Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 5199–5219.
DOI: 10.3934/math.2022290
Received: 09 September 2021
Revised: 01 December 2021
Accepted: 02 December 2021
Published: 04 January 2022

Research article

The ordered implicit relations and related fixed point problems in the cone b-metric spaces

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Abstract: In this paper, we introduce an ordered implicit relation. We present some examples for the illustration of the ordered implicit relation. We investigate conditions for the existence of the fixed points of an implicit contraction. We obtain some fixed point theorems in the cone b-metric spaces and hence answer a fixed-point problem. We present several examples and consequences to explain the obtained theorems. We solve an homotopy problem and show existence of solution to a Urysohn Integral Equation as applications of the obtained fixed point theorem.

Keywords: fixed point; implicit relation; cone b-metric space; application
Mathematics Subject Classification: 47H09, 47H10, 54H25

1. Introduction

The Banach Contraction Principle [7] for ordered metric spaces was given by Ran and Reuring [39]. Subsequently, Nieto and Rodriguez-Lopez [34], Ó Regan and Petrusel [40] and Agarwal et al. [1] extended this work. Popa [36] introduced an implicit relation and proved a fixed-point theorem for the self-mappings on the complete metric spaces. Altun and Simsek [2] presented a generalization of the results in [1, 34, 36, 39] by using implicit relation in ordered metric space as follow:

Theorem 1.1. [2] Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose $S : X \to X$ is a non-decreasing mapping such that for all $x, y \in X$ with $x \preceq y$

$$\mathcal{T}(d(Sx, Sy), d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)) \leq 0, \tag{1.1}$$

where $\mathcal{T} : [0, \infty)^6 \to (-\infty, \infty)$. Also suppose that either S is continuous or $(X, d, \preceq)$ is regular, if there exists an element $x_0 \in X$ with $x_0 \preceq S(x_0)$, then S admits a fixed point.

We can obtain several contractive conditions from (1.1), for example, defining $\mathcal{T} : [0,\infty)^6 \to (-\infty,\infty)$ by

$$\mathcal{T}(x_1,x_2,x_3,x_4,x_5,x_6) = x_1 - \psi\left(\max\left\{x_2,x_3,x_4,\frac{1}{2}(x_5+x_6)\right\}\right),$$

we have the main result presented in [1]. Similarly, if we choose

$$\mathcal{T}(x_1,x_2,x_3,x_4,x_5,x_6) = x_1 - kx_2;\ \ k \in [0,1)$$

in (1.1), we have the main result presented in [39]. Thus, different definitions of $\mathcal{T} : [0,\infty)^6 \to (-\infty,\infty)$ produce different contractive conditions. Moreover, the investigation of fixed points of implicit contractions was done by Popa [37,38], Beg et al. [8,9], Berinde et al. [11,12] and Sedghi [43].

The graphical metric spaces [20], partial metric spaces [31], dualistic partial metric spaces [32], b-metric spaces, multivalued contractions [27, 33], $\alpha - \psi$-contractions [42], F-contractions [25], (ψ,ϕ)-contractions [35] are extensively being used in metric fixed point theory. Motivated by the above discoveries, Huang and Zhang [16] presented the idea of cone metric by replacing the set of positive real numbers with ordered Banach space, and utilized this idea to universalize Banach contraction principle. Huang and Zhang [16] worked with the concept of normal cone, however, Rezapour et al. [41] neglected normality of the cone and improved the various theorems presented by Huang. The idea of cone b-metric space [17] was directly influenced by b-metric space [14]. Huang and Xu [18] applied the cone b-metric axioms to prove some fixed point theorems. For detail readings about cone metric, b-metric and cone b-metric, we suggest [3,13,14,21–23,26,29].

We observe that the implicit relation defined in Popa [37,38], can be generalized to vector spaces. So, in this research paper, motivated by Beg et al. [8,9], Berinde et al. [11,12] and Sedghi [43], we define an ordered implicit relation in a cone b-metric space and contribute a fixed point problem. We answer the proposed problem subject to monotone mappings satisfying an implicit contraction. We solve an homotopy problem and show existence of solution to a Urysohn Integral Equation as applications of the obtained fixed point theorem. The obtained fixed point theorems are independent of the observations presented by Ercan [15]. These observations apply on linear contractions and in this paper, we considered nonlinear contractions (implicit relation involving the contraction mappings). So, the obtained results are real generalizations and could not be followed from known ones in literature.

2. Preliminaries

In this section, we recall cone, cone metric space and some related properties. Let $\mathcal{E}$ represent the real Banach space.

Definition 2.1. [19] The set $\aleph \subseteq \mathcal{E}$ is called a cone if and only if the following axioms hold:

(1) $\aleph$ is closed, non empty and $\aleph \neq \{\mathbf{0}_{\mathcal{E}}\}$;
(2) $\alpha v + \beta w \in \aleph$, for all $v,w \in \aleph$ and $\alpha,\beta \in \mathbb{R}$ such that $\alpha,\beta \geq 0$;
(3) $\aleph \cap (-\aleph) = \{\mathbf{0}_{\mathcal{E}}\}$.

The partial order $\preceq$ with respect to $\aleph$ is defined as follows:

$$v \preceq w \Leftrightarrow w - v \in \aleph \ \text{ for all } v,w \in \mathcal{E}.$$

$v \prec w$ serves as $v \preceq w$ but $v \neq w$, and $v \ll w$ represents that $w - v \in \aleph^\circ$(interior of $\aleph$).

Definition 2.2. [19] The cone $\aleph \subseteq \mathcal{E}$ is *normal* if for all $v, w \in \aleph$, there exists $\mathcal{S} > 0$ such that,

$$\mathbf{0}_{\mathcal{E}} \preceq v \preceq w \text{ implies } \|v\| \leq \mathcal{S}\|w\|.$$

Let $\mathfrak{R}$ be a partial order in any ordinary set X and $\preceq$ be a partial order in cone $\aleph \subseteq \mathcal{E}$. If $X \subseteq \mathcal{E}$ then $\mathfrak{R}$ and $\preceq$ would be considered as identical.

Definition 2.3. [19] Let $X \neq \phi$ be a set, and the mapping $d_c : X \times X \mapsto \mathcal{E}$ satisfies the following axioms:

(dc1) $d_c(x, y) \succeq \mathbf{0}_{\mathcal{E}}$, for all $x, y \in X$ and $d_c(x, y) = \mathbf{0}_{\mathcal{E}}$ if and only if $x = y$;
(dc2) $d_c(x, y) = d_c(y, x)$;
(dc3) $d_c(x, \xi) \preceq d_c(x, y) + d_c(y, \xi)$, for all $x, y, \xi \in X$.

Then d_c is known as a cone metric on X, and (X, d_c) is called a cone metric space.

Example 2.4. [6] Let $X = R$, $\mathcal{E} = R^2$, and $\aleph = \{(x, y) \in \mathcal{E} : x, y \geq 0\} \subset R^2$. Define the mapping $d_c : X \times X \to \mathcal{E}$ by

$$d_c(x, y) = (\mid x - y \mid, \alpha \mid x - y \mid),$$

where $\alpha \geq 0$ is to be taken as a constant. Then d_c defines a cone metric on X.

Proposition 2.5. [6] Suppose that (X, d_c) is a cone metric space, with cone $\aleph$. Then for $w, \zeta, \xi \in \mathcal{E}$, we have

(1) If $w \preceq \beta w$ and $\beta \in [0, 1)$, then $w = 0_{\mathcal{E}}$.
(2) If $0_{\mathcal{E}} \preceq w \ll \zeta$ for each $0_{\mathcal{E}} \ll \zeta$, then $w = 0_{\mathcal{E}}$.
(3) If $w \preceq \zeta$ and $\zeta \ll \xi$, then $w \ll \xi$.

Definition 2.6. [19] Let X be a non-empty set and $x, y, \upsilon \in X$. The mapping $c : X \times X \mapsto \mathcal{E}$ satisfying the following axioms:

(cb1) $\mathbf{0}_{\mathcal{E}} \preceq c(x, y)$ and $c(x, y) = \mathbf{0}_{\mathcal{E}}$ if and only if $x = y$;
(cb2) $c(x, y) = c(y, x)$;
(cb3) $c(x, \upsilon) \preceq s[c(x, \xi) + c(\xi, \upsilon)]$ for $s \geq 1$,

is known as a b-cone metric on X, and (X, c) is called a cone b-metric space.

Example 2.7. [6] Let $\mathcal{E} = R^2$, $\aleph = \{(x, y) \in \mathcal{E} : x, y \geq 0\} \subset R^2$, $X = \{1, 2, 3, 4\}$. Define the mapping $c : X \times X \to \mathcal{E}$ by

$$c(x, y) = \begin{cases} (|x - y|^{-1}, |x - y|^{-1}) & \text{if } x \neq y \\ \mathbf{0}_{\mathcal{E}} & \text{if } x = y, \end{cases}$$

and the partial order on $\mathcal{E}$ by

$$v \preceq w \text{ if and only if } v - w \in \aleph \text{ for all } v, w \in \mathcal{E}.$$

Then (X, c) is a cone b-metric space for $s = \frac{6}{5}$. We note that $c(1, 2) \succ c(1, 4) + c(4, 2)$, this shows that c is not a cone metric.

Remark 2.8. Every cone metric space is a cone b-metric space, but converse is not true in general as seen in Example 2.7.

Definition 2.9. [10] Let $\mathcal{E}$ be a real Banach space and (X, c) be a cone b-metric space. Then for every $\epsilon \in \mathcal{E}$ with $\mathbf{0}_{\mathcal{E}} \ll \epsilon$, we have the following information.

(1) A sequence $\{x_n\}$ is said to be a Cauchy sequence, if there exists a natural number $K \in \mathbb{N}$ so that $c(x_n, x_m) \ll \epsilon$ for all $n, m \geq K$.
(2) A sequence $\{x_n\}$ is said to be a convergent sequence (converging to $x \in X$), if there exists $K \in \mathbb{N}$ so that $c(x_n, x) \ll \epsilon$ for all $n \geq K$.
(3) A cone b-metric space (X, c) is complete if each Cauchy sequence converges in X.

3. Ordered implicit relations

Let $(\mathcal{E}, \|.\|)$ be a real Banach space and $B(\mathcal{E}, \mathcal{E})$ be a space of all bounded linear operators $S : \mathcal{E} \to \mathcal{E}$ such that $\|S\|_1 < 1$ and $\|.\|_1$ is taken as usual norm in $B(\mathcal{E}, \mathcal{E})$.

Following the implicit relations presented in [4, 5, 8, 9, 11, 12], we define a new ordered implicit relation as follows:

Definition 3.1. Let $(\mathcal{E}, \|.\|)$ be a real Banach space. The relation $\mathcal{L} : \mathcal{E}^6 \to \mathcal{E}$ is said to be an ordered implicit relation, if it is continuous on $\mathcal{E}^6$ and satisfies the following axioms:

$(\mathcal{L}_1)$ $v_1 \preceq \upsilon_1$, $v_5 \preceq \upsilon_5$ and $v_6 \preceq \upsilon_6$
implies $\mathcal{L}(\upsilon_1, v_2, v_3, v_4, \upsilon_5, \upsilon_6) \preceq \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6)$;
$(\mathcal{L}_2)$ if $\mathcal{L}(v_1, v_2, v_2, v_1, \alpha[v_1 + v_2], \mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}}$
or,
if $\mathcal{L}(v_1, v_2, v_1, v_2, \mathbf{0}_{\mathcal{E}}, \alpha[v_1 + v_2]) \preceq \mathbf{0}_{\mathcal{E}}$, then there exists $S \in B(\mathcal{E}, \mathcal{E})$ such that $v_1 \preceq S(v_2)$ (for all $v_1, v_2 \in \mathcal{E}$) and $\alpha \geq 1$;
$(\mathcal{L}_3)$ $\mathcal{L}(\alpha v, \mathbf{0}_{\mathcal{E}}, \mathbf{0}_{\mathcal{E}}, v, \alpha v, \mathbf{0}_{\mathcal{E}}) \succ \mathbf{0}_{\mathcal{E}}$ whenever $\|v\| > 0$ and $\alpha \geq 1$.

Let $\mathcal{G} = \{\mathcal{L} : \mathcal{E}^6 \to \mathcal{E} \,|\, \mathcal{L}$ satisfies conditions $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$.

Example 3.2. Let $\preceq$ be a partial order with respect to cone $\aleph$ as defined in Section 2 and $(\mathcal{E}, \|.\|)$, be a real Banach space. For $v_i \in \mathcal{E}(i = 1$ to $6)$, $\alpha > 2$ and $\gamma > 1$, we define the relation $\mathcal{L} : \mathcal{E}^6 \to \mathcal{E}$ by

$$\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) = \alpha v_1 - \{v_5 + \gamma v_6\}.$$

Then $\mathcal{L} \in \mathcal{G}$. Indeed
$(\mathcal{L}_1)$: Let $v_1 \preceq \gamma_2$, $v_5 \preceq \gamma_5$ and $v_6 \preceq \gamma_6$, then $\gamma_5 - v_5 \in \aleph$ and $\gamma_6 - v_6 \in \aleph$, we show that $\mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) - \mathcal{L}(\gamma_1, v_2, v_3, v_4, \gamma_5, \gamma_6) \in \aleph$. Consider,

$$\begin{aligned} & \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6) - \mathcal{L}(\gamma_1, v_2, v_3, v_4, \gamma_5, \gamma_6) \\ = \; & \alpha\gamma_1 - \{v_5 + \gamma v_6\} - \alpha\gamma_1 + \{\gamma_5 + \gamma\gamma_6\} \\ = \; & \gamma_5 - v_5 + \gamma(\gamma_6 - v_6) \in \aleph. \end{aligned}$$

Thus, $\mathcal{L}(\gamma_1, v_2, v_3, v_4, \gamma_5, \gamma_6) \preceq \mathcal{L}(v_1, v_2, v_3, v_4, v_5, v_6)$.
$(\mathcal{L}_2)$: Let $v_1, v_2, v_3 \in \mathcal{E}$ be such that $\mathbf{0}_{\mathcal{E}} \preceq v_1, \mathbf{0}_{\mathcal{E}} \preceq v_2$. If $\mathcal{L}(v_1, v_2, v_2, v_1, s[v_1 + v_2], \mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}}$ then, we have $-\alpha v_1 + s\{v_1 + v_2\} \in \aleph$.

$$\text{So, } sv_2 - (\alpha - s)v_1 \in \aleph. \text{ implies} \tag{3.1}$$

$$v_2 - \frac{(\alpha - s)}{s} v_1 \in \aleph. \tag{3.2}$$

If $v_1 = \mathbf{0}_{\mathcal{E}}$, then $v_2 \in \aleph$ (by (3.1)). Thus, there exists $T : \mathcal{E} \to \mathcal{E}$ defined by $T(v_2) = \eta v_2$ (where $\eta = s$) such that $\| T \| = s > 1$ (not possible). Now if $v_1 \neq \mathbf{0}_{\mathcal{E}}$, then, (3.2) implies $v_2 \preceq \frac{s}{(\alpha-s)} v_1$. So for $\frac{\alpha}{2} > s \geq 1$, there exists $T : \mathcal{E} \to \mathcal{E}$ defined by $T(v_2) = \eta v_2$ ($\eta = \frac{s}{(\alpha-s)}$ is a scalar) such that $v_1 \preceq T(v_2)$, for $\alpha > 2$.
$(\mathcal{L}_3)$: Let $v \in \mathcal{E}$ be such that $\|v\| > 0$ and consider, $\mathbf{0}_{\mathcal{E}} \preceq \mathcal{L}(sv, \mathbf{0}_{\mathcal{E}}, \mathbf{0}_{\mathcal{E}}, v, sv, \mathbf{0}_{\mathcal{E}})$ then $(\alpha s - 1)v \in \aleph$, which hold whenever $\|v\| > 0$.

Example 3.3. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 : \mathcal{E}^6 \to \mathcal{E}$ be defined by

(i) $\mathcal{L}_1(v_1, v_2, v_3, v_4, v_5, v_6) = \alpha v_1 - v_2; \alpha < 1$.
(ii) $\mathcal{L}_2(v_1, v_2, v_3, v_4, v_5, v_6) = v_3 - \alpha\{v_3 + v_4\} - (1 - \alpha)\beta v_5; \alpha < \frac{1}{2}, \beta \in R$.
(iii) $\mathcal{L}_3(v_1, v_2, v_3, v_4, v_5, v_6) = v_1 + v_5 - \alpha\{v_2 + v_4\}; \alpha > \frac{1}{2}$.

Then $\mathcal{L}_i$ defines an ordered implicit relation for each $i = 1, 2, 3$.

In the next section, we employ this implicit relation in association with a few other conditions to construct an iterative sequence and hence to answer the following fixed-point problem:
"find $p \in (X, c)$ such that $f(p) = p$" where $T \in B(\mathcal{E}, \mathcal{E})$, $I : \mathcal{E} \to \mathcal{E}$ an identity operator , $\mathcal{L} \in \mathcal{G}$ and $f : X \to X$ satisfies (3.3), for all comparable $x, y \in X$ and $s \geq 1$.

$$(I - T)(c(x, f(x))) \preceq sc(x, y) \textit{ implies}$$

$$\mathcal{L}(c(f(x), f(y)), c(x, y), c(x, f(x)), c(y, f(y)), c(x, f(y)), c(y, f(x))) \preceq \mathbf{0}_{\mathcal{E}}. \tag{3.3}$$

The following assertion is essential in the sequel.

Remark 3.4. If $S \in B(\mathcal{E}, \mathcal{E})$, then Neumann series $I + S + S^2 + \cdots + S^n + \cdots$ converges whenever $\|S\|_1 < 1$ and diverge if $\|S\|_1 > 1$. Also for $\|S\|_1 < 1$ there exists $\mu > 0$ so that $\|S\|_1 < \mu < 1$ and $\|S^n\|_1 \leq \mu^n < 1$.

4. The new results

Popa [36] applied implicit type contractive conditions on a self-mapping to establish some fixed point results. Altun and Simsek [2] extended the research work in [36] to partially ordered metric spaces. Nazam et al. [30] have further generalized the results given in [2] by using the concept of the cone metric space [19]. In this section, we shall address the proposed fixed-point problem that generalizes the results in [2, 30, 36]. For that purpose, we have the following theorem.

Theorem 4.1. *Let (X, c) be a complete cone b-metric space with $\aleph \subset \mathcal{E}$ as a cone and $f : X \to X$. If $T \in B(\mathcal{E}, \mathcal{E})$, $I : \mathcal{E} \to \mathcal{E}$ is an identity operator and $\mathcal{L} \in \mathcal{G}$ such that, for all comparable elements $x, y \in X$ and $s \geq 1$, we have*

$$(I - T)(c(x, f(x))) \preceq sc(x, y) \textit{ implies}$$

$$\mathcal{L}(c(f(x), f(y)), c(x, y), c(x, f(x)), c(y, f(y)), c(x, f(y)), c(y, f(x))) \preceq \mathbf{0}_{\mathcal{E}}. \tag{4.1}$$

If,

(1) There exists $x_0 \in X$, so that $x_0 \Re f(x_0)$;
(2) For all $x, y \in X$, $x \Re y$ implies $f(x) \Re f(y)$;
(3) The sequence $\{x_n\}$ satisfies $x_{n-1} \Re x_n$ and $x_n \to p$, then $x_n \Re p$ for all $n \in \mathbb{N}$.

Then, there exists a point $p \in X$ such that $p = f(p)$.

Proof. Suppose that $x_0 \in X$ be an arbitrary point such that $x_0 \Re f(x_0)$. We construct a sequence $\{x_n\}$ by $f(x_{n-1}) = x_n$ taking x_0 as an initial guess. Since, $x_0 \Re x_1$, by assumption (2) we have $x_1 \Re x_2$, $x_2 \Re x_3 \cdots$ $x_{n-1} \Re x_n$. Since, $x_0 \Re x_1$, by (4.1) we have

$$(I - T)(c(x_0, f(x_0))) = (I - T)(c(x_0, x_1)) \preceq sc(x_0, x_1) \text{ implies}$$

$$\mathcal{L}(c(f(x_0), f(x_1)), c(x_0, x_1), c(x_0, f(x_0)), c(x_1, f(x_1)), c(x_0, f(x_1)), \mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}},$$

that is,

$$\mathcal{L}(c(x_1, x_2), c(x_0, x_1), c(x_0, x_1), c(x_1, x_2), c(x_0, x_2), \mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}}. \tag{4.2}$$

By triangle property of the cone b-metric, we have

$$c(x_0, x_2) \preceq s[c(x_0, x_1) + c(x_1, x_2)].$$

Rewriting (4.2) and using condition ($\mathcal{L}_1$), we get:

$$\mathcal{L}(c(x_1, x_2), c(x_0, x_1), c(x_0, x_1), c(x_1, x_2), s[c(x_0, x_1) + c(x_1, x_2)], \mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}}.$$

By using ($\mathcal{L}_2$), there exists $K \in B(\mathcal{E}, \mathcal{E})$ with $\|K\|_1 < 1$ such that

$$c(x_1, x_2) \preceq K(c(x_0, x_1)).$$

Again since, $x_1 \Re x_2$, by (4.1) we obtain

$$(I - T)(c(x_1, f(x_1))) = (I - T)(c(x_1, x_2)) \preceq sc(x_1, x_2) \text{ implies}$$

$$\mathcal{L}(c(f(x_1), f(x_2)), c(x_1, x_2), c(x_1, f(x_1)), c(x_2, f(x_2)), c(x_1, f(x_2)), c(x_2, f(x_1))) \preceq \mathbf{0}_{\mathcal{E}},$$

that is,

$$\mathcal{L}(c(x_2, x_3), c(x_1, x_2), c(x_1, x_2), c(x_2, x_3), c(x_1, x_3), \mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}}.$$

By (cb3), we have

$$c(x_1, x_3) \preceq s[c(x_1, x_2) + c(x_2, x_3)]$$

using ($\mathcal{L}_1$) we get

$$\mathcal{L}(c(x_2, x_3), c(x_1, x_2), c(x_1, x_2), c(x_2, x_3), s[c(x_1, x_2) + c(x_2, x_3)], \mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}}.$$

By ($\mathcal{L}_2$) there exists $K \in B(\mathcal{E}, \mathcal{E})$ with $\|K\|_1 < 1$ such that

$$c(x_2, x_3) \preceq K(c(x_1, x_2)) \preceq K^2(c(x_0, x_1)).$$

Now keeping in view the above pattern and with the relation $x_n \Re x_{n+1}$ $n \geq 1$, we can construct a sequence $\{x_n\}$ so that $x_{n+1} = f(x_n)$ and

$$(I - T)(c(x_{n-1}, f(x_{n-1}))) = (I - T)(c(x_{n-1}, x_n)) \preceq sc(x_{n-1}, x_n)$$

implies

$$c(x_n, x_{n+1}) \preceq K(c(x_{n-1}, x_n)) \preceq K^2(c(x_{n-2}, x_{n-1})) \preceq \cdots \preceq K^n(c(x_0, x_1)).$$

For $\hbar, n \in \mathbb{N}$ and $s \geq 1$, consider

$$\begin{aligned}
c(x_{n+\hbar}, x_n) &\preceq s[c(x_{n+\hbar}, x_{n+\hbar-1}) + c(x_{n+\hbar-1}, x_n)] \\
&\preceq sc(x_{n+\hbar}, x_{n+\hbar-1} + s^2[c(x_{n+\hbar-1}, x_{n+\hbar-2}) + c(x_{n+\hbar-2}, x_n)] \\
&\preceq sc(x_{n+\hbar}, x_{n+\hbar-1}) + s^2 c(x_{n+\hbar-1}, x_{n+\hbar-2}) + ... + s^{(\hbar-1)} c(x_{n+1}, x_n) \\
&\preceq sK^{n+\hbar-1}(c(x_0, x_1)) + s^2 K^{n+\hbar-2}(c(x_0, x_1)) + ... + s^{(\hbar-1)} K^n(c(x_0, x_1)) \\
&= \frac{sK^{n+\hbar}((sK^{-1})^{\hbar-1} - 1)}{s - K}(c(x_0, x_1)) + s^{(\hbar-1)} K^n(c(x_0, x_1)) \\
&\preceq \frac{s^{\hbar} K^{n+1}}{s - K}(c(x_0, x_1)) + s^{(\hbar-1)} K^n(c(x_0, x_1)).
\end{aligned}$$

Since $\|K\|_1 < 1$, so, $K^n \to \mathbf{0}_{\mathcal{E}}$ as $n \to \infty$ (Remark 3.4). Hence, $\lim_{n\to\infty} c(x_n, x_m) = \mathbf{0}_{\mathcal{E}}$, this shows that $\{x_n\}$ is a Cauchy sequence in X. Since, (X, c) is a complete cone b-metric space, so there exists $p \in X$ so that $x_n \to p$ for large n, alternately, for a given $0 \ll \epsilon$, there is a natural number N_2 so that

$$c(x_n, p) \ll \epsilon \text{ for all } n \geq N_2.$$

Now since $x_{n-1} \mathfrak{R} x_n$ and $x_n \to p$, by the assumption (3), we have $x_n \mathfrak{R} p$ for all $n \in \mathbb{N}$. We claim that,

$$(I - T)(c(x_n, f(x_n))) \preceq sc(x_n, p).$$

Suppose on the contrary that

$$(I - T)(c(x_n, f(x_n))) \succ sc(x_n, p) \text{ and}$$

$$(I - T)(c(x_{n+1}, f(x_{n+1}))) \succ sc(x_{n+1}, p) \text{ for } n \in \mathbb{N}.$$

By (cb3) and (4.1), we get

$$\begin{aligned}
c(x_n, f(x_n)) &\preceq s[c(x_n, p) + c(p, x_{n+1})] \\
&\prec s[\frac{1}{s}(I - T)(c(x_n, f(x_n))) + \frac{1}{s}(I - T)c(x_{n+1}, f(x_{n+1})))] \\
&\prec [(I - T)(c(x_n, f(x_n))) + (I - T)T(c(x_n, f(x_n)))] \\
&\prec (I - T)(I + T)(c(x_n, f(x_n))) \\
&\prec (I - T^2)(c(x_n, f(x_n))).
\end{aligned}$$

Thus,

$$T^2(c(x_n, f(x_n))) \prec \mathbf{0}_{\mathcal{E}},$$

which leads to a contradiction. So, for each $n \geq 1$ and $s \geq 1$, we get

$$(I - T)(c(x_n, f(x_n))) \preceq sc(x_n, p),$$

thus, by (4.1), we have

$$\mathcal{L}\begin{pmatrix} c(f(x_n), f(p)), c(x_n, p), c(x_n, f(x_n)), c(p, f(p)), \\ c(x_n, f(p)), c(p, f(x_n)) \end{pmatrix} \preceq \mathbf{0}_{\mathcal{E}}. \tag{4.3}$$

We need to show that $\|c(p, f(p))\| = 0$. If $\|c(p, f(p))\| > 0$, then we have the following information.

$$\begin{aligned} c(f(x_n), f(p)) &\preceq s[c(f(x_n), p) + c(p, f(p))] \\ \lim_{n\to\infty} c(f(x_n), f(p)) &\preceq s \lim_{n\to\infty}[c(x_{n+1}, p) + c(p, f(p))] \\ &= sc(p, f(p)), \text{ and} \end{aligned}$$

$$\begin{aligned} c(x_n, f(p)) &\preceq s[c(x_n, p) + c(p, f(p))] \\ \lim_{n\to\infty} c(x_n, f(p)) &\preceq s \lim_{n\to\infty}[c(x_n, p) + c(p, f(p))] \\ \lim_{n\to\infty} c(x_n, f(p)) &\preceq sc(p, f(p)). \end{aligned}$$

In view of the condition ($\mathcal{L}_1$) and (4.3), we have

$$\mathcal{L}(s(c(p, f(p))), \mathbf{0}_{\mathcal{E}}, \mathbf{0}_{\mathcal{E}}, c(p, f(p)), s(c(p, f(p))), \mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}}.$$

This contradicts the condition ($\mathcal{L}_3$). Thus, $\|c(p, f(p))\| = 0$. So, $c(p, f(p)) = \mathbf{0}_{\mathcal{E}}$, and hence $p = f(p)$. □

Remark 4.2. If the operator $\mathcal{L} : \mathcal{E}^6 \to \mathcal{E}$ is defined by

$$\mathcal{L}(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 - \psi\left(\max\left\{x_2, x_3, x_4, \frac{1}{2}(x_5 + x_6)\right\}\right), \text{ for all } x_i \in \mathcal{E}$$

where, $\psi : \mathcal{E} \to \mathcal{E}$ is a non-decreasing operator satisfying $\lim_{n\to\infty} \psi^n(v) = \mathbf{0}_{\mathcal{E}}$. Then Theorem (4.1), generalizes the corresponding result in [1]. If we define $\mathcal{L} : \mathcal{E}^6 \to \mathcal{E}$ by

$$\mathcal{L}(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 - kx_2; \ k \in [0, 1)$$

in Theorem (4.1), we obtain a generalization of the corresponding result in [39]. Thus, different definitions of $\mathcal{L} : \mathcal{E}^6 \to \mathcal{E}$ produce different ordered contractive conditions. Moreover, Theorem (4.1) generalizes the results in Popa [37, 38], Beg et al. [8, 9], Berinde et al. [11, 12].

The following theorem is for decreasing self-mappings.

Theorem 4.3. *Let (X, c) be a complete cone b-metric space with $\aleph \subset \mathcal{E}$ as a cone and $f : X \to X$. Let $T \in B(\mathcal{E}, \mathcal{E})$, $I : \mathcal{E} \to \mathcal{E}$ be an identity operator, $\mathcal{L} \in \mathcal{G}$ so that, for all comparable elements $x, y \in X$ and $s \geq 1$*

$$(I - T)(c(x, f(x))) \preceq sc(x, y) \text{ implies}$$

$$\mathcal{L}(c(f(x), f(y)), c(x, y), c(x, f(x)), c(y, f(y)), c(x, f(y)), c(k, f(x))) \preceq \mathbf{0}_{\mathcal{E}}. \quad (4.4)$$

If,

(1) There exists $x_0 \in X$, so that $f(x_0)\mathfrak{R}x_0$;
(2) For all $x, y \in X$, $x\mathfrak{R}y$ implies $f(y)\mathfrak{R}f(x)$;
(3) The sequence $\{x_n\}$ satisfies $x_{n-1}\mathfrak{R}x_n$ and $x_n \to x^*$, then $x_n\mathfrak{R}x^*$ for all $n \in \mathbb{N}$.

Then, there exists a point $x^* \in X$ such that $x^* = f(x^*)$.

Proof. Let $x_0 \in X$ be arbitrary satisfying assumption (1). We define a sequence $\{x_n\}$ by $x_n = f(x_{n-1})$ for all n. As $x_1 = f(x_0)\mathfrak{R}x_0$ and by condition (2) $x_1 = f(x_0)\mathfrak{R}f(x_1) = x_2$ and repeated implementation of hypothesis (2) leads to have $x_n\mathfrak{R}x_{n-1}$. Since, $x = x_1\mathfrak{R}x_0$, by (4.4), we get

$$(I-T)(c(f(x_0),x_0) = (I-T)(c(x_1,x_0)) \preceq sc(x_1,x_0) \text{ implies}$$
$$\mathcal{L}(c(f(x_1),f(x_0)),c(x_1,x_0),c(x_1,f(x_1)),c(x_0,f(x_0)),c(x_1,f(x_0)),c(x_0,f(x_1))) \preceq \mathbf{0}_{\mathcal{E}}$$
$$\Rightarrow \mathcal{L}(c(x_1,x_2),c(x_0,x_1),c(x_1,x_2),c(x_0,x_1),\mathbf{0},c(x_0,x_2)) \preceq \mathbf{0}_{\mathcal{E}}.$$

By (cb3), we have

$$c(x_0,x_2) \preceq s[c(x_0,x_1)+c(x_1,x_2)]$$

and then using $\mathcal{L}_1$, we obtain

$$\mathcal{L}(c(x_1,x_2),c(x_0,x_1),c(x_1,x_2),c(x_0,x_1),\mathbf{0}_{\mathcal{E}},s[c(x_0,x_1)+c(x_1,x_2)] \preceq \mathbf{0}_{\mathcal{E}}.$$

By ($\mathcal{L}_2$), there exists $K \in B(\mathcal{E},\mathcal{E})$ with $\|K\|_1 < 1$ such that

$$c(x_1,x_2) \preceq K(c(x_0,x_1)).$$

Again since, $x = x_1\mathfrak{R}x_2$, by (4.4)

$$(I-T)(c(x_1,f(x_1)) = (I-T)(c(x_1,x_2)) \preceq sc(x_1,x_2) \text{ implies}$$
$$\mathcal{L}(c(f(x_1),f(x_2)),c(x_1,x_2),c(x_1,f(x_1)),c(x_2,f(x_2)),c(x_1,f(x_2)),c(x_2,f(x_1))) \preceq \mathbf{0}_{\mathcal{E}}$$
$$\Rightarrow \mathcal{L}(c(x_2,x_3),c(x_1,x_2),c(x_1,x_2),c(x_2,x_3),c(x_1,x_3),\mathbf{0}_{\mathcal{E}}) \preceq \mathbf{0}_{\mathcal{E}}.$$

By (cb3), ($\mathcal{L}_1$) and ($\mathcal{L}_2$), we get

$$c(x_2,x_3) \preceq T(c(x_1,x_2)) \preceq T^2(c(x_0,x_1)).$$

By following same steps, we can construct a sequence $\{x_n\}$ such that

$$c(x_n,x_{n+1}) \preceq K(c(x_{n-1},x_n)) \preceq K^2(c(x_{n-2},x_{n-1})) \preceq \cdots \preceq K^n(c(x_0,x_1)).$$

Hence copying the arguments for the proof of Theorem 4.1, we get $x^* = f(x^*)$. □

The following theorem encapsulate the statements of Theorem 4.1 and Theorem 4.3.

Theorem 4.4. *Let (X,c) be a complete cone b-metric space with $\aleph \subset \mathcal{E}$ as a cone and $f : X \to X$ be monotone mapping. Let $T \in B(\mathcal{E},\mathcal{E})$, $I : \mathcal{E} \to \mathcal{E}$ be an identity operator. If there exists $\mathcal{L} \in \mathcal{G}$ so that, for all comparable $x,y \in X$ and $s \geq 1$*

$$(I-T)(c(x,f(x))) \preceq sc(x,y) \text{ implies}$$
$$\mathcal{L}(c(f(x),f(y)),c(x,y),c(x,f(x)),c(y,f(y)),c(x,f(y)),c(k,f(x))) \preceq \mathbf{0}_{\mathcal{E}}, \tag{4.5}$$

and,

(1) There exists $x_0 \in X$, so that $x_0\mathfrak{R}f(x_0)$ or $f(x_0)\mathfrak{R}x_0$,
(2) The sequence $\{x_n\}$ satisfies $x_{n-1}\mathfrak{R}x_n$ and $x_n \to a^*$, then $x_n\mathfrak{R}a^*$ for all $n \in \mathbb{N}$.

Then, there exists a point $a^* \in X$ such that $a^* = f(a^*)$.

Proof. Proof is obvious. □

Remark 4.5. We can get a unique fixed point in Theorem 4.1, Theorem 4.3 and Theorem 4.4 by taking an additional condition, "for each pair $x,y \in X$, we have either an upper bound or lower bound." (2). The cone is assumed as non-normal.

5. Examples and consequences

We illustrate above theorems with the help of the following examples.

Example 5.1. Let $\mathcal{E} = C_R^1[0,1]$, and $\|\zeta\| = \|\zeta\|_\infty + \|\acute{\zeta}\|_\infty$, $\aleph = \{\zeta \in \mathcal{E} : \zeta(t) > 0, t \in [0,1]\}$. For each $K \geq 1$, take $\zeta = x$ and $y = x^{2K}$. By definition $\|\zeta\| = 1$ and $\|y\| = 2k+1$. Clearly $\zeta \preceq y$, and $K\|\zeta\| \; \leq \|y\|$. Hence $\aleph$ is a non-normal cone. Define the operator $T : \mathcal{E} \to \mathcal{E}$ by

$$(T\zeta)(t) = \frac{1}{2}\int_0^t \zeta(s)ds. \text{ Thus, } T \text{ is linear and bounded as shown below.}$$

$$(T(a\zeta + by))(t) = \frac{1}{2}\int_0^t (a\zeta + by)(s)ds = \frac{a}{2}\int_0^t \zeta(s)ds + \frac{b}{2}\int_0^t y(s)ds, \text{ and}$$

$$\|T^n\zeta\| \leq \frac{\|\zeta\|^{n+1}}{2(n+1)!} \text{ for each } n \geq 1.$$

So

$$\|(T^n\zeta)\|_\infty \leq \frac{1}{2(n+1)!}.$$

$$\|(T^n\zeta)'\|_\infty \leq \frac{\|\zeta\|^n}{2(n)!} \leq \frac{1}{2(n)!} \text{ for } n \geq 1.$$

$$\|(T^n\zeta)\| = \|(T^n\zeta)\|_\infty + \|(T^n\zeta)'\|_\infty \leq \frac{1}{2(n+1)!} + \frac{1}{2(n)!} \text{ for } n \geq 1$$

$$\|(T^n\zeta)\| = 0 \text{ when } n \geq n_1 \text{ for } n_1 \in N.$$

Hence $T \in B(\mathcal{E}, \mathcal{E})$. Let $X = \{1,2,3\}$ and $f : X \to X$ defined by $f(1) = f(2) = 1$ and $f(3) = 2$, then f is increasing with respect to usual order. Define the mapping c by

$$c(x_1, x_2) = \begin{cases} \mathbf{0} & \text{if } x_1 = x_2 \\ \frac{\zeta}{3} & \text{if } x_1, x_2 \in \{1,2\}; \zeta \in \mathcal{E} \\ \frac{\zeta}{8} & \text{otherwise.} \end{cases}$$

$$\frac{\zeta}{3} = c(1,2) \succ c(1,3) + c(3,2) = \frac{\zeta}{8} + \frac{\zeta}{8},$$

since triangular inequality does not hold, so c is not a cone metric space, but one can check that c is a cone b-metric space for $s = \frac{4}{3}$. Now, for $x = 1$ and $y = 2$ ($x \leq y$), we have

$$c(x,y) = \frac{\zeta}{3} = c(y, f(x)) = c(y, f(y))$$

$$c(x, f(x)) = \mathbf{0}_\mathcal{E} = c(f(x), f(y)) = c(x, f(y)).$$

For $x = 2$ and $y = 3$

$$c(x,y) = \frac{\zeta}{8}, \;\; c(x, f(x)) = \frac{\zeta}{3} = c(f(x), f(y))$$

$$c(x, f(y)) = \mathbf{0}_\mathcal{E}, \;\; c(y, f(x) = \frac{\zeta}{8} = c(y, f(y)).$$

For $\alpha > 2$ and $\gamma > \frac{16}{3}$, define

$$\mathcal{L}\left(\begin{array}{l} c(f(x), f(y)), c(x, y), c(x, f(x)), c(y, f(y)), \\ c(x, f(y)), c(y, f(x)) \end{array}\right) = \alpha c(f(x), f(y)) - [c(x, f(y)) + \gamma c(y, f(x)].$$

Clearly

$$(I - T)c(x, f(x)) \preceq sc(x, y) \text{ implies}$$

$$\alpha c(f(x), f(y)) \preceq c(x, f(y)) + \gamma c(y, f(x).$$

Thus, by Theorem 4.1, f has a fixed point which is given by $f(1) = 1$.

Remark 5.2. Since, c is not a cone metric, this shows that Theorem 4.1 does not hold in a cone metric space. The Example 5.1 also endorses the choice of cone b-metric space for this paper.

In the following, we have some consequences of the main results given above.

Corollary 5.3. *Let (X, c) be a complete cone b-metric space with $\aleph \subset \mathcal{E}$ as a cone and $f : X \to X$. If $T \in B(\mathcal{E}, \mathcal{E})$, $I : \mathcal{E} \to \mathcal{E}$ an identity operator and there exist $\mathcal{L} \in \mathcal{G}$ so that, for all comparable elements $x, y \in X$ and $s \geq 1$*

$$(I - T)(c(x, f(x))) \preceq sc(x, y) \text{ implies}$$

$$c(f(x), f(y)) \preceq T(c(x, y)), \tag{5.1}$$

and,

(1) $x \in X$ so that $x_0 \mathfrak{R} f(x_0)$ or $f(x_0) \mathfrak{R} x_0$;
(2) For all $x, y \in X$, $x \mathfrak{R} y$ implies $f(x) \mathfrak{R} f(y)$ or $f(y) \mathfrak{R} f(x)$;
(3) The sequence $\{x_n\}$ satisfies $x_{n-1} \mathfrak{R} x_n$ and $x_n \to b^*$, then $x_n \mathfrak{R} b^*$ for all $n \in \mathbb{N}$.

Then, there exists a point $b^* \in X$ such that $b^* = f(b^*)$.

Proof. Define $\mathcal{L}$ as in Example 3.3 (i), and the operator $T : \mathcal{E} \to \mathcal{E}$ by $T(v) = \alpha v$ for all $v \in \mathcal{E}$ and $0 \leq \alpha < 1$, then $T \in B(\mathcal{E}, \mathcal{E}$ and application of Theorem 4.4 provide the proof. □

Corollary 5.4. *Let (X, c) be a complete cone b-metric space with $\aleph \subset \mathcal{E}$ as a cone and $f : X \to X$. If $T \in B(\mathcal{E}, \mathcal{E})$, $I : \mathcal{E} \to \mathcal{E}$ an identity operator and there exist $\mathcal{L} \in \mathcal{G}$ so that, for any comparable $x, y \in X$ and $s \geq 1$*

$$(I - T)(c(x, f(x))) \preceq sc(x, y) \text{ implies}$$

$$c(f(x), f(y)) \preceq \frac{1}{s} T(c(x, y)), \tag{5.2}$$

and,

(1) $x \in X$ so that $x_0 \mathfrak{R} f(x_0)$ or $f(x_0) \mathfrak{R} x_0$;
(2) For all $x, y \in X$, $x \mathfrak{R} y$ implies $f(x) \mathfrak{R} f(y)$ or $f(y) \mathfrak{R} f(x)$;
(3) The sequence $\{x_n\}$ satisfies $x_{n-1} \mathfrak{R} x_n$ and $x_n \to r^*$, then $x_n \mathfrak{R} r^*$ for all $n \in \mathbb{N}$.

Then, there exists a point $r^* \in X$ such that $r^* = f(r^*)$.

Proof. Define the operator $T : \mathcal{E} \to \mathcal{E}$ by $T(v) = v$ for all $v \in \mathcal{E}$ and following the proof of Corollary 5.3, we receive the result. □

The following examples illustrates Corollary 5.3 and Corollary 5.4.

Example 5.5. Let $\mathcal{E} = (\mathbb{R}, \|\cdot\|)$ be a real Banach space. Define $\aleph = \{x \in \mathbb{R} : x \geq 0\}$, then, it is a cone in $\mathcal{E}$. Let $X = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$, define $f : X \to X$ so that, $f(\frac{2}{3}) = f(\frac{3}{4}) = \frac{4}{5}$ and $f(\frac{1}{2}) = f(\frac{4}{5}) = \frac{1}{2}$, then f is deceasing with respect to usual order. Let $T \in B(\mathcal{E}, \mathcal{E})$ be defined by $T(x) = \frac{x}{2}$. Define the mapping c by

$$c(x_1, x_2) = \begin{cases} 0.3 & \text{if } (x_1, x_2) = \left(\frac{1}{2}, \frac{2}{3}\right) \\ 0.1 & \text{if } (x_1, x_2) = \left(\frac{1}{2}, \frac{4}{5}\right) \\ 0.01 & \text{if } (x_1, x_2) = \left(\frac{1}{2}, \frac{3}{4}\right) \\ 0.8 & \text{if } (x_1, x_2) = \left(\frac{3}{4}, \frac{4}{5}\right), (x_1, x_2) = \left(\frac{2}{3}, \frac{3}{4}\right) \text{ or } (x_1, x_2) = \left(\frac{2}{3}, \frac{4}{5}\right) \\ \mid x_1 - x_2 \mid & \text{otherwise.} \end{cases}$$

Observe that:

$$0.8 = c\left(\frac{2}{3}, \frac{3}{4}\right) \geq c\left(\frac{2}{3}, \frac{1}{2}\right) + c\left(\frac{1}{2}, \frac{3}{4}\right) = 0.3 + 0.01 = 0.31$$

$$0.8 = c\left(\frac{3}{4}, \frac{4}{5}\right) \geq c\left(\frac{3}{4}, \frac{1}{2}\right) + c\left(\frac{1}{2}, \frac{4}{5}\right) = 0.01 + 0.1 = 0.11.$$

So (cb3) holds for $s = 10$, and c is a cone b-metric, but not a cone metric, as (dc3) does not hold. Consider $x = \frac{1}{2}$ and $y = \frac{2}{3}$. Then

$$c\,(f(x), f(y)) = c\,(x, f(y)) = 0.1, \quad c(x, f(x)) = 0$$

$$c(x, y) = c\,(y, f(x)) = 0.3, \quad c(y, f(y)) = 0.8$$

$$(I - T)c(x, f(x)) = 0.$$

Now take $x = \frac{2}{3}$ and $y = \frac{3}{4}$. Then

$$c(f(x), f(y)) = 0, \quad c(x, y) = c(x, f(x)) = 0.8$$

$$(I - T)c(x, f(x)) = 0.4, \quad sc(x, f(x)) = 10(0.8) = 8.$$

For $x = \frac{3}{4}$ and $y = \frac{4}{5}$. Then

$$c(f(x), f(y)) = 0.1, \quad c(x, y) = c(x, f(x)) = 0.8$$

$$(I - T)c(x, f(x)) = 0.4, \quad sc(x, f(x)) = 10(0.8) = 8.$$

Thus, for all $x, y \in X$ such that $x \leq y$, we have

$$(I - T)(c(x, f(x))) \leq sc(x, y) \text{ implies } c(f(x), f(y)) \leq T(c(x, y)).$$

Hence, Corollary 5.3 holds for all comparable $x, y \in X$. Notice that $\frac{1}{2}$ is a fixed point of f.

Remark 5.6. Since, c is not a cone metric, this shows that Corollary 5.3 does not hold in a cone metric space. The Example 5.5 also endorses the choice of cone b-metric space for this paper.

Example 5.7. Consider $\mathcal{E} = (\mathbb{R}, \|\cdot\|)$ be a real Banach space. Define $\aleph = \{x \in \mathbb{R} : x \geq 0\}$, then, it is a cone in $\mathcal{E}$. Take $X = \{0, 1, 2\}$, and $f : X \to X$, $f(0) = f(2) = 0$, $f(1) = 2$. Define the mapping c by

$$c(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 3 & \text{if } x_1, x_2 \in \{1, 2\} \\ 10 & \text{if } x_1, x_2 \in \{0, 1\} \\ 0.5 & \text{otherwise .} \end{cases}$$

Notice that:

$$10 = c(0, 1) \geq c(0, 2) + c(2, 1) = 0.5 + 3 = 3.5.$$

For $s = 3$, c is a cone b-metric space, but not cone metric space, as (dc3) does not hold. Define $T(x) = \frac{x}{2}$ Consider $x = 0$, $y = 1$. Then

$$c(x, y) = 10, \quad c(f(x), f(y)) = 0.5, \quad c(x, f(x)) = 0$$

$$Tc(x, y) = \frac{10}{2} = 5, \quad \frac{T}{s}c(x, y) = \frac{5}{3} = 1.6667.$$

Take $x = 1$, $y = 2$. Then

$$c(x, y) = 3, \quad c(f(x), f(y)) = 0.5, \quad c(x, f(x)) = 3$$

$$Tc(x, y) = \frac{3}{2} = 1.5, \quad \frac{T}{s}c(x, y) = \frac{1.5}{3} = 0.5$$

$$(I - T)c(x, f(x)) = 0.25.$$

Clearly for all $x, y \in X$,

$$(I - T)c(x, f(x)) \leq sc(x, y)$$

implies

$$c(f(x), f(y)) \leq \frac{1}{s}T(c(x, y)).$$

So for all values of $x, y \in X$, the Corollary 5.4 holds. Here 0 is a fixed point of f.

6. A homotopy result

This section consists of a homotopy theorem as an application of Corollary 5.4.

Theorem 6.1. *Let $(\mathcal{E}, \|.\|)$ be a real Banach space with $\aleph \subset \mathcal{E}$ taken as a cone and (X, c) be a complete cone b-metric space with open set $U \subset X$. Suppose that $T \in B(\mathcal{E}, \mathcal{E})$ such that $\|T\|_1 < 1$ with $T(\aleph) \subset \aleph$. If the mapping $h : \overline{U} \times [0, 1] \to X$ admits conditions of Corollary 5.4 in the first variable and*

(1) $x \neq h(x, \theta)$ for each $x \in \partial U$ (∂U represents the boundary of U in X);
(2) There exists $M \geq 0$ so that

$$\|c(h(x, \mu_1), h(x, \mu_2))\| \leq M|\mu_1 - \mu_2|$$

for some $x \in \overline{U}$ and $\mu_1, \mu_2 \in [0, 1]$;

(3) For any $x \in U$ there is $y \in X$ such that $\|c(x,y)\| \leq r$, then $x\mathfrak{R}y$, here r represents radius of U.

Then, whenever $h(\cdot, 0)$ possesses a fixed point in U, $h(\cdot, 1)$ also possesses a fixed point in U.

Proof. Let

$$\mathcal{B} = \{t \in [0,1] \mid x = h(x,t);\ \text{for} x \in U\}.$$

Define the partial order $\preceq$ in $\mathcal{E}$ by $u \preceq v \Leftrightarrow \|u\| \leq \|v\|$ for all $u, v \in \mathcal{E}$. Clearly $0 \in \mathcal{B}$, since $h(.,0)$ possesses a fixed point in U. So $\mathcal{B} \neq \phi$, In consideration of $c(x, h(x,\theta)) = c(x,y)$, $(I-T)(c(x,h(x,\theta))) \preceq sc(x,y)$ for all $x\mathfrak{R}y$ and $s \geq 1$, by Corollary 5.4, we get

$$c(h(x,\theta), h(y,\theta)) \preceq \frac{1}{s}T(c(x,y)).$$

Firstly, we prove that $\mathcal{B}$ is closed in $[0,1]$. For this, let $\{\theta_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$ with $\theta_n \to \theta \in [0,1]$ as $n \to \infty$. It is necessary to prove that $\theta \in \mathcal{B}$. Since, $\theta_n \in \mathcal{B}$ for $n \in \mathbb{N}$, there exists $x_n \in U$ with $x_n = h(x_n, \theta_n)$. Since, $h(\cdot, \theta)$ is monotone, so, for $n, m \in \mathbb{N}$, we have $x_m\mathfrak{R}x_n$. Since for $s \geq 1$

$$(I-T)(c(x_n, h(x_m, \theta_m))) = (I-T)(c(x_n, x_m)) \preceq sc(x_n, x_m),$$

we have

$$c(h(x_n,\theta_m), h(x_m,\theta_m)) \preceq \frac{1}{s}T(c(x_n x_m)),$$

and

$$\begin{aligned}
c(x_n, x_m) &= c(h(x_n,\theta_n), h(x_m,\theta_m)) \\
&\preceq s[c(h(x_n,\theta_n), h(x_n,\theta_m)) + c(h(x_n,\theta_m), h(x_m,\theta_m))] \\
\|c(x_n,x_m)\| &\leq sM|\theta_n - \theta_m| + \frac{s}{s}\|T(c(x_n,x_m))\| \\
\|c(x_n,x_m)\| &\leq \frac{sM}{1-\|T\|}[|\theta_n - \theta_m|].
\end{aligned}$$

As $\{\theta_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $[0,1]$, we have

$$\lim_{n,m\to\infty} c(x_n, x_m) = \mathbf{0}_{\mathcal{E}}.$$

So $\{x_n\}$ is a Cauchy sequence in X. As X is a complete cone b-metric space, so we have $x \in \overline{U}$ such that $\lim_{n\to\infty} c(x_n, x) \ll \epsilon$. Hence $x_n\mathfrak{R}x$ for all $n \in \mathbb{N}$. By triangle property, we have

$$\begin{aligned}
c(x, h(x,\theta)) &\preceq s[c(x,x_n) + c(x_n, h(x,\theta))] \\
&\preceq sc(x,x_n) + s^2[c(x_n, h(x_n,\theta)) + c(h(x_n,\theta), h(x,\theta))] \\
&\preceq sc(x,x_n) + s^2[c(h(x_n,\theta_n), h(x_n,\theta)) + c(h(x_n,\theta), h(x,\theta))] \\
\|c(x,h(x,\theta))\| &\leq \|sc(x,x_n)\| + s^2M|\theta_n - \theta| + \frac{s^2}{s}\|T(c(x_n,x))\|.
\end{aligned}$$

Thus, $c(x, h(x,\theta)) = \mathbf{0}$ as $n \to \infty$. So $\theta \in \mathcal{B}$, hence $\mathcal{B}$ is closed in $[0,1]$. Now we show that $\mathcal{B}$ is open in $[0,1]$. Let $\theta_1 \in \mathcal{B}$, so, there exists $x_1 \in U$ such that $h(\theta_1, x_1) = x_1$. As U is open, we have $r > 0$ so that $\mathcal{B}(x_1, r) \subseteq U$. Consider

$$l = \|c(x_1, \partial U)\| = \inf\{\|c(x_1,\xi)\| : \xi \in \partial U\}.$$

Then $r = l > 0$. Given $\epsilon > 0$ such that $\omega < \frac{(1-\|T\|)l}{sM}$ for $s \geq 1$. Let $\theta \in (\theta_1 - \omega, \theta_1 + \omega)$. Then

$$x \in \overline{\mathcal{B}(x_1, r)} = \{x \in X : \|c(x, x_1)\| \leq r\}, \text{ so that } x\mathfrak{R}x_1.$$

Consider

$$\begin{aligned} c(h(x,\theta), x_1) &= c(h(x,\theta), h(x_1, \theta_1)) \\ &\leq s[c(h(x,\theta), h(x,\theta_1)) + c(h(x,\theta_1), h(x_1,\theta_1))] \\ \|c(h(x,\theta), x_1)\| &\leq sM|\theta_1 - \theta| + \frac{s}{s}\|T(c(x_1, x))\| \\ &\leq sM\omega + \|T\|\, l = sM\omega + \|T\|\, l < l. \end{aligned}$$

Thus, for each $\theta \in (\theta_1 - \omega, \theta_1 + \omega)$, $h(\cdot, \theta) : \overline{\mathcal{B}(x, r)} \to \overline{\mathcal{B}(x, r)}$ has a fixed point in $\overline{U}$ by applying Corollary 5.4. Hence $\theta \in \mathcal{B}$ for any $\theta \in (\theta_1 - \omega, \theta_1 + \omega)$ and so $\mathcal{B}$ is open in $[0, 1]$. Thus, $\mathcal{B}$ is open as well as closed in $[0, 1]$ and by connectedness, $\mathcal{B} = [0, 1]$. Hence $h(\cdot, 1)$ has a fixed point in U. □

7. Application of homotopy to human aging process

In this section, we use the homotopy to describe the process of aging of human body. The aging process is considered by choosing suitable values for the time parameters t of the homotopy $a(t, x)$. The values of the parameters t and x in the function $a(t, x)$ are adjusted to control the process of the aging. For example, if $t \in [0, 1]$, and there is a homotopy $a(t, x)$ from $f(x)$ to $g(x)$ such that $a(0, x) = f(x)$ and $a(1, x) = g(x)$ then the body is only one year old. Thus, if $t \in [l, n]$, where $n > l$, and there is a continuous function $a(t, x)$ called homotopy from one function from $f(x)$ to $w(x)$ satisfying the condition $a(l, x) = f(x)$ and $a(n, x) = w(x)$, then the body is described as being n years old. It is observed that for the interval $t \in [l, n]$, the supremum $a(n, x) = y(x)$ is the actual age of the body. Topologically the infant is equal to the adult since the infant continuously grows into the adult. The study found an algebraic way of relating homotopy to the process of aging of human body. The compact connected human body with boundary is assumed to be topologically equivalent to a cylinder $X = S \times I$, where S is a circle and $I = [0, \alpha]$. The initial state of the body $X = S \times I$ is the topological shape of the infant. The aging process, called homotopy, is the family of continuous functions $a(t, x)$ on the interval $I = [0, \alpha]$. It is an increasing sequence of the function $a(t, x)$ of the body X. The homotopy relates the topological shape of the infant to the topological shape of the adult. For the human body X, let $x \in X$ and $t \in I$ define the growth of the body and the age of the body respectively. Since the final age of the human body is not known let $t = \infty$ represent the final age of the body such that $t \in [\theta, \infty]$ denotes the age interval of the body from $t = \theta$ to $t = \infty$. The time $t = \infty$ is the age threshold value of the human body. The aging process for all $t \in [0, \infty]$ is the family or the sequence of the functions $a(t, x)$ such that $a(0, x) = f(x)$ and $a(\infty, x) = y(x)$.

Theorem 7.1. *Let $X = S \times I$ be a cylinder. Let $a(t, x)$ be a homotopy related to the process of aging of human body. Then, whenever $a(0, x)$ possesses a fixed point in X, Then, $a(\infty, x)$ also possesses a fixed point in X.*

Proof. Since, the human body is compact connected and bounded, so, human body is topologically equivalent to a cylinder $X = S \times I$. It is known that continuous reshaping of a cylinder possesses many invariant points. Thus, whenever $a(0, x)$ possesses a fixed point in X, Then, $a(\infty, x)$ possesses a fixed point in X. □

8. The existence of the solution to Urysohn Integral Equation (UIE)

In this section, we will apply Theorem 4.1 for the existence of the unique solution to UIE:

$$\ell(\hbar) = f(\hbar) + \int_{IR} K_1(\hbar, s, \ell(s))ds. \tag{8.1}$$

This integral equation encapsulates both Volterra Integral Equation (VIE) and Fredholm Integral Equation (FIE), depending upon the region of integration (IR). If IR $= (a, x)$ where a is fixed, then UIE is VIE and for IR $= (a, b)$ where a, b are fixed, UIE is FIE. In the literature, one can find many approaches to find a unique solution to UIE (see [24, 28, 44] and references therein). We are interested to use a fixed-point technique for this purpose. The fixed-point technique is simple and elegant to show the existence of a unique solution to further mathematical models.

Let IR be a set of finite measure and $\mathcal{L}^2_{\text{IR}} = \left\{\ell \mid \int_{\text{IR}} |\ell(s)|^2 ds < \infty\right\}$. Define the norm $\|.\| : \mathcal{L}^2_{\text{IR}} \to [0, \infty)$ by

$$\|\ell\|_2 = \sqrt{\int_{\text{IR}} |\ell(s)|^2 ds}, \text{ for all } \ell, \jmath \in \mathcal{L}^2_{\text{IR}}.$$

An equivalent norm can be defined as follows:

$$\|\ell\|_{2,\nu} = \sqrt{\sup\{e^{-\nu \int_{\text{IR}} \alpha(s)ds} \int_{\text{IR}} |\ell(s)|^2 ds\}}, \text{ for all } \ell \in \mathcal{L}^2_{\text{IR}}, \nu > 1.$$

Then $\mathcal{E} = (\mathcal{L}^2_{\text{IR}}, \|.\|_{2,\nu})$ is a Banach space. Let $\mathcal{A} = \{\ell \in \mathcal{L}^2_{\text{IR}} : \ell(s) > 0$ for almost every s$\}$ be a cone in $\mathcal{E}$. The cone b-metric c_ν associated to norm $\|.\|_{2,\nu}$ is given by $c_\nu(\ell, \jmath) = \ell \, \|\ell - \jmath\|^2_{2,\nu}$ for all $\ell, \jmath \in \mathcal{A}$. Define a partial order $\preceq$ on $\mathcal{E}$ by

$$a \preceq \upsilon \text{ if and only if } a(s)\upsilon(s) \geq \upsilon(s), \text{ for all } a, \upsilon \in \mathcal{E}.$$

Then $(\mathcal{E}, \preceq, c_\nu)$ is a complete cone b-metric space. Let

(A1) The kernel $K_1 : \text{IR} \times \text{IR} \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and

$$|K_1(\hbar, s, \ell(s))| \leq w(\hbar, s) + e(\hbar, s)\,|\ell(s)|\,;\; w, e \in \mathcal{L}^2(\text{IR} \times \text{IR}), e(\hbar, s) > 0.$$

A2) The function $f : \text{IR} \to [1, \infty)$ is continuous and bounded on IR.

(A3) There exists a positive constant C such that

$$\sup_{\hbar \in \text{IR}} \int_{IR} |K_1(\hbar, s)|\, ds \leq C.$$

(A4) For any $\ell_0 \in \mathcal{L}^2_{\text{IR}}$, there is $\ell_1 = R(\ell_0)$ such that $\ell_1 \preceq \ell_0$ or $\ell_0 \preceq \ell_1$.

(A4') The sequence $\{\ell_n\}$ satisfies $\ell_{n-1} \preceq \ell_n$ and $\ell_n \to p$, then $\ell_n \preceq p$ for all $n \in \mathbb{N}$.

(A5) There exists a non-negative and measurable function $q : \text{IR} \times \text{IR} \to \mathbb{R}$ such that

$$\alpha(\hbar) := \int_{\text{IR}} q^2(\hbar, s)\, ds \leq \frac{1}{\nu}$$

and integrable over IR with

$$|K_1(\hbar, s, \ell(s)) - K_1(\hbar, s, \jmath(s))| \leq q(\hbar, s)|\ell(s) - \jmath(s)|$$

for all $\hbar, s \in \mathrm{IR}$ and $\ell, \jmath \in \mathcal{E}$ with $\ell \preceq \jmath$.

Theorem 8.1. *Suppose that the mappings f and K_1 mentioned above satisfy the conditions* (A1)–(A5)*, then the* UIE *(8.1) has a unique solution.*

Proof. Define the mapping $R : \mathcal{E} \to \mathcal{E}$, in accordance with the above-mentioned notations, by

$$(R\ell)(\hbar) = f(\hbar) + \int_{\mathrm{IR}} K_1(\hbar, s, \ell(s))ds.$$

The operator R is $\preceq$-preserving:

Let $\ell, \jmath \in \mathcal{E}$ with $\ell \preceq \jmath$, then $\ell(s)\jmath(s) \geq \jmath(s)$. Since, for almost every $\hbar \in \mathrm{IR}$,

$$(R\ell)(\hbar) = f(\hbar) + \int_{\mathrm{IR}} K_1(\hbar, s, \ell(s))ds \geq 1,$$

this implies that $(R\ell)(\hbar)(R\jmath)(\hbar) \geq (R\jmath)(\hbar)$. Thus, $(R\ell) \preceq (R\jmath)$.

Self-operator:

The conditions (A1) and (A3) imply that R is continuous and compact mapping from $\mathcal{A}$ to $\mathcal{A}$ (see [24, Lemma 3]).

By (A4), for any $\ell_0 \in \mathcal{A}$ there is $\ell_1 = R(\ell_0)$ such that $\ell_1 \preceq \ell_0$ or $\ell_0 \preceq \ell_1$ and using the fact that R is $\preceq$-preserving, we have $\ell_n = R^n(\ell_0))$ with $\ell_n \preceq \ell_{n+1}$ or $\ell_{n+1} \preceq \ell_n$ for all $n \geq 0$. We will check the contractive condition of Theorem 4.1 in the next lines. By (A5) and Holder inequality, we have

$$\begin{aligned}
\ell\,|(R\ell)(\hbar) - (R\jmath)(\hbar)|^2 &= \ell\left|\int_{IR} K_1(\hbar, s, \ell(s))\,ds - \int_{IR} K_1(\hbar, s, \jmath(s))\,ds\right|^2 \\
&\leq \ell\left(\int_{IR} |K_1(\hbar, s, \ell(s)) - K_1(\hbar, s, \jmath(s))|\;ds\right)^2 \\
&\leq \ell\left(\int_{IR} q(\hbar, s)|\ell(s) - \jmath(s)|\,ds\right)^2 \\
&\leq \ell\int_{IR} q^2(\hbar, s)\,ds \cdot \int_{IR} |\ell(s) - \jmath(s)|^2\,ds \\
&= \ell\alpha(\hbar)\int_{IR} |\ell(s) - \jmath(s)|^2\,ds.
\end{aligned}$$

This implies, by integrating with respect to $\hbar$,

$$\begin{aligned}
\ell \int_{IR} |(R\ell)(\hbar) - (R\jmath)(\hbar)|^2 \, d\hbar &\leq \ell \int_{IR} \left(\alpha(\hbar) \int_{IR} |\ell(s) - \jmath(s)|^2 \, ds \right) d\hbar \\
&= \ell \int_{IR} \left(\alpha(\hbar) e^{\nu \int_{\mathrm{IR}} \alpha(s)ds} \cdot e^{-\nu \int_{\mathrm{IR}} \alpha(s)ds} \int_{IR} |\ell(s) - \jmath(s)|^2 \, ds \right) d\hbar \\
&\leq \ell \, ||\ell - \jmath||_{2,\nu}^2 \int_{IR} \alpha(\hbar) e^{\nu \int_{\mathrm{IR}} \alpha(s)ds} \, d\hbar \\
&\leq \ell \frac{1}{\nu} \, ||\ell - \jmath||_{2,\nu}^2 \, e^{\nu \int_{\mathrm{IR}} \alpha(s)ds}.
\end{aligned}$$

Thus, we have

$$\ell e^{-\nu \int_{\mathrm{IR}} \alpha(s)ds} \int_{IR} |(R\ell)(\hbar) - (R\jmath)(\hbar)|^2 \, d\hbar \leq \ell \frac{1}{\nu} \, ||\ell - \jmath||_{2,\nu}^2 \, .$$

This implies that

$$\ell \, ||R\ell - R\jmath||_{2,\nu}^2 \leq \ell \frac{1}{\nu} \, ||\ell - \jmath||_{2,\nu}^2 \, .$$

That is,

$$c_\nu(R\ell, R\jmath) \leq \frac{1}{\nu} c_\nu(\ell, \jmath).$$

Thus, defining $\mathcal{L} : \mathcal{E}^6 \to \mathcal{E}$ by

$$\mathcal{L}(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 - kx_2; \;\; k \in [0, 1),$$

we have

$$\mathcal{L}\left(c(R\ell, R\jmath), c(\ell, \jmath), c(\ell, R\ell), c(\jmath, R\jmath), c(\ell, R\jmath), c(\jmath, R\ell)\right) \preceq \mathbf{0}_{\mathcal{E}}.$$

Hence, by Theorem 4.1, the operator R has a unique fixed point. This means that the UIE (8.1) has a unique solution. □

9. Conclusions

The ordered implicit relation in relation with implicit contraction is useful to obtain fixed point theorems that unify many corresponding fixed point theorems. These results can be applied to show the existence of the solutions to DE's and FDE's. The ordered implicit relation can be generalized to orthogonal implicit relation. The study of fixed point theorems is valid in cone metric spaces and hence in the cone b-metric spaces for the nonlinear contractions.

Conflict of interest

The authors declare that they have no competing interests.

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