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Research article

Radial distributions of Julia sets of difference operators of entire solutions of complex differential equations

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Abstract: In this paper, we mainly investigate the radial distribution of Julia sets of difference operators of entire solutions of complex differential equation $F(z)f^n(z) + P(z, f) = 0$, where $F(z)$ is a transcendental entire function and $P(z, f)$ is a differential polynomial in f and its derivatives. We obtain that the set of common limiting directions of Julia sets of non-trivial entire solutions, their shifts have a definite range of measure. Moreover, an estimate of lower bound of measure of the set of limiting directions of Jackson difference operators of non-trivial entire solutions is given.

Keywords: Julia sets; radial distribution; entire function; difference operator; complex differential equations
Mathematics Subject Classification: 30D30, 30D35

1. Introduction and main results

In this paper, we investigate the radial distribution of Julia sets of non-trivial entire solutions of equation

$$F(z)f^n(z) + P(z, f) = 0, \tag{1.1}$$

where $F(z)$ is a transcendental entire function, $P(z, f) = \sum_{j=1}^{s} \alpha_j(z) f^{n_{0j}} (f')^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}$ is a differential polynomial in $f(z)$ and its derivatives. The powers $n_{0j}, n_{1j}, \cdots, n_{kj}$ are non-negative integers and satisfy $\gamma_p = \min\limits_{1\le j\le s}(\sum_{i=0}^{k} n_{ij}) \ge n$. Meromorphic functions $\alpha_j(z) (j = 1, 2, \cdots, s)$ are small functions of $F(z)$.

The Nevanlinna theory is an important tool in this paper, and its standard notations as well as well-known theorems can be found in [6, 8]. Let f be a meromorphic function in the complex plane. For example, we denote by $m(r, f)$, $N(r, f)$ and $T(r, f)$ the proximity function, counting function of poles and Nevanlinna characteristic function with respect to f, respectively. The order $\sigma(f)$ and lower order

$\mu(f)$ are defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r},$$

respectively, and the deficiency of the value a is defined by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r,f)}.$$

We say that a is a Nevanlinna deficient value of $f(z)$ if $\delta(a,f) > 0$. And when $a = \infty$, we have

$$\delta(\infty,f) = \liminf_{r\to\infty} \frac{m(r,f)}{N(r,f)}.$$

We define f^n, $n \in \mathbb{N}$ as the nth iterate of f, that is, $f^1 = f, \cdots, f^n = f \circ (f^{n-1})$. The Fatou set $\mathcal{F}(f)$ of transcendental meromorphic function f is the subset of the complex plane $\mathbb{C}$, where the iterates f^n of f form a normal family. The complement of $\mathcal{F}(f)$ in $\mathbb{C}$ is called the Julia set $\mathcal{J}(f)$ of f. It is well known that $\mathcal{F}(f)$ is open, $\mathcal{J}(f)$ is closed and non-empty. For an introduction to the dynamics of meromorphic functions, we refer the reader to see Bergweiler's paper [4] and Zheng's book [21].

Suppose that $f(z)$ is a transcendental meromorphic function in $\mathbb{C}$ and $argz = \theta$ is a ray from the origin. The ray $\arg z = \theta(\theta \in [0, 2\pi])$ is said to be the limiting direction of $\mathcal{J}(f)$ if

$$\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap \mathcal{J}(f)$$

is unbounded for any $\varepsilon > 0$, where $\Omega(\theta - \varepsilon, \theta + \varepsilon) = \{z \in \mathbb{C} | \arg z \in (\theta - \varepsilon, \theta + \varepsilon)\}$. And we define

$$\Delta(f) = \{\theta \in [0, 2\pi) | \text{ the ray } \arg z = \theta \text{ is a limiting direction of } \mathcal{J}(f)\}.$$

Obviously, $\Delta(f)$ is closed and measurable, we use mes$\Delta(f)$ to stand for its linear measure.

There are a lot of works around the radial distributions of Julia sets of meromorphic functions, see [2, 13–15, 17, 20]. When f is transcendental entire, Baker [2] observed that $\mathcal{J}(f)$ cannot be contained in any finite union of straight lines. Furthermore, Qiao [13] proved that $mes\Delta(f) = 2\pi$ when $\mu(f) < 1/2$ and $mes\Delta(f) \geq \pi/\mu(f)$ when $\mu(f) \geq 1/2$, where $f(z)$ is a transcendental entire function with finite lower order. Then, for entire functions with infinite order, what is sufficient condition for the existence of lower bound of the measure of the limit directions?

Huang and Wang [9, 10] considered this problem. They first studied the radial distribution of Julia sets of a solution base of complex linear differential equations and obtained the following result.

Theorem A. [9] Let $\{f_1, f_2, \cdots, f_n\}$ be a solution base of

$$f^{(n)} + A(z)f = 0, \tag{1.2}$$

where $A(z)$ is a transcendental entire function with finite order, and denote $E = f_1 f_2 \cdots f_n$. Then

$$mes\Delta(E) \geq min\{2\pi, \frac{\pi}{\sigma(A)}\}.$$

After that, Huang and Wang [10] directly studied the limiting direction of Julia sets of solutions of a class of higher order linear differential equations.

Theorem B. [10] Let $A_i(z)(i = 0, 1, 2, \cdots, n-1)$ be entire functions of finite lower order such that A_0 is transcendental and $m(r, A_i) = o(m(r, A_0))(i = 1, 2, \cdots, n-1)$ as $r \to \infty$. Then every non-trivial solution f of the equation

$$f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_0 f = 0, \tag{1.3}$$

satisfies $mes\Delta(f) \geq \{2\pi, \frac{\pi}{\mu(A_0)}\}$.

Since then, inspired by the research of Huang and Wang, many scholars have studied the above problem. Especially, under the hypothesis of Theorem B, Zhang et al. [19] proved that $mes(\Delta(f) \cap (\Delta(f^{(k)}))) \geq min\{2\pi, \pi/\mu(A_0)\}$, where $f^{(k)}(k \in \mathbb{N})$ denote the derivatives for k and $f^{(0)} = f$.

Theorem C. [19] Let $A_i(z)(i = 0, 1, 2, \cdots, n-1)$ be entire functions of finite lower order such that A_0 is transcendental and $m(r, A_i) = o(m(r, A_0))(i = 1, 2, \cdots, n-1)$ as $r \to \infty$. Then every non-trivial solution f of Eq (1.3) satisfies

$$mes(\Delta(f) \cap (\Delta(f^{(k)})) \geq min\{2\pi, \pi/\mu(A_0)\},$$

where k is a positive integer.

In 2021, Wang et al. [16] introduced the definition of transcendental directions to describe such directions in which f grows fast, and studied the relation between transcendental directions and limiting directions of entire solutions of Eq (1.1).

Theorem D. Suppose that n, k are integers, $F(z)$ is a transcendental entire function of finite lower order, and that $P(z, f)$ is a differential polynomial in f with $\gamma_p \geq n$, where all coeffcients $\alpha_j (j = 1, 2, \cdots, s)$ are polynomials if $\mu(F) = 0$, or all $\alpha_j (j = 1, 2, \cdots, s)$ are entire and $\rho(r, \alpha_j) < \mu(F)$. Then for every nonzero transcendental entire solution f of the differential Eq (1.1), we have $TD(f^{(k)}) \cap TD(F) \subseteq \Delta(f^{(k)})$ and

$$mes(\Delta(f^{(k)})) \geq mes(TD(f^{(k)}) \cap TD(F)) \geq min\{2\pi, \frac{\pi}{\mu(F)}\}.$$

Here, the notation $TD(f)$ denoted by the union of all transcendental directions of f, where a value $\theta \in [0, 2\pi]$ is said to be a transcendental direction of f if there exists an unbounded sequence $\{z_n\}$ such that

$$\lim_{n\to\infty} \arg z_n = \theta \quad and \quad \lim_{n\to\infty} \frac{\log |f(z_n)|}{\log |z_n|} = +\infty.$$

In recent years, value distribution in difference analogues of meromorphic functions has become a subject of great interest. The difference analogues of the lemma on the logarithmic derivatives, the Clunie lemma and etc. are applicable to study large classes of difference equations, often by using methods similar to the case of differential equations, see [5, 7]. Inspired by Theorem A–D and the progress on the difference analogues of classical Nevanlinna theory of meromorphic functions, it is quite natural to investigate the limit directions of difference operators of meromorphic functions. This paper is an attempt in this direction. Set

$$E(f) = \bigcap_{k\in\mathbb{Z}} \bigcap_{i\in L} \Delta(f^{(k)}(z + \eta_i)),$$

where $k \in \mathbb{Z}$, $f^{(k)}$ denotes the $k-th$ derivative of $f(z)$ for $k \geq 0$ or $k-th$ integra primitive of $f(z)$ for $k < 0$, L is a set of positive integers and $\{\eta_i : i \in L\}$ is a countable set of distinct complex numbers.

Theorem 1.1. *Suppose that n, k are integers, $F(z)$ is a transcendental entire function of finite lower order, and that $P(z, f)$ is a differential polynomial in f with $\gamma_p \geq n$, where all coeffcients $\alpha_j(j = 1, 2, \cdots, s)$ are small functions of $F(z)$. Then every non-trivial entire solution $f(z)$ of Eq (1.1) satisfies*

$$mes(E(f)) \geq min\{2\pi, \frac{\pi}{\mu(F)}\}. \tag{1.4}$$

Remark 1.1. *Clearly, when $n = 1$, $F = A_0(z)$ and $P(z, f) = f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_1 f'$, then Theorem C is a corollary of Theorem 1.1.*

Next, we recall the Jackson difference operator

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, z \in \mathbb{C}\backslash\{0\}, q \in \mathbb{C}\backslash\{0, 1\}.$$

For $k \in \mathbb{N} \cup \{0\}$, the Jackson k-th difference operator is denoted by

$$D_q^0 f(z) := f(z), \quad D_q^k f(z) := D_q(D_q^{k-1} f(z)).$$

Clearly, if f is differentiable,

$$\lim_{q \to 1} D_q^k f(z) = f^{(k)}(z).$$

Therefore, a natural question arises: for Eq (1.1), if we consider the Jackson difference operators of f, does the conclusion $mes(\bigcap_{k\in\mathbb{N}\cup\{0\}} \Delta(D_q^k f(z))) \geq min\{2\pi, \frac{\pi}{\mu(F)}\}$ still hold? Set $R(f) = \bigcap_{k\in\mathbb{N}\cup\{0\}} \Delta(D_q^k f(z))$, where $q \in (0, +\infty)\backslash\{1\}$ and $D_q^k f(z)$ denotes the $k-$th Jackson difference operators of $f(z)$. Our result can be stated as follows.

Theorem 1.2. *Under the hypothesis of Theorem 1.1, we have*

$$mesR(f) \geq min\{2\pi, \frac{\pi}{\mu(F)}\} \tag{1.5}$$

for every non-trivial entire solution $f(z)$ of Eq (1.1).

2. Preliminary lemmas

Before introducing lemmas and completing the proof of Theorems, we recall the Nevanlinna characteristic in an angle, see [8, 11]. Assuming $0 < \alpha < \beta < 2\pi$, $k = \pi/(\beta - \alpha)$, we denote

$$\Omega(\alpha, \beta) = \{z \in \mathbb{C} | \arg z \in (\alpha, \beta)\},$$

$$\Omega(\alpha, \beta, r) = \{z \in \mathbb{C} | z \in \Omega(\alpha, \beta), |z| < r\},$$

$$\Omega(r, \alpha, \beta) = \{z \in \mathbb{C} | z \in \Omega(\alpha, \beta), |z| > r\},$$

and use $\overline{\Omega}(\alpha, \beta)$ to denote the closure of $\Omega(\alpha, \beta)$.

Let $f(z)$ be meromorphic on the angular $\overline{\Omega}(\alpha,\beta)$, we define

$$A_{\alpha,\beta}(r,f) = \frac{k}{\pi}\int_1^r\left(\frac{1}{t^k}-\frac{t^k}{r^{2k}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right| + \log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f) = \frac{2k}{\pi r^k}\int_\alpha^\beta \log^+\left|f\left(re^{i\theta}\right)\right|\sin k(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f) = 2\sum_{1<|b_v|<r}\left(\frac{1}{|b_v|^k}-\frac{|b_v|^k}{r^{2k}}\right)\sin k\left(\beta_v-\alpha\right),$$

where $b_v = |b_v|e^{i\beta_v}(v = 1,2,\cdots)$ are the poles of $f(z)$ in $\overline{\Omega}(\alpha,\beta)$, counting multiplicities. The Nevanlinna angular characteristic function is defined by

$$S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f).$$

Especially, we use $\sigma_{\alpha,\beta}(f) = \limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\log r}$ to denote the order of $S_{\alpha,\beta}(r,f)$.

Lemma 2.1. *[3] If f is a transcendental entire function, then the Fatou set of f has no unbounded multiply connected component.*

Lemma 2.2. *[20] Suppose $f(z)$ is analytic in $\Omega(r_0,\theta_1,\theta_2)$, U is a hyperbolic domain and f : $\Omega(r_0,\theta_1,\theta_2)\to U$. If there exists a point $a\in\partial U\backslash\{\infty\}$ such that $C_U(a)>0$, then there exists a constant $d>0$ such that for sufficiently small $\varepsilon>0$, we have*

$$|f(z)| = O(|z|^d), z\in\Omega(r_0,\theta_1+\varepsilon,\theta_2-\varepsilon), |z|\to\infty.$$

Remark 2.1. *The open set W is called a hyperbolic domain if $\overline{\mathbb{C}}\backslash W$ has greater than two points. For an $a\in\mathbb{C}\backslash W$, we set*

$$\mathbb{C}_W(a) = \inf\{\lambda_W(z)|z-a| : \forall z\in W\},$$

where $\lambda_W(z)$ is the hyperbolic density on W. It is well known that if every component of W is simply connected, then $C_W(a)\geq\frac{1}{2}$.

Before stating the following lemma, we recall the definition of R-set. Suppose that the set $B(z_n,r_n) = \{z\in\mathbb{C} : |z-z_n|<r_n\}$, if $\sum_{n=1}^\infty r_n<\infty, z_n\to\infty$, then we call $\bigcup_{n=1}^\infty B(z_n,r_n)$ a R-set. Obviously, set $\{|z| : z\in\bigcup_{n=1}^\infty B(z_n,r_n)\}$ is set of finite linear measure.

Lemma 2.3. *[10] Let $z = r\exp(i\psi), r_0+1<r$ and $\alpha\leq\psi\leq\beta$, where $0<\beta-\alpha\leq 2\pi$. Suppose that $n(\geq 2)$ is an integer, and that $f(z)$ is analytic in $\Omega(r_0,\alpha,\beta)$ with $\sigma_{\alpha,\beta}<\infty$. Choose $\alpha<\alpha_1<\beta_1<\beta$. Then, for every $\varepsilon\in(0,\frac{\beta_j-\alpha_j}{2})(j=1,2,...,n-1)$ outside a set of linear measure zero with*

$$\alpha_j = \alpha+\sum_{s=1}^{j-1}\varepsilon_s \quad and \quad \beta_j = \beta+\sum_{s=1}^{j-1}\varepsilon_s, (j=2,3,...,n-1)$$

there exist $K>0$ and $M>0$ only depending f, $\varepsilon_1,...,\varepsilon_{n-1}$ and $\Omega(\alpha_{n-1},\beta_{n-1})$, and not depending on z such that

$$\left|\frac{f'(z)}{f(z)}\right|\leq Kr^M(\sin k(\psi-\alpha))^{-2}$$

and

$$\left|\frac{f^{(n)}(z)}{f(z)}\right| \leq Kr^M \left(\sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_j \left(\psi - \alpha_j\right)\right)^{-2}$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ *outside an R-set H, where* $k = \pi/(\beta - \alpha)$ *and* $k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j (j = 1, 2, ..., n-1))$.

Remark 2.2. *Mokhon'ko [12] proved that Lemma 2.2 holds when n=1; Wu [18] proved that the case of* $n = 2$*; and Huang and Wang [10] proved that the case of* $n > 2$.

Lemma 2.4. *[21] Suppose that* $f(z)$ *is a meromorphic function on* $\Omega(\alpha - \varepsilon, \beta + \varepsilon)$ *for* $\varepsilon > 0$ *and* $0 < \alpha < \beta < 2\pi$. *Then*

$$A_{\alpha,\beta}(r, \frac{f'}{f}) + B_{\alpha,\beta}(r, \frac{f'}{f}) \leq K(\log^+ S_{\alpha-\varepsilon,\beta+\varepsilon}(r, f) + \log\log r + 1),$$

for $r > 1$ *possibly except a set with finite linear measure.*

Lemma 2.5. *[1] Let* $f(z)$ *be a transcendental meromorphic function with positive order and finite lower order* μ, *and have one deficient value a. Let* $\Lambda(r)$ *be a positive function with* $\Lambda(r) = o(T(r, f))$ *as* $r \to \infty$. *Then for any fixed sequence of Pólya peaks* $\{r_n\}$ *of order* $\lambda > 0, \mu(f) \leq \lambda \leq \sigma(f)$, *we have*

$$\liminf_{r_n \to \infty} \operatorname{mes} D_\Lambda(r_n, a) \geq \min\left\{2\pi, \frac{4}{\lambda} \arcsin \sqrt{\frac{\delta(a, f)}{2}}\right\},$$

where $D_\Lambda(r_n, a)$ *is defined by*

$$D_\Lambda(r_n, \infty) = \{\theta \in [0, 2\pi) : \log^+ |f(re^{i\theta})| > \Lambda(r)T(r, f)\},$$

and for finite a

$$D_\Lambda(r_n, a) = \{\theta \in [0, 2\pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \Lambda(r)T(r, f)\}.$$

3. Proof of Theorem 1.1

Clearly, every nontrivial entire solution f of Eq (1.1) is transcendental. Suppose on the contary that $\operatorname{mes} E(f) < \sigma := \min\{2\pi, \pi/\mu(F\}$. Then $\xi := \sigma - \operatorname{mes} E(f) > 0$. For every $i \in L$ and $k \in \mathbb{Z}$, $\Delta(f^{(k)}(z + \eta_i))$ is closed, and so $E(f)$ is closed. Denoted by $S := [0, 2\pi)\backslash E(f)$ the complement of $E(f)$. Then S is open and contains at most countably many open intervals. Therefore, we can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)(i = 1, 2, ..., m)$ in S such that

$$\operatorname{mes}(S \setminus \bigcup_{i=1}^{m} I_i) < \frac{\xi}{4}. \tag{3.1}$$

For every $\theta_i \in I_i$, there exist $m_{\theta_i} \in L$ and $k_{\theta_i} \in \mathbb{Z}$ such that $\arg z = \theta_i$ is not a limiting direction of the Julia set of some $f^{(k_{\theta_i})}(z + \eta_{m_{\theta_i}})$, where $m_{\theta_i} \in L$ and $k_{\theta_i} \in \mathbb{Z}$ only depending on θ_i. Then there exists some angular domain $\Omega(\theta_i - \zeta_{\theta_i}, \theta_i + \zeta_{\theta_i})$ such that

$$(\theta_i - \zeta_{\theta_i}, \theta_i + \zeta_{\theta_i}) \subset I_i \quad and \quad \Omega(r, \theta_i - \zeta_{\theta_i}, \theta_i + \zeta_{\theta_i}) \cap \mathcal{J}(f^{(k_{\theta_i})}(z + \eta_{m_{\theta_i}})) = \emptyset \tag{3.2}$$

for sufficiently large r, where $\zeta_{\theta_i} > 0$ is a constant only depending on θ_i. Hence, $\bigcup_{\theta_i \in I_i}(\theta_i - \zeta_{\theta_i}, \theta_i + \zeta_{\theta_i})$ is an open covering of $[\alpha_i + \varepsilon, \beta_i - \varepsilon]$ with $0 < \varepsilon < \min\{(\beta_i - \alpha_i)/6, i = 1, 2, ..., m\}$. By Heine-Borel theorem, we can choose finitely many θ_{ij}, such that

$$[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset \bigcup_{j=1}^{p_i}(\theta_{ij} - \zeta_{\theta_{ij}}, \theta_{ij} + \zeta_{\theta_{ij}}).$$

From (3.2) and Lemma 2.1, there exist a related r_{ij} and an unbounded Fatou component U_{ij} of $\mathcal{F}(f^{(k_{\theta_{ij}})}(z + \eta_{m_{\theta_{ij}}}))$ such that $\Omega(r_{ij}, \theta_{ij} - \zeta_{\theta_{ij}}, \theta_{ij} + \zeta_{\theta_{ij}}) \subset U_{ij}$, see [3]. We take an unbounded and connected closed section Γ_{ij} on boundary ∂U_{ij} such that $\mathbb{C}\backslash\Gamma_{ij}$ is simply connected. Clearly, $\mathbb{C}\backslash\Gamma_{ij}$ is hyperbolic and open. By remark 2.1, there exists a $a \in \mathbb{C}\backslash\Gamma_{ij}$ such that $C_{\mathbb{C}\backslash\Gamma_{ij}}(a) \geq 1/2$. Since the mapping $f^{(k_{\theta_{ij}})}(z + \eta_{m_{\theta_{ij}}}) : \Omega(r_{ij}, \theta_{ij} - \zeta_{\theta_{ij}}, \theta_{ij} + \zeta_{\theta_{ij}}) \to \mathbb{C}\backslash\Gamma_{ij}$ is analytic, it follows from Lemma 2.2 that there exists a positive constant d_1 such that

$$|f^{(k_{\theta_{ij}})}(z + \eta_{m_{\theta_{ij}}})| = O(|z|^{d_1}) \quad as \quad |z| \to \infty \tag{3.3}$$

for $z \in \Omega(r_{ij}, \theta_{ij} - \zeta_{\theta_{ij}} + \varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - \varepsilon)$. Selecting $r^*_{ij} > r_{ij}$ such that $z - \eta_{m_{\theta_{ij}}} \in \Omega(r_{ij}, \theta_{ij} - \zeta_{\theta_{ij}} + \varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - \varepsilon)$, when $z \in \Omega(r^*_{ij}, \theta_{ij} - \zeta_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - 2\varepsilon)$. Thus,

$$|f^{(k_{\theta_{ij}})}(z)| = O(|z - \eta_{m_{\theta_{ij}}}|^{d_1}) = O(|z|^{d_1}) \quad as \quad |z| \to \infty \tag{3.4}$$

holds for $z \in \Omega(r^*_{ij}, \theta_{ij} - \zeta_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - 2\varepsilon)$.

Case 1. Suppose $k_{\theta_{ij}} \geq 0$. By integration, we have

$$|f^{(k_{\theta_{ij}}-1)}(z)| = \int_0^z |f^{(k_{\theta_{ij}})}(\gamma)||d\gamma| + c_{k_{\theta_{ij}}}, \tag{3.5}$$

where $c_{k_{\theta_{ij}}}$ is is a constant, and the integral path is the segment of a straight line from 0 to z. From this and (3.4), we can deduce $|f^{(k_{\theta_{ij}}-1)}(z)| = O(|z|^{d_1+1})$ for $z \in \Omega(r^*_{ij}, \theta_{ij} - \zeta_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - 2\varepsilon)$. Repeating the discussion $k_{\theta_{ij}}$ times, we can obtain

$$|f(z)| = O(|z|^{d_1 + k_{\theta_{ij}}}), \quad z \in \Omega(r^*_{ij}, \theta_{ij} - \zeta_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - 2\varepsilon). \tag{3.6}$$

From the definition of angular characteristic, we have

$$S_{\theta_{ij} - \zeta_{\theta_{ij}} + 2\varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - 2\varepsilon}(r, f) = O(\log r). \tag{3.7}$$

Case 2. Suppose $k_{\theta_{ij}} < 0$. For any angular $\Omega(\alpha, \beta)$, we have

$$S_{\alpha,\beta}(f^{(k_{\theta_{ij}}+1)}) \leq S_{\alpha,\beta}(r, \frac{f^{(k_{\theta_{ij}}+1)}}{f^{(k_{\theta_{ij}})}}) + S_{\alpha,\beta}(r, f^{(k_{\theta_{ij}})}). \tag{3.8}$$

By Lemma 2.4, we obtain

$$S_{\alpha,\beta}(r, \frac{f^{(k_{\theta_{ij}}+1)}}{f^{(k_{\theta_{ij}})}}) \leq K_1(\log^+ S_{\alpha+\epsilon,\beta-\epsilon}(r, f^{(k_{\theta_{ij}})}) + \log r + 1), \tag{3.9}$$

where $\epsilon = \frac{\varepsilon}{|k_{\theta_{ij}}|}$, K_1 is a positive constant. Combining (3.4), (3.8) and (3.9), we easy to have

$$S_{\theta_{ij}-\zeta_{\theta_{ij}}+2\varepsilon+\epsilon,\theta_{ij}+\zeta_{\theta_{ij}}-2\varepsilon-\epsilon}(r, f^{(k_{\theta_{ij}}+1)}) = O(\log r). \tag{3.10}$$

Similar to the above, repeating the discussion $|k_{\theta_{ij}}|$ times, we get

$$S_{\theta_{ij}-\zeta_{\theta_{ij}}+3\varepsilon,\theta_{ij}+\zeta_{\theta_{ij}}-3\varepsilon}(r, f) = O(\log r). \tag{3.11}$$

This means that whatever $k_{\theta_{ij}}$ is positive or not, we always have

$$S_{\theta_{ij}-\zeta_{\theta_{ij}}+3\varepsilon,\theta_{ij}+\zeta_{\theta_{ij}}-3\varepsilon}(r, f) = O(\log r). \tag{3.12}$$

Therefore, $\sigma_{\theta_{ij}-\zeta_{\theta_{ij}}+3\varepsilon,\theta_{ij}+\zeta_{\theta_{ij}}-3\varepsilon} < \infty$. According to Lemma 2.3, there exist two constants $K > 0$ and $N > 0$ such that

$$|\frac{f^{(s)}(z)}{f(z)}| \leq Kr^N, s = 1, 2, \cdots, k. \tag{3.13}$$

for all $z \in \Omega(r^*_{ij}, \theta_{ij} - \zeta_{\theta_{ij}} + 3\varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - 3\varepsilon)$ outside a R-set H. Next, we define

$$\Lambda(r) = \max\{\sqrt{\log r}, \sqrt{T(r, \alpha_j)}\} \sqrt{T(r, F)}. \tag{3.14}$$

Since $T(r, \alpha_j) = S(r, F)$ and F is transcendental, we obtain

$$\Lambda(r) = o(T(r, F)) \quad \textit{and} \quad T(r, \alpha_j) = o(\Lambda(r))(j = 1, 2, \cdots, s).$$

Since F is entire, ∞ is a deficient value of F and $\delta(\infty, F) = 1$. By Lemma 2.5, there exists an increasing and unbounded sequence $\{r_n\}$ such that

$$\operatorname{mes} D_\Lambda(r_n) \geq \sigma - \xi/4, \tag{3.15}$$

where

$$D_\Lambda(r) := D_\Lambda(r, \infty) = \left\{\theta \in [-\pi, \pi) : \log^+ \left|F\left(re^{i\theta}\right)\right| > \Lambda(r)\right\}, \tag{3.16}$$

and all $r_n \notin \{|z| : z \in H\}$. Clearly,

$$\begin{aligned} \operatorname{mes}\left(\left(\bigcup_{i=1}^{m} I_i\right) \cap D_\Lambda(r_n)\right) &= \operatorname{mes}(S \cap D_\Lambda(r_n)) - \operatorname{mes}\left(\left(S \setminus \bigcup_{i=1}^{m} I_i\right) \cap D_\Lambda(r_n)\right) \\ &\geq \operatorname{mes}(D_\Lambda(r_n)) - \operatorname{mes} E(f) - \operatorname{mes}\left(S \setminus \bigcup_{i=1}^{m} I_i\right) \\ &\geq \sigma - \frac{\xi}{4} - \operatorname{mes} E(f) - \frac{\xi}{4} = \frac{\xi}{2}. \end{aligned} \tag{3.17}$$

Let $M_{ij} = (\theta_{ij} - \zeta_{\theta_{ij}} + 3\varepsilon, \theta_{ij} + \zeta_{\theta_{ij}} - 3\varepsilon)$, then

$$\operatorname{mes}\left(\bigcup_{i=1}^{m}\bigcup_{j=1}^{p_i} M_{ij}\right) \geq \operatorname{mes}\left(\bigcup_{i=1}^{m} I_i\right) - (3m + 6\nu)\varepsilon,$$

where $\nu = \sum_{i=1}^{m} p_i$. Choosing ε small enough, we can deduce

$$\operatorname{mes}\left(\left(\bigcup_{i=1}^{m}\bigcup_{j=1}^{p_i} M_{ij}\right)\cap D_{\Lambda}(r_n)\right) \geq \frac{\xi}{4}.$$

Thus, there exists an open interval $M_{i_0j_0}$ of all M_{ij} such that for every k,

$$\operatorname{mes}\left(M_{i_0j_0}\cap D_{\Lambda}(r_n)\right) > \frac{\xi}{4\nu} > 0. \tag{3.18}$$

Let $G = M_{i_0j_0} \cap D_{\Lambda}(r_n)$. Then by (3.16), we have

$$\int_G \log^+ \left|F\left(r_n \mathrm{e}^{\mathrm{i}\theta}\right)\right| \mathrm{d}\theta \geq \frac{\xi}{4\nu}\Lambda(r_n). \tag{3.19}$$

On the other hand, from (1.1), we have

$$|F(z)| = \sum_{j=1}^{s} |\alpha_j(z)(\frac{f'}{f})^{n_{1j}}(\frac{f''}{f})^{n_{2j}}\cdots(\frac{f^{(k)}}{f})^{n_{kj}} f^{n_{0j}+n_{1j}+\cdots+n_{kj}-n}|. \tag{3.20}$$

Since $n_{0j} + n_{1j} + \cdots + n_{kj} - n \geq 0$ and substituting (3.4)–(3.13) into Eq (1.1), we obtain

$$\begin{aligned}\int_G \log^+ \left|F\left(r_n e^{i\theta}\right)\right| d\theta &\leq \int_G \left(\sum_{j=1}^{s}\log^+\left|\alpha_j\left(r_n e^{i\theta}\right)\right|\right) d\theta + O(\log r_n)\\ &\leq \sum_{j=1}^{s} m\left(r_n, \alpha_j\right) + O(\log r_n)\\ &\leq \sum_{j=1}^{s} T\left(r_n, \alpha_j\right) + O(\log r_n).\end{aligned} \tag{3.21}$$

Combining (3.19) and (3.21), it is found that

$$\frac{\xi}{4\nu}\Lambda(r_n) \leq \sum_{j=1}^{s} T\left(r_n, \alpha_j\right) + O(\log r_n), \tag{3.22}$$

which is impossible since $T\left(r, \alpha_j\right) = o(\Lambda(r))\,(j = 1, ..., s)$ as $r \to \infty$. Therefore,

$$mes(E(f)) \geq min\{2\pi, \frac{\pi}{\mu(F)}\}.$$

4. Proof of Theorem 1.2

In the following, we shall obtain the assertion by reduction to contraction. Assuming that $mesR(f) < \tau = min\{2\pi, \frac{\pi}{\mu(F)}\}$, so $\upsilon = \tau - mesR(f) > 0$. Since $\Delta(D_q^k f(z))$ is closed, clearly $S = [0, 2\pi)\backslash R(f)$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)(i = 1, 2, \cdots, s) \subset S$ satisfying

$$mes(S \setminus \bigcup_{i=1}^{s} I_i) < \frac{\upsilon}{4}.$$

For every $\theta_i \in I_i$, $argz = \theta_i$ is not a limiting direction of the Julia set of $D_q^k f(z)$ for some $k \in \mathbb{N} \cup \{0\}$. Then there exists an angular domain $\Omega(\theta_i - \phi_{\theta_i}, \theta_i + \phi_{\theta_i})$ such that

$$(\theta_i - \phi_{\theta_i}, \theta_i + \phi_{\theta_i}) \subset I_i \quad and \quad \Omega(\theta_i - \phi_{\theta_i}, \theta_i + \phi_{\theta_i}) \cap \Delta(D_q^k f(z)) = \emptyset, \tag{4.1}$$

where $\phi_{\theta_i} > 0$ is a constant only depending on θ_i. Take $0 < \varepsilon < min\{(\beta_i - \alpha_i)/6, i = 1, 2, \cdots, s\}$, then $\bigcup_{\theta_i \in I_i}(\theta_i - \phi_{\theta_i}, \theta_i + \phi_{\theta_i})$ is an open covering of $[\alpha_i + \varepsilon, \beta_i - \varepsilon]$. By Heine-Borel theorem, we can choose finitely many θ_{ij}, such that

$$[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset \bigcup_{j=1}^{s_i}(\theta_{ij} - \phi_{\theta_{ij}}, \theta_{ij} + \phi_{\theta_{ij}}).$$

From (4.1) and Lemma 2.1, there exists an unbounded Fatou component U of $\mathcal{F}(D_q^k f(z))$ such that $\Omega(\theta_i - \phi_{\theta_i}, \theta_i + \phi_{\theta_i}) \subset U$. Taking an unbounded connected set $\Gamma \subset \partial U$ and the mapping $D_q^k f(z) : \Omega(\theta_i - \phi_{\theta_i}, \theta_i + \phi_{\theta_i}) \to \mathbb{C} \backslash \Gamma$ is analytic. Since $\mathbb{C} \backslash \Gamma$ is simply connected, then for arbitrary $a \in \Gamma \backslash \{\infty\}$, we have $C_{\mathbb{C} \backslash \Gamma(a)} \geq \frac{1}{2}$. Thus, for sufficiently small $\varepsilon > 0$, there exists a constant $d_2 > 0$ such that

$$|D_q^k f(z)| = O(|z|^{d_2}), \quad z \in \Omega(\alpha_{ij}^*, \beta_{ij}^*), \tag{4.2}$$

where $\alpha_{ij}^* = \theta_{ij} - \phi_{\theta_{ij}} + \varepsilon$ and $\beta_{ij}^* = \theta_{ij} + \phi_{\theta_{ij}} - \varepsilon$.

By the definition of Jackson $k-$th difference operator,

$$|D_q^k f(z)| = \frac{|D_q^{k-1} f(qz) - D_q^{k-1} f(z)|}{|qz - z|} = O(|z|^{d_2}), \quad z \in \Omega(\alpha_{ij}^*, \beta_{ij}^*). \tag{4.3}$$

Therefore,

$$|D_q^{k-1} f(qz) - D_q^{k-1} f(z)| = O(|z|^{d_2+1}), \quad z \in \Omega(\alpha_{ij}^*, \beta_{ij}^*). \tag{4.4}$$

Thus, there exists a positive constants C such that

$$|D_q^{k-1} f(qz) - D_q^{k-1} f(z)| \leq C(|z|^{d_2+1}), \quad z \in \Omega(\alpha_{ij}^*, \beta_{ij}^*). \tag{4.5}$$

Case 1. Suppose $q \in (0, 1)$. If $|z|$ is sufficiently large, there exists a positive integer r such that $(\frac{1}{q})^r \leq |z| \leq (\frac{1}{q})^{r+1}$. Therefore, $1 \leq |q^r z| \leq \frac{1}{q}$. Then there exists a positive constant M_1 such that $|D_q^{k-1} f(q^r z)| \leq M_1$ for all $z \in \{z | 1 \leq |q^r z| \leq \frac{1}{q}\}$. Using inequality (4.5) repeatedly, we have

$$\begin{aligned} |D_q^{k-1} f(z) - D_q^{k-1} f(qz)| &\leq C(|z|^{d_2+1}), \\ |D_q^{k-1} f(qz) - D_q^{k-1} f(q^2 z)| &\leq C(|qz|^{d_2+1}), \\ &\cdots \\ |D_q^{k-1} f(q^{r-1} z) - D_q^{k-1} f(q^r z)| &\leq C(|q^{r-1} z|^{d_2+1}). \end{aligned} \tag{4.6}$$

Taking the sum of all inequalities, we obtain

$$\begin{aligned}|D_q^{k-1}f(z)| &\le |D_q^{k-1}f(z)-D_q^{k-1}f(qz)|+|D_q^{k-1}f(qz)-D_q^{k-1}f(q^2z)|+\cdots\\ &\quad+|D_q^{k-1}f(q^{r-1}z)-D_q^{k-1}f(q^rz)|+|D_q^{k-1}f(q^rz)|\\ &\le C(|z|^{d_2+1})+C(|qz|^{d_2+1})+\cdots+C(|q^{r-1}z|^{d_2+1})+M_1\\ &\le rC(1+q^{d_2+1}+\cdots+q^{(r-1)(d_2+1)})|z|^{d_2+1}+M_1\\ &=O(|z|^{d_2+1}),\quad z\in\Omega(\alpha_{ij}^*,\beta_{ij}^*).\end{aligned}\tag{4.7}$$

Thus,

$$|D_q^{k-1}f(z)|=O(|z|^{d_2+1}),\quad z\in\Omega(\alpha_{ij}^*,\beta_{ij}^*).\tag{4.8}$$

Repeating the operations from (4.2) to (4.8), we get

$$|f(z)|=O(|z|^{d_2+k-1}),\quad z\in\Omega(\alpha_{ij}^*,\beta_{ij}^*).\tag{4.9}$$

Case 2. Suppose $q\in(1,+\infty)$. Obviously, there exists a positive integer t such that $q^t\le|z|\le q^{t+1}$ for sufficiently large $|z|$. And this is exactly $1\le|\frac{z}{q^t}|\le q$. Therefore, there exists a positive constant M_2 such that $|D_q^{k-1}f(\frac{z}{q^t})|\le M_2$ for all $z\in\{z|1\le|\frac{z}{q^t}|\le q\}$.

Using inequality (4.5) repeatedly, we have

$$\begin{aligned}|D_q^{k-1}f(z)-D_q^{k-1}f(\frac{z}{q})| &\le C(|\frac{z}{q}|^{d_2+1}),\\ |D_q^{k-1}f(\frac{z}{q})-D_q^{k-1}f(\frac{z}{q^2})| &\le C(|\frac{z}{q^2}|^{d_2+1}),\\ &\cdots\\ |D_q^{k-1}f(\frac{z}{q^{t-1}})-D_q^{k-1}f(\frac{z}{q^t})| &\le C(|\frac{z}{q^t}|^{d_2+1}).\end{aligned}\tag{4.10}$$

Taking the sum of all inequalities, we obtain

$$\begin{aligned}|D_q^{k-1}f(z)| &\le |D_q^{k-1}f(z)-D_q^{k-1}f(\frac{z}{q})|+|D_q^{k-1}f(\frac{z}{q})-D_q^{k-1}f(\frac{z}{q^2})|+\cdots\\ &\quad+|D_q^{k-1}f(\frac{z}{q^{t-1}})-D_q^{k-1}f(\frac{z}{q^t})|+|D_q^{k-1}f(\frac{z}{q^t})|\\ &\le C(|\frac{z}{q}|^{d_2+1})+C(|\frac{z}{q^2}|^{d_2+1})+\cdots+C(|\frac{z}{q^t}|^{d_2+1})+M_2\\ &\le tC(\frac{1}{q^{d_2+1}}+\frac{1}{q^{2(d_2+1)}}+\cdots+\frac{1}{q^{t(d_2+1)}})|z|^{d_2+1}+M_2\\ &=O(|z|^{d_2+1}),\quad z\in\Omega(\alpha_{ij}^*,\beta_{ij}^*).\end{aligned}\tag{4.11}$$

Therefore,

$$|D_q^{k-1}f(z)|=O(|z|^{d_2+1}),\quad z\in\Omega(\alpha_{ij}^*,\beta_{ij}^*).\tag{4.12}$$

Similarly, we can deduce

$$|f(z)|=O(|z|^{d_2+k-1}),\quad z\in\Omega(\alpha_{ij}^*,\beta_{ij}^*),\tag{4.13}$$

which implies that

$$S_{\alpha_{ij}^*,\beta_{ij}^*}(r,f)=O(\log r).\tag{4.14}$$

By the similar proof in (3.12) to (3.22), we can get a contradiction. Therefore,

$$mesR(f)\ge min\{2\pi,\frac{\pi}{\mu(F)}\}.$$

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Conflict of interest

The authors declare no conflict of interest.

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