{f} - BU_{hp}\right)^2 dxdt \\
&\quad + C(\delta)\|\hat{f} - f\|^2_{L^2(0,T;L^2(\Omega))} + C(\delta)\sum_{\tau\in\mathcal{T}}\int_0^T\int_\tau |A\nabla(Y_{hp} - \hat{Y}_{hp})|^2 dxdt \\
&\quad + C(\delta)\|Y_{hp}(x,0) - y_0(x)\|^2_{L^2(\Omega)} \\
&\quad + \delta\sum_{\tau\in\mathcal{T}}\int_0^T \|\zeta\|^2_{H^2(\tau)} + \delta\|\zeta(x,0)\|^2_{L^2(\Omega)} \\
&\le C(\delta)\sum_5^9 \hat{\eta}_i^2 + \delta\|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

Hence, let δ be small enough, we have

$$
\|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} \le C\sum_5^9 \hat{\eta}_i^2. \tag{4.15}
$$

Then, (4.7) follows from (4.14) and (4.15). □

From Theorem 3.1 and Lemma 4.1, we have the following a $L^2(L^2)$−$L^2(L^2)$ posteriori error estimate.

Theorem 4.2. *Let (y, p, u) and (Y_{hp}, P_{hp}, U_{hp}) be the solutions of (2.8)–(2.10) and (2.29)–(2.33), respectively. Assume that all the conditions in Theorem 3.1 are valid. Then*

$$
\|Y_{hp} - y\|^2_{L^2(0,T;L^2(\Omega))} + \|P_{hp} - p\|^2_{L^2(0,T;L^2(\Omega))} + \|U_{hp} - u\|^2_{L^2(0,T;L^2(\Omega))} \le C\eta_1 + C\sum_{i=2}^9 \hat{\eta}_i^2, \tag{4.16}
$$

where η_1^2 and $\hat{\eta}_i^2$, $i = 2, \cdots, 9$ are defined in Theorem 3.1 and Theorem 4.1.

Proof. Applying Theorem 3.1 and Theorem 4.1, we derive

$$
\begin{aligned}
\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} &\le C\eta_1^2 + C\|\tilde{P}_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} \\
&\le C\eta_1^2 + C\|\tilde{P}_{hp} - P_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + C\|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} \\
&\le C\eta_1^2 + C\sum_{i=2}^9 \hat{\eta}_i^2 + C\|\tilde{P}_{hp} - P_{hp}\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned} \tag{4.17}
$$

Note that A is positive definite, it follows from the Poincaré inequality that

$$
\|\tilde{P}_{hp} - P_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \le C\sum_{\tau\in\mathcal{T}}\int_0^T\int_\tau |A^*\nabla(\tilde{P}_{hp} - P_{hp})|^2 dxdt = C\hat{\eta}_3^2. \tag{4.18}
$$

Employing representation (4.17) and (4.18), it turns out that

$$
\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \le C\eta_1^2 + C\sum_{i=2}^9 \hat{\eta}_i^2. \tag{4.19}
$$

Note that

$$\|Y_{hp} - y\|^2_{L^2(0,T;L^2(\Omega))} \le \|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} + \|y(U_{hp}) - y\|^2_{L^2(0,T;L^2(\Omega))}, \tag{4.20}$$

$$\|P_{hp} - p\|^2_{L^2(0,T;L^2(\Omega))} \le \|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} + \|p(U_{hp}) - p\|^2_{L^2(0,T;L^2(\Omega))}, \tag{4.21}$$

and

$$\|y(U_{hp}) - y\|^2_{L^2(0,T;L^2(\Omega))} \le C\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))}, \tag{4.22}$$

$$\|p(U_{hp}) - p\|^2_{L^2(0,T;L^2(\Omega))} \le \|y(U_{hp}) - y\|^2_{L^2(0,T;L^2(\Omega))} \le C\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))}. \tag{4.23}$$

From (4.19), (4.20), (4.22), and Theorem 4.1, we derive

$$\begin{aligned}\|Y_{hp} - y\|^2_{L^2(0,T;L^2(\Omega))} \le& \|Y_{hp} - y(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} + C\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \\ \le& C\eta_1^2 + C\sum_{i=2}^{9}\hat{\eta}_i^2.\end{aligned} \tag{4.24}$$

Similarly, from (4.19), (4.21), (4.23), and Theorem 4.1, we derive

$$\begin{aligned}\|P_{hp} - p\|^2_{L^2(0,T;L^2(\Omega))} \le& \|P_{hp} - p(U_{hp})\|^2_{L^2(0,T;L^2(\Omega))} + C\|u - U_{hp}\|^2_{L^2(0,T;L^2(\Omega))} \\ \le& C\eta_1^2 + C\sum_{i=2}^{9}\hat{\eta}_i^2.\end{aligned} \tag{4.25}$$

Therefore, we obtain (4.16) follows from (4.19), (4.24) and (4.25). □

5. Conclusions

In this paper, a completely discrete scheme is proposed, which uses the inverse Euler scheme in time and the hp spectral element approximation in space to solve the parabolic optimal control problem (2.5)–(2.7). By using the Scott-Zhang type quasi-interpolation operator, we obtain a $L^2(H^1) - L^2(L^2)$ posteriori error estimates of the hp spectral element approximated solutions for both the state variables and the control variable. And two auxiliary equations are introduced, we derive a $L^2(L^2) - L^2(L^2)$ posteriori error estimates for parabolic optimal control problems.

A fully discrete scheme is proposed for improve the accuracy and construct an adaptive finite element algorithm in this paper, which uses the inverse Euler scheme in time and the hp spectral element approximation in space to solve the parabolic optimal control problem (2.5)–(2.7). Our main results as follows: (1) We extend the elliptic optimal control problem to the parabolic optimal control problem, by using the Scott-Zhang type quasi-interpolation operator and get two kind of posteriori error estimates for parabolic optimal control problems. (2) For the general elliptic problem, only a $L^2(H^1) - L^2(L^2)$ posteriori error estimate of the elliptic optimal control problem is deduced, however, we derive a $L^2(H^1) - L^2(L^2)$ and $L^2(L^2) - L^2(L^2)$ posteriori error estimates for parabolic optimal control problem. (3) The two kinds of error estimates we obtained are very useful for us to construct adaptive finite element approximation.

These results and the techniques used can be generalized to optimal control problems with more general objective functions. Furthermore, we well consider the hp spectral element approximation for a posteriori error estimates of nonlinear optimal control problems, nonlinear parabolic optimal control problems and hyperbolic optimal control problems and etc.

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Conflict of interest

The authors declare that they have no competing interests.

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