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Research article

Monotonicity, convexity properties and inequalities involving Gaussian hypergeometric functions with applications

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Abstract: In this paper, we mainly prove monotonicity and convexity properties of certain functions involving zero-balanced Gaussian hypergeometric function $F(a, b; a+b; x)$. We generalize conclusions of elliptic integral to Gaussian hypergeometric function, and get some accurate inequalities about Gaussian hypergeometric function.

Keywords: Gaussian hypergeometric function; monotonicity; convexity
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1. Introduction

For $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by [1, 3]

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} x^n, x \in (-1, 1), \tag{1.1}$$

where (a, n) is the shifted fractional function, namely $(a, n) = a(a + 1)(a + 2) \ldots (a + n - 1)$ for $n = 1, 2, 3, \ldots,$ and $(a, 0) = 1$ for $a \neq 0$. The function $F(a, b; c; x)$ is called zero-balanced when $c = a + b$.

Firstly, we introduce the following properties of the function $F(a, b; c; x)$ at $x = 1$:

(1) For $a + b < c$ (See [13]),

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{1.2}$$

(2) For $a + b = c$ (See [1, 6]),

$$B(a, b)F(a, b; c; x) + \log(1 - x) = R(a, b) + O((1 - x) \log(1 - x)), x \to 1. \tag{1.3}$$

The above asymptotical formula was raised by Ramanujan, where

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

is the classical beta function and

$$R(a,b) = -\psi(a) - \psi(b) - 2\gamma,$$

$\psi(z) = \Gamma^{'}(z)/\Gamma(z)$, $Re(z) > 0$ is the digamma function and γ is called Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577156649\ldots.$$

For $a + b < c$, we have (See [2, Theorem 1.19(10)]),

$$F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x). \tag{1.4}$$

From the uniform convergence of the termwise first derivative of (1, 1) in [1], it follows that

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x), x \in (-1,1). \tag{1.5}$$

It is well known that $F(a,b;c;x)$ has been widely applied in many fields of mathematics and physics. Many special functions in mathematical physics and even some common elementary functions are particular or limiting cases of $F(a,b;c;x)$ in [1,10]. For example, Legendre's complete elliptic integrals of the first kind, are defined by

$$\mathscr{K}(r) = \int_0^{\pi/2}\frac{1}{(1-r^2sin^2\theta)^{1/2}}d\theta = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right),$$

$$\mathscr{K}^{'}(r) = \mathscr{K}(r^{'}).$$

Here and hereafter we always let $r^{'} = \sqrt{1-r^2}$ for $r \in (0,1)$.

For $a \in (0,1)$, we can use zero-balanced Gaussian hypergeometric function to define generalized elliptic integrals $\mathscr{K}_a(r)$ and $\mathscr{K}_a^{'}(r)$ of the first kind on (0, 1) as follow [7]:

$$\mathscr{K}_a(r) = \frac{\pi}{2}F(a,1-a;1;r^2), \tag{1.6}$$

$$\mathscr{K}_a^{'}(r) = \mathscr{K}_a(r^{'}). \tag{1.7}$$

If $a = 1/2$, then we get complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{K}^{'}(r)$.

During the past decades, many properties are revealed for $F(a,b;c;x)$ (See [5,17,18,21,22,28,30]), $\mathscr{K}(r)$ and $\mathscr{K}_a(r)$ (See [4, 8, 9, 15, 16, 19, 20, 23–27, 29]) by showing the monotonicity and cancavity properties of certain combinations defined in terms of these special functions and other elementary functions. From these analytic properties, we can get some inequalities of $F(a,b;c;x)$ and $\mathscr{K}_a(r)$.

For $r \in (0,1)$, one kind of known elegant functional inequalities for $\mathscr{K}_a(r)$ are of the following form

$$\frac{sin(\pi a)}{c_1 + (1-c_1)r^2} < \frac{\mathscr{K}_a(r)}{\log[e^{R(a)/2}/r^{'}]} < \frac{sin(\pi a)}{c_2 + (1-c_2)r^2}, \tag{1.8}$$

where $R(a) = R(a, 1-a) = -2\gamma - \psi(a) - \psi(1-a)$, with constants $c_1, c_2 \in (0,1)$. For example, Wang and Chu [19] proved that

$$\frac{sin(\pi a)}{A_1 + (1-A_1)r^2} < \frac{\mathscr{K}_a(r)}{\log[e^{R(a)/2}/r']} < \frac{sin(\pi a)}{A_2 + (1-A_2)r^2} \tag{1.9}$$

for $r \in (0,1)$, where $A_1 = R(a)/B(a)$ and $A_2 = 1 - a(1-a)$.

In present paper, we try to generalize the above inequality (1.9) to zero-balanced hypergeometric function.

In 1999, Qiu and Vuorinen [12] considered the ratio function $x \mapsto F(a,b;a+b;x)/\log[e^R/(1-x)]$, and obtained the following theorem:

Theorem 1.1. *([12, Theorem 2.1]) Let $a, b \in (0,\infty)$ with $R(a,b) \geq 0$. Then the function*

$$F(x) = \frac{F(a,b;a+b;x)}{R(a,b) - \log(1-x)} \tag{1.10}$$

is strictly decreasing from $(0,1)$ onto $(1/B(a,b), 1/R(a,b))$. In particular, $\mathscr{K}(\sqrt{(}1-r^2))/\log(4/r)$ is strictly decreasing from $(0,1)$ onto $(1, \pi/\log 16)$.

It is natural to think about the monotonicity of the reciprocal of $F(x)$, that is

$$f(x) = 1/F(x) = \frac{R(a,b) - \log(1-x)}{F(a,b;a+b;x)}, \tag{1.11}$$

and the relationship between the monotonicity of the function $f(x)$ and the value of $R(a,b)$. Thus we consider the following questions:

Question 1.2. *For $a, b > 0$, is the function $f(x)$ stictly increasing or decreasing on $(0,1)$? What's the relationship between the monotonicity of the function $f(x)$ and the value of $R(a,b)$?*

Let $f(x)$ be in (1.11) and

$$f_1(x) = \frac{f(x) - R(a,b)}{x}, \tag{1.12}$$

$$f_2(x) = \frac{B(a,b) - f(x)}{1-x}, \tag{1.13}$$

$$f_3(x) = \frac{c+x}{f(x)}. \tag{1.14}$$

In [7], Huang, Qiu and Ma considered the above functions for the particular case of $a + b = 1$ and obtained the following theorems:

Theorem 1.3. *([7, Theorem 1.1]) Let $f(x)$ be as in (1.11), $f_1(x)$ be as in (1.12), and $f_2(x)$ be as in (1.13). If $a + b = 1$, then we have the following conclusions:*

(1) *$f(x)$ is convex on $(0,1)$.*

(2) *$f_1(x)$ is strictly increasing from (0,1) onto $(1 - abR(a,b), B(a,b) - R(a,b))$.*

(3) *$f_2(x)$ is strictly increasing from (0,1) onto $(B(a,b) - R(a,b), abB(a,b))$.*

Theorem 1.4. *([7, Theorem 1.2]) Let $f(x)$ be as in (1.11) and $f_3(x)$ be as in (1.14). If $a+b=1$, then we have the following conclusions:*

(1) $f_3(x)$ is strictly decreasing from $(0,1)$ onto $((c+1)/B(a,b), c/R(a,b))$ if and only if $c \geq R(a,b)/(1-abR(a,b))$.

(2) $f_3(x)$ is strictly increasing form $(0,1)$ onto $(c/R(a,b),(c+1)/B(a,b))$ if and only if $0 < c \leq 1/(ab)-1$. Moreover $f_3(x)$ is concave on $(0,1)$ provided that $0 < c \leq 1/(ab)-1$.

(3) If $1/(ab)-1 < c < R(a,b)/(1-abR(a,b)$, then there exists a unique number $x_0 = x_0(a,c) \in (0,1)$, depending on a and c, such that $f_3(x)$ is strictly increasing on $(0,x_0)$, and decreasing on $(x_0,1)$.

For the particular case of $a+b=1$, Theorem 1.3 and 1.4 actually obtain the conclusions of generalized elliptical integral $\mathscr{K}_a(r)$. In light of the above results, we are trying to extend the above theorem to the zero-balanced hypergeometric function and it is natural to consider the following questions:

Question 1.5. *Whether Theorem 1.3 and 1.4 can be extended to zero-balanced hypergeometric function?*

The purpose of this paper is to give complete answers to Question 1.2 and 1.5. This paper is organized as follows. The preliminaries we needed are listed in Section 2, and the main results and their complete proofs of this paper are listed in Section 3. As applications, inequalities of hypergeometric function are displayed in Section 4.

2. Preliminaries

Before proving our main results, we firstly introduce the following important lemmas, which will be used in the proofs of main results.

Lemma 2.1. *(See [11, Lemma 2.1]) Let $-\infty < a < b < \infty$, $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *(See [14, Lemma 1.1]) Suppose that the power series $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r>0$ and that $b_n>0$ for all $n \in \{0,1,2,\cdots\}$. Let $h(x)=f(x)/g(x)$. If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0,r)$.*

Lemma 2.3. *For $a \in (0,\infty)$, the following function*

$$R(a,1/a) = -2\gamma - \psi(a) - \psi(1/a)$$

is decreasing from $(0,1]$ onto $[0,\infty)$, and increasing from $(1,\infty)$ onto $(0,\infty)$.

Proof. Firstly, we consider the monotonicity of $\phi(x) = x\psi'(x)$, $x \in (0, +\infty)$. From the formula

$$\psi^n(x) = (-1)^{n+1}\int_0^{+\infty}\frac{t^n e^{-xt}}{1-e^{-t}}dt,$$

we have

$$\phi(x) = x\psi'(x) = \int_0^{+\infty}\frac{xte^{-xt}}{1-e^{-t}}dt \overset{v=xt}{=\!=} \int_0^{+\infty} e^{-v}\frac{v/x}{1-e^{-v/x}}dv.$$

Differentiating $\phi(x)$ gives

$$\phi'(x) = \int_0^{+\infty} e^{-v}\frac{v/x^2}{(1-e^{-v/x})^2}\left(\frac{v/x+1}{e^{v/x}} - 1\right)dv.$$

Let $y = v/x$, it is not difficult to find $e^y - y - 1 > 0$ for $y > 0$. It implies $(v/x+1)/e^{v/x} - 1 < 0$ and $\phi'(x) < 0$. Hence $\phi(x)$ is a decreasing function on $(0, +\infty)$.

Next by the monotonicity of $\phi(x) = x\psi'(x)$ and

$$\frac{dR(a, 1/a)}{da} = \frac{-a\psi'(a) + \psi'(1/a)/a}{a},$$

we have

$$\frac{dR(a, 1/a)}{da} = \frac{-\phi(a) + \phi(1/a)}{a}\begin{cases}\le 0, & 0 < a \le 1,\\ > 0, & a > 1.\end{cases}$$

The proof of this lemma is completed. □

Lemma 2.4. *For $a, b > 0$ and $ab \le 1$, we have $R(a, b) \ge 0$.*

Proof. Firstly, we let $b = 1/a$, then $R(a, 1/a) = -2\gamma - \psi(a) - \psi(1/a)$. By Lemma 2.3, we obtain $R(a, 1/a)$ is decreasing from $(0, 1]$ onto $[0, \infty)$, and increasing from $(1, \infty)$ onto $(0, \infty)$. Hence the equation $R(a, 1/a) = 0$ has only one zero point $a = 1$, that is $ab = 1$ and $R(a, b) = 0$ have only one intersection point $(a, b) = (1, 1)$.

Next, we can easily find some special points (a, b) such that $ab > 1$ and $R(a, b) \le 0$, such as $(3, 0.4)$ and $(4, 0.4)$. Since the symmetry of the two regions, $ab \le 1$ and $R(a, b) \ge 0$, we obtain that if $ab \le 1$, then $R(a, b) \ge 0$. □

Lemma 2.5. *For $a, b > 0$ and $ab \le 1$. If $0 \le c \le 1/(ab) - 1$, then the function*

$$\begin{aligned} f_4(x) = &-F(a, b; a+b; x) + \frac{2ab}{a+b}F(a, b; a+b+1; x)\\ &+ \frac{a^2b^2(c+x)}{(a+b)(a+b+1)}F(a+1, b+1; a+b+2; x) \end{aligned} \tag{2.1}$$

is a decreasing function and $f_4(x) < 0$ on $(0, 1)$.

Proof. According to (1.1)

$$
\begin{aligned}
f_4(x) &= -\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(a+b,n)n!}x^n + \frac{2ab}{a+b}\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(a+b+1,n)n!}x^n \\
&+ \frac{a^2b^2}{(a+b)(a+b+1)}\left[c\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{(a+b+2,n)n!}x^n + x\sum_{n=0}^{\infty}\frac{(a+1,n)(b+1,n)}{(a+b+2,n)n!}x^n\right] \\
&= -\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(a+b,n)n!}x^n + \sum_{n=0}^{\infty}\frac{2(a,n+1)(b,n+1)}{(a+b,n+1)n!}x^n \\
&+ \sum_{n=0}^{\infty}\frac{abc(a,n+1)(b,n+1)}{(a+b,n+2)n!}x^n + \sum_{n=0}^{\infty}\frac{ab(a,n+1)(b,n+1)}{(a+b,n+2)n!}\frac{n^2+(a+b+1)n}{(a+n)(b+n)}x^n \\
&= \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(a+b,n+2)n!}\Big\{[ab(c+1)-1]n^2 + [ab(c+1)(a+b)+3ab-2a-2b-1]n \\
&+ a^2b^2c + 2ab(a+b) - (a^2+b^2) - (a+b)\Big\}x^n.
\end{aligned} \tag{2.2}
$$

Since $0 \le c \le 1/(ab) - 1$, we have $ab(c+1) - 1 \le 0$. Hence

$$
\begin{aligned}
ab(c+1)(a+b) + 3ab - 2a - 2b - 1 &\le ab[ab/(a+b) + 3] - 2a - 2b - 1 \\
&= 3ab - a - b - 1 := u(a,b)
\end{aligned}
$$

and

$$
\begin{aligned}
a^2b^2c + 2ab(a+b) - (a^2+b^2) - (a+b) &\le a^2b^2[1/(ab) - 1] + 2ab(a+b) - (a^2+b^2) - (a+b) \\
&= ab - a^2b^2 + 2ab(a+b) - (a^2+b^2) - (a+b) := v(a,b).
\end{aligned}
$$

Since

$$
\frac{\partial u(a,b)}{\partial a} = 3b - 1 = 0, \frac{\partial u(a,b)}{\partial b} = 3a - 1 = 0,
$$

there is a extremal point $(1/3, 1/3)$ of $u(a,b)$, and $u(1/3,1/3) = -4/3 < 0$. Since

$$
u(a,0) = -a - 1 < 0,\ u(a,1/a) = 2 - a - 1/a \le 0,
$$

we have $u(a,b) \le 0$. Hence $ab(c+1)(a+b) + 3ab - 2a - 2b - 1 \le 0$.

Similarly, since

$$
\begin{aligned}
\frac{\partial v(a,b)}{\partial a} &= b - 2ab^2 + 2b(a+b) + 2ab - 2a - 1 = 0 \\
\frac{\partial v(a,b)}{\partial b} &= a - 2a^2b + 2a(a+b) + 2ab - 2b - 1 = 0,
\end{aligned}
$$

we have that $(a-b)[2(a+b) - 2ab + 3] = 0$. Since $2(a+b) - 2ab + 3 > 0$, $a = b$, there is an extremal point $(a,b) = (a_0, a_0)$, where a_0 is the solution of equation $-2a^3 + 6a^2 - a - 1 = 0$ and $v(a_0,a_0) < 0$. Since

$$
v(a,0) = -a^2 - a < 0,\ v(a,1/a) = -(a+1/a)^2 + (a+1/a) + 2 \le 0,
$$

we have $v(a,b) \le 0$. Hence $a^2b^2c + 2ab(a+b) - (a^2+b^2) - (a+b) \le 0$. Therefore, $f_4(x)$ is a decreasing function on $(0,1)$, and

$$\begin{aligned} f_4(0^+) &= -1 + \frac{2ab}{a+b} + \frac{a^2b^2c}{(a+b)(a+b+1)} \\ &= \frac{-(a+b)(a+b+1) + 2ab(a+b+1) + a^2b^2c}{(a+b)(a+b+1)} \\ &\le \frac{v(a,b)}{(a+b)(a+b+1)} \le 0. \end{aligned}$$

Hence $f_4(x) < 0$ on $(0,1)$. □

3. Main results and proofs

In the following statement, we always let $R = R(a,b)$, $B = B(a,b)$ for $(a,b) \in (0,\infty)$.

Theorem 3.1. *Let $a, b \in (0,\infty)$. The function*

$$f(x) = \frac{R - \log(1-x)}{F(a,b;a+b;x)} \tag{3.1}$$

is strictly increasing from $(0,1)$ onto (R,B).

Proof. Differentiating $f(x)$ gives

$$\begin{aligned} f'(x) &= \frac{F(a,b;a+b;x)/(1-x) - \frac{ab}{a+b}F(a+1,b+1;a+b+1;x)\log\frac{e^R}{1-x}}{F(a,b;a+b;x)^2} \\ &= \frac{F(a,b;a+b;x) - \frac{ab}{a+b}F(a,b;a+b+1;x)\log\frac{e^R}{1-x}}{(1-x)F(a,b;a+b;x)^2}. \\ &= \frac{g_1(x)}{g_2(x)}, \end{aligned} \tag{3.2}$$

where $g_1(x) = F(a,b;a+b;x) - [ab/(a+b)]F(a,b;a+b+1;x)\log[e^R/(1-x)]$, $g_2(x) = (1-x)F(a,b;a+b;x)^2$.

For $x \in (0,1)$, $g_2(x) > 0$. If $g_1(x) > 0$, then $f'(x)$ is a ratio of two positive functions, and $f(x)$ is strictly increasing on $(0,1)$.

By (1.1) and (1.3), $g_1(0) = 1 - abR/(a+b)$ and

$$\begin{aligned} g_1(1^-) &= \lim_{x\to 1^-} F(a,b;a+b;x) - \frac{ab}{a+b}F(a,b;a+b+1;x)\log\frac{e^R}{1-x} \\ &= \lim_{x\to 1^-} \frac{R-\log(1-x)}{B} - \frac{ab}{a+b}\frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}\log\frac{e^R}{1-x} = 0. \end{aligned}$$

By (1.5), we obtain

$$g_1'(x) = -\frac{a^2b^2}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x)\log\frac{e^R}{1-x}.$$

Case 1. If $R \geq 0$, then $\log[e^R/(1-x)] \geq 0$, $g_1^{'}(x) \leq 0$ on $(0,1)$. Therefore, $g_1(x)$ is decreasing and positive and $f^{'}(x) = g_1(x)/g_2(x) > 0$.

Case 2. If $R < 0$, then $\log[e^R/(1-x)]$ is increasing on $(0,1)$, $\log[e^R/(1-x)] < 0$ on $(0, 1-e^R)$ and $\log[e^R/(1-x)] > 0$ on $[1-e^R, 1)$. Therefore, $g_1^{'}(x) > 0$ on $(0, 1-e^R)$ and $g_1^{'}(x) < 0$ on $[1-e^R, 1)$. Moreover, $g_1(x)$ is increasing on $(0, 1-e^R)$ and decreasing on $[1-e^R, 1)$. Meanwhile, $g_1(0) = 1 - abR/(a+b) > 0$ and $g_1(1^-) = 0$. Hence $g_1(x) > 0$ and $f^{'}(x) = g_1(x)/g_2(x) > 0$ on $(0,1)$.

In summary, no matter what value R takes, $f^{'}(x) > 0$ on $(0,1)$. By (1.1) and (1.3), we obtain $f(0) = R/F(a,b;a+b;0) = R$ and

$$\begin{aligned} f(1^-) &= \lim_{x\to 1^-} \frac{R-\log(1-x)}{F(a,b;a+b;x)} \\ &= \lim_{x\to 1^-} \frac{R-\log(1-x)}{\frac{R-\log(1-x)}{B}} = B. \end{aligned}$$

Hence $f(x)$ is strictly increasing from $(0,1)$ onto (R, B). □

Remark 3.2. *From the above proof, it can be known that no matter what value R takes, $f(x)$ is an increasing function. Hence Theorem 3.1 answers Question 1.2.*

Remark 3.3. (1) *For $R \geq 0$, $R - \log(1-x) > 0$ and $f(x) > 0$ for $x \in (0,1)$. Acorrding to Theorem 3.1, it is easy to know that $F(x) = 1/f(x)$ in Theorem 1.1 is strictly decresing on* $(0,1)$.

(2) *For $R < 0$, there exists unique point $x_0 \in (0,1)$ such that $R - \log(1-x) = 0$, that is $x_0 = 1 - e^R$. Acorrding to Theorem 3.1, we can know that $F(x) = 1/f(x)$ is strictly decreasing from $(0, x_0)$ onto $(-\infty, 1/R)$ and strictly decreasing from $(x_0, 1)$ onto $(1/B, +\infty)$.*

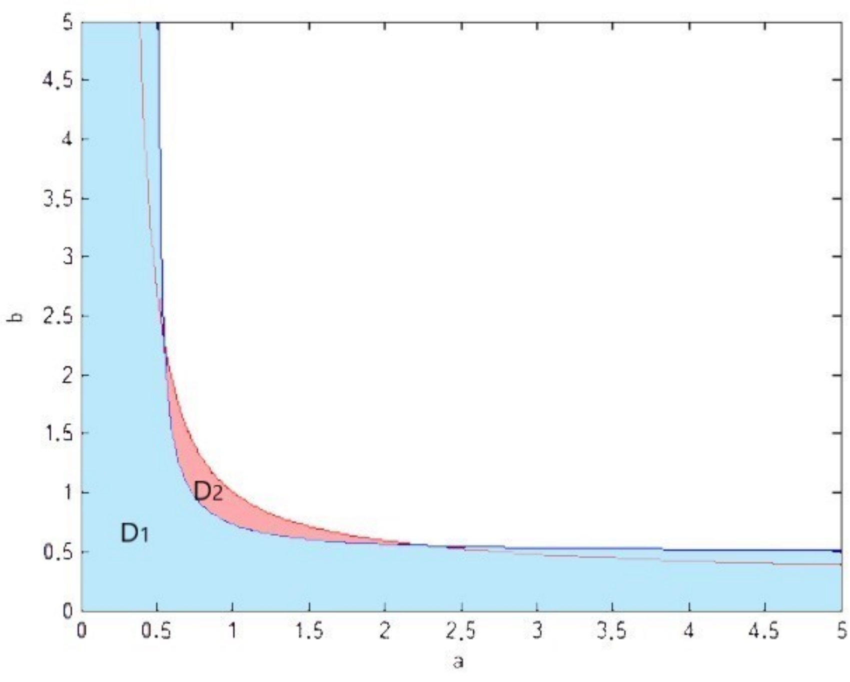

Figure 1. The regions D_1, D_2.

Theorem 3.4. *For* $(a,b) \in D_1 = \{(a,b)|a,b>0,(a+b)(a+b+1)-ab(2a+2b+3)>0\}$ *as shown in the Figure 1, the function* $f(x)$ *is convex on* $(0,1)$.

Proof. By (1.1) − (1.5), we obtain $g_1(1^-)=0$ and

$$\begin{aligned} g_2(1^-) &= \lim_{x\to 1^-}\frac{F(a,b;a+b;x)^2}{\frac{1}{1-x}} \\ &= \lim_{x\to 1^-}\frac{2ab}{a+b}\frac{\frac{1}{1-x}F(a,b;a+b;x)F(a,b;a+b+1;x)}{\frac{1}{(1-x)^2}} \\ &= \lim_{x\to 1^-}\frac{2}{B}\frac{F(a,b;a+b;x)}{\frac{1}{1-x}} \\ &= \lim_{x\to 1^-}\frac{2ab}{(a+b)B}\frac{\frac{1}{1-x}F(a,b;a+b+1;x)}{\frac{1}{(1-x)^2}} \\ &= \lim_{x\to 1^-}\frac{2}{B^2}(1-x)=0. \end{aligned}$$

and

$$\begin{aligned} \frac{g_1'(x)}{g_2'(x)} &= \frac{-\frac{a^2b^2}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x)\log\frac{e^R}{1-x}}{-F(a,b;a+b;x)^2+\frac{2ab}{a+b}F(a,b;a+b;x)F(a,b;a+b+1;x)} \\ &= ab\frac{\log\frac{e^R}{1-x}}{F(a,b;a+b;x)}\frac{\frac{ab}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x)}{F(a,b;a+b;x)-\frac{2ab}{a+b}F(a,b;a+b+1;x)} \\ &= abf(x)h(x), \end{aligned}$$

where

$$h(x)=\frac{\frac{ab}{(a+b)(a+b+1)}F(a+1,b+1;a+b+2;x)}{F(a,b;a+b;x)-\frac{2ab}{a+b}F(a,b;a+b+1;x)}.$$

By (1.1), $h(x)$ can be written as

$$\begin{aligned} h(x) &= \sum_{n=0}^{\infty}\frac{(a,n+1)(b,n+1)}{(a+b,n+2)n!}x^n\Big/\sum_{n=0}^{\infty}\left(1-\frac{2ab}{a+b+n}\right)\frac{(a,n)(b,n)}{(a+b,n)n!}x^n \\ &= \sum_{n=0}^{\infty}A_nx^n\Big/\sum_{n=0}^{\infty}B_nx^n, \end{aligned}$$

where

$$A_n=\frac{(a,n+1)(b,n+1)}{(a+b,n+2)n!},\ B_n=\left(1-\frac{2ab}{a+b+n}\right)\frac{(a,n)(b,n)}{(a+b,n)n!}.$$

Let

$$\begin{aligned} C_n=\frac{A_n}{B_n} &= \frac{(a+n)(b+n)}{(a+b+n+1)(a+b-2ab+n)} \\ &= 1-\frac{(1+a+b-2ab)n+(a+b)(a+b+1)-ab(2a+2b+3)}{n^2+(1+2a+2b-2ab)n+(a+b+1)(a+b-2ab)} \end{aligned}$$

$$= 1 - \frac{p(n)}{q(n)},$$

where $p(n) = (1 + a + b - 2ab)n + (a + b)(a + b + 1) - ab(2a + 2b + 3)$, $q(n) = n^2 + (1 + 2a + 2b - 2ab)n + (a + b + 1)(a + b - 2ab)$.

We can easily obtain $a + b - 2ab > 0$ by $(a + b)(a + b + 1) - ab(2a + 2b + 3) > 0$. Hence $C_n > 0$, $q(n)$ is positive and increasing and $p(n)$ is positive and increasing for $n \in \mathbb{N}$. Meanwhile, $q(n) - p(n) = n^2 + (a + b)n + ab$ is positive and increasing for $n \in \mathbb{N}$. It implies $0 < p(n)/q(n) < 1$ and $p(n)/q(n)$ is decreasing for $n \in \mathbb{N}$. Hence $h(x)$ is positive and increasing on $(0, 1)$ by Lemma 2.2

Case 1. For $R \geq 0$, we have $g_1^{'}(x)/g_2^{'}(x) = abf(x)h(x)$ is positive and increasing on $(0, 1)$. According to Lemma 2.1, $f^{'}(x) = g_1(x)/g_2(x)$ is increasing and $f(x)$ is convex on $(0, 1)$.

Case 2. For $R < 0$, we have

$$f(x)\begin{cases} < 0, & 0 < x < 1 - e^R, \\ \geq 0, & x \geq 1 - e^R. \end{cases}, g_1^{'}(x)\begin{cases} > 0, & 0 < x < 1 - e^R, \\ \leq 0, & x \geq 1 - e^R. \end{cases}$$

Hence $g_1(x)$ is positive and increasing on $(0, 1 - e^R)$.

Differentiating $g_2(x)$ gives

$$\begin{aligned} g_2^{'}(x) &= -F(a, b; a + b; x)^2 + 2ab/(a + b)F(a, b; a + b; x)F(a, b; a + b + 1; x) \\ &= -F(a, b; a + b; x)[F(a, b; a + b; x)^2 - 2ab/(a + b)F(a, b; a + b + 1; x)] \\ &= -F(a, b; a + b; x)\sum_{n=0}^{\infty}\left(1 - \frac{2ab}{a + b + n}\right)\frac{(a, n)(b, n)}{(a + b, n)n!}x^n < 0 \end{aligned}$$

Hence $g_2(x)$ is positive and decreasing on $(0, 1)$. Moreover, $f^{'}(x) = g_1(x)/g_2(x)$ is positive and increasing on $(0, 1 - e^R)$. For $x \in [1 - e^R, 1)$, $f(x) \geq 0$ and $g_1^{'}(x)/g_2^{'}(x) = abf(x)h(x)$ is non-negative and increasing. Hence $f^{'}(x)$ is increasing on $[1 - e^R, 1)$. Therefore, $f^{'}(x)$ is increasing and $f(x)$ is convex on $(0, 1)$.

□

Remark 3.5. *Theorem 3.4 is a generalization of Theorem 1.3(1) for $a \in (0, 1)$ and $a + b = 1$.*

Theorem 3.6. *Let $f_1(x)$ be as in* (1.12) *and $f_2(x)$ be in (1.13). If $(a, b) \in D_1$, then we have the following conclusions:*

(1) *$f_1(x)$ is strictly increasing from $(0, 1)$ onto $(1 - abR/(a + b), B - R)$.*

(2) *$f_2(x)$ is strictly increasing from $(0, 1)$ onto $(B - R, abB)$.*

Proof. (1) By (1.3) and (1.5), we obtain

$$f_1(0^+) = \lim_{x \to 0^+}\frac{f(x) - R}{x} = \lim_{x \to 0^+} f^{'}(x) = 1 - \frac{abR}{a + b}.$$

and

$$f_1(1^-) = \lim_{x \to 1^-}\frac{f(x) - R}{x} = f(1^-) - R = B - R.$$

By Lemma 2.1, we obtain

$$f_1'(x) = \frac{xf'(x) - f(x) + R}{x^2}$$

and

$$(xf'(x) - f(x) + R)' = xf''(x) + f'(x) - f'(x) = xf''(x).$$

If $(a,b) \in D_1$, then $f''(x) > 0$ and $xf'(x) - f(x) + R$ is strictly increasing on $(0,1)$. Since $xf'(x) - f(x) + R \to 0$ as $x \to 0$. Hence $xf'(x) - f(x) + R > 0$ on $(0,1)$, and $f_1(x)$ is strictly increasing from $(0,1)$ onto $(1 - ab/(a+b)R, B - R)$.

(2) By (1.3) and l'Hopital's Rule, we obtain

$$f_2(0^+) = \lim_{x\to 0^+} \frac{B - f(x)}{1 - x} = B - f(0^+) = B - R,$$

and

$$f_2(1^-) = \lim_{x\to 1^-} \frac{B - f(x)}{1 - x} = f'(1^-) = abB.$$

Since $B - f(x) \to 0$ and $1 - x \to 0$ as $x \to 1^-$ and

$$\frac{\frac{d}{dx}(B - f(x))}{\frac{d}{dx}(1 - x)} = \frac{-f'(x)}{-1} = f'(x),$$

the monotonicity of $f_2(x)$ follows from Lemma 3.1 and the convexity of $f(x)$ when $(a,b) \in D_1$. Hence $f_2(x)$ is strictly increasing from $(0,1)$ onto $(B - R, abB)$. □

Remark 3.7. *If $(a,b) \in D_1$, then we have a general conclusion of Theorem 1.3.*

Theorem 3.8. *If $c \geq 0$ and $(a,b) \in D_1 \cap D_2$, where $D_2 = \{(a,b)|a,b > 0, R(a,b) > 0\}$ as shown in the Figure 1, the function*

$$f_3(x) = \frac{c + x}{f(x)} = \frac{(c + x)F(a,b;a+b;x)}{R - \log(1 - x)}$$

is strictly decreasing from $(0,1)$ onto $((c+1)/B, c/R)$ if and only if $c \geq R/(1 - abR/(a+b)) > 0$.

Proof. If $(a,b) \in D_2$, then $R > 0$ and $f_3(x) = (c + x)/f(x) > 0$ on $(0,1)$, Next by Theorem 3.1, we obtain

$$f_3(0^+) = \lim_{x\to 0^+} \frac{c + x}{f(x)} = \frac{c}{f(0^+)} = \frac{c}{R}.$$

and

$$f_3(1^-) = \lim_{x\to 1^-} \frac{c + x}{f(x)} = \frac{c + 1}{f(1^-)} = \frac{c + 1}{B}.$$

Let $g_3(x) = f(x) - (c + x)f'(x)$. Then $f_3'(x) = g_3(x)/[f(x)]^2$ and $g_3'(x) = -(c + x)f''(x)$. By Theorem 2.2, if $(a,b) \in D_1$, then $f''(x) > 0$ and $g_3(x) < 0$ for $x \in (0,1)$. $f'(0) = 1 - abR/(a+b)$ and $f'(1^-) = abB$. It follows from the above that $g_3(x)$ is strictly decreasing from $(0,1)$ onto $(B - (c+1)abB, R - c(1 - abR/(a+b)))$. Hence $g_3(x) < 0$ if and only if $R - c(1 - abR/(a+b)) \leq 0$. Therefore, $f_3(x)$ is strictly decreasing on $(0,1)$ if and only if $R - c(1 - abR/(a+b)) \leq 0$. □

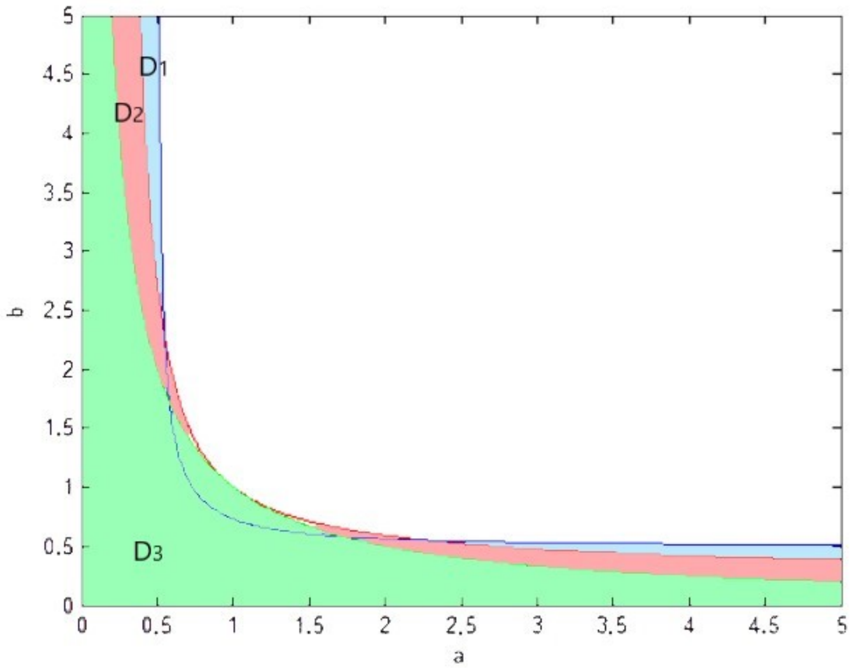

Figure 2. The regions D_1, D_2, D_3.

Theorem 3.9. *If $c \geq 0$ and $(a, b) \in D_3$, where $D_3 = \{(a, b)|a, b > 0, ab < 1\}$ as shown in the Figure 2, the function*

$$f_3(x) = \frac{(c + x)F(a, b; a + b; x)}{R - \log(1 - x)}$$

is strictly increasing from $(0, 1)$ onto $(c/R, (c + 1)/B)$ if and only if $c \leq 1/(ab) - 1$.

Proof. By Lemma 2.4, if $(a, b) \in D_3$, then $(a, b) \in D_2$.

$\Leftarrow$ Differentiating $f_3(x)$ gives

$$f_3^{'}(x) = \frac{f_5(x)}{(1 - x)(\log \frac{e^R}{1-x})^2},$$

where

$$f_5(x) = \log \frac{e^R}{1 - x}[(1 - x)F(a, b; a + b; x) + \frac{ab}{a + b}(c + x)F(a, b; a + b + 1; x)] - (c + x)F(a, b; a + b; x).$$

Next, differentiating $f_5(x)$ gives

$$f_5^{'}(x) = \log \frac{e^R}{1 - x} f_4(x),$$

where

$$f_4(x) = -F(a, b; a + b; x) + \frac{2ab}{a + b}F(a, b; a + b + 1; x) + \frac{a^2b^2(c + x)}{(a + b)(a + b + 1)}F(a + 1, b + 1; a + b + 2; x).$$

By Lemma 2.4 and 2.5,

$$f_5'(x) = \log \frac{e^R}{1-x} f_4(x) \leq 0,$$

hence $f_5(x)$ is a decreasing function on $(0, 1)$

By (1.2),

$$f_5(1^-) = \lim_{x\to 1^-} \log \frac{e^R}{1-x} \frac{ab(c+1)}{a+b} \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} - \frac{c+1}{B} \log \frac{e^R}{1-x} = 0$$

Hence we have $f_5(x) > 0$ and $f_3'(x) > 0$ on $(0, 1)$. It implies $f_3(x)$ is strictly increasing from $(0, 1)$ onto $(c/R, (c+1)/B)$.

$\Rightarrow$ Suppose $c > 1/(ab) - 1$,

$$\begin{aligned}
&\lim_{x\to 1^-} \frac{a^2b^2}{(a+b)(a+b+1)} \frac{(c+x)F(a+1, b+1; a+b+2; x)}{F(a, b; a+b; x)} - 1\\
&= \lim_{x\to 1^-} \frac{a^2b^2(c+1)}{(a+b)(a+b+1)} \frac{B(a, b)}{B(a+1, b+1)} \frac{R(a+1, b+1) - \log(1-x)}{R(a, b) - \log(1-x)}\\
&= ab(c+1) - 1 > 0.
\end{aligned}$$

For the function

$$\begin{aligned}
f_4(x) = &- F(a, b; a+b; x) + \frac{2ab}{a+b} F(a, b; a+b+1; x)\\
&+ \frac{a^2b^2(c+x)}{(a+b)(a+b+1)} F(a+1, b+1; a+b+2; x)
\end{aligned}$$

we have $\lim_{x\to 1^-} f_4(x) = +\infty$. It implies there is a $\delta > 0$ such that $f_5'(x) = \log[e^R/(1-x)]f_4(x) > 0$ on $(1-\delta, 1)$. So $f_5(x)$ is increasing on $(1-\delta, 1)$. Since $f_5(1^-) = 0$, $f_5(x) < 0$ and $f_3'(x) = f_5(x)/[(1-x)\log[e^R/(1-x)]]^2 < 0$ on $(1-\delta, 1)$. Hence $f_3(x)$ is decreasing on $(1-\delta, 1)$. This is a contradiction, since $f(x)$ is an increasing function on $(0, 1)$. Hence $c \leq 1/(ab) - 1$.

□

Remark 3.10. *Theorem 3.8 is a general conclusion of Theorem 1.4 (1) in region $D_1 \cap D_2$ and Theorem 3.9 is a general conclusion of Theorem 1.4 (2) in region D_3.*

4. Applications for inequalities

From our main Theorems, we can easily obtain several asymptotically sharp inequalities for $F(a, b; a+b; x)$. Let $x = r^2$, we can get the inequalities of generalized elliptic integral $\mathscr{K}_a(r)$.

Corollary 4.1. *Let $(a, b) \in D_1 \cap D_2$ and $c > 0$. Then for the inequalities*

$$\frac{\log[e^R/(1-x)]}{RF(a, b; a+b; x)} \leq \frac{c+x}{c}, \tag{4.1}$$

$$\frac{\log[e^R/(1-x)]}{RF(a, b; a+b; x)} \geq \frac{c+x}{c}, \tag{4.2}$$

we have the following conclusions:

(1) (4.1) *holds if and only if* $c \le R/(B-R)$.

(2) (4.2) *holds if and only if* $c \ge R/(1-abR/(a+b))$.

Proof. Since

$$(4.1) \Leftrightarrow \frac{f(x)}{R} \le \frac{c+x}{c},\ (4.2) \Leftrightarrow \frac{f(x)}{R} \ge \frac{c+x}{c},$$

By Theorem 3.1, $f(x)$ is strictly increasing from $(0,1)$ onto (R,B). Hence $f_1(x) = (f(x)-R)/x > 0$ on $(0,1)$.

For $(a,b) \in D_1 \cap D_2$, $R > 0$,

$$\begin{aligned}(4.1) \Leftrightarrow cf_1(x) \le R \Leftrightarrow c \le \frac{R}{f_1(x)}\\ \Leftrightarrow c \le \frac{R}{f_1(1^-)} \Leftrightarrow c \le \frac{R}{B-R};\end{aligned}$$

$$\begin{aligned}(4.2) \Leftrightarrow cf_1(x) \ge R \Leftrightarrow c \ge \frac{R}{f_1(x)}\\ \Leftrightarrow c \ge \frac{R}{f_1(0^+)} \Leftrightarrow c \ge \frac{R}{1-\frac{abR}{a+b}}.\end{aligned}$$

□

Corollary 4.2. *Let* $(a,b) \in D_1$ *and* $c > 0$. *Then for the inequalities*

$$\frac{\log[e^R/(1-x)]}{BF(a,b;a+b;x)} \le \frac{c+x}{c+1}, \tag{4.3}$$

$$\frac{\log[e^R/(1-x)]}{BF(a,b;a+b;x)} \ge \frac{c+x}{c+1}, \tag{4.4}$$

we have the following conclusions:

(1) (4.3) *holds if and only if* $c \ge R/(B-R)$.

(2) (4.4) *holds if and only if* $c \le 1/(ab) - 1$.

Proof. Since

$$(4.3) \Leftrightarrow \frac{f(x)}{B} \le \frac{c+x}{c+1},\ (4.4) \Leftrightarrow \frac{f(x)}{B} \ge \frac{c+x}{c+1},$$

By Theorem 3.1, $f(x)$ is strictly increasing from $(0,1)$ onto (R,B). Hence $f_2(x) = (B-f(x))/(1-x) > 0$ on $(0,1)$.

$$\begin{aligned}(4.3) \Leftrightarrow B(1-x) \le (c+1)(B-f(x)) \Leftrightarrow B \le (c+1)\frac{B-f(x)}{1-x} = (c+1)f_2(x)\\ \Leftrightarrow c \ge \frac{B}{f_2(x)} - 1 \Leftrightarrow c \ge \frac{B}{f_2(0^+)} - 1 = \frac{R}{B-R};\end{aligned}$$

Similary,

$$(4.4) \Leftrightarrow B \geq (c+1)f_2(x) \Leftrightarrow c \leq \frac{B}{f_2(x)} - 1$$
$$\Leftrightarrow c \leq \frac{B}{f_2(1^-)} - 1 = \frac{1}{ab} - 1.$$

□

Corollary 4.3. *Let* $(a,b) \in D_1$ *and* $c > 0$*. Then*

$$\max\left\{abBx + (1-ab)B, (1-abR/(a+b))x + R\right\} < \frac{\log\left[e^R/(1-x^2)\right]}{F(a,b;a+b;x)} < (B-R)x + R \tag{4.5}$$

for $x \in (0,1)$*, with equality in each instance if and only if* $x \to 0$*.*

Proof. It follows from the monotonicity properties of $f_1(x)$ and $f_2(x)$ given in Theorem 3.5 that

$$(1-abR/(a+b))x + R < \frac{\log[e^R/(1-x)]}{F(a,b;a+b;x)} < (B-R)x + R,$$

$$abBx + (1-ab)B < \frac{\log[e^R/(1-x)]}{F(a,b;a+b;x)} < (B-R)x + R.$$

Hence we have (4.5) for $x \in (0,1)$, with equality in each instance if and only if $x \to 0$. □

Corollary 4.4. *Let* $(a,b) \in D_1 \cap D_2$ *and* $C_1 = R/(1-abR/(a+b))$*. Then*

$$\frac{C_1+1}{B(C_1+x)} < \frac{F(a,b;a+b;x)}{\log[e^R/(1-x)]} < \frac{C_1}{R(C_1+x)} \tag{4.6}$$

for $x \in (0,1)$*.*

Proof. It follows from the monotonicity properties of $f_3(x)$ of Theorem 3.7 that

$$\frac{c+1}{B(c+x)} < \frac{F(a,b;a+b;x)}{\log[e^R/(1-x)]} < \frac{c}{R(c+x)}$$

for $c \geq R/(1-abR/(a+b))$ and $x \in (0,1)$. Let $c = C_1$, we have (4.6). □

Corollary 4.5. *Let* $(a,b) \in D_3$ *and* $C_2 = 1/(ab) - 1$*. Then*

$$\frac{C_2}{R(C_2+x)} < \frac{F(a,b;a+b;x)}{\log[e^R/(1-x)]} < \frac{C_2+1}{B(C_2+x)} \tag{4.7}$$

for $r \in (0,1)$*. In particular, if* $(a,b) \in D_1 \cap D_3$*, then*

$$\max\left\{\frac{C_1+1}{B(C_1+x)}, \frac{C_2}{R(C_2+x)}\right\} < \frac{F(a,b;a+b;x)}{\log[e^R/(1-x)]} < \min\left\{\frac{C_1}{R(C_1+x)}, \frac{(C_2+1)}{B(C_2+x)}\right\} \tag{4.8}$$

for $x \in (0,1)$*, with equality in each instance if and only if* $x \to 0$*.*

Proof. The proof is similar to the proof of Corollary 4.4. Using the monotonicity properties of $f_3(x)$ of Theorem 3.8 that

$$\frac{c}{R(c+x)} < \frac{F(a,b;a+b;x)}{\log[e^R/(1-x)]} < \frac{c+1}{B(c+x)}$$

for $c \le 1/(ab) - 1$ and $x \in (0,1)$. And let $c = C_2$, we have (4.7).

If $(a,b) \in D_1 \cap D_3$, then (4.6) and (4.7) both holds. Hence we have (4.8) for $x \in (0,1)$, with equality in each instance if and only if $x \to 0$. □

Remark 4.6. *Acorrding to the above Corollaries, let $a + b = 1$ and $x = r^2$, we can obtain the conculsions of genelized ellptic integral $\mathscr{K}_a(r)$ in [7].*

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Conflict of interest

The authors declare that they have no competing interests.

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