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Research article

Long time behavior of higher-order delay differential equation with vanishing proportional delay and its convergence analysis using spectral method

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Abstract: Delay differential equations (DDEs) are used to model some realistic systems as they provide some information about the past state of the systems in addition to the current state. These DDEs are used to analyze the long-time behavior of the system at both present and past state of such systems. Due to the oscillatory nature of DDEs their explicit solution is not possible and therefore one need to use some numerical approaches. In this article, we developed a higher-order numerical scheme for the approximate solution of higher-order functional differential equations of pantograph type with vanishing proportional delays. Some linear and functional transformations are used to change the given interval $[0, T]$ into standard interval $[-1, 1]$ in order to fully use the properties of orthogonal polynomials. It is assumed that the solution of the equation is smooth on the entire domain of interval of integration. The proposed scheme is employed to the equivalent integrated form of the given equation. A Legendre spectral collocation method relative to Gauss-Legendre quadrature formula is used to evaluate the integral term efficiently. A detail theoretical convergence analysis in L_∞ norm is provided. Several numerical experiments were performed to confirm the theoretical results.

Keywords: Long time behavior; delay differential equations; Legendre quadrature formula; convergence analysis; numerical examples
Mathematics Subject Classification: 65M70, 34K28

1. Introduction

In nature there are so many physical phenomena where the state of the system not only depends on the current state but also depends on the history of the function. In this case it is more natural to model such type of phenomena using the delay differential equations (DDEs), where the state of the function depends on the current state as well as on the history of the function. Among the many available delay differential system, pantograph type delay differential equation is more commonly used in mathematical modeling of chemical and pharmaceutical kinetics, control problems and ships aircraft where they are used in navigational control electronic systems. It is called pantograph type equation because it was first used to investigate that how an electric current is obtained by the electric locomotive of a pantograph. Consider the kth-order functional differential equation of pantograph type of the form:

$$y^{(k)}(x) = \sum_{l=0}^{k-1} \lambda_l(x) y^{(l)}(x) + \sum_{l=0}^{k-1} \mu_l(x) y^{(l)}(\alpha_l x) + g(x), \; x \in [0, T] \tag{1}$$

subject to

$$y^{(m)}(0) = y_0^{(m)}, \; (m = 0, 1, \ldots, k-1). \tag{2}$$

For $k \geq 2$, spectral method will be based on the integrated form of given equation. The functions $\lambda_l(x)$ and $\mu_l(x)$ are given analytical functions on $I := [0, T]$ and $\alpha_l \in (0,1), l = 0, 1$ is a fixed constant known as proportional delay. For simplicity, we employed and analyzed spectral method for the second order functional differential equation, that is $k = 2.$ Eq (1) will take the form:

$$y''(x) = \sum_{l=0}^{1} \lambda_l(x) y^{(l)}(x) + \sum_{l=0}^{1} \mu_l(x) y^{(l)}(\alpha_l x) + g(x), \; x \in [0, T] \tag{3}$$

subject to

$$y(0) = y_0, \; y'(0) = y_1. \tag{4}$$

Equation (3) plays an important role in modeling of many physical phenomena like, for example in electrodynamics and in nonlinear dynamical systems. In practice it is arises in the problems when $\lambda_l(x)$and $\mu_l(x)$ are real constant and $\alpha_l\,(x)$ is real valued function. The case $\lambda_l(x) = 0,$ is also important in the number theory in the context of partitioning [12, 14]. Numerical solution of first order pantograph type delay differential equation has been studied extensively in numerous papers such as [1–5, 9–15, 17–28]. A spectral collocation method is applied to integro-delay differential equation with proportional delay in [6, 29, 30]. A comprehensive list of references for the solution of DDEs can be found in [16]. A very limited work is available regarding the approximate solution of higher-order pantograph equation, especially using the higher-order schemes. To this end, we will use spectral discretization based on Legendre spectral collocation method to solve Eq (3) subject to Eq (4) numerically in order to get an accurate solution for a very few numbers of collocation points, as spectral methods are well known for their exponential rate of convergence.

The rest of the paper is organized as follows. In Section 2, we discuss the spectral method for the approximate solution of Eq (3). Section 3 includes some useful lemmas and convergence analysis of our proposed scheme in L_∞ norm. In Section 4, we perform some numerical test to confirm the spectral accuracy and Section 5 consist of conclusion.

2. Spectral collocation method

In order to fully use the properties of orthogonal polynomials for ease of convergence analysis, spectral methods will be employed on the standard interval $[-1,1]$. For this reason, we use the following transformation

$$x = \frac{T}{2}(1+t), \qquad t = \frac{2x}{T} - 1.$$

Using this transformation, Eqs (3) and (4) will take the form:

$$u''(t) = \sum_{l=0}^{1} A_l(t)u^{(l)}(t) + \sum_{l=0}^{1} B_l(t)u^{(l)}(\alpha_l t + \alpha_l - 1) + G(t),\ t \in [-1,1] \tag{5}$$

subject to

$$u(-1) = y_0,\ u'(1) = \left(\frac{T}{2}\right)y_1. \tag{6}$$

Where

$$u(t) = y\left(\frac{T}{2}(1+t)\right), G(t) = \left(\frac{T}{2}\right)^2 g\left(\frac{T}{2}(1+t)\right),$$

$$A_0 = \left(\frac{T}{2}\right)^2 \lambda_0\left(\frac{T}{2}(1+t)\right), A_1 = \left(\frac{T}{2}\right)\lambda_1\left(\frac{T}{2}(1+t)\right),$$

$$B_0 = \left(\frac{T}{2}\right)^2 \mu_0\left(\frac{T}{2}(1+t)\right), B_1 = \left(\frac{T}{2}\right)\mu_1\left(\frac{T}{2}(1+t)\right),$$

with

$$u(-1) = u_{-1}, \qquad u'(-1) = u'_{-1},$$

where $u_{-1} = y_0,\ u'_{-1} = \left(\frac{T}{2}\right)y_1$.

For any given positive integer N, we denote the collocation points by $\{t_j\}_{j=0}^{N}$, which is the set of $(N+1)$, Legendre Gauss points corresponding to weights ω_i . Let P_N denote the space of all polynomials of degree not exceeding N. For any $v \in C[-1,1]$, we define the Lagrange interpolating polynomial

$$I_N v(t) = \sum_{j=0}^{N} v(t_j)F_j(t), \tag{7}$$

where $\{F(t_j)\}_{j=0}^{N}$ is the Lagrange interpolation polynomial associated with the Legendre collocation points $\{t_j\}_{j=0}^{N}$. Spectral method will be employed to the integrated form. For this reason, integrate Eq (5), and using $u_{-1} = y_0,\ u'_{-1} = \left(\frac{T}{2}\right)y_1$, we get,

$$u'(t) = u'_{-1} + \sum_{l=0}^{1} \int_{-1}^{t} A_l(s)u^{(l)}(s)ds + \sum_{l=0}^{1} \int_{-1}^{t} B_l(s)u^{(l)}(\alpha_l s + \alpha_l - 1)ds + \int_{-1}^{t} G(s)ds, \tag{8}$$

$$u(t) = u_{-1} + \int_{-1}^{t} u'(s)ds. \tag{9}$$

We assume that Eqs (8) and (9) holds at collocation points $\{t_j\}_{j=0}^{N}$, to get

$$u'(t_j) = u'_{-1} + \sum_{l=0}^{1} \int_{-1}^{t_j} \mathrm{A}_l(s)u^{(l)}(s)ds + \sum_{l=0}^{1} \int_{-1}^{t_j} B_l(s)u^{(l)}(\alpha_l s + \alpha_l - 1)ds + \int_{-1}^{t_j} G(s)ds, \tag{10}$$

$$u(t_j) = u_{-1} + \int_{-1}^{t_j} u'(s)ds, t \in [-1, 1] \tag{11}$$

for $0 \le j \le N$. In order to compute all these integral terms efficiently for the higher-order accuracy, we transform the interval of integral from $[-1, t_j]$ to $[-1, 1]$, as we have a very little information available for both $u(s)$ and $u'(s)$, we get

$$u'(t_j) = u'_{-1} + \frac{t_j+1}{2}\sum_{l=0}^{1} \int_{-1}^{1} \mathrm{A}_l\left(s(t_j,\varphi)\right) u^{(l)}\left(s(t_j,\varphi)\right) d\varphi + \frac{t_j+1}{2}\sum_{l=0}^{1} \int_{-1}^{1} B_l\left(s(t_j,\varphi)\right) u^{(l)}(\alpha_l s(t_j,\varphi) + \alpha_l - 1)d\varphi + \frac{t_j+1}{2}\int_{-1}^{1} G\left(s(t_j,\varphi)\right) d\varphi , \tag{12}$$

$$u(t_j) = u_{-1} + \frac{t_j+1}{2}\int_{-1}^{1} u'\left(s(t_j,\varphi)\right) d\varphi,\ t \in [-1, 1] \tag{13}$$

where, we use $s = \frac{1+t_j}{2}\varphi + \frac{t_j-1}{2} \approx s(t_j,\varphi)$.

Now using the $(N+1)$ Gauss quadrature rule relative to the Legendre weight to approximate the integral term, we get

$$u'_j = u'_{-1} + \frac{t_j+1}{2}\sum_{l=0}^{1}\left(\sum_{k=0}^{N} \mathrm{A}_l\left(s(t_j,\varphi_k)\right) u^{(l)}\left(s(t_j,\varphi_k)\right)\omega_k\right) + \frac{t_j+1}{2}\sum_{l=0}^{1}\left(\sum_{k=0}^{N} B_l\left(s'(t_j,\varphi_k)\right) u^{(l)}(\alpha_l s(t_j,\varphi_k) + \alpha_l - 1)\omega_k\right) + \frac{t_j+1}{2}\sum_{k=0}^{N} G\left(s(t_j,\varphi_k)\right)\omega_k, \tag{14}$$

$$u_j = u_{-1} + \frac{t_j+1}{2}\sum_{k=0}^{N} u'\left(s(t_j,\varphi_k)\right)\omega_k. \tag{15}$$

Let $u'_j \approx u'(t_j)$, $u_j \approx u(t_j)$, $0 \le j \le N$. The set $\{\varphi_k\}_{k=0}^{N}$ coincide with the collocation points $\{t_j\}_{j=0}^{N}$.

We expand u', u and G using Lagrange interpolating polynomials, that is

$$u'(s) \approx \sum_{p=0}^{N} u'_p F_p(s),\ u(s) = \sum_{p=0}^{N} u_p F_p(s),\ G(s) \approx \sum_{p=0}^{N} G_p F_p(s), \tag{16}$$

The Legendre spectral collocation method is to seek $\{u'_j\}_{j=0}^{N}, \{u_j\}_{j=0}^{N}, \{G_j\}_{j=0}^{N}$, holds at colocation points, then the spectral approximation to Eqs (3) and (4) is given by:

$$u'_j = u'_{-1} + \frac{t_j+1}{2}\sum_{l=0}^{1}\left(\sum_{p=0}^{N} u_p^{(l)} \sum_{k=0}^{N} \mathrm{A}_l\left(s(t_j,\varphi_k)\right) F_p\left(s(t_j,\varphi_k)\right)\omega_k\right) + \frac{t_j+1}{2}\sum_{l=0}^{1}\left(\sum_{p=0}^{N} u_p^{(l)} \sum_{k=0}^{N} B_l\left(s(t_j,\varphi_k)\right) F_p(\alpha_l s(t_j,\varphi_k) + \alpha_l - 1)\omega_k\right) + \frac{t_j+1}{2}\sum_{p=0}^{N} G_p \sum_{k=0}^{N} F_p\left(s(t_j,\varphi_k)\right)\omega_k, \tag{17}$$

$$u_j = u_{-1} + \frac{t_j+1}{2}\sum_{p=0}^{N} u_p' \ \sum_{k=0}^{N} F_p\left(s\left(t_j, \varphi_k\right)\right)\omega_k. \tag{18}$$

To compute $F_p(s)$ efficiently, we express it in terms of the Legendre functions of the form [8].

$$F_p(s) = \sum_{m=0}^{N} f_{pm} L_m(s), \tag{19}$$

where f_{pm} is called the discrete polynomial coefficients of F_p. The inverse relation is

$$f_{pm} = \frac{1}{\gamma_m}\sum_{m=0}^{N} F_m(x_i)L_m(x_i)\omega_i = L_m(x_i)/\gamma_m\,,\ \gamma_m = \left(m + {}^{1}\!/_{2}\right)^{-1}, m < N, \tag{20}$$

and $\gamma_N = \left(N + {}^{1}\!/_{2}\right)^{-1}$ for the Gauss and Gauss-Radau formulas.

3. Convergence analysis

To prove the convergence analysis of our method we first introduce the following useful lemmas.

3.1. Lemma 1

Assume that a $(N+1)$ −point Gauss, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight is used to integrate the product $u\varphi$, where $u \in H^m(I)$ with $I := (-1, 1)$ for some $m \geq 1$ and $\varphi \in p_N$. Then there exist a constant C independent of N such that [7].

$$\left|\int_{-1}^{1} u(x)\varphi(x)dx - \langle u, \varphi\rangle_N\right| \leq CN^{-m}|u|_{\hat{H},N(I)}\|\varphi\|_{L^2(I)},$$

where

$$|u|_{\hat{H},N(I)} = \left(\sum_{k=min\,(m,N)}^{m} \|u\|_{L^2(I)}.\right)^{1/2} \ and\ \langle u, \varphi\rangle_N = \sum_{k=0}^{N} \omega_k u(x_k)\varphi(x_k).$$

3.2. Lemma 2

Assume that $u \in H^m(I)$ and denote $I_N u$ the interpolation polynomial associated with the$(N+1)$ −point Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{x_k\}_{k=0}^{N}$. Then

$$\|u - I_N u\|_{L^2(I)} \leq CN^{-m}|u|_{\hat{H},N(I)} \tag{21}$$

$$\|u - I_N u\|_{L^\infty(I)} \leq CN^{3/4-m}|u|_{\hat{H},N(I)} \tag{22}$$

Proof. The estimation in Eq (21) is given on p. 289 of [7]. The following estimate is also given in [7].

$$\|u - I_N u\|_{H^s(I)} \le C N^{2s-1/2-m} |u|_{\tilde{H},N(I)}, \qquad 1 \le s \le m,$$

using the above estimate and the inequality

$$\|v\|_{L^\infty(a,b)} \le \sqrt{\frac{1}{b-a} + 2\|v\|_{L^2(a,b)}^{1/2} \|v\|_{H^1(a,b)}^{1/2}}, \qquad \forall\, v \in H^1(a,b),$$

one obtains the estimation given in Eq (22).

3.3. Lemma 3

Assume that $F_j(t)$ be the $j-th$ Lagrange interpolation polynomial with the $(N+1)$ −point Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{t_k\}_{k=0}^N$. Then

$$max_{t\in I} \sum_{j=0}^N \left| F_j(t) \right| \le C\sqrt{N}. \tag{23}$$

3.4. Lemma 4 (Gronwall inequality)

Let $T > 0$ and $C_1, C_2 \ge 0$. If a non-negative integrable function $E(t)$ satisfies

$$E(t) \le C_1 \int_0^t E(\alpha s)ds' + C_2 \int_0^t E(s)ds + G(t),\ \forall\, t\ \in [0,T], \tag{24}$$

where $0 < \alpha < 1$ is a constant and $G(t)$ is a nonnegative function, then

$$\|E\|_{L^\infty(I)} \le C\|G\|_{L^\infty(I)} \tag{25}$$

Proof. It follows from Eq (23) and a simple change of variable that

$$E(t) \le C_1 \frac{1}{\alpha} \int_0^t E(s)ds' + C_2 \int_0^t E(s)ds + G(t),\ \forall\, t\ \in [0,T], \tag{26}$$

since $0 < \alpha < 1$ and where $G(s) \ge 0$, we have

$$E(t) \le C\alpha^{-1} \int_0^t E(s)ds + G(t),$$

which is a standard Gronwall inequality. This leads to the estimate given in Eq (26).

3.5. Theorem 1

Consider the pantograph Eqs (3) and (4) and its spectral approximations Eqs (17) and (18). If the functions $\lambda_l(t)$ and $\mu_l(t)$ are smooth (which implies that the solution of Eqs (3) and (4) is also smooth), then

$$\left\|U^{(l)}-u^{(l)}\right\|_{L^{\infty}(I)} \leq CN^{3/4-m}\sum_{l=0}^{1}|\mathrm{A}_l u|_{\widehat{H}m-1,N(I)}+CN^{3/4-m}\sum_{l=0}^{1}|\mathrm{B}_l u|_{\widehat{H}m-1,N(I)}$$

$$+CN^{1/2-m}\textstyle\sum_{l=0}^{1}|\mathrm{B}_l|_{\widehat{H}m,,N(I)}\|u\|_{L^2(I)} \tag{27}$$

where U is the polynomial of degree N associated with the spectral approximation Eqs (17) and (18) and C is a constant independent of N.

Proof. Following the notation in Lemma 1, the numerical scheme given in Eqs (17) and (18), can be written as

$$u_j' = u_{-1}' + \frac{t_j+1}{2}\sum_{l=0}^{1}\left(\mathrm{A}_l\left(s(t_j,\varphi_k)\right)U^{(l)}\left(s(t_j,\varphi_k)\right)\right)_{N,t_j}$$

$$+\frac{t_j+1}{2}\sum_{l=0}^{1}\left(\mathrm{B}_l\left(s(t_j,\varphi_k)\right)U^{(l)}\left(\alpha_l s(t_j,\varphi_k)+\alpha_l-1\right)\right)_{N,t_j}$$

$$+\textstyle\int_{-1}^{t_j}G(s)ds \tag{28}$$

$$u_j = u_{-1} + \textstyle\int_{-1}^{t_j}U'(s)ds. \tag{29}$$

In order to use Lemma 1, we write Eqs (28) and (29) as

$$u_j' = u_{-1}' + \frac{t_j+1}{2}\sum_{l=0}^{1}\int_{-1}^{1}\left(\mathrm{A}_l\left(s(t_j,\varphi_k)\right)U^{(l)}\left(s(t_j,\varphi_k)\right)\right)d\varphi$$

$$+\frac{t_j+1}{2}\sum_{l=0}^{1}\int_{-1}^{1}\left(\mathrm{B}_l\left(s(t_j,\varphi_k)\right)U^{(l)}\left(\alpha_l s(t_j,\varphi_k)+\alpha_l-1\right)\right)d\varphi$$

$$+\textstyle\int_{-1}^{t_j}G(s)ds - \frac{t_j+1}{2}\sum_{l=0}^{1}\ I_l(t_j), \tag{30}$$

where

$$I_l(t) = u_{-1}' + \frac{t_j+1}{2}\int_{-1}^{1}\left(\mathrm{A}_l\left(s(t_j,\varphi_k)\right)U^{(l)}\left(s(t_j,\varphi_k)\right)\right)d\varphi - \left(\mathrm{A}_l\left(s(t_j,\varphi_k)\right)U^{(l)}\left(s(t_j,\varphi_k)\right)\right)_{N,t'}$$
$$+\int_{-1}^{1}\left(\mathrm{B}_l\left(s(t_j,\varphi_k)\right)U^{(l)}\left(\alpha_l s(t_j,\varphi_k)+\alpha_l-1\right)\right)d\varphi$$
$$-\left(\mathrm{B}_l\left(s(t_j,\varphi_k)\right)U^{(l)}\left(\alpha_l s(t_j,\varphi_k)+\alpha_l-1\right)\right)_{N,t'}.$$

Using the estimation given in Lemma 2, we get

$$\|I_l\|_{L^{\infty}(I)} \leq CN^{3/4-m}|A_l|_{\widehat{H}m,N(I)} + CN^{3/4-m}|B_l|_{\widehat{H}m,N(I)}\left\|U^{(l)}\right\|_{L^2(I)}$$

Multiplying $F_j(t)$ on both sides of Eqs (30) and (31) summing up from 0 to N yield

$$U'(t) = u_{-1}' + \textstyle\sum_{l=0}^{1} I_N\int_{-1}^{t}\mathrm{A}_l(s)U^{(l)}(s)ds + \sum_{l=0}^{1} I_N\int_{-1}^{t}\mathrm{B}_l(s)U^{(l)}(\alpha_l s+\alpha_l -$$
$$1)ds + I_N\textstyle\int_{-1}^{t}G(s)ds, \tag{31}$$

$$U(t) = u_{-1} + I_N \int_{-1}^{t_j} U'(s)ds. \tag{32}$$

Similarly, multiplying $F_j(t)$ on both sides of Eqs (10) and (11) summing up from 0 to N yield

$$I_N\, u'(t) = u'_{-1} + \sum_{l=0}^{1} I_N \int_{-1}^{t} \mathrm{A}_l(s)u^{(l)}(s)ds + \sum_{l=0}^{1} I_N \int_{-1}^{t} \mathrm{B}_l(s)u^{(l)}(\alpha_l s + \alpha_l - 1)ds + I_N \int_{-1}^{t} G(s)ds, \tag{33}$$

$$I_N u(t) = u_{-1} + I_N \int_{-1}^{t_j} u'(s)ds. \tag{34}$$

It follows from Eqs (31)–(34) that

$$\begin{aligned} e_{u'}(t) + I_N u'(t) - u'(t) &= \sum_{l=0}^{1} I_N \int_{-1}^{t} \mathrm{A}_l(s)e_{u'}(s)ds \\ &+ \sum_{l=0}^{1} I_N \int_{-1}^{t} \mathrm{B}_l(s)e_{u'}(\alpha_l s + \alpha_l - 1)ds + I_N \int_{-1}^{t} e_v(s)ds + \sum_{l=0}^{1} J_l(t), \end{aligned}$$

$$e_{u(t)} + I_N u(t) - u(t) = I_N \int_{-1}^{t} e_{u'}(s)ds,$$

where

$$e_{u'}(t) = u'(t) - U'(t),\ e_u(t) = u(t) - U(t),$$

Consequently,

$$e_{u'}(t) = \int_{-1}^{t} e_v(s)ds + \sum_{l=0}^{1} \int_{-1}^{t} \mathrm{A}_l(s)e_{u'}(s)ds + \sum_{l=0}^{1} \int_{-1}^{t} \mathrm{B}_l(s)e_{u'}(\alpha_l s + \alpha_l - 1)ds$$
$$+ \sum_{l=0}^{1} J_l(t) + f_1(t) + f_2(t) + \sum_{l=0}^{1} H_l(t)$$

$$e_u(t) = \int_{-1}^{t} e_{u'}(s)ds + f_3(t) + f_4(t),$$

where $f_1(t) = u'(t) - I_N U'(t),\ f_3(t) = u(t) - I_N u(t)$

$$f_2(t) = I_N \int_{-1}^{t} e_v(s)ds - \int_{-1}^{t} e_v(s)ds,\ f_4(t) = I_N \int_{-1}^{t} e_{u'}(s)ds - \int_{-1}^{t} e_{u'}(s)ds,$$

$$\begin{aligned} H_l(t) = I_N \int_{-1}^{t} \mathrm{A}_l(s)e_{u'}(s)ds - \int_{-1}^{t} \mathrm{A}_l(s)e_{u'}(s)ds + I_N \int_{-1}^{t} \mathrm{B}_l(s)e_{u'}(\alpha_l s + \alpha_l - 1)ds \\ - \int_{-1}^{t} \mathrm{B}_l(s)e_{u'}(\alpha_l s + \alpha_l - 1)ds \end{aligned}$$

Using Lemma 2,

$$\|f_1(t)\|_{L^\infty(I)} \le CN^{3/4-m}|u'|_{\hat{H}m,N(I)},\ \|f_3(t)\|_{L^\infty(I)} \le CN^{3/4-m}|u'|_{\hat{H}m,N(I)}$$

Using Lemma 2, with $m = 1$,

$$\|f_2(t)\|_{L^\infty(I)} \le CN^{-1/4}|e_v(t)|_{L^\infty(I)},\ \ \|f_4(t)\|_{L^\infty(I)} \le CN^{-1/4}|e_{u'}(t)|_{L^\infty(I)}$$

$$\|H_l(t)\|_{L^\infty(I)} \le CN^{-1/4}|e_{u'}(t)|_{L^\infty(I)}.$$

It follows from the Gronwall inequality presented in Lemma 4, to get

$$\left\|e_{u^{(1)}}\right\|_{L^\infty(I)} \le C\left(\|J_1\|_{L^\infty(I)} + \|J_2\|_{L^\infty(I)}\right)$$

Our next concern is the estimation of $\|J_1\|_{L^\infty(I)}\ and\ \|J_2\|_{L^\infty(I)}$. First

$$\|J_1\|_{L^\infty(I)} \le C\left\|I_{i,1}\right\|_{L^\infty(I)} + \left\|I_{i,2}\right\|_{L^\infty(I)} max_{t\in I}\sum_{j=0}^{N} F_j(t) \le CN^{1/2-m}\|U\|_{L^\infty(I)}$$

$$\le CN^{1/2-m}\left(\left\|e_{u^{(1)}}\right\|_{L^\infty(I)} + \|u\|_{L^\infty(I)}\right). \tag{35}$$

$$\|J_2\|_{L^\infty(I)} \le CN^{3/4-m}|\mathrm{A}_l u|_{\hat{H}m,N(I)} + CN^{3/4-m}|\mathrm{B}_l u(\alpha_l t)|_{\hat{H}m,N(I)}$$

The above two estimates, together with Eq (35), yields:

$$\left\|e_{u^{(1)}}\right\|_{L^\infty(I)} \le CN^{1/2-m}\left(\|J_1\|_{L^\infty(I)} + \|J_2\|_{L^\infty(I)}\right) + CN^{3/4-m}|\mathrm{A}_l u|_{\hat{H}m,N(I)}$$
$$+ CN^{3/4-m}|\mathrm{B}_l u(\alpha_l t)|_{\hat{H}m,N(I)},$$

which leads to the result of Theorem 1.

4. Numerical examples

In order to confirm the theoretical results, we perform some numerical test to illustrate the accuracy and efficiency of the proposed scheme.

4.1. Example 1

Consider the following constructed example [1]

$$\begin{cases} y''(x) = \dfrac{3}{4}y(x) + y\left(\dfrac{x}{2}\right) - x^2 + 2, & 0 \le x \le 1 \\ \qquad\qquad y(0) = y'(0) = 0. \end{cases}$$

The exact solution is given by $y(x) = x^2$. The maximum point-wise error between numerical solution and exact solution for different value of N is shown in Figure 1, Table 1.

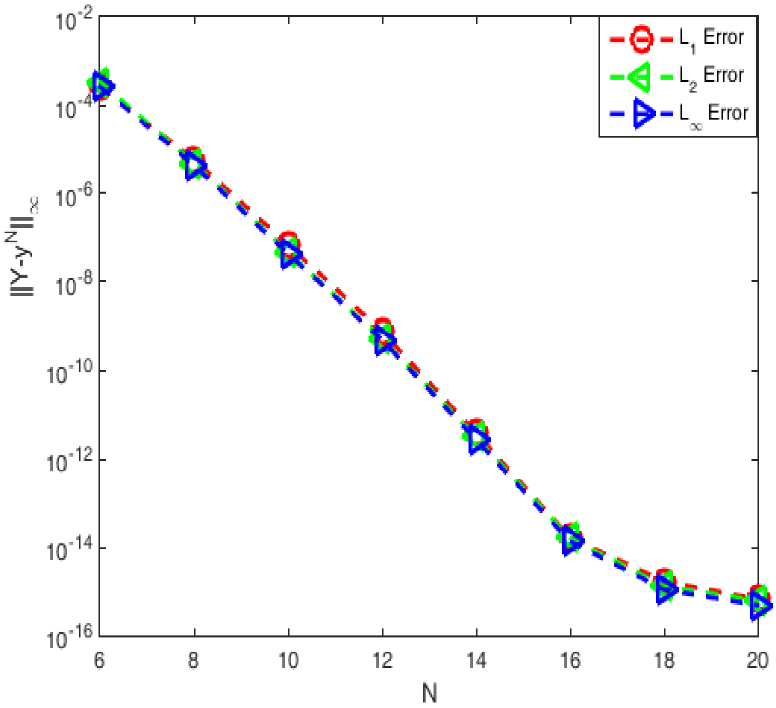

Figure 1. Example 1: The error behavior in different norm.

Table 1. Example 1: The point-wise error in different norms.

N	L_1 $error$	L_2 $error$	L_∞ $error$
6	$2.588e^{-04}$	$3.730e^{-04}$	$2.630e^{-04}$
8	$5.850e^{-06}$	$4.881e^{-06}$	$4.373e^{-06}$
10	$6.988e^{-08}$	$4.736e^{-08}$	$4.926e^{-08}$
12	$7.704e^{-10}$	$5.233e^{-10}$	$2.701e^{-10}$
14	$3.842e^{-12}$	$3.101e^{-12}$	$1.362e^{-12}$
16	$1.688e^{-14}$	$1.609e^{-14}$	$1.688e^{-14}$
18	$1.665e^{-15}$	$1.305e^{-15}$	$1.108e^{-15}$
20	$6.661e^{-16}$	$6.186e^{-16}$	$4.856e^{-16}$

4.2. Example 2

Consider the following nonlinear second-order equation of pantograph type

$$\begin{cases} y''(x) = -y(x) + 5y^2\left(\dfrac{x}{2}\right), & x \geq 0 \\ \quad y(0) = 1, \quad y(0) = -2. & \end{cases}$$

The exact solution is given by $y(x) = e^{-2x}$. The maximum point-wise error between numerical solution and exact solution for with respect to different N is shown in Figure 2, Table 2.

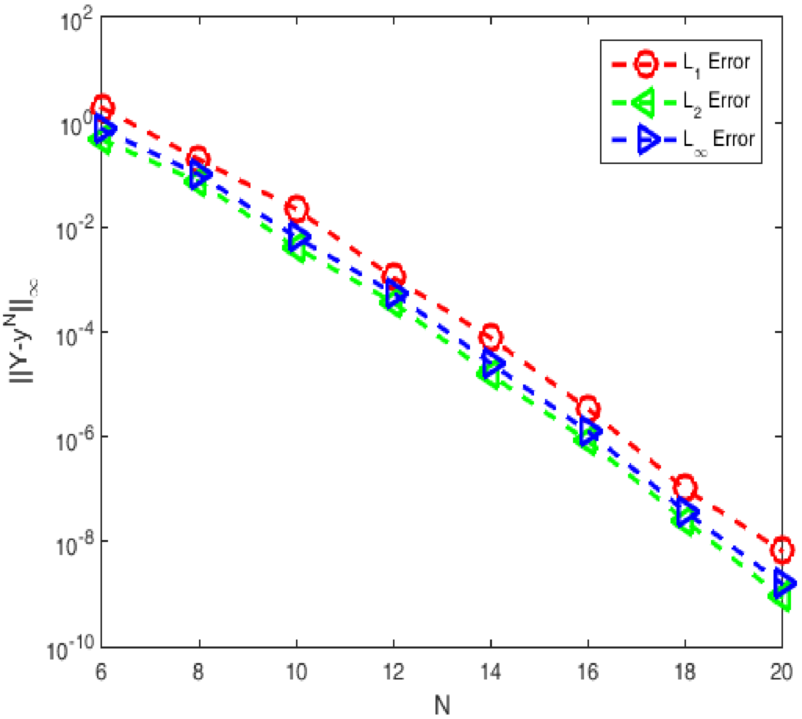

Figure 2. Example 2: The error behavior in different norm.

Table 2. Example 2: The point-wise error in different norms.

N	L_1 $error$	L_2 $error$	L_∞ $error$
6	$1.901e^{+00}$	$4.650e^{-01}$	$7.400e^{-01}$
8	$5.850e^{-01}$	$7.255e^{-02}$	$1.009e^{-01}$
10	$2.218e^{-02}$	$4.024e^{-03}$	$6.456e^{-03}$
12	$1.093e^{-03}$	$3.674e^{04}$	$5.585e^{-04}$
14	$7.824e^{-05}$	$1.537e^{-05}$	$2.584e^{-05}$
16	$3.480e^{-06}$	$8.481e^{-07}$	$1.295e^{-06}$
18	$1.039e^{-07}$	$2.590e^{-08}$	$3.863e^{-08}$
20	$6.763e^{-08}$	$8.806e^{-09}$	$1.602e^{-09}$

4.3. Example 3

Consider the following second-order equation

$$\begin{cases} y''(x) = -y\left(\frac{x}{2}\right) - y^2(t) + sin^4(x) + sin^2\left(\frac{x}{2}\right) + 8, \qquad x \geq 0 \\ y(0) = 2, \qquad y'(0) = 0. \end{cases}$$

The exact solution is given by $y(x) = \frac{5-cos2x}{2}$. The error behavior relative to N is displayed in Figure 3, Table 3.

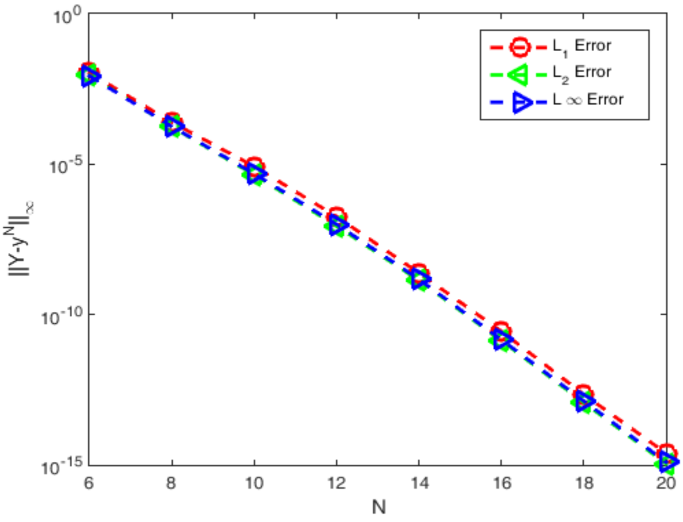

Figure 3. Example 3: The error behavior in different norms.

Table 3. Example 3: The point-wise error in different norms.

N	L_1 $error$	L_2 $error$	L_∞ $error$
6	$1.081e^{-02}$	$8.895e^{-03}$	$8.615e^{-03}$
8	$2.474e^{-04}$	$1.734e^{-04}$	$1.324e^{-04}$
10	$8.401e^{-06}$	$4.221e^{-06}$	$4.782e^{-06}$
12	$1.809e^{-07}$	$8.732e^{-08}$	$1.002e^{-07}$
14	$2.396e^{-09}$	$1.368e^{-09}$	$1.480e^{-09}$
16	$2.776e^{-11}$	$1.361e^{-11}$	$1.463e^{-11}$
18	$2.223e^{-13}$	$1.201e^{-13}$	$1.339e^{-13}$
20	$2.442e^{-15}$	$1.098e^{-15}$	$1.245e^{-15}$

5. Conclusions

A spectral method was introduced for the numerical solution of higher-order delay differential equation of pantograph equation to achieve the high order accuracy. A detail analysis of the proposed scheme is provided in L_∞ norm. By solving some numerical examples, it is shown that the error between the exact and numerical solution decays exponentially, which further authenticates our theoretical results.

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Conflict of interest

The author declares that they have no conflicts of interest to report regarding the present study.

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