Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(4): 4887–4897.
DOI:10.3934/math.2022272
Received: 25 October 2021
Revised: 09 December 2021
Accepted: 14 December 2021
Published: 28 December 2021

Research article

The Green's function for Caputo fractional boundary value problem with a convection term

Youyu Wang*, **Xianfei Li and Yue Huang**

Department of Mathematics, Tianjin University of Finance and Economics, Tianjin 300222, China

* **Correspondence:** Email: wang_youyu@163.com.

Abstract: By using the operator theory, we establish the Green's function for Caputo fractional differential equation under Sturm-Liouville boundary conditions. The results are new, the method used in this paper will provide some new ideas for the study of this kind of problems and easy to be generalized to solving other problems.

Keywords: Caputo derivative; Mittag-Leffler function; Green's function; convection term
Mathematics Subject Classification: 34A08, 34B27, 35B50

1. Introduction

In this paper, we give a new approach to construct Green's function for the following Caputo two-point boundary value problems with a constant convection coefficient

$$\begin{cases} -({}^C D_{a^+}^{\alpha} u)(t) + \lambda u'(t) = h(t), \quad a < t < b, \ 1 < \alpha < 2, & (1.1) \\ u(a) - \beta_0 u'(a) = \gamma_0, \ u(b) + \beta_1 u'(b) = \gamma_1, & (1.2) \end{cases}$$

where the constants $\lambda, \beta_0, \beta_1, \gamma_0, \gamma_1$ and the function $h \in C[a, b]$ are given.

In recent years, fractional differential equations are becoming a powerful tool to describe real-world phenomena, enormous numbers of very interesting and novel applications of fractional differential equations in physics, chemistry, engineering, finance, and other sciences have been developed. The Caputo derivative is especially suitable to describe real phenomena since in many ways it behaves like the usual derivative of integer order. In particular, the Caputo derivative of constant functions is zero, which is not true for the Riemann-Liouville derivative. However, the study of fractional differential equations with Caputo derivative is far from enough, there are still many basic problems to be solved, for example, the research on the expression of Green's function and its sign in the problem with a constant convection coefficient has aroused the interest of experts.

Papers such as [1–3] examine the non-negativity of Green's functions for Caputo two-point boundary value problems, but these papers contain no convection terms, Papers [4,5] consider Caputo two-point boundary value problems with convection, when the convection term is constant, the non-negative condition for Green's function is sufficient but not a necessary condition.

As a special case of problems (1.1)–(1.2), X. Meng and M. Stynes [6] consider the following Caputo two-point boundary value problems with a constant convection coefficient

$$\begin{cases} -({}^C D_{a^+}^{\alpha} u)(t) + \lambda u'(t) = h(t), \quad 0 < t < 1, \ 1 < \alpha < 2, & (1.3) \\ u(0) - \beta_0 u'(0) = \gamma_0, \ u(1) + \beta_1 u'(1) = \gamma_1, & (1.4) \end{cases}$$

an explicit formula for the associated Green's function is obtained by applying two-parameter Mittag-Leffler functions, and the necessary and sufficient conditions that ensure non-negativity of the Green's function can be deduce. This is the first derivation in the research literature of an explicit Green's function for a Caputo two-point boundary value problem with a convection term.

Recently, Z. Bai et al. [7] restudied problem (1.3)–(1.4), they constructed the Green's function by use of the Laplace transform.

Motivated by the works [6,7], in this paper, we will give the Green's function of boundary value problems (1.1)–(1.2) and generalize the results of [6,7]. Compared with these two articles, this paper includes the following features. Firstly, the operator theory is used in the process of solving the problem. By using operator theory, we generalize the conclusions of [6,7], these results cannot be obtained by using the methods provided in [6,7]. Secondly, the method provided in this paper may be more straightforward and easy to be generalized to solving other problems.

The paper is organized as follows. Some fundamental concepts and lemmas are described in Section 2 while Section 3 is devoted to the construction of Green's function for problem (1.1)–(1.2). In Section 4, we give the positive property of Green's function. Finally, the Green's function for multi-point boundary value problem is shown in Section 5.

2. Basic definitions and preliminaries

In this section, we will recall some of the necessary definitions and results that will be used in the main results.

Definition 2.1. [8] Let $\alpha \geq 0$ and f be a real function defined on $[a, b]$. The Riemann-Liouville fractional integral of order α is defined by $(I_{a^+}^0 f) \equiv f$ and

$$(I_{a^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, \ t \in [a, b].$$

Definition 2.2. [8] The Caputo fractional derivative of order $\alpha \geq 0$ is defined by $({}^C D_{a^+}^0 f) \equiv f$ and

$$({}^C D_{a^+}^{\alpha} f)(t) = (I_{a^+}^{m-\alpha} D^m f)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds,$$

for $\alpha > 0$, where m is the smallest integer greater or equal to α.

Lemma 2.3. *If* $\alpha \geq 0$ *and* $\beta > 0$*, then*

$$I_{a+}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}.$$

Lemma 2.4. *[8] Let $\alpha > 0$ and $n = [\alpha] + 1$, then*

$$I_{a^+}^{\alpha}({}^C D_{a^+}^{\alpha} u)(t) = u(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_n(t-a)^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \cdots, n$.

Definition 2.5. [8] The Mittag-Leffler function is defined by:

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)},\ \alpha > 0.$$

The two-parameter Mittag-Leffler function is defined by:

$$E_{\alpha,\gamma}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \gamma)},\ \alpha > 0.$$

For convenience, we denote $F_\beta(x) = x^{\beta-1}E_{\alpha-1,\beta}[\lambda x^{\alpha-1}]$, the following properties of F_β have been deduced in [6].
$(P_1) : [F_{\beta+1}(x)]' = F_\beta(x)$ for $\beta \geq 0, x \geq 0$;
$(P_2) : F_1(0) = 1, F_\beta(0) = 0$ for $\beta > 1$;
$(P_3) : F_1(x) > 0$ for $x > 0$, $F_2(x)$ is increasing for $x \geq 0$;
$(P_4) : F_{\alpha-1}(x) > 0$ for $x > 0$, $F_\alpha(x)$ is increasing for $x > 0$.
Now, we prove the important result used in this paper.

Lemma 2.6. *For any $\lambda \in R, \alpha > 0$, we have the following results.*
(1) For any $r \in C([a,b],\mathbb{R})$, series $\sum_{k=0}^{\infty} \lambda^k I_{a^+}^{k\alpha} r(t)$ is convergent and the sum is

$$\sum_{k=0}^{\infty} \lambda^k I_{a^+}^{k\alpha} r(t) = r(t) + \lambda \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[\lambda(t-s)^\alpha] r(s) ds.$$

(2) The operator $I - \lambda I_{a^+}^{\alpha} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is reversible and:

$$(I - \lambda I_{a^+}^{\alpha})^{-1} r(t) = \sum_{k=0}^{\infty} \lambda^k I_{a^+}^{k\alpha} r(t).$$

Proof. (1) By the properties of Mittag-Leffler function and two-parameter Mittag-Leffler function, we have

$$\begin{aligned}\lambda \int_a^t (t-s)^{\alpha-1} E_{\alpha,\alpha}[\lambda(t-s)^\alpha] r(s) ds &= \lambda \int_a^t (t-s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k (t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)} r(s) ds \\ &= \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{\Gamma(k\alpha+\alpha)} \int_a^t (t-s)^{k\alpha+\alpha-1} r(s) ds = \sum_{k=0}^{\infty} \lambda^{k+1} I_{a^+}^{k\alpha+\alpha} r(t) = \sum_{k=1}^{\infty} \lambda^k I_{a^+}^{k\alpha} r(t),\end{aligned}$$

and

$$r(t) + \sum_{k=1}^{\infty} \lambda^k I_{a^+}^{k\alpha} r(t) = \sum_{k=0}^{\infty} \lambda^k I_{a^+}^{k\alpha} r(t).$$

Thus, (1) is proved.
(2) First, we show that $(I-\lambda I_{a^+}^{\alpha})\left(\sum_{k=0}^{\infty}\lambda^k I_{a^+}^{k\alpha}r(t)\right)=r(t)$. In fact, it is easy to see

$$\begin{aligned}(I-\lambda I_{a^+}^{\alpha})\left(\sum_{k=0}^{\infty}\lambda^k I_{a^+}^{k\alpha}r(t)\right)&=\sum_{k=0}^{\infty}\lambda^k I_{a^+}^{k\alpha}r(t)-\lambda I_{a^+}^{\alpha}\sum_{k=0}^{\infty}\lambda^k I_{a^+}^{k\alpha}r(t)\\&=\sum_{k=0}^{\infty}\lambda^k I_{a^+}^{k\alpha}r(t)-\sum_{k=0}^{\infty}\lambda^{k+1} I_{a^+}^{k\alpha+\alpha}r(t)\\&=\sum_{k=0}^{\infty}\lambda^k I_{a^+}^{k\alpha}r(t)-\sum_{k=1}^{\infty}\lambda^k I_{a^+}^{k\alpha}r(t)=r(t).\end{aligned}$$

Similarly, we can easily prove the fact that:

$$\sum_{k=0}^{\infty}\lambda^k I_{a^+}^{k\alpha}(I-\lambda I_{a^+}^{\alpha})r(t)=r(t).$$

□

3. The Green's function of Problem (1.1)–(1.2)

Theorem 3.1. *Assume that $\beta_0\geq 0,\beta_1\geq 0$. The boundary value problem (1.1)–(1.2) has a unique solution*

$$u(t)=\int_a^b G(t,s)h(s)ds+\gamma_1\sigma(t)+\gamma_0[1-\sigma(t)],$$

where

$$G(t,s)=\begin{cases}\sigma(t)[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)]-F_\alpha(t-s), & a\leq s\leq t\leq b,\\ \sigma(t)[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)], & a\leq t\leq s\leq b,\end{cases}$$
$$\sigma(t)=\frac{\beta_0+F_2(t-a)}{\beta_0+\beta_1F_1(b-a)+F_2(b-a)}.$$

Proof. Applying $I_{a^+}^{\alpha}$ to the both sides of the equation (1.1), we have:

$$-I_{a^+}^{\alpha}({}^CD_{a^+}^{\alpha}u)(t)+\lambda I_{a^+}^{\alpha}u'(t)=I_{a^+}^{\alpha}h(t),$$

From Lemmas 2.4,

$$-u(t)+c_0+c_1(t-a)-\frac{\lambda}{\Gamma(\alpha)}u(a)(t-a)^{\alpha-1}+\lambda I_{a^+}^{\alpha-1}u(t)=I_{a^+}^{\alpha}h(t),$$

let $t=a$, we obtain $u(a)=c_0$, so

$$-u(t)+c_0+c_1(t-a)-\frac{\lambda}{\Gamma(\alpha)}c_0(t-a)^{\alpha-1}+\lambda I_{a^+}^{\alpha-1}u(t)=I_{a^+}^{\alpha}h(t),$$

or

$$((I-\lambda I_{a^+}^{\alpha-1})u)(t)=c_0+c_1(t-a)-\frac{\lambda}{\Gamma(\alpha)}c_0(t-a)^{\alpha-1}-I_{a^+}^{\alpha}h(t),$$

by Lemma 2.6, we have

$$\begin{aligned}
u(t) =&(I-\lambda I_{a^+}^{\alpha-1})^{-1}\left(c_0+c_1(t-a)-\frac{\lambda}{\Gamma(\alpha)}c_0(t-a)^{\alpha-1}-I_{a^+}^{\alpha}h(t)\right)\\
=&\sum_{k=0}^{\infty}\lambda^k I_{a^+}^{k(\alpha-1)}\left(c_0+c_1(t-a)-\frac{\lambda}{\Gamma(\alpha)}c_0(t-a)^{\alpha-1}-I_{a^+}^{\alpha}h(t)\right)\\
=&c_0E_{\alpha-1,1}[\lambda(t-a)^{\alpha-1}]+c_1(t-a)E_{\alpha-1,2}[\lambda(t-a)^{\alpha-1}]\\
&-\lambda c_0(t-a)^{\alpha-1}E_{\alpha-1,\alpha}[\lambda(t-a)^{\alpha-1}]-\int_a^t(t-s)^{\alpha-1}E_{\alpha-1,\alpha}[\lambda(t-s)^{\alpha-1}]h(s)ds\\
=&c_0F_1(t-a)+c_1F_2(t-a)-\lambda c_0F_\alpha(t-a)-\int_a^t F_\alpha(t-s)h(s)ds\\
=&c_0+c_1F_2(t-a)-\int_a^t F_\alpha(t-s)h(s)ds,
\end{aligned}$$

and

$$u'(t)=c_1F_1(t-a)-\int_a^t F_{\alpha-1}(t-s)h(s)ds,$$

by the boundary conditions $u(a)-\beta_0u'(a)=\gamma_0$ and $u(b)+\beta_1u'(b)=\gamma_1$, we can get

$$\begin{cases} c_0-\beta_0c_1=\gamma_0,\\ c_0+[\beta_1F_1(b-a)+F_2(b-a)]c_1=\gamma_1+\int_a^b[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)]h(s)ds,\end{cases}$$

thus,

$$\begin{aligned}
c_0&=\gamma_0+\frac{\beta_0(\gamma_1-\gamma_0)+\beta_0\int_a^b[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)]h(s)ds}{\beta_0+\beta_1F_1(b-a)+F_2(b-a)},\\
c_1&=\frac{\gamma_1-\gamma_0+\int_a^b[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)]h(s)ds}{\beta_0+\beta_1F_1(b-a)+F_2(b-a)},
\end{aligned}$$

therefore,

$$\begin{aligned}
u(t)=&c_0+c_1F_2(t-a)-\int_a^t F_\alpha(t-s)h(s)ds\\
=&(\beta_0+F_2(t-a))\frac{\int_a^b[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)]h(s)ds}{\beta_0+\beta_1F_1(b-a)+F_2(b-a)}-\int_a^t F_\alpha(t-s)h(s)ds\\
&+\gamma_0+\frac{\beta_0(\gamma_1-\gamma_0)}{\beta_0+\beta_1F_1(b-a)+F_2(b-a)}+\frac{\gamma_1-\gamma_0}{\beta_0+\beta_1F_1(b-a)+F_2(b-a)}F_2(t-a)\\
=&\sigma(t)\int_a^b[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)]h(s)ds-\int_a^t F_\alpha(t-s)h(s)ds+\gamma_1\sigma(t)+\gamma_0[1-\sigma(t)]\\
=&\int_a^b G(t,s)h(s)ds+\gamma_1\sigma(t)+\gamma_0[1-\sigma(t)].
\end{aligned}$$

□

4. The positive property of the Green's function

A function $h(x)$ is said to be *log-concave* if $\ln h(x)$ is concave, i.e., $(\ln h(x))'' \leq 0$. Similarly, a function $h(x)$ is said to be *log-convex* if $\ln h(x)$ is convex, i.e., $(\ln h(x))'' \geq 0$.

Lemma 4.1. *[6] Fix $\tau \in (0,1]$. Then for $x > 0$, the functions $x^\tau E_{\tau,\tau+1}(x^\tau)$ and $E_{\tau,1}(x^\tau)$ are log-concave.*

Lemma 4.2. *[6] Fix $\tau \in (0,1]$. Then for $x > 0$, the functions $x^\tau E_{\tau,\tau+1}(-x^\tau)$ is log-concave; $E_{\tau,1}(-x^\tau)$ and $x^{\tau-1}E_{\tau,\tau}(-x^\tau)$ are log-convex.*

Theorem 4.3. *Fix $t \in [a,b]$. Then for $a \leq s \leq t$,*

$$f_t(s) := \frac{F_\alpha(t-s)}{F_\alpha(b-s) + \beta_1 F_{\alpha-1}(b-s)}$$

is a decreasing function of s.

Proof. **(I).** If $\lambda = 0$, then

$$f_t(s) = \frac{(t-s)^{\alpha-1}}{(b-s)^{\alpha-1} + \beta_1(\alpha-1)(b-s)^{\alpha-2}},$$

it is easy to check that

$$\begin{aligned} f_t'(s) &= f_t(s)\left[-\frac{\alpha-1}{t-s} + \frac{\alpha-2}{b-s} + \frac{1}{b-s+(\alpha-1)\beta_1}\right] \\ &< f_t(s)\left[-\frac{\alpha-1}{b-s+(\alpha-1)\beta_1} + \frac{\alpha-2}{b-s+(\alpha-1)\beta_1} + \frac{1}{b-s+(\alpha-1)\beta_1}\right] \\ &= 0, \end{aligned}$$

so the conclusion holds.

(II). If $\lambda \neq 0$. By Lemma 4.1 and 4.2, the function $|\lambda| F_\alpha(x)$ is log-concave when $x > 0$. Thus, one can infer from the property (P_1) that for $x > 0$,

$$\left(\frac{F_{\alpha-1}(x)}{F_\alpha(x)}\right)' = \left(\frac{|\lambda| F_{\alpha-1}(x)}{|\lambda| F_\alpha(x)}\right)' = [\ln(|\lambda| F_\alpha(x))]'' \leq 0.$$

This shows that $\frac{F_{\alpha-1}(x)}{F_\alpha(x)}$ is a decreasing function for $x > 0$. Consequently

$$\frac{F_{\alpha-1}(t-s)}{F_\alpha(t-s)} \geq \frac{F_{\alpha-1}(b-s)}{F_\alpha(b-s)}, \text{ for } a \leq s < t,$$

or

$$F_\alpha(t-s)F_{\alpha-1}(b-s) - F_{\alpha-1}(t-s)F_\alpha(b-s) \leq 0, \text{ for } a \leq s < t. \tag{4.1}$$

That is, the numerator of $\frac{\partial}{\partial s}\left(\frac{F_\alpha(b-s)}{F_\alpha(t-s)}\right)$ is non-negative. So, $\frac{F_\alpha(b-s)}{F_\alpha(t-s)}$ is an increasing function of s. As $\frac{F_{\alpha-1}(x)}{F_\alpha(x)}$ is a decreasing function for $x > 0$, the function $\frac{F_{\alpha-1}(b-s)}{F_\alpha(b-s)}$ is an increasing function of s. Consequently,

$\frac{F_{\alpha-1}(b-s)}{F_\alpha(b-s)} \cdot \frac{F_\alpha(b-s)}{F_\alpha(t-s)} = \frac{F_{\alpha-1}(b-s)}{F_\alpha(t-s)}$ is also increasing on s. By considering its derivative with respect to s, we obtain

$$F_\alpha(t-s)F_{\alpha-2}(b-s) - F_{\alpha-1}(t-s)F_{\alpha-1}(b-s) \le 0. \tag{4.2}$$

By (4.1) and (4.2), the numerator of $f_t'(s)$ is

$$\begin{aligned}
& F_\alpha(t-s)[F_{\alpha-1}(b-s) + \beta_1 F_{\alpha-2}(b-s)] - F_{\alpha-1}(t-s)[F_\alpha(b-s) + \beta_1 F_{\alpha-1}(b-s)] \\
=& F_\alpha(t-s)F_{\alpha-1}(b-s) - F_{\alpha-1}(t-s)F_\alpha(b-s) \\
& + \beta_1[F_\alpha(t-s)F_{\alpha-2}(b-s) - F_{\alpha-1}(t-s)F_{\alpha-1}(b-s)] \\
\le& 0,
\end{aligned}$$

hence $f_t(s)$ is a decreasing function of $s \in [a,t]$. □

Lemma 4.4. *For $a \le t \le b$, the function*

$$g(t) := \frac{\beta_1 F_1(b-a) + F_2(b-a) - F_2(t-a)}{\beta_1 F_{\alpha-1}(b-a) + F_\alpha(b-a) - F_\alpha(t-a)}$$

is an increasing function of t.

Proof. **(I).** If $\lambda = 0$, then for $\beta \ge 0$, $F_\beta(x) = \frac{1}{\Gamma(\beta)}x^{\beta-1}$, thus

$$\begin{aligned}
g(t) &= \Gamma(\alpha-1)\frac{\beta_1 + (b-a) - (t-a)}{\beta_1(b-a)^{\alpha-2} + \frac{(b-a)^{\alpha-1}-(t-a)^{\alpha-1}}{\alpha-1}} \\
&= \Gamma(\alpha-1)\frac{\beta_1 + (b-a)[1-\frac{t-a}{b-a}]}{\beta_1(b-a)^{\alpha-2} + \frac{(b-a)^{\alpha-1}}{\alpha-1}[1-(\frac{t-a}{b-a})^{\alpha-1}]},
\end{aligned}$$

it is easy to check that

$$\left(\frac{t-a}{b-a}\right)^{2-\alpha} < 1, \quad \frac{1}{\alpha-1}\left(\frac{t-a}{b-a}\right)^{2-\alpha}\left[1-\left(\frac{t-a}{b-a}\right)^{\alpha-1}\right] < 1 - \frac{t-a}{b-a},$$

we have

$$\begin{aligned}
\frac{1}{\Gamma(\alpha-1)}g'(t) &= g(t)\left[\frac{-1}{\beta_1 + (b-a)[1-\frac{t-a}{b-a}]} + \frac{1}{\beta_1(\frac{t-a}{b-a})^{2-\alpha} + \frac{b-a}{\alpha-1}(\frac{t-a}{b-a})^{2-\alpha}[1-(\frac{t-a}{b-a})^{\alpha-1}]}\right] \\
&> 0,
\end{aligned}$$

hence $g(t)$ is an increasing function of t.

(II). If $\lambda > 0$, then by Lemma 4.1 the function $F_1(x) = \lambda F_\alpha(x) + 1 = x^{\alpha-1}E_{\alpha-1,\alpha}(\lambda x^{\alpha-1}) + 1$ is log-concave when $x > 0$, so

$$\left(\frac{(\lambda F_\alpha(x)+1)'}{\lambda F_\alpha(x)+1}\right)' \le 0, \text{ for } x > 0,$$

which imply

$$\left(\frac{F_1(x)}{F_{\alpha-1}(x)}\right)' = \left(\frac{\lambda F_\alpha(x)+1}{(F_\alpha(x))'}\right)' = \lambda\left(\frac{\lambda F_\alpha(x)+1}{(\lambda F_\alpha(x)+1)'}\right)' \geq 0, \text{ for } x>0.$$

(III). If $\lambda<0$, we can similarly prove that $\left(\frac{F_1(x)}{F_{\alpha-1}(x)}\right)' \geq 0$.

Suppose $t\in(a,b)$, By the Cauchy mean value theorem, there exists $\xi\in(t-a,b-a)$ such that

$$\frac{F_2(b-a)-F_2(t-a)}{F_\alpha(b-a)-F_\alpha(t-a)} = \frac{F_1(\xi)}{F_{\alpha-1}(\xi)} \geq \frac{F_1(t-a)}{F_{\alpha-1}(t-a)}.$$

Equivalently,

$$[F_2(b-a)-F_2(t-a)]F_{\alpha-1}(t-a)-F_1(t-a)[F_\alpha(b-a)-F_\alpha(t-a)] \geq 0. \tag{4.3}$$

Furthermore, $\left(\frac{F_1(x)}{F_{\alpha-1}(x)}\right)' \geq 0$ implies that

$$\frac{F_1(b-a)}{F_{\alpha-1}(b-a)} \geq \frac{F_1(t-a)}{F_{\alpha-1}(t-a)},$$

or

$$F_1(b-a)F_{\alpha-1}(t-a)-F_1(t-a)F_{\alpha-1}(b-a) \geq 0. \tag{4.4}$$

By (4.3) and (4.4), the sign of the numerator of $g'(t)$ is

$$\begin{aligned}
&[\beta_1F_1(b-a)+F_2(b-a)-F_2(t-a)]F_{\alpha-1}(t-a)\\
&\quad -F_1(t-a)[\beta_1F_{\alpha-1}(b-a)+F_\alpha(b-a)-F_\alpha(t-a)]\\
=&[F_2(b-a)-F_2(t-a)]F_{\alpha-1}(t-a)-F_1(t-a)[F_\alpha(b-a)-F_\alpha(t-a)]\\
&\quad +\beta_1[F_1(b-a)F_{\alpha-1}(t-a)-F_1(t-a)F_{\alpha-1}(b-a)]\\
\geq&0.
\end{aligned}$$

Hence $g(t)$ is an increasing function of $t\in[a,b]$. □

The main result of this paper is as follow.

Theorem 4.5. *Assume $\beta_1\geq 0$. Then the Green's function $G(t,s)$ is nonnegative if and only if*

$$\beta_0 \geq -F_2(b-a)+F_1(b-a)\frac{F_\alpha(b-a)}{F_{\alpha-1}(b-a)}.$$

Proof. The Green's function $G(t,s)$ is nonnegative on $[a,b]\times[a,b]$ if and only if

$$\sigma(t)[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)]-F_\alpha(t-s)\geq 0, \qquad \text{for } a\leq s\leq t\leq b.$$

Equivalently,

$$\beta_0 \geq -\beta_1F_1(b-a)-F_2(b-a)+\frac{\beta_1F_1(b-a)+F_2(b-a)-F_2(t-a)}{1-\frac{F_\alpha(t-s)}{F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)}}, \text{ for } a\leq s\leq t\leq b. \tag{4.5}$$

by Theorem 4.3 and Lemma 4.4, inequality (4.5) is equivalent to

$$\begin{aligned}
\beta_0 &\geq \max_{a\leq s\leq t\leq b}\left[-\beta_1F_1(b-a)-F_2(b-a)+\frac{\beta_1F_1(b-a)+F_2(b-a)-F_2(t-a)}{1-f_t(s)}\right]\\
&\overset{s=a}{=} \max_{a\leq t\leq b}\left[-\beta_1F_1(b-a)-F_2(b-a)+\frac{\beta_1F_1(b-a)+F_2(b-a)-F_2(t-a)}{1-f_t(a)}\right]\\
&= \max_{a\leq t\leq b}\left[-\beta_1F_1(b-a)-F_2(b-a)+[F_\alpha(b-a)+\beta_1F_{\alpha-1}(b-a)]g(t)\right]\\
&\overset{t=b}{=} -\beta_1F_1(b-a)-F_2(b-a)+[F_\alpha(b-a)+\beta_1F_{\alpha-1}(b-a)]\frac{F_1(b-a)}{F_{\alpha-1}(b-a)}\\
&= -F_2(b-a)+F_1(b-a)\frac{F_\alpha(b-a)}{F_{\alpha-1}(b-a)}.
\end{aligned}$$

The proof is complete. □

5. The Green's function for multi-point boundary condition

In this section, we present the Green's function for the following multi-point boundary value problem

$$\begin{cases} -({}^CD^\alpha_{a^+}u)(t)+\lambda u'(t)=h(t), \quad a<t<b,\ 1<\alpha<2, & (5.1)\\ u(a)-\beta_0u'(a)=0,\ u(b)+\beta_1u'(b)=\sum_{i=1}^{m-2}\gamma_iu(\xi_i), & (5.2)\end{cases}$$

where the constants $\lambda,\beta_0,\beta_1,\gamma_i>0 (i=1,2,\cdots,m-2), a<\xi_1<\cdots<\xi_{m-2}<b$ and the function $h\in C[a,b]$ are given.

Theorem 5.1. *Assume that $\beta_0\geq 0,\beta_1\geq 0$. The boundary value problem (5.1)–(5.2) has a unique solution*

$$u(t)=\int_a^b\left[G(t,s)+\frac{\sigma(t)}{1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i)}\sum_{i=1}^{m-2}\gamma_iG(\xi_i,s)\right]h(s)ds,$$

where

$$G(t,s)=\begin{cases}\sigma(t)[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)]-F_\alpha(t-s), & a\leq s\leq t\leq b,\\ \sigma(t)[F_\alpha(b-s)+\beta_1F_{\alpha-1}(b-s)], & a\leq t\leq s\leq b,\end{cases}$$

$$\sigma(t)=\frac{\beta_0+F_2(t-a)}{\beta_0+\beta_1F_1(b-a)+F_2(b-a)}.$$

Proof. For convenience, we denote $\rho=\beta_0+\beta_1F_1(b-a)+F_2(b-a)$, then we have the relations

$$\begin{aligned}
&\beta_0+F_2(t-a)=\rho\sigma(t),\\
&\beta_1F_1(b-a)+F_2(b-a)-\sum_{i=1}^{m-2}\gamma_iF_2(\xi_i-a)=\rho\left[1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i)\right]-\beta_0\left(1-\sum_{i=1}^{m-2}\gamma_i\right),\\
&\beta_0(1-\sum_{i=1}^{m-2}\gamma_i)+\beta_1F_1(b-a)+F_2(b-a)-\sum_{i=1}^{m-2}\gamma_iF_2(\xi_i-a)=\rho\left[1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i)\right].
\end{aligned}$$

According to the proof of Theorem 3.1, $u(t)$ and $u'(t)$ satisfy

$$u(t) = c_0 + c_1 F_2(t-a) - \int_a^t F_\alpha(t-s)h(s)ds,$$
$$u'(t) = c_1 F_1(t-a) - \int_a^t F_{\alpha-1}(t-s)h(s)ds,$$

by the boundary conditions $u(a)-\beta_0 u'(a) = 0$ and $u(b)+\beta_1 u'(b) = \sum_{i=1}^{m-2}\gamma_i u(\xi_i)$, we can get $c_0-\beta_0 c_1 = 0$ and

$$\begin{aligned}&\left(1-\sum_{i=1}^{m-2}\gamma_i\right)c_0 + \left(\rho(1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i)) - \beta_0(1-\sum_{i=1}^{m-2}\gamma_i)\right)c_1\\ &= \int_a^b [F_\alpha(b-s)+\beta_1 F_{\alpha-1}(b-s)]h(s)ds - \sum_{i=1}^{m-2}\gamma_i \int_a^{\xi_i} F_\alpha(\xi_i - s)h(s)ds,\end{aligned}$$

thus,

$$\begin{aligned}c_0 =& \frac{\beta_0 \int_a^b [F_\alpha(b-s)+\beta_1 F_{\alpha-1}(b-s)]h(s)ds - \beta_0\sum_{i=1}^{m-2}\gamma_i\int_a^{\xi_i} F_\alpha(\xi_i-s)h(s)ds}{\rho(1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i))}\\ =& \frac{\beta_0 \int_a^b [F_\alpha(b-s)+\beta_1 F_{\alpha-1}(b-s)]h(s)ds}{\rho}\\ &+\beta_0\frac{\sum_{i=1}^{m-2}\gamma_i\int_a^b \sigma(\xi_i)[F_\alpha(b-s)+\beta_1 F_{\alpha-1}(b-s)]h(s)ds - \sum_{i=1}^{m-2}\gamma_i\int_a^{\xi_i} F_\alpha(\xi_i-s)h(s)ds}{\rho(1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i))}\\ =& \frac{\beta_0 \int_a^b [F_\alpha(b-s)+\beta_1 F_{\alpha-1}(b-s)]h(s)ds}{\rho} + \beta_0\frac{\sum_{i=1}^{m-2}\gamma_i\int_a^b G(\xi_i,s)h(s)ds}{\rho(1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i))},\\ c_1 =& \frac{\int_a^b [F_\alpha(b-s)+\beta_1 F_{\alpha-1}(b-s)]h(s)ds - \sum_{i=1}^{m-2}\gamma_i\int_a^{\xi_i} F_\alpha(\xi_i-s)h(s)ds}{\rho(1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i))}\\ =& \frac{\int_a^b [F_\alpha(b-s)+\beta_1 F_{\alpha-1}(b-s)]h(s)ds}{\rho} + \frac{\sum_{i=1}^{m-2}\gamma_i\int_a^b G(\xi_i,s)h(s)ds}{\rho(1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i))},\end{aligned}$$

therefore,

$$\begin{aligned}u(t) &= c_0 + c_1 F_2(t-a) - \int_a^t F_\alpha(t-s)h(s)ds\\ &= \int_a^b G(t,s)h(s)ds + \frac{\sigma(t)}{1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i)}\sum_{i=1}^{m-2}\gamma_i\int_a^b G(\xi_i,s)h(s)ds\\ &= \int_a^b \left[G(t,s) + \frac{\sigma(t)}{1-\sum_{i=1}^{m-2}\gamma_i\sigma(\xi_i)}\sum_{i=1}^{m-2}\gamma_i G(\xi_i,s)\right]h(s)ds.\end{aligned}$$

□

6. Conclusions

In this paper, by use of the operator theory, the Green's function for a class of Sturm-Liouville fractional boundary value problems is obtained. Compare with other literature, our method provide some new ideas for the study of this kind of problems and easy to be generalized to solving other related problems.

Acknowledgments

The authors would like to thank the referees for their helpful comments and suggestions. This work is supported by the Tianjin Natural Science Foundation (grant no. 20JCYBJC00210).

References

1. R. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta. Appl. Math.*, **109** (2010), 973–1033. https://doi.org/10.1007/s10440-008-9356-6
2. M. Benchohra, J. Graef, S. Hamani, Existence results for boundary value problems with non-linear fractional differential equations, *Appl. Anal.*, **87** (2008), 851–863. https://doi.org/10.1080/00036810802307579
3. S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electron. J. Differ. Eq.*, **2006** (2006), 36.
4. M. Al-Refai, Basic results on nonlinear eigenvalue problems of fractional order, *Electron. J. Differ. Eq.*, **2012** (2012), 191.
5. M. Al-Refai, On the fractional derivatives at extreme points, *Electron. J. Qual. Theory Differ. Equ.*, **2012** (2012), 55.
6. X. Meng, M. Stynes, The Green function and a maximum principle for a Caputo two-point boundary value problem with a convection term, *J. Math. Anal. Appl.*, **461** (2018), 198–218. https://doi.org/10.1016/j.jmaa.2018.01.004
7. Z. Bai, S. Sun, Z. Du, Y. Chen, The Green function for a class of Caputo fractional differential equations with a convection term, *Fract. Calc. Appl. Anal.*, **23** (2020), 787–798. https://doi.org/10.1515/fca-2020-0039
8. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.