AIMS Mathematics, 7(4): 5146–5155.
DOI:10.3934/math.2022287
Received: 14 December 2020
Revised: 10 August 2021
Accepted: 11 August 2021
Published: 04 January 2022

Mathematics

http://www.aimspress.com/journal/Math

Research article

Based on a family of bi-univalent functions introduced through the Faber polynomial expansions and Noor integral operator

F. Müge Sakar[1,*] **and Arzu Akgül**[2]

[1] Department of Management, Faculty of Economics and Administrative Sciences, Dicle University, Diyarbakır, Turkey

[2] Department of Mathematics, Faculty of Arts and Science, Umuttepe Campus, Kocaeli University, Kocaeli, Turkey

* **Correspondence:** Email: mugesakar@hotmail.com; Tel: +904122411000; Fax: +904122488348.

Abstract: In this study, by using q-analogue of Noor integral operator, we present an analytic and bi-univalent functions family in $\mathfrak{D}$. We also derive upper coefficient bounds and some important inequalities for the functions in this family by using the Faber polynomial expansions. Furthermore, some relevant corollaries are also presented.

Keywords: bi-univalent functions; Faber polynomials; q-analogue of Noor integral operator; coefficient inequalities
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

Conformal mapping is used in electromagnetic theory as well as in heat transfer theory. Univalent functions have wide application in heat transfer problems (see [16]). Special functions contain a very old branch of mathematics. In addition, in recent years, special functions and inequalities are widely used for solving some problems in physics, integer-order differential equations and systems, electromagnetizm, heat-transfer problems, mathematical models, etc [24]. Specially harmonic, analytical functions and inequalities of coefficients are widely used in thermodynamics, electricity and magnetism and quantum physics. In electricity, current and impedance equations can be expressed in a complex plane, and basic electrical relations become complex functions. However, in this study, we consider only upper coefficient bounds and some important inequalities for analytic and bi-univalent functions family by using special functions.

Let's denote by $\mathbb{C}$ which is the complex plane in the open unit disk $\mathfrak{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Additionaly, $\mathcal{A}$ denotes the family of functions $s(z)$ which are *analytic* in the open unit disk and normalized by $s(0) = s'(0) - 1 = 0$ and having the style:

$$s(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let $\mathcal{S}$ be a subfamily of $\mathcal{A}$ which is univalent in $\mathfrak{D}$ (for details, see [11]). Furthermore, $\mathcal{P}$ be the family of functions, formed:

$$\varphi(z) = 1 + \sum_{k=1}^{\infty} \varphi_k z^k \qquad (z \in \mathfrak{D})$$

$\mathfrak{D}$ and hold the necessity $\Re(\varphi(z)) > 0$ in $\mathfrak{D}$. By the Carathéodory's Lemma (e.g., see [11]), we get $|\varphi_k| \leq 2$.

In accordance with the Koebe Theorem (e.g.,see [11]), each univalent function $s(z) \in \mathcal{A}$ has an inverse s^{-1} fulfilling

$$s^{-1}(s(z)) = z \qquad (z \in \mathfrak{D})$$

and

$$s\left(s^{-1}(w)\right) = w \qquad \left(|w| < r_0(s)\ r_0(s) \geq \frac{1}{4}\right).$$

Actually, the inverse function s^{-1} is denoted by

$$r(w) = s^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

If both $s(z)$ and $s^{-1}(z)$ are univalent, we can say that, $s \in \mathcal{A}$ is be *bi-univalent* in $\mathfrak{D}$. All families of bi-univalent functions in $\mathfrak{D}$ with Taylor-Maclaurin series expansion (1.1) are presented by Σ.

For both of some knowledges and different examples for functions belong to Σ, see the following references [9, 17, 18, 22, 23, 25]. Also, see references by Ali et al. [5], Jahangiri and Hamidi [15], and other studies such as [6, 7, 10, 13, 21].

Definition 1. For analytic functions s and r, s is subordinate to r, presented by

$$s(z) \prec r(z), \tag{1.3}$$

if there is an analytic function w such that

$$w(0) = 0\ ,\ |w(z)| < 1 \text{ and } s(z) = r(w(z)).$$

The following definition gives us the knowledge about fractional q-calculus operators (see, [22]).

Definition 2. [22] For $q \in (0, 1)$, the q-derivative of $s \in \mathcal{A}$ is given by

$$\partial_q s(z) = \frac{s(qz) - s(z)}{(q-1)z}, z \neq 0 \tag{1.4}$$

and

$$\partial_q s(0) = s'(0).$$

Thus we have

$$\partial_q s(z) = 1 + \sum_{k=2}^{\infty} [k, q]\, a_k z^{k-1} \tag{1.5}$$

where $[k,q]$ is presented by

$$[k,q] = \frac{1-q^k}{1-q}, \quad [0,q] = 0 \tag{1.6}$$

and define the q-fractional by

$$[k,q]! = \begin{cases} \prod_{n=1}^{k} [n,q], & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{1.7}$$

Furthermore, Pochhammer symbol which is $q-$generalized for $\mathfrak{p} \geq 0$ is denoted by

$$[\mathfrak{p},q]_k = \begin{cases} \prod_{n=1}^{k} [\mathfrak{p}+n-1,q], & k \in \mathbb{N} \\ 1, & k = 0 \end{cases}. \tag{1.8}$$

In addition, as $q \to 1^-$, $[k,q] \to k$, if we select $r(z) = z^k$, then we obtain

$$\partial_q r(z) = \partial_q z^k = [k,q]\, z^{k-1} = r'(z),$$

where r' is the ordinary derivative.

Recently, $F^{-1}_{q,\mu+1}(z)$, given with the following relation, was defined by Arif et al. (see [8])

$$F^{-1}_{q,\mu+1}(z) * F_{q,\mu+1}(z) = z\partial_q s(z), \quad (\mu > -1) \tag{1.9}$$

where

$$F_{q,\mu+1}(z) = z + \sum_{k=2}^{\infty} \frac{[\mu+1,q]_{k-1}}{[k-1,q]!} z^k, \quad z \in \mathfrak{D}. \tag{1.10}$$

Due to the fact that series given in (1.10) is convergent absolutely in $z \in \mathfrak{D}$, by taking advantage of the characterization of q-derivative via convolution, one can define the integral operator $\zeta_q^\mu : \mathfrak{D} \to \mathfrak{D}$ by

$$\zeta_q^\mu s(z) = F^{-1}_{q,\mu+1}(z) * s(z) = z + \sum_{k=2}^{\infty} \phi_{k-1} a_k z^k, \quad (z \in \mathfrak{D}) \tag{1.11}$$

where

$$\phi_{k-1} = \frac{[k,q]!}{[\mu+1,q]_{k-1}}. \tag{1.12}$$

We note that

$$\zeta_q^0 s(z) = z\partial_q s(z), \; \zeta_q' s(z) = s(z) \tag{1.13}$$

and

$$\lim_{q\to 1} \zeta_q^\mu s(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k. \tag{1.14}$$

Equation (1.14) means that the operator denoted by (1.11) reduces to the known Noor integral operator by getting $q \to 1$, which is presented in (see [19, 20]). For further informations on the q-analogue of differential-integral operators, see the study of Aldweby and Darus (see [4]).

This work was motivated by Akgül and Sakar's study [3]. The basic purpose of this study is to give a new subfamily, which is in Σ and provide general coefficient bound $|a_n|$ by using Faber polynomial technics for this subfamily. Additionaly, we derive bounds of the $|a_2|$ and $|a_3|$ which are the first two coefficients of this subfamily.

2. The family $W_{\Sigma}^{\mu,q}(\alpha,\tau;\varphi)$

In this part, firstly we will introduce the class $W_{\Sigma}^{\mu,q}(\alpha,\tau;\varphi)$ and then give the knowledgements about Faber polynomial expansions.

Definition 3. A function $s \in \Sigma$ is known in the class $W_{\Sigma}^{\mu,q}(\alpha,\tau;\varphi)$ if the requirements given below hold:

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}s(z)}{z}+\alpha\partial_q(\zeta_q^{\mu}s(z))-1\right]\prec\varphi(z) \qquad (z\in\mathfrak{D}), \tag{2.1}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}r(w)}{w}+\alpha\partial_q(\zeta_q^{\mu}r(w))-1\right]\prec\varphi(w) \qquad (w\in\mathfrak{D}) \tag{2.2}$$

where $(\mu>-1, 0<q<1, \tau>0, \alpha\geq 0)$ and $s=r^{-1}(w)$ is given by (1.2).

It is clear from Definition 3 that upon setting $q\to 1^-$, for $\tau=1$, $\alpha=1$ and $\mu=1$, one can easily see that $s\in\Sigma$ is in

$$W_{\Sigma}^{1}(1,1;\varphi)=\mathcal{H}_{\sigma}(\varphi)$$

if the conditions given below hold true:

$$s'(z)\prec\varphi(z) \qquad (z\in\mathfrak{D}),$$

and

$$r'(w)\prec\varphi(w) \qquad (w\in\mathfrak{D}),$$

where $r=s^{-1}$ is given by (1.2) . The class $\mathcal{H}_{\sigma}(\varphi)$ was investigated by Ali et al. [5].

The Faber polynomials act effective role in several fields of mathematical sciences, specially, in the Theory of Geometric Function [12]. Also, Grunsky [14] gave some sufficient conditions for the univalency.

To obtain our main results, we need to following knowledgements owing to Airault and Bouali [1].

Using the Faber polynomial expansion of function $s\in\mathcal{A}$ given in (1.1), $s^{-1}=g$ may be given as

$$r(w)=s^{-1}(w)=w+\sum_{k=2}^{\infty}\frac{1}{k}K_{k-1}^{-k}(a_2,a_3,\ldots)w^k,$$

where

$$\begin{aligned}K_{k-1}^{-k}&=\frac{(-k)!}{(-2k+1)!(k-1)!}a_2^{k-1}+\frac{(-k)!}{[2(-k+1)]!(k-3)!}a_2^{k-3}a_3\\&+\frac{(-k)!}{(-2k+3)!(k-4)!}a_2^{k-4}a_4\\&+\frac{(-k)!}{[2(-k+2)]!(k-5)!}a_2^{k-5}[a_5+(-k+2)a_3^2]\\&+\frac{(-k)!}{(-2k+5)!(k-6)!}a_2^{k-6}[a_6+(-2k+5)a_3a_4]\end{aligned}$$

$$+\sum_{j\geq 7} a_2^{k-j} V_j,$$

symbolically such term $(-k!) \equiv \Gamma(1-k) := (-k)(-k-1)(-k-2)\cdots (k \in \mathbb{N}_0,\ \mathbb{N} := \{1,2,3,\cdots\})$ and V_j with $7 \leq j \leq k$ is a homologous polynomial in $a_2, a_3, \ldots a_k$, [2]. Particularly, some initial terms of K_{k-1}^{-k} are

$$K_1^{-2} = -2a_2,$$

$$K_2^{-3} = 3(2a_2^2 - a_3),$$

$$K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

Generally, for any $p \in \mathbb{N} := \{1,2,3\ldots\}$, an expansion of K_k^p is given, [1],

$$K_k^p = pa_k + \frac{p(p-1)}{2}D_k^2 + \frac{p!}{(p-3)!3!}D_k^3 + \ldots + \frac{p!}{(p-k)!(k)!}D_k^k,$$

where $D_k^p = D_k^p(a_1, a_2, a_3, \ldots, a_k)$, and by [26],

$$D_k^m(a_1, a_2, \ldots a_k) = \sum_{m=2}^{\infty} \frac{m!}{i_1! \ldots i_k!} a_1^{i_1} \ldots a_k^{i_k} \qquad for \quad m \leq k$$

while $a_1 = 1$, and non-negative integers $i_1, \ldots, i_k$ satisfying

$$i_1 + i_2 + ... + i_k = m,$$

$$i_1 + 2i_2 + ... + ki_k = k.$$

It is obvious that $D_k^k(a_1, a_2, ...a_k) = a_1^k$.

As a result, for $s \in W_\Sigma^{\mu,q}(\alpha, \tau; \varphi)$ given by (1.1) , we can write

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^\mu s(z)}{z} + \alpha\partial_q(\zeta_q^\mu s(z)) - 1\right] = 1 + \frac{1}{\tau}\sum_{k=2}^{\infty}[k, \alpha q]\phi_{k-1}a_k z^k \tag{2.3}$$

where

$$[k, \alpha q] = 1 + \sum_{l=1}^{k-1} \alpha q^l.$$

Theorem 4. For $\alpha \geq 1, \mu > -1, 0 < q < 1, \tau > 0$, let the function given by (1.1) $s \in W_\Sigma^{\mu,q}(\alpha, \tau; \varphi)$. If $a_m = 0$ for $2 \leq m \leq k-1$, then

$$|a_k| \leq \frac{2\tau}{\left[1 + \sum\limits_{l=1}^{k-1} \alpha q^l\right]\phi_{k-1}}.$$

Proof. For analytic functions s given by (1.1), we get

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^\mu s(z)}{z} + \alpha\partial_q(\zeta_q^\mu s(z)) - 1\right] = 1 + \frac{1}{\tau}\sum_{k=2}^{\infty}\left[1 + \sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}a_k z^{k-1} \tag{2.4}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}r(w)}{w}+\alpha\partial_q(\zeta_q^{\mu}r(w))-1\right]=1+\frac{1}{\tau}\sum_{k=2}^{\infty}\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}b_kw^{k-1}$$

$$=\ 1+\frac{1}{\tau}\sum_{k=2}^{\infty}\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}\times\frac{1}{k}K_{k-1}^{-k}(a_2,a_3,\ldots a_k)w^{k-1}. \tag{2.5}$$

Moreover, the correlations (2.1) and (2.2) refer to the presence of Schwartz functions

$$u(z)=\sum_{k=2}^{\infty}c_kz^k \quad \text{and} \quad \vartheta(w)=\sum_{k=2}^{\infty}d_kw^k \tag{2.6}$$

so that

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}s(z)}{z}+\alpha\partial_q(\zeta_q^{\mu}s(z))-1\right]=\varphi\left(u(z)\right) \tag{2.7}$$

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}r(w)}{w}+\alpha\partial_q(\zeta_q^{\mu}r(w))-1\right]=\varphi\left(\vartheta(w)\right) \tag{2.8}$$

where

$$\varphi\left(u(z)\right)=1+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\varphi_nD_k^n(c_1,c_2,\ldots,c_k)z^k \tag{2.9}$$

$$\varphi\left(\vartheta(w)\right)=1+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\varphi_nD_k^n(d_1,d_2,\ldots,d_k)w^k. \tag{2.10}$$

Thus, from (2.4), (2.6) and (2.9) we have

$$\frac{1}{\tau}\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}a_k=\sum_{n=1}^{k}\varphi_nD_k^n(c_1,c_2,\ldots,c_k), \quad (k\geq 2). \tag{2.11}$$

Similarly, by using (2.5), (2.6) and (2.10) we find that

$$\frac{1}{\tau}\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}b_k=\sum_{n=1}^{k}\varphi_nD_k^n(d_1,d_2,\ldots,d_k), \quad (k\geq 2). \tag{2.12}$$

For $a_n=0$ $(2\leq n\leq k-1)$, we get

$$b_k=-a_k$$

and so

$$\begin{aligned}\frac{1}{\tau}\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}a_k &= \varphi_1c_{k-1},\\ -\frac{1}{\tau}\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}a_k &= \varphi_1d_{k-1}.\end{aligned}$$

When we take the absolute values of either of the above two equalities and using $|\varphi_1| \leq 2, |c_{k-1}| \leq 1$ and $|d_{k-1}| \leq 1$, we obtain

$$a_k = \frac{|\varphi_1 c_{k-1}|\,\tau}{\left|\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}\right|} = \frac{|\varphi_1 d_{k-1}|\,\tau}{\left|\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}\right|} \leq \frac{2\tau}{\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}}$$

which evidently completes the proof of theorem.

We have the Corollary 5, when we choose $\tau = 1$ in Theorem 4.

Corollary 5. For $\alpha \geq 1$, $\mu > -1$ $0 < q < 1$, let s in the form (1.1) be in $W_{\Sigma}^{\mu,q}(\alpha;\varphi)$. If $a_m = 0$ for $2 \leq m \leq k-1$, then

$$|a_k| \leq \frac{2}{\left[1+\sum_{l=1}^{k-1}\alpha q^l\right]\phi_{k-1}}.$$

Comforting the coeefficient restricts produced in Theorem 4, we get coefficients given early of $s \in W_{\Sigma}^{\mu,q}(\alpha,\tau;\varphi)$ given below.

Theorem 6. Let $s \in W_{\Sigma}^{\mu,q}(\alpha,\tau;\varphi)$ and for $\alpha \geq 1$, $\mu > -1$, $0 < q < 1$, $\tau > 0$. Then

$$(i) \quad |a_2| \leq \min\left\{\frac{2\tau}{(1+\alpha q)\phi_1},\ \frac{2\sqrt{\tau}}{\sqrt{(1+\alpha q+\alpha q^2)\phi_2}}\right\}$$

$$(ii) \quad |a_3| \leq \min\left\{\frac{4\tau^2}{(1+\alpha q)^2\phi_1^2}+\frac{2|\tau|}{(1+\alpha q+\alpha q^2)\phi_2},\ \frac{6\tau}{(1+\alpha q+\alpha q^2)\phi_2}\right\}$$

and

$$(iii) \quad \left|a_3 - 2a_2^2\right| \leq \frac{4\tau}{(1+\alpha q+\alpha q^2)\phi_2}.$$

Proof. we obtain following equalities by replacing k by 2 and 3 in (2.11) and (2.12), respectively,

$$\frac{1}{\tau}(1+\alpha q)\phi_1 a_2 = \varphi_1 c_1 \tag{2.13}$$

$$\frac{1}{\tau}(1+\alpha q+\alpha q^2)\phi_2 a_3 = \varphi_1 c_2 + \varphi_2 c_1^2 \tag{2.14}$$

$$-\frac{1}{\tau}(1+\alpha q)\phi_1 a_2 = \varphi_1 d_1 \tag{2.15}$$

$$\frac{1}{\tau}(1+\alpha q+\alpha q^2)\phi_2(2a_2^2 - a_3) = \varphi_1 d_2 + \varphi_2 d_1^2. \tag{2.16}$$

From (2.13) and (2.15), we obtain,

$$d_1 = -c_1$$

and taking their absolute values,

$$|a_2| = \frac{|\varphi_1 c_1|\,\tau}{|(1+\alpha q)\phi_1|} = \frac{|\varphi_1 d_1|\,\tau}{|(1+\alpha q)\phi_1|} \leq \frac{2\tau}{(1+\alpha q)\phi_1}. \tag{2.17}$$

Now, by adding (2.14) and (2.16), implies that

$$\frac{2}{\tau}\left[(1+\alpha q+\alpha q^2)\phi_2\right]a_2^2=\varphi_1(c_2+d_2)+\varphi_2(c_1^2+d_1^2)$$

or equivalently, (by taking the square roots and using Caratheodary Lemma)

$$|a_2|\leq\frac{2\sqrt{\tau}}{\sqrt{(1+\alpha q+\alpha q^2)\phi_2}} \tag{2.18}$$

Next, in order to obtain the coefficient estimate of $|a_3|$, we subtract (2.16) from (2.14). Thus we get

$$\frac{2}{\tau}[\tau](a_3-a_2^2)=\varphi_1(c_2-d_2)+\varphi_2(c_1^2-d_1^2)$$

or equivalently,

$$|a_3|\leq|a_2^2|+\frac{|\varphi_1(c_2-d_2)|\tau}{2|(1+\alpha q+\alpha q^2)\phi_2|}. \tag{2.19}$$

By replacing $|a_2^2|$ from (2.17) and (2.18) into (2.19), we get,

$$|a_3|\leq\frac{4\tau^2}{(1+\alpha q)^2\phi_1^2}+\frac{2\tau}{(1+\alpha q+\alpha q^2)\phi_2}$$

and

$$|a_3|\leq\frac{6\tau}{(1+\alpha q+\alpha q^2)\phi_2}.$$

Finally, from (2.16), we deduce that (by Caratheodary Lemma)

$$\left|a_3-2a_2^2\right|=\frac{|\varphi_1 d_2+\varphi_2 d_1^2|\tau}{|(1+\alpha q+\alpha q^2)\phi_2|}\leq\frac{4\tau}{(1+\alpha q+\alpha q^2)\phi_2}.$$

So, the proof is over.

By letting $q\to 1^-$ in Theorem 6, we get the Corollary 7.

Corollary 7. Let s presented by (1.1) be in the family $W_\Sigma^\mu(\alpha,\tau;\varphi)$ if $a_m=0$ for $2\leq m\leq k-1$, then

$$(i)\quad |a_2|\leq\frac{2\tau}{(1+\alpha)\phi_1},\quad \frac{2\sqrt{\tau}}{\sqrt{(1+2\alpha)\phi_2}}$$

$$(ii)\quad |a_3|\leq\frac{4\tau^2}{(1+\alpha)^2\phi_1^2}+\frac{2\tau}{(1+2\alpha)\phi_2},\quad \frac{6\tau}{(1+2\alpha)\phi_2}$$

and

$$(iii)\quad \left|a_3-2a_2^2\right|\leq\frac{4\tau}{(1+2\alpha)\phi_2}.$$

By letting $\tau=1$ in Corrollary 7, we obtain Corollary 8.

Corollary 8. Let s indicated by (1.1) be in the family $W_\Sigma^\mu(\alpha;\varphi)$ if $a_m=0$ for $2\leq m\leq k-1$, then

$$(i)\quad |a_2|\leq\frac{2}{(1+\alpha)\phi_1},\quad \frac{2}{\sqrt{(1+2\alpha)\phi_2}}$$

$$(ii)\quad |a_3|\leq\frac{4}{(1+\alpha)^2\phi_1^2}+\frac{2}{(1+2\alpha)\phi_2},\quad \frac{6}{(1+2\alpha)\phi_2}$$

and

$$(iii)\quad \left|a_3-2a_2^2\right|\leq\frac{4}{(1+2\alpha)\phi_2}.$$

Conflict of interest

The authors declare no conflict of interest.

References

1. H. Airault, A. Bouali, Differential calculus on the Faber polynomials, *B. Sci. Math.*, **130** (2006), 179–222. http://dx.doi.org/10.1016/j.bulsci.2005.10.002
2. H. Airault, J. Ren, An algebra of differential operators and generating functions on the set of univalent functions, *B. Sci. Math.*, **126** (2002), 343–367. http://dx.doi.org/10.1016/S0007-4497(02)01115-6
3. A. Akgül, F. M. Sakar, A certain subclass of bi-univalent analytic functions introduced by means of the q -analogue of Noor integral operator and Horadam polynomials, *Turk. J. Math.*, **43** (2019), 2275–2286. http://dx.doi.org/10.3906/mat-1905-17
4. H. Aldweby, M. Darus, A subclass of harmonic univalent functions associated with q-analogue of Dziok-Srivastava operator, *International Scholarly Research Notices*, **2013** (2013), 382312. http://dx.doi.org/10.1155/2013/382312
5. R. M. Ali, S. K. Lee, V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, *Appl. Math. Lett.*, **25** (2012), 344–351. http://dx.doi.org/10.1016/j.aml.2011.09.012
6. Ş. Altınkaya, Bounds for a new subclass of bi-univalent functions subordinate to the Fibonacci numbers, *Turk. J. Math.*, **44** (2020), 553–560. http://dx.doi.org/10.3906/mat-1910-41
7. Ş. Altınkaya, S. Yalçın, S. Çakmak, A subclass of bi-univalent functions based on the Faber polynomial expansions and the Fibonacci numbers, *Mathematics*, **7** (2019), 160. http://dx.doi.org/10.3390/math7020160
8. M. Arif, M. Ul Haq, J.-L. Liu, Subfamily of univalent functions associated with q-analogue of Noor integral operator, *J. Funct. Space.*, **2018** (2018), 3818915. http://dx.doi.org/10.1155/2018/3818915
9. D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, In: S. M. Mazhar, A. Hamoui, N. S. Faour, Editors, *Mathematical analysis and its applications*, Oxford: Pergamon Press, Elsevier Science Limited, 1988, 53–60.
10. E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, *J. Class. Anal.*, **2** (2013), 49–60. http://dx.doi.org/10.7153/jca-02-05
11. P. L. Duren, *Univalent functions*, New York, Berlin, Heidelberg and Tokyo: Springer-Verlag, 1983.
12. G. Faber, Über polynomische entwickelungen, *Math. Ann.*, **57** (1903), 389–408. http://dx.doi.org/10.1007/BF01444293
13. B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.*, **24** (2011), 1569–1573. http://dx.doi.org/10.1016/j.aml.2011.03.048
14. H. Grunsky, Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen, *Math. Z.*, **45** (1939), 29–61. http://dx.doi.org/10.1007/BF01580272

15. J. M. Jahangiri, S. G. Hamidi, Coefficient estimates for certain classes of bi-univalent functions, *Int. J. Math. Math. Sci.*, **2013** (2013), 190560. http://dx.doi.org/10.1155/2013/190560

16. P. A. A. Laura, A survey of modern applıcatıons of the method of conformal mapping, *Rev. Unión Mat. Argent.*, **27** (1975), 167–179.

17. M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.*, **18** (1967), 63–68.

18. M. E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Rational Mech. Anal.*, **32** (1969), 100–112. http://dx.doi.org/10.1007/BF00247676

19. K. I. Noor, On new classes of integral operators, *J. Natur. Geom.*, **16** (1999), 71–80.

20. K. I. Noor, M. A. Noor, On integral operators, *J. Math. Anal. Appl.*, **238** (1999), 341–352. http://dx.doi.org/10.1006/jmaa.1999.6501

21. Z. G. Peng, Q. Q. Han, On the coefficients of several classes of bi-univalent functions, *Acta Math. Sci.*, **34** (2014), 228–240. http://dx.doi.org/10.1016/S0252-9602(13)60140-X

22. H. M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, In: *Univalent functions; fractional calculus, and their applications*, New York, Chichester, Brisbane and Toronto: Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, 1989, 329–354.

23. H. M. Srivastava, Some inequalities and other results associated with certain subclasses of univalent and bi-univalent analytic functions, In: *Nonlinear analysis*, New York: Springer, 2012, 607–630. http://dx.doi.org/10.1007/978-1-4614-3498-6_38

24. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran J. Sci. Technol. Trans. Sci.*, **44** (2020), 327–344. http://dx.doi.org/10.1007/s40995-019-00815-0

25. T. S. Taha, Topics in univalent function theory, Ph.D. Thesis of University of London, 1981.

26. P. G. Todorov, On the Faber polynomials of the univalent functions of class $\sum$, *J. Math. Anal. Appl.*, **162** (1991), 268–276. http://dx.doi.org/10.1016/0022-247X(91)90193-4

AIMS Press