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Research article

A strong convergence theorem for solving pseudo-monotone variational inequalities and fixed point problems using subgradient extragradient method in Banach spaces

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Abstract: In this paper, we introduce an algorithm for solving variational inequalities problem with Lipschitz continuous and pseudomonotone mapping in Banach space. We modify the subgradient extragradient method with a new and simple iterative step size, and the strong convergence to a common solution of the variational inequalities and fixed point problems is established without the knowledge of the Lipschitz constant. Finally, a numerical experiment is given in support of our results.

Keywords: variational inequalities; subgradient extragradient method; fixed point problem; Banach space; strong convergence
Mathematics Subject Classification: 65J15, 47C25, 90C33, 90C52

1. Introduction

In 1959, A. Signorini [1] proposed an interesting contact problem which was well known as Signorini Problem. Since then, many researchers have carried on the research to this problem and reformulated as the variational inequality problem (the VI in short) [2]. A key step for the solution of the VI was introduced by Hartman and Stampacchia [3] in 1966, which produce the VI as an important tool in studying optimization theory, engineering mechanics, economics and applied sciences in a unified and general framework (see [4,5]).

Under appropriate conditions, there are two general methods for solving the VI problem: The projection method and the regularized method. Many projection-type algorithms for solving the VI problem can be found in [6–11]. The gradient method is the simplest algorithm in which only one projection on feasible set is performed, but a strongly monotonicity is required to obtain the convergence of the method. To avoid the hypothesis of the strongly monotonicity, Korpelevich [6] proposed a decisive algorithm for solving the variational inequalities in Euclidean space, which was called the extragradient-type method. In 2011, the subgradient extragradient-type method was introduced by Censor et al. [7], which for solving variational inequalities in real Hilbert space. Very recently, Liu [11] proposed an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities.

It is natural to consider the algorithm for solving the variational inequalities in the setting of Banach spaces or Hilbert spaces. Several results were obtained in the case of various iterative algorithms for finding a common element of the fixed points set and the set of solutions of the variational inequality problem in Hilbert spaces or Banach spaces (see [12–24]). Especially, the fixed point technique was introduced by Browder [12] in 1967. Then Liu and Kong [19] provided a algorithm for finding a common element of fixed points set and variational inequality in Banach space. Recently, Ceng [24] introduced two subgradient extragradient methods for solving pseudomonotone variational inequalities and fixed point problems.

Motivated by the works mentioned, in the present paper, we extend subgradient extragradient algorithm proposed by [22] for solving a common solution of variational inequalities and fixed point problems in Banach spaces. It is worth stressing that our algorithm has a simple structure and the convergence of algorithms is not required to know the Lipschitz constant of the mapping.

The paper is organized as follows. In Section 2, we present some preliminaries that will be needed in the sequel. In Section 3, we propose an algorithm and analyze its convergence. Finally, in Section 4 we present a numerical example and comparison.

2. Mathematical preliminaries

Assume that X is a real Banach space with its dual X^*, $\|\cdot\|$ and $\|\cdot\|_*$ denote the norms of X and X^*, respectively, $\langle x^*, x\rangle$ is the duality coupling in $X^* \times X$, and $x_n \longrightarrow x$ ($x_n \rightharpoonup x$) is called a sequence $\{x_n\}$ convergence to x strongly (weakly). Let C be a nonempty closed convex subset of X, and $F : C \longrightarrow X^*$ be a continuous mapping. Consider with the following variational inequality (for short, $VI(F, C)$) which consists in finding a point $x \in C$ such that

$$\langle F(x), y - x\rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

Let S be the solution set of (2.1).

Definition 2.1. A mapping $F : C \longrightarrow X^*$ is said as follows:
$(A1)$ *Monotone*, if $\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in C$;
$(A2)$ *Pseudomonotone*, if $\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq 0, \quad \forall x, y \in C$;
$(A3)$ *Lipschitz-continuous* with constant $L > 0$, if there exists $L > 0$ such that $\| F(x) - F(y) \| \leq L \| x - y \|, \quad \forall x, y \in C$.

Recall that a point $x \in C$ is called fixed point of an operator $T : C \to C$, if $Tx = x$. We shall denote the set of fixed points of T by $F(T)$. It is well known that in a real Hilbert space, x is the solution of the $VI(F, C)$ if and only if x is the solution of the fixed point equation $x = P_C(x - \lambda F(x))$, where λ is an arbitrary positive constant. Therefore, fixed point algorithms can be used to solve $VI(F, C)$. The mapping $T : C \to C$ is called *nonexpansive*, if,

$$\| T(x) - T(y) \| \leq \| x - y \|, \quad \forall x, y \in C.$$

The *normalized duality mapping* J_X (usually write by J) of X into 2^{X^*} is defined by

$$J(x) = \{x^* \in X^* | \langle x, x^* \rangle = \| x^* \|^2 = \| x \|^2\}$$

for all $x \in X$. Let $q \in (0, 2]$. The *generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined (the definitions and properties, see [15]) by

$$J_q(x) = \{j_q(x) \in X^* | \langle j_q(x), x \rangle = \| x \| \| j_q(x) \|, \| j_q(x) \| = \| x \|^{q-1}\}$$

for all $x \in X$. More details, can be found in [25].

Let $U = \{x \in X : \|x\| = 1\}$, and the norm of X is called *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t} \tag{2.2}$$

exists. In this case, the space X is also called *smooth*. It is well known that if X is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection, and furthermore, there exists inverse mapping J^{-1} which coincides with the duality mapping J^* on X^*. X is said to be uniformly smooth if (2.2) converges uniformly for $x, y \in U$. It is strictly convex if $\| \frac{x+y}{2} \| < 1$ for all $x, y \in U$ and $x \neq y$. The modulus δ_X of convexity is defined by

$$\delta_X(\varepsilon) = inf\{1 - \| \frac{x + y}{2} \| \, | x, y \in B_X, \| x - y \| \geq \varepsilon\},$$

for all $\varepsilon \in [0, 2]$, where B_X is the closed unit ball of X. A Banach space X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$. A Banach space X is uniformly convex iff for any two sequences $\{x_n\}$, $\{y_n\} \in X$,

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|x_n + y_n\| = 2, \lim_{n\to\infty} \|x_n - y_n\| = 0$$

hold. Moreover, X is called *2-uniformly convex* if there exists $c > 0$ such that for all $\varepsilon \in [0, 2]$, $\delta_X(\varepsilon) > c\varepsilon^2$. Obviously, every 2-uniformly convex Banach space is uniformly convex.

Alber [25] introduces a functional $V(x^*, y) : X^* \times X \longrightarrow R$ by

$$V(x^*, y) = \| x^* \|_*^2 - 2\langle x^*, y \rangle + \| y \|^2 . \tag{2.3}$$

The operator $P_C : X^* \longrightarrow C \subseteq X$ is called *the generalized projection operator* if it associates to an arbitrary fixed point $x^* \in X^*$, where x^* is the solution to the minimization problem

$$V(x^*, \tilde{x^*}) = \inf_{y \in C} V(x^*, y),$$

and $\tilde{x^*} = P_C x^* \in C \subset X$ is called a *generalized projection* of the point x^*. For more results about P_C refer to [25]. The next lemma can describe the properties of P_C.

Lemma 2.1. [25] Let C be a nonempty closed convex set in X and $x^*, y^* \in X^*$, $\tilde{x^*} = P_C x^*$. Then
(1) $\langle J\tilde{x^*} - x^*, y - \tilde{x^*} \rangle \geq 0, \quad \forall y \in C$;
(2) $V(J\tilde{x^*}, y) \leq V(x^*, y) - V(x^*, \tilde{x^*}), \quad \forall y \in C$;
(3) $V(x^*, z) + 2\langle y^*, J^{-1}x^* - z \rangle \leq V(x^* + y^*, z), \forall z \in X$.

By the definition of V, it is easy to check the following lemma.

Lemma 2.2. For any $x, y, z \in X$ and $\alpha \in (0, 1)$,
(1) $(\| x \| - \| y \|)^2 \leq V(Jx, y) \leq (\| x \| + \| y \|)^2$;
(2) $V(\alpha Jx + (1-\alpha)Jy, z) \leq \alpha V(Jx, z) + (1-\alpha)V(Jy, z)$;
(3) $V(Jx, z) = V(Jx, y) + V(Jy, z) + 2\langle Jz - Jy, y - x\rangle$;
(4) $V(Jx, y) \leq \| x \| \| Jx - Jy \| + \| y \| \| x - y \|$.

In [26], they prove following lemma.

Lemma 2.3. [26] Let X be a real 2-uniformly convex Banach space. Then, there exists $\mu \geq 1$ such that for all $x, y \in X$,

$$\frac{1}{\tau}\|x - y\|^2 \leq \varphi(x, y).$$

The minimum value of the set of all τ is denoted by τ_X(also write by τ) and is called the 2-uniform convexity constant of X.

The following Lemma which will be useful to our subsequent convergence analysis.

Lemma 2.4. [27] Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ which satisfies $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:

$$\tau(n) = max\{k \leq n : a_k < a_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : a_k < a_{k+1}\}$ is nonempty. Then the following hold:
(1) $\tau(n) \leq \tau(n+1) < \cdots$, and $\tau(n) \longrightarrow \infty$;
(2) $a_{\tau(n)} \leq a_{\tau(n)+1}$ and $a_n \leq a_{\tau(n)+1}$.

3. Main results

In this section, we introduce a new subgradient extragradient algorithm for solving pseudomonotone variational inequality and fixed point problems in Banach spaces. At first, let's make the following assumptions.

Assumption 3.1:

(a) X is a real 2-uniformly convex Banach space and C is its nonempty closed convex subset.

(b) $F : X \to X^*$ is pseudomonotone on C, L-Lipschitz continuous on X, and T is a nonexpansive mapping of C into itself such that $S \bigcap F(T) \neq \emptyset$.

(c) The mapping F is sequentially weakly continuous, i.e., for each sequence $\{x_n\} \in C$: if $x_n \rightharpoonup x$, then $F(x_n) \rightharpoonup F(x)$.

Our algorithm has the following forms:

Algorithm 3.1:

(Step 0) Take $\lambda_0 > 0$, $x_0 \in X$, $\mu \in (0, 1)$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=0}^{\infty} \theta_n < \infty$.

(Step 1) Given the current iterate x_n, compute

$$y_n = P_C(Jx_n - \lambda_n F(x_n)).$$

If $x_n = y_n$, and $Tx_n = x_n$, then stop: x_n is a solution. Otherwise, (Step 2) Construct $T_n = \{x \in X | \langle Jx_n - \lambda_n F(x_n) - Jy_n, x - y_n\rangle \leq 0\}$ and compute

$$z_n = P_{T_n}(Jx_n - \lambda_n F(y_n)), \ \ t_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n),$$

$$x_{n+1} = J^{-1}(\beta_n Jz_n + (1-\beta_n)J(Tt_n)).$$

(Step 3) Compute

$$\lambda_{n+1} = \begin{cases} min\{\frac{\mu(\|x_n-y_n\|^2+\|z_n-y_n\|^2)}{2\langle F(x_n)-F(y_n), z_n-y_n\rangle}, \lambda_n + \theta_n\}, & if \ \ \langle F(x_n) - F(y_n), z_n - y_n\rangle > 0, \\ \lambda_n + \theta_n, & otherwise. \end{cases}$$

Set $n := n + 1$ and return to step 1.

We prove the strong convergence theorem for Algorithm 3.1. Firstly, we give the following lemma, which plays a crucial role in the proof of the main theorem.

Lemma 3.1. Assume that x_n, y_n, λ_n are the sequences generated by Algorithm 3.1 and Assumption 3.1 holds, then

(1) If $x_n = y_n$ and $Tx_n = x_n$, for some $n \in N$, then $x_n \in S \bigcap F(T)$;

(2) $\lim_{n\to\infty} \lambda_n = \lambda \in [min\{\frac{\mu}{L}, \lambda_0\}, \lambda_0 + \theta]$, where $\theta = \sum_{n=0}^{\infty} \theta_n$.

Proof. (1) If $x_n = y_n$, by Algorithm 3.1, we have $x_n = P_C(Jx_n - \lambda_n F(x_n))$, and thus $x_n \in C$. By the definition of P_C, we have

$$\langle Jx_n - \lambda_n F(x_n) - Jx_n, x_n - x\rangle \geq 0 \quad \forall x \in C.$$

Therefore,

$$\langle -\lambda_n F(x_n), x_n - x\rangle = \lambda_n \langle F(x_n), x - x_n\rangle \geq 0 \quad \forall x \in C.$$

Since $\lambda_n > 0$, we have $x_n \in S$. Combining $Tx_n = x_n$, we obtain $x_n \in S \bigcap F(T)$.

(2) Since F is a Lipschitz-continuous mapping with positive constant L, in the case of $\langle F(x_n) - F(y_n), z_n - y_n\rangle > 0$, we get

$$\frac{\mu(\|x_n - y_n\|^2 + \|z_n - y_n\|^2)}{2\langle F(x_n) - F(y_n), z_n - y_n\rangle} \geq \frac{2\mu\|x_n - y_n\|\|z_n - y_n\|}{2\|F(x_n) - F(y_n)\|\|z_n - y_n\|} \geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}.$$

Thus, $\{\lambda_n\}$ has the upper bound $\lambda_0 + \theta$, and lower bound $min\{\frac{\mu}{L}, \lambda_0\}$. Similar to the proof of Lemma 3.1 in [21], we have that

$$\lim_{n\to\infty} \lambda_n = \lambda \in [min\{\frac{\mu}{L}, \lambda_0\}, \lambda_0 + \theta].$$

The proof is complete. □

Theorem 3.1. Assume that Assumption 3.1 holds, the sequence $\{\alpha_n\}$ satisfies $\{\alpha_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\beta_n \in (0,1)$. Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then $\{x_n\}$ strongly converges to a solution $x^* = P_{S \bigcap F(T)} Jx_0$.

Proof. We divide the proof into two steps.

Step 1. The sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{t_n\}$ generated by Algorithm 3.1 are bounded.

To observe this, take $u \in S \bigcap F(T)$. Noting that $y_n \in C$, we have $\langle F(u), y_n - u\rangle \geq 0$, for all $n \in \mathbb{N}$. Since F is pseudomonotone, we have $\langle F(y_n), y_n - u\rangle \geq 0, \quad \forall n \in \mathbb{N}$. Then,

$$0 \leq \langle F(y_n), y_n - u + z_n - z_n\rangle = \langle F(y_n), y_n - z_n\rangle - \langle F(y_n), u - z_n\rangle.$$

This implies that

$$\langle F(y_n), y_n - z_n\rangle \geq \langle F(y_n), u - z_n\rangle, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

By the definition of T_n, we know $\langle Jx_n - \lambda_n F(x_n) - Jy_n, z_n - y_n\rangle \leq 0$. Then

$$\begin{aligned}
&\langle Jx_n - \lambda_n F(y_n) - Jy_n, z_n - y_n\rangle \\
= &\langle Jx_n - \lambda_n F(x_n) - Jy_n, z_n - y_n\rangle + \lambda_n\langle F(x_n) - F(y_n), z_n - y_n\rangle \\
\leq &\lambda_n\langle F(x_n) - F(y_n), z_n - y_n\rangle.
\end{aligned} \tag{3.2}$$

By Lemma 2.1(2), the definition of λ_{n+1} and combining (3.1), (3.2), we obtain

$$\begin{aligned}
V(Jz_n, u) &= V(JP_{T_n}(Jx_n - \lambda_n F(y_n)), u) \\
&\leq V(Jx_n - \lambda_n F(y_n), u) - V(Jx_n - \lambda_n F(y_n), z_n) \\
&= \|Jx_n - \lambda_n F(y_n)\|^2 - 2\langle Jx_n - \lambda_n F(y_n), u\rangle + \|u\|^2 \\
&- \|Jx_n - \lambda_n F(y_n)\|^2 + 2\langle Jx_n - \lambda_n F(y_n), z_n\rangle - \|z_n\|^2 \\
&= -2\langle Jx_n, u\rangle + 2\lambda_n\langle F(y_n), u - z_n\rangle + 2\langle Jx_n, z_n\rangle + \|u\|^2 - \|z_n\|^2 \\
&= V(Jx_n, u) - V(Jx_n, z_n) + 2\lambda_n\langle F(y_n), u - z_n\rangle \\
&\leq V(Jx_n, u) - V(Jx_n, z_n) + 2\lambda_n\langle F(y_n), y_n - z_n\rangle \\
&= V(Jx_n, u) - V(Jx_n, y_n) - V(Jy_n, z_n) + 2\langle Jx_n - \lambda_n F(y_n) - Jy_n, z_n - y_n\rangle \\
&\leq V(Jx_n, u) - V(Jx_n, y_n) - V(Jy_n, z_n) + 2\lambda_n\langle F(x_n) - F(y_n), z_n - y_n\rangle \\
&\leq V(Jx_n, u) - V(Jx_n, y_n) - V(Jy_n, z_n) + \lambda_n\frac{\mu}{\lambda_{n+1}}(\|x_n - y_n\|^2 + \|z_n - y_n\|^2).
\end{aligned} \tag{3.3}$$

From Lemma 3.1(2), we obtain $\lim_{n\to\infty} \lambda_n \frac{\mu}{\lambda_{n+1}} = \mu (0 < \mu < 1)$. It means that there exists a positive integer number N_0, such that for all $n > N_0$, $0 < \lambda_n \frac{\mu}{\lambda_{n+1}} < 1$. Combining Lemma 2.3 and (3.3), we know that there exits a 2-uniformly convex constant τ, such that when $n > N_0$,

$$\begin{aligned}
V(Jz_n, u) &\leq V(Jx_n, u) - V(Jx_n, y_n) - V(Jy_n, z_n) + \lambda_n\tfrac{\mu}{\lambda_{n+1}}(\|x_n - y_n\|^2 + \|z_n - y_n\|^2) \\
&\leq V(Jx_n, u) - (1 - \mu\tau)(V(Jx_n, y_n) + V(Jy_n, z_n)) \\
&\leq V(Jx_n, u).
\end{aligned}$$

Then, by Lemma 2.2(2) and the definition of x_{n+1}, we obtain for every $n > N_0$,

$$\begin{aligned}
V(Jx_{n+1}, u) &= V(\beta_n Jz_n + (1 - \beta_n)J(Tt_n), u) \\
&= \|\beta_n Jz_n + (1 - \beta_n)J(Tt_n)\|^2 - 2\langle\beta_n Jz_n + (1 - \beta_n)J(Tt_n), u\rangle + \|u\|^2 \\
&\leq \beta_n\|Jz_n\|^2 - 2\beta_n\langle Jz_n, u\rangle + \beta_n\|u\|^2 \\
&\quad +(1 - \beta_n)\|J(Tt_n)\|^2 - 2(1 - \beta_n)\langle J(Tt_n), u\rangle + (1 - \beta_n)\|u\|^2 \\
&= \beta_n V(Jz_n, u) + (1 - \beta_n)V(J(Tt_n), u) \\
&\leq \beta_n V(Jz_n, u) + (1 - \beta_n)V(Jt_n, u) \\
&= \beta_n V(Jz_n, u) + (1 - \beta_n)V(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n, u) \\
&\leq \beta_n V(Jz_n, u) + (1 - \beta_n)(\alpha_n V(Jx_0, u) + (1 - \alpha_n)V(Jz_n, u)) \\
&= (\beta_n + (1 - \beta_n)(1 - \alpha_n))V(Jz_n, u) + (1 - \beta_n)\alpha_n V(Jx_0, u) \\
&\leq (\beta_n + (1 - \beta_n)(1 - \alpha_n))V(Jx_n, u) + (1 - \beta_n)\alpha_n V(Jx_0, u) \\
&= (1 - (1 - \beta_n)\alpha_n)V(Jx_n, u) + (1 - \beta_n)\alpha_n V(Jx_0, u) \\
&\leq max\{V(Jx_0, u), V(Jx_n, u)\} \\
&\leq \cdots \leq max\{V(Jx_0, u), V(Jx_{N_0}, u)\}.
\end{aligned}$$

Thus, $\{V(Jx_n, u)\}$ is bounded. Combining $V(Jx_n, u) \geq \frac{1}{\tau}\|x_n - u\|^2$, we get $\{x_n\}$ is bounded. Furthermore, from (3.3), we have the fact that $\{y_n\}$, $\{z_n\}$, and $\{t_n\}$ are bounded.

Step 2. $\{x_n\}$ strongly converges to a point $x^* = P_{S \bigcap F(T)} Jx_0$.

Let $x^* = P_{S \bigcap F(T)} Jx_0$. From Lemma 2.1(1), we can obtain

$$\langle Jx_0 - Jx^*, z - x^* \rangle \leq 0,\ \forall z \in S \bigcap F(T).$$

From Step 1, we know that there exists $\exists N_0 \geq 0$, such that $\forall n \geq N_0$, $V(Jz_n, x^*) \leq V(Jx_n, x^*)$, and the sequences $\{x_n\}$, $\{y_n\}$ $\{z_n\}$ and $\{t_n\}$ are bounded. Moreover, by Lemma 2.1(3) and Lemma 2.2, exists $N_0 \geq 0$, such that for every $n \geq N_0$,

$$\begin{aligned}
&V(Jx_{n+1}, x^*) = V(\beta_n Jz_n + (1 - \beta_n)J(Tt_n), x^*) \\
&\leq \beta_n V(Jz_n, x^*) + (1 - \beta_n)V(J(Tt_n), x^*) \\
&\leq \beta_n V(Jz_n, x^*) + (1 - \beta_n)V(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n, x^*) \\
&\leq \beta_n V(Jz_n, x^*) + (1 - \beta_n)(2\alpha_n \langle Jx_0 - Jx^*, t_n - x^* \rangle + (1 - \alpha_n)V(Jz_n, x^*)) \\
&= (\beta_n + (1 - \beta_n)(1 - \alpha_n))V(Jz_n, x^*) + 2(1 - \beta_n)\alpha_n \langle Jx_0 - Jx^*, t_n - x^* \rangle.
\end{aligned} \tag{3.4}$$

By (3.3), (3.4) and Lemma 2.3, Lemma 3.1(2), we can obtain that for every $n \geq N_0$,

$$\begin{aligned}
&V(Jx_{n+1}, x^*) \leq (\beta_n + (1 - \beta_n)(1 - \alpha_n))V(Jz_n, x^*) + 2(1 - \beta_n)\alpha_n \langle Jx_0 - Jx^*, t_n - x^* \rangle \\
&\leq (\beta_n + (1 - \beta_n)(1 - \alpha_n))(V(Jx_n, x^*) - (1 - \lambda_n \frac{\mu}{\lambda_{n+1}})(V(Jx_n, y_n) + V(Jy_n, z_n))) \\
&+ 2(1 - \beta_n)\alpha_n \langle Jx_0 - Jx^*, t_n - x^* \rangle \\
&\leq V(Jx_n, x^*) - (1 - \mu\tau)(V(Jx_n, y_n) + V(Jy_n, z_n)) + 2(1 - \beta_n)\alpha_n \langle Jx_0 - Jx^*, t_n - x^* \rangle.
\end{aligned} \tag{3.5}$$

Two cases arise:

Case 1. From the result of Lemma 2.5 in [28], set $a_n = \varphi(x_n, x^*) = V(Jx_n, x^*)$. By the proof of Step 1, there exists $N_1 \in \mathbb{N}(N_1 \geq N_0)$, such that $\{\varphi(x_n, x^*)\}_{n=N_1}^{\infty}$ is nonincreasing sequence. Then $\{a_n\}_{n=1}^{\infty}$ converges. By using this in (3.5), when $n > N_1 \geq N_0$, we have

$$(1 - \mu\tau)(V(Jx_n, y_n) + \varphi(y_n, z_n)) \leq V(Jx_n, x^*) - V(Jx_{n+1}, x^*) + 2(1 - \beta_n)\alpha_n \langle Jx_0 - Jx^*, t_n - x^* \rangle.$$

By $V(Jx_0 - Jx_n, x^*)$ is bounded and $\{a_n\}_{n=1}^{\infty}$ converges, we have that when $n \longrightarrow \infty$,

$$V(Jx_n, x^*) - V(Jx_{n+1}, x^*) + 2(1 - \beta_n)\alpha_n \langle Jx_0 - Jx^*, t_n - x^* \rangle \longrightarrow 0.$$

Combining $\varphi(x_n, y_n) \geq 0$ and $0 < \mu, \alpha_n < 1$, we have that when $n \longrightarrow \infty$,

$$\|x_n - y_n\|^2 \longrightarrow 0 \ \text{ and } \ \|y_n - z_n\|^2 \longrightarrow 0. \tag{3.6}$$

Thus, when $n \longrightarrow \infty$,

$$\begin{aligned}
&\|Jx_{n+1} - Jz_n\| = \|\beta_n Jz_n + (1-\beta_n)J(Tt_n) - Jz_n\| \\
&= (1-\beta_n)\|J(Tt_n) - Jz_n\| = (1-\beta_n)(\|J(Tt_n) - Jt_n\|\|Jt_n - Jz_n\| \\
&\leq (1-\beta_n)\|Jt_n - Jz_n\| = (1-\beta_n)\alpha_n\|Jx_0 - Jz_n\| \leq (1-\beta_n)\alpha_n M_1 \longrightarrow 0,
\end{aligned} \tag{3.7}$$

for some $M_1 > 0$. By (3.7), we also can see that $\|J(Tt_n) - Jz_n\| \longrightarrow 0, \|Jt_n - Jz_n\| \longrightarrow 0$. From $\|J(Tt_n) - Jt_n\| \leq \|J(Tt_n) - Jz_n\| + \|J(z_n) - Jt_n\|$, we have $\|J(Tt_n) - Jt_n\| \longrightarrow 0$.

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subset of X^*, we have $\|x_{n+1} - z_n\| \longrightarrow 0$. Therefore, we get that when $n \longrightarrow \infty$,

$$\|Tt_n - t_n\| \longrightarrow 0. \tag{3.8}$$

Thus, when $n \longrightarrow \infty$,

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - y_n\| + \|y_n - x_n\| \longrightarrow 0,$$

and

$$\|x_n - t_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| + \|z_n - t_n\| \longrightarrow 0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_k}\}$ that converges weakly to some $z_0 \in X$, such that $x_{n_k} \rightharpoonup z_0$. By (3.9), we also have $\{t_{n_k}\}$ converges weakly to z_0. It follows from (3.8) and the definition of the nonexpansive mapping T that $z_0 \in F(T)$.

Now, we show that $z_0 \in S$.

Since $\{x_{n_k}\}$ converges weakly to z_0, we have

$$\limsup_{n\to\infty}\langle Jx_0 - Jx^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle Jx_0 - Jx^*, x_{n_k} - x^*\rangle = \langle Jx_0 - Jx^*, z_0 - x^*\rangle. \tag{3.10}$$

Since $\|x_n - y_n\|^2 \longrightarrow 0$, we know that $y_{n_k} \rightharpoonup z_0$ and $z_0 \in C$. Since $y_{n_k} = P_C(x_{n_k} - \lambda_{n_k}F(x_{n_k}))$, by Lemma 2.1(1), we have that for all $z \in C$, $\langle Jx_{n_k} - \lambda_{n_k}F(x_{n_k}) - Jy_{n_k}, z - y_{n_k}\rangle \leq 0$. This implies that

$$\langle Jx_{n_k} - Jy_{n_k}, z - y_{n_k}\rangle \leq \lambda_{n_k}\langle F(x_{n_k}), z - y_{n_k}\rangle.$$

Therefore, we have that for all $z \in C$,

$$\frac{1}{\lambda_{n_k}}\langle Jx_{n_k} - Jy_{n_k}, z - y_{n_k}\rangle + \langle F(x_{n_k}), y_{n_k} - x_{n_k}\rangle \leq \langle F(x_{n_k}), z - x_{n_k}\rangle.$$

Fixing $z \in C$, according to (3.6), and considering that $\{x_{n_k}\}$ is bounded, we can obtain

$$\liminf_{k\to\infty}\langle F(x_{n_k}), z - x_{n_k}\rangle \geq 0.$$

Choose a decreasing nonnegative sequence $\{\varepsilon_k\}$, such that $\lim_{k\to\infty}\varepsilon_k = 0$. By definition of the lower limit, for each ε_k, there exists a smallest positive integer M_k such that for all $k \geq M_k$,

$$\langle F(x_{n_k}), z - x_{n_k}\rangle + \varepsilon_k \geq 0. \tag{3.11}$$

Clearly, as $\{\varepsilon_k\}$ is decreasing, $\{M_k\}$ is increasing.

If there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$, such that for every i, $F(x_{n_{k_i}}) = 0$, then

$$\langle F(z_0, z - z_0\rangle = \lim_{i\to\infty}\langle F(x_{n_{k_i}}), z - x_{n_{k_i}}\rangle = 0.$$

It means $z_0 \in S$.

If there exists a positive integer $N_2 \in \mathbb{N}$ such that for all positive integer $n_{k_i} \geq N_2$, $F(x_{n_{k_i}}) \neq 0$. Let $u_{n_{k_i}} = F(x_{n_{k_i}})/\|F(x_{n_{k_i}})\|^2$. Then for each positive integer $n_{k_i} \geq N_2$, $\langle F(x_{n_{k_i}}), u_{n_{k_i}}\rangle = 1$. Thus, from (3.11), we have that for all positive integer $n_{k_i} \geq N_2$,

$$\langle F(x_{n_{k_i}}), z + \varepsilon_k u_{n_{k_i}} - x_{n_{k_i}}\rangle \geq 0. \tag{3.12}$$

Since F is pseudomonotone, then we have from (3.12) that $\langle F(z + \varepsilon_k u_{n_{k_i}}), z + \varepsilon_k u_{n_{k_i}} - x_{n_{k_i}}\rangle \geq 0$. This implies that

$$\langle F(z), z - x_{n_{k_i}}\rangle \geq \langle F(z) - F(z + \varepsilon_k u_{n_{k_i}}), z + \varepsilon_k u_{n_{k_i}} - x_{n_{k_i}}\rangle - \varepsilon_k\langle F(z), u_{n_{k_i}}\rangle. \tag{3.13}$$

Since $\{x_{n_k}\}$ converges weakly to $z_0 \in C$, and F is sequentially weakly continuous on C, we get $F(x_{n_k})$ converges weakly to $F(z_0)$. If $F(z_0) = 0$, then $z_0 \in S$. Now, assume that $F(z_0) \neq 0$. Combining $\|F(z_0)\| \leq \liminf_{k\to\infty} \|F(x_{n_k})\|$ and $\lim_{k\to\infty} \varepsilon_k = 0$, we get that the right-hand of (3.13) tends to zero. Thus, we obtain that for all $z \in C$,

$$\langle F(z), z - z_0\rangle = \lim_{k\to\infty}\langle F(z), z - x_{n_{k_i}}\rangle \geq 0.$$

By the result of Lemma 3.1 in [29], we also have $z_0 \in S$. combine Lemma 2.1(1) and (3.10), we can obtain,

$$\limsup_{n\to\infty}\langle Jx_0 - Jx^*, x_n - x^*\rangle \leq 0.$$

By Lemma 2.1(3) and (3.6), we have that for all positive integer $n > max\{N_1, N_2\}$,

$$\begin{aligned} V(Jx_{n+1}, x^*) &\leq (\beta_n + (1-\beta_n)(1-\alpha_n))V(Jx_n, x^*) + 2(1-\beta_n)\alpha_n\langle Jx_0 - Jx^*, t_n - x^*\rangle \\ &= (1 - (1-\beta_n)\alpha_n)V(Jx_n, x^*) + 2(1-\beta_n)\alpha_n\langle Jx_0 - Jx^*, t_n - x^*\rangle. \end{aligned}$$

It follows from the result of [14, Lemma 3.3] and [28, Lemma 2.5], we obtain $\lim_{n\to\infty} \varphi(x_n, x^*) = 0$, that means

$$\lim_{n\to\infty} x_n = x^*.$$

Case 2. Assume that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that for all $j \in \mathbb{N}$, $\varphi(x_{m_j}, x^*) < \varphi(x_{m_j+1}, x^*)$. Then, from Lemma 2.4, we know that there exists a nondecreasing sequence $m_k \in \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and for every $k \in \mathbb{N}$:

$$\varphi(x_{m_k}, x^*) < \varphi(x_{m_k+1}, x^*) \text{ and } \varphi(x_k, x^*) < \varphi(x_{m_k+1}, x^*). \tag{3.14}$$

By (3.5) and Lemma 3.1(2), we have

$$
\begin{aligned}
&(1-\mu\tau)(\varphi(x_{m_k},y_{m_k})+\varphi(y_{m_k},z_{m_k}))\\
&\le V(Jx_{m_k},x^*)-V(Jx_{m_k+1},x^*)+2(1-\beta_n)\alpha_n\langle Jx_0-Jx^*,t_n-x^*\rangle.
\end{aligned} \tag{3.15}
$$

Since $\{x_n\}$ is bounded, then exists a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ which converges weakly to $z_0 \in X$. Use the same argument as shown in Case 1, and Combining (3.15) we obtain

$$\lim_{k\to\infty}\|x_{m_k}-y_{m_k}\|=0,\ \ \lim_{k\to\infty}\|z_{m_k}-y_{m_k}\|=0,\ \ \lim_{k\to\infty}\|x_{m_k+1}-x_{m_k}\|=0.$$

Similarly we can obtain

$$\limsup_{k\to\infty}\langle Jx_0-Jx^*,t_{m_k+1}-x^*\rangle=\limsup_{k\to\infty}\langle Jx_0-Jx^*,t_{m_k}-x^*\rangle\le 0.$$

It follows from (3.15) and the proof of case 1, for all $m_k \ge N_0$, we have

$$
\begin{aligned}
\varphi(x_{m_k+1},x^*) &\le (1-\alpha_{m_k}(1-\beta_{m_k}))\varphi(x_{m_k},x^*)+2(1-\beta_{m_k})\alpha_{m_k}\langle Jx_0-Jx^*,t_{m_k}-x^*\rangle\\
&\le (1-\alpha_{m_k}(1-\beta_{m_k}))\varphi(x_{m_k+1},x^*)+2(1-\beta_{m_k})\alpha_{m_k}\langle Jx_0-Jx^*,t_{m_k}-x^*\rangle.
\end{aligned}
$$

Since $0<\alpha_n,\beta_n<1$, this implies that $\forall\ m_k \ge N_1$, we have

$$\varphi(x_{m_k},x^*)\le\varphi(x_{m_k+1},x^*)\le 2\langle Jx_0-Jx^*,t_{m_k}-x^*\rangle.$$

And then

$$\limsup_{k\to\infty}\varphi(x_{m_k},x^*)\le\limsup_{k\to\infty}2\langle Jx_0-Jx^*,x_{m_k+1}-x^*\rangle\le 0,$$

we obtain $\limsup_{k\to\infty}\varphi(x_{m_k},x^*)=0$, that means $\lim_{k\to\infty}\|x_{m_k}-x^*\|^2=0$. Since $\|x_k-x^*\|\le\|x_{m_k+1}-x^*\|$, we have $\lim_{k\to\infty}\|x_k-x^*\|=0$. Therefore $x_k\to x^*$. The proof is complete. □

4. Numerical experiments

In this section, we give a numerical experiment to demonstrate the convergence and efficiency of the proposed algorithm. We will compare Algorithm 3.1 with a strongly convergent algorithms as HSEGM proposed in [30, Theorem 4.2].

Example 4.1 Let $H=L^2([0,1])$. We apply our problem in H with norm $\|x\|=(\int_0^1|x(t)|^2dt)^{\frac{1}{2}}$ and inner product $\langle x,y\rangle=\int_0^1 x(t)y(t)dt$, $x,y\in H$. The operator $F:H\to H$ is of form

$$Fx(t)=\max(0,x(t)),\ t\in[0,1],$$

for all $x\in H$. Clearly, F is Lipschitz-continuous and monotone (so is also pseudomonotone). The feasible set is $C=\{x\in H:\|x\|\le 1\}$. Observe that $0\in S$ and so $S\neq\emptyset$. Let $T:H\to H$ is defined by

$$Tx(t)=\int_0^1 tx(s)ds,\ t\in[0,1].$$

Clearly, $0 \in F(T)$ and so $F(T) \neq \emptyset$. Since

$$|Tx(t) - Ty(t)|^2 = |\int_0^1 t(x(s) - y(s))ds|^2 \leq \int_0^1 |(x(s) - y(s))|^2 ds = \|x - y\|^2.$$

This means that T is nonexpansive and therefore

$$\|Tx - Ty\|^2 = \int_0^1 |Tx(t) - Ty(t))|^2 dt \leq \|x - y\|^2.$$

Hence, the solution of the problem is $x^* = 0$. To terminate the algorithms, we use the condition $\|x_n - x^*\| \leq \varepsilon$ and $\varepsilon = 10^{-3}$ for all the algorithms. We take $\alpha_n = \frac{1}{100n}$, $\beta_n = \frac{1}{2n+1}$, $\theta_n = 0$, $\lambda_0 = 0.7$ and $\mu = 0.9$ for Algorithm 3.1. For Theorem 4.2 in [30], we take $\alpha_n = \frac{1}{100n}$, $\beta_n = \frac{1}{2n+1}$ and $\tau = 0.7$. The numerical results are showed in Table 1 and Figure 1.

Table 1. Comparison between the Algorithm 3.1 and Theorem 4.2 in [30].

	x_0	Algorithm A		Theorem 4.2 in [30]	
		iter.	time	iter.	time
case 1	$\frac{1}{4}t^2 e^{-4t}$	4	8.58	4	8.27
case 2	$\frac{1}{120}(1 - t^2)$	4	8.34	4	8.06
case 3	$\frac{1}{100} sin(t)$	4	8.34	4	8.14

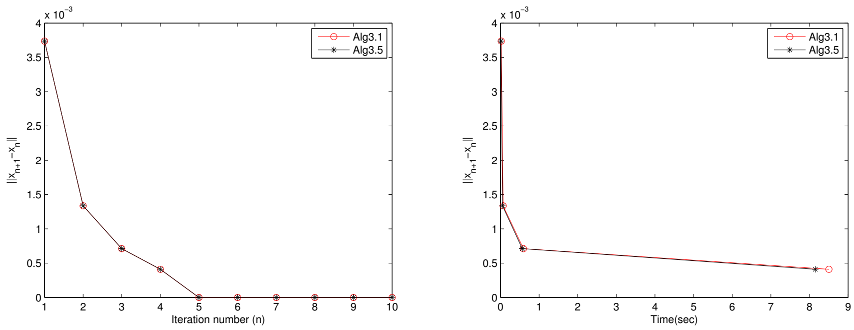

Figure 1. Comparison between the Algorithm 3.1 and Theorem 4.2 in [30] with case 2.

Remark 4.1. Observing from the numerical results of the example presented above, the conclusion that our algorithm is consistent, stable, effective and easy to implement is obtained. This example shows that Algorithm 3.1 converges slightly slower than that of Theorem 4.2 in [30], but it has some advantages. First, it is done without any information of the Lipschitz constant of the cost operator F. Second, the step size is variable in Algorithm 3.1. Third, it can be applied to solve fixed point problem for non-expansive mappings in a real 2-uniformly convex Banach space, and furthermore, it is more useful in the infinite-dimensional spaces.

5. Conclusions

In this paper, we consider a strong convergence result for solving a common solution of pseudomonotone Variational Inequalities and fixed point problems in convex Banach spaces. Our algorithm is based on the subgradient extragradient methods with a new step size, the convergence of algorithm is established without the knowledge of the Lipschitz constant of the mapping. Finally, some numerical experiments are given to illustrate the convergence of our algorithms and compared with other known methods.

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Conflict of interest

The authors declare no conflicts of interest.

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