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*Research article*

## Weyl almost periodic solutions for quaternion-valued shunting inhibitory cellular neural networks with time-varying delays

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**Abstract:** We consider the existence and stability of Weyl almost periodic solutions for a class of quaternion-valued shunting inhibitory cellular neural networks with time-varying delays. In order to overcome the incompleteness of the space composed of Weyl almost periodic functions, we first obtain the existence of a bounded continuous solution of the system under consideration by using the fixed point theorem, and then prove that the bounded solution is Weyl almost periodic by using a variant of Gronwall inequality. Then we study the global exponential stability of the Weyl almost periodic solution by using the inequality technique. Even when the system we consider degenerates into a real-valued one, our results are new. A numerical example is given to illustrate the feasibility of our results.

**Keywords:** quaternion-valued neural network; Weyl almost periodic solution; global exponential stability; time-varying delays

**Mathematics Subject Classification:** 34K14, 34K20, 92B20

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### 1. Introduction

Since shunting inhibitory cellular neural networks were proposed by Bouzerdoum and Pinter [1] as a new type of neural networks, they have received more and more attention and have been widely applied in optimisation, psychophysics, speech and other fields. At the same time, since time delays are ubiquitous, many research results have been obtained on the dynamics of shunting inhibitory cellular neural networks with time delays [2–6].

On the one hand, the quaternion is a generalization of real and complex numbers [7]. The skew field of quaternions is defined by

$$\mathbb{H} := \{q = q^R + iq^I + jq^J + kq^K\},$$

where  $q^R, q^I, q^J, q^K \in \mathbb{R}$  and the elements  $i, j$  and  $k$  obey the Hamilton's multiplication rules:

$$ij = -jk = k, \quad jk = -ki = i, \quad ki = -ij = j, \quad i^2 = j^2 = k^2 = -1.$$

For  $q = q^R + iq^I + jq^J + kq^K$ , we denote  $\check{q} = iq^I + jq^J + kq^K$  and  $q^R = q - \check{q}$ . The norm of  $q$  is defined by  $\|q\|_{\mathbb{H}} = \sqrt{(q^R)^2 + (q^I)^2 + (q^J)^2 + (q^K)^2}$ . For  $y = (y_1, y_2, \dots, y_n)^T \in \mathbb{H}^n$ , we define  $\|y\|_{\mathbb{H}^n} = \max_{1 \leq p \leq n} (\|y_p\|_{\mathbb{H}})$ , then  $(\mathbb{H}^n, \|\cdot\|_{\mathbb{H}^n})$  is a Banach space. As we all know, quaternion-valued neural networks include real-valued neural networks and complex-valued neural networks as their special cases. Compared with complex-valued neural networks, quaternion-valued neural networks only needs half of the connection weight parameters of complex-valued neural networks when dealing with multi-level information [8]. In recent years, quaternion-valued neural networks have attracted the attention of many researchers, and their various dynamic behaviors, including fractional-order and stochastic quaternion-valued neural networks, have been extensively studied [9–24].

On the other hand, because periodic and almost periodic oscillations are important dynamics of neural networks, the periodic and almost periodic oscillations of neural networks have been studied a lot in the past few decades [25–33]. Weyl almost periodicity is a generalization of Bohr almost periodicity and Stepanov almost periodicity [34–37]. It is a more complex recurrent oscillation. Because the spaces composed of Bohr almost periodic functions and Stepanov almost periodic functions are Banach spaces, it brings some convenience to study the existence of almost periodic solutions in these two senses of differential equations. Therefore, many results have been obtained on the Bohr almost periodic oscillation and Stepanov almost periodic oscillation of neural networks. However, the space composed of Weyl almost periodic functions is incomplete [38]. Therefore, the results of Weyl almost periodic solutions of neural networks are still very rare. Therefore, it is a meaningful and challenging work to study the existence of Weyl almost periodic solutions of neural networks.

Motivated by the above, in this paper, we consider the following shunting inhibitory cellular neural networks with time-varying delays:

$$\begin{aligned} \dot{x}_{ij}(t) = & -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(t)f_{ij}(x_{kl}(t))x_{ij}(t) \\ & - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(t)g_{ij}(x_{kl}(t - \tau_{kl}(t)))x_{ij}(t) + I_{ij}(t), \end{aligned} \quad (1.1)$$

where  $ij \in \{11, 12, \dots, 1n, \dots, m1, m2, \dots, mn\} := \Lambda$ ,  $C_{ij}$  denotes the cell at the  $(i, j)$  position of the lattice. The  $r$ -neighborhood  $N_r(i, j)$  of  $C_{ij}$  is given as

$$N_r(i, j) = \{C_{kl} : \max(|k - i|, |l - j|) \leq r, 1 \leq k \leq m, 1 \leq l \leq n\},$$

and  $N_s(i, j)$  is similarly specified;  $x_{ij}(t) \in \mathbb{H}$  denotes the activity of the cell of  $C_{ij}$ ,  $I_{ij}(t) \in \mathbb{H}$  is the external input to  $C_{ij}$ ,  $a_{ij}(t) \in \mathbb{H}$  is the coefficient of the leakage term, which represents the passive decay rate of the activity of the cell  $C_{ij}$ ,  $B_{ij}^{kl}(t) \geq 0$  and  $C_{ij}^{kl}(t) \geq 0$  represent the connection or coupling strength of postsynaptic of activity of the cell transmitted to the cell  $C_{ij}$ , the activity functions  $f_{ij}, g_{ij} : \mathbb{H} \rightarrow \mathbb{H}$  are continuous functions representing the output or firing rate of the cell  $C_{ij}$ ,  $\tau_{kl}(t)$  corresponds to the transmission delay and satisfies  $0 \leq \tau_{kl}(t) \leq \tau$ .

The purpose of this paper is to use the fixed point theorem and a variant of Gronwall inequality to establish the existence and global exponential stability of Weyl almost periodic solutions for a class

of quaternion-valued shunting inhibitory cellular neural networks whose coefficients of the leakage terms are quaternions. This is the first paper to study the existence and global exponential stability of Weyl almost periodic solutions of system (1.1) by using the fixed point theorem and a variant of Gronwall inequality. Our result of this paper is new, and our method can be used to study other types of quaternion-valued neural networks.

For convenience, we introduce the following notations:

$$a^m = \min_{ij \in \Lambda} \left\{ \inf_{t \in \mathbb{R}} \{a_{ij}(t)\} \right\}, \quad a^M = \max_{ij \in \Lambda} \left\{ \sup_{t \in \mathbb{R}} \{a_{ij}(t)\} \right\}, \quad \check{a}_{ij}^M = \sup_{t \in \mathbb{R}} \|\check{a}_{ij}(t)\|_{\mathbb{H}},$$

$$B_{ij}^{klM} = \sup_{t \in \mathbb{R}} \{B_{ij}^{kl}(t)\}, \quad C_{ij}^{klM} = \sup_{t \in \mathbb{R}} \{C_{ij}^{kl}(t)\}, \quad \tau_{ij}^M = \sup_{t \in \mathbb{R}} \{\tau_{ij}(t)\}, \quad \tau' = \max_{kl \in \Lambda} \left\{ \sup_{t \in \mathbb{R}} \{\tau'_{kl}(t)\} \right\}.$$

The initial condition of system (1.1) is given by

$$x_{ij}(s) = \varphi_{ij}(s), \quad s \in [-\tau, 0],$$

where  $\varphi_{ij} \in C(\mathbb{R}, \mathbb{H})$ ,  $ij \in \Lambda$ .

Throughout this paper, we assume that:

- (H<sub>1</sub>) For  $ij, kl \in \Lambda$ , functions  $a_{ij}^R, B_{ij}^{kl}, C_{ij}^{kl} \in AP(\mathbb{R}, \mathbb{R}^+)$ ,  $\check{a}_{ij} \in AP(\mathbb{R}, \mathbb{H})$ ,  $I_{ij} \in APW^p(\mathbb{R}, \mathbb{H})$ ,  $\tau_{kl} \in AP(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R})$  and  $\tau' < 1$ .
- (H<sub>2</sub>) For  $ij \in \Lambda$ , there exist positive constants  $L_{ij}^f$  and  $L_{ij}^g$  such that for all  $x, y \in \mathbb{H}$ ,

$$\|f_{ij}(x) - f_{ij}(y)\|_{\mathbb{H}} \leq L_{ij}^f \|x - y\|_{\mathbb{H}}, \quad \|g_{ij}(x) - g_{ij}(y)\|_{\mathbb{H}} \leq L_{ij}^g \|x - y\|_{\mathbb{H}},$$

$$\text{and } f_{ij}(\mathbf{0}) = g_{ij}(\mathbf{0}) = \mathbf{0}.$$

The rest of this paper is arranged as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, we study the existence and global exponential stability of Weyl almost periodic solutions of (1.1). In Section 4, an example is given to verify the theoretical results. This paper ends with a brief conclusion in Section 5.

## 2. Preliminaries

Let  $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$  be a Banach space and  $BC(\mathbb{R}, \mathbb{X})$  be the set of all bounded continuous functions from  $\mathbb{R}$  to  $\mathbb{X}$ .

**Definition 2.1.** [38] A function  $f \in BC(\mathbb{R}, \mathbb{X})$  is said to be almost periodic, if for every  $\epsilon > 0$ , there exists a constant  $l = l(\epsilon) > 0$  such that in every interval of length  $l(\epsilon)$  contains at least one  $\sigma$  such that

$$\|f(t + \sigma) - f(t)\|_{\mathbb{X}} < \epsilon, \quad t \in \mathbb{R}.$$

Denote by  $AP(\mathbb{R}, \mathbb{X})$  the set of all such functions.

For  $p \in [1, \infty)$ , we denote by  $L_{loc}^p(\mathbb{R}, \mathbb{X})$  the space of all functions from  $\mathbb{R}$  into  $\mathbb{X}$  which are locally  $p$ -integrable. For  $f \in L_{loc}^p(\mathbb{R}, \mathbb{X})$ , we define the following seminorm:

$$\|f\|_{W^p} = \lim_{r \rightarrow +\infty} \sup_{\beta \in \mathbb{R}} \left( \frac{1}{r} \int_{\beta}^{\beta+r} \|f(t)\|_{\mathbb{X}}^p dt \right)^{\frac{1}{p}}.$$

**Definition 2.2.** [38] A function  $f \in L_{loc}^p(\mathbb{R}, \mathbb{X})$  is said to be  $p$ -th Weyl almost periodic ( $W^p$ -almost periodic for short), if for every  $\epsilon > 0$ , there exists a constant  $l = l(\epsilon) > 0$  such that in every interval of length  $l(\epsilon)$  contains at least one  $\sigma$  such that

$$\|f(t + \sigma) - f(t)\|_{W^p} < \epsilon.$$

This  $\sigma$  is called on  $\epsilon$ -translation number of  $f$ . The set of all such functions will be denoted by  $APW^p(\mathbb{R}, \mathbb{X})$ .

**Remark 2.1.** By Definitions 2.1 and 2.2, it is easy to see that if  $f \in AP(\mathbb{R}, \mathbb{X})$ , then  $f \in APW^p(\mathbb{R}, \mathbb{X})$ .

Similar to the proofs of the lemma on page 83 and the lemma on page 84 of [39], it is not difficult to prove the following two lemmas.

**Lemma 2.1.** If  $f \in APW^p(\mathbb{R}, \mathbb{X})$ , then  $f$  is bounded and uniformly continuous on  $\mathbb{R}$  with respect to the seminorm  $\|\cdot\|_{W^p}$ .

Using the argumentation contained in the proof of Proposition 3.21 in [38], one can easily prove the following.

**Lemma 2.2.** If  $f_k \in APW^p(\mathbb{R}, \mathbb{X})$ ,  $k = 1, 2, \dots, n$ . Then, for every  $\epsilon > 0$ , there exist common  $\epsilon$ -translation numbers for these functions.

**Lemma 2.3.** [40] Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that, for every  $t \in \mathbb{R}$ ,

$$0 \leq g(t) \leq \rho(t) + \gamma_1 \int_{-\infty}^t e^{-\eta_1(t-s)} g(s) ds + \dots + \gamma_n \int_{-\infty}^t e^{-\eta_n(t-s)} g(s) ds \quad (2.1)$$

for some locally integrable function  $\rho : \mathbb{R} \rightarrow \mathbb{R}$ , and for some constants  $\gamma_1, \dots, \gamma_n \geq 0$ , and some constants  $\eta_1, \dots, \eta_n \geq \gamma$ , where  $\gamma = \sum_{p=1}^n \gamma_p$ . We assume that the integrals in the right hand side of

(2.1) are convergent. Let  $\eta = \min\{\eta_1, \dots, \eta_n\}$ . Then, for every  $\xi \in (0, \eta - \gamma]$  such that  $\int_{-\infty}^0 e^{\xi s} \rho(s) ds$  converges, we have, for every  $t \in \mathbb{R}$ ,

$$g(t) \leq \rho(t) + \gamma \int_{-\infty}^t e^{-\xi(t-s)} \rho(s) ds.$$

In particular, if  $\rho(t)$  is constant, we have

$$g(t) \leq \rho \frac{\eta}{\eta - \gamma}.$$

### 3. Main results

Let  $BUC(\mathbb{R}, \mathbb{H}^{m \times n})$  be a collection of bounded and uniformly continuous functions from  $\mathbb{R}$  to  $\mathbb{H}^{m \times n}$ , then, the space  $BUC(\mathbb{R}, \mathbb{H}^{m \times n})$  with the norm  $\|x\|_{\infty} = \sup_{t \in \mathbb{R}} \|x(t)\|_{\mathbb{H}^{m \times n}}$  is a Banach space, where  $x \in BUC(\mathbb{R}, \mathbb{H}^{m \times n})$ .

Denote  $\phi^0 = (\phi_{11}^0, \dots, \phi_{1n}^0, \phi_{21}^0, \dots, \phi_{2n}^0, \dots, \phi_{m1}^0, \dots, \phi_{mn}^0)^T$ , where

$$\phi_{ij}^0(t) = \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v) dv} I_{ij}(s) ds, \quad ij \in \Lambda.$$

We will show that  $\phi^0$  is well defined under assumption  $(H_1)$ . In fact, by  $I_{ij} \in APW^p(\mathbb{R}, \mathbb{H})$  and Lemma 2.1, there exists a constant  $M > 0$  such that  $\|I_{ij}\|_{W^p} \leq M$  for all  $ij \in \Lambda$ . According to the Hölder inequality, one has

$$\begin{aligned} \|\phi_{ij}^0(t)\|_{\mathbb{H}} &\leq \left\| \int_{-\infty}^t e^{-a^m(t-s)} I_{ij}(s) ds \right\|_{\mathbb{H}} \\ &\leq \sum_{r=0}^{\infty} \left( \int_{t-(r+1)}^{t-r} e^{-a^m q(t-s)} ds \right)^{\frac{1}{q}} \left( \int_{t-(r+1)}^{t-r} \|I_{ij}(s)\|_{\mathbb{H}}^p ds \right)^{\frac{1}{p}} \\ &\leq \sum_{r=0}^{\infty} e^{-a^m r} M < +\infty, \end{aligned} \quad (3.1)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ , which means that  $\phi^0$  is well defined.

Take a positive constant  $\alpha \geq \|\phi^0\|_{\infty}$ . Let

$$\Omega = \{\phi \in BUC(\mathbb{R}, \mathbb{H}^{m \times n}) \mid \|\phi - \phi^0\|_{\infty} \leq \alpha\}.$$

Then, for every  $\phi \in \Omega$ , one has

$$\|\phi\|_{\infty} \leq \|\phi - \phi^0\|_{\infty} + \|\phi^0\|_{\infty} \leq 2\alpha.$$

**Theorem 3.1.** Assume  $(H_1)$ – $(H_2)$  hold. Furthermore, suppose that

$(H_3)$

$$\kappa = \max_{ij \in \Lambda} \left\{ \frac{2}{a^m} \left[ \check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right] \right\} < 1,$$

$(H_4)$  for  $p > 2$ ,

$$\begin{aligned} &\max_{ij \in \Lambda} \left\{ 24 \left( \frac{2p-4}{a^m p} \right)^{p-2} \left( \frac{4}{a^m p} \right)^2 \left[ 2(\check{\alpha}_{ij}^M)^p + 2 \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^p \right. \right. \\ &\left. \left. + \left( 1 + \frac{2e^{\frac{p}{4} a^m \tau}}{1 - \tau'} \right) \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^p \right] \right\} < 1, \end{aligned}$$

and, for  $p = 2$ ,

$$\begin{aligned} &\max_{ij \in \Lambda} \left\{ 24 \left( \frac{2}{a^m} \right)^2 \left[ 2(\check{\alpha}_{ij}^M)^2 + 2 \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^2 \right. \right. \\ &\left. \left. + \left( 1 + \frac{2e^{\frac{1}{2} a^m \tau}}{1 - \tau'} \right) \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^2 \right] \right\} < 1, \end{aligned}$$

then system (1.1) has a unique  $W^p$ -almost periodic solution in  $\Omega$ .

*Proof.* It is easy to check that if  $x = (x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn})^T \in \Omega$  is a solution of the integral equation

$$\begin{aligned} x_{ij}(t) = & \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \left[ -\check{a}_{ij}(s)x_{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)f_{ij}(x_{kl}(s))x_{ij}(s) \right. \\ & \left. - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s)g_{ij}(x_{kl}(s - \tau_{kl}(s)))x_{ij}(s) + I_{ij}(s) \right] ds, \quad ij \in \Lambda, \end{aligned} \quad (3.2)$$

then  $x$  is a solution of system (1.1).

Define an operator  $T : \Omega \rightarrow \mathbb{H}^{m \times n}$  by

$$(T\phi)(t) = ((T_{11}\phi)(t), \dots, (T_{1n}\phi)(t), (T_{21}\phi)(t), \dots, (T_{2n}\phi)(t), \dots, (T_{m1}\phi)(t), \dots, (T_{mn}\phi)(t))^T,$$

where

$$\begin{aligned} (T_{ij}\phi)(t) = & \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \left[ -\check{a}_{ij}(s)\phi_{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)f_{ij}(\phi_{kl}(s))\phi_{ij}(s) \right. \\ & \left. - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s)g_{ij}(\phi_{kl}(s - \tau_{kl}(s)))\phi_{ij}(s) + I_{ij}(s) \right] ds, \quad ij \in \Lambda. \end{aligned}$$

Now, we will prove that  $T\phi$  is well defined. Actually, by  $(H_1)$ – $(H_3)$  and (3.1), for  $ij \in \Lambda$ , one deduces that

$$\begin{aligned} \|(T_{ij}\phi)(t)\|_{\mathbb{H}} & \leq \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \left\| -\check{a}_{ij}(s)\phi_{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)(f_{ij}(\phi_{kl}(s)) - f_{ij}(\mathbf{0}))\phi_{ij}(s) \right. \\ & \quad \left. - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s)(g_{ij}(\phi_{kl}(s - \tau_{kl}(s))) - g_{ij}(\mathbf{0}))\phi_{ij}(s) \right\|_{\mathbb{H}} ds + \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \|I_{ij}(s)\|_{\mathbb{H}} ds \\ & \leq \frac{1}{\alpha^m} \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} L_{ij}^f \|\phi\|_{\infty} + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} L_{ij}^g \|\phi\|_{\infty} \right) \|\phi\|_{\infty} \\ & \quad + \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \|I_{ij}(s)\|_{\mathbb{H}} ds \\ & < +\infty. \end{aligned} \quad (3.3)$$

That is,  $T\phi$  is well defined.

We will divide the rest of the proof into four steps.

*Step 1*, we will prove that  $T\phi \in BUC(\mathbb{R}, \mathbb{H}^{m \times n})$ , for every  $\phi \in \Omega$ .

In fact, by (3.3), we see that  $T\phi$  is bounded on  $\mathbb{R}$ . So, we only need to show that  $T\phi$  is uniformly continuous on  $\mathbb{R}$ . Based on the Hölder inequality for  $0 \leq h \leq 1$  and  $q \geq 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , one has

$$\begin{aligned} & \|(T_{ij}\phi)(t+h) - (T_{ij}\phi)(t)\|_{\mathbb{H}} \\ = & \left\| \int_{-\infty}^{t+h} e^{-\int_s^{t+h} a_{ij}^R(v)dv} \left( -\check{a}_{ij}(s)\phi_{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)f_{ij}(\phi_{kl}(s))\phi_{ij}(s) - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s) \right. \right. \end{aligned}$$

$$\begin{aligned}
 & \times g_{ij}(\phi_{kl}(s - \tau_{kl}(s)))\phi_{ij}(s) + I_{ij}(s) \Big) ds - \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \left( -\check{a}_{ij}(s)\phi_{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s) \right. \\
 & \times f_{ij}(\phi_{kl}(s))\phi_{ij}(s) - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s)g_{ij}(\phi_{kl}(s - \tau_{kl}(s)))\phi_{ij}(s) + I_{ij}(s) \Big) ds \Big\|_{\mathbb{H}} \\
 \leq & \int_{-\infty}^t \left| e^{-\int_s^{t+h} a_{ij}^R(v)dv} - e^{-\int_s^t a_{ij}^R(v)dv} \right| \left\| \check{a}_{ij}(s)\phi_{ij}(s) + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)f_{ij}(\phi_{kl}(s))\phi_{ij}(s) + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s) \right. \\
 & \times g_{ij}(\phi_{kl}(s - \tau_{kl}(s)))\phi_{ij}(s) \Big\|_{\mathbb{H}} ds + \int_{-\infty}^t \left| e^{-\int_s^{t+h} a_{ij}^R(v)dv} - e^{-\int_s^t a_{ij}^R(v)dv} \right| \|I_{ij}(s)\|_{\mathbb{H}} ds \\
 & + \int_t^{t+h} e^{-\int_s^{t+h} a_{ij}^R(v)dv} \left\| \check{a}_{ij}(s)\phi_{ij}(s) + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)f_{ij}(\phi_{kl}(s))\phi_{ij}(s) + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s) \right. \\
 & \times g_{ij}(\phi_{kl}(s - \tau_{kl}(s)))\phi_{ij}(s) \Big\|_{\mathbb{H}} ds + \int_t^{t+h} e^{-\int_s^{t+h} a_{ij}^R(v)dv} \|I_{ij}(s)\|_{\mathbb{H}} ds \\
 \leq & \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} L_{ij}^f \|\phi\|_{\infty} + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} L_{ij}^g \|\phi\|_{\infty} \right) \|\phi\|_{\infty} \int_{-\infty}^t \left( e^{-\int_s^t a_{ij}^R(v)dv} \int_t^{t+h} a_{ij}^R(v)dv \right) ds \\
 & + \int_{-\infty}^t \left( e^{-\int_s^t a_{ij}^R(v)dv} \int_t^{t+h} a_{ij}^R(v)dv \right) \|I_{ij}(s)\|_{\mathbb{H}} ds + \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} L_{ij}^f \|\phi\|_{\infty} \right. \\
 & \left. + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} L_{ij}^g \|\phi\|_{\infty} \right) \|\phi\|_{\infty} \int_t^{t+h} e^{-\int_s^t a_{ij}^R(v)dv} ds + \int_t^{t+h} e^{-\int_s^t a_{ij}^R(v)dv} \|I_{ij}(s)\|_{\mathbb{H}} ds \\
 \leq & a^M h \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} L_{ij}^f \|\phi\|_{\infty} + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} L_{ij}^g \|\phi\|_{\infty} \right) \|\phi\|_{\infty} \int_{-\infty}^t e^{-a^m(t-s)} ds \\
 & + a^M h \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \|I_{ij}(s)\|_{\mathbb{H}} ds + \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} L_{ij}^f \|\phi\|_{\infty} + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} L_{ij}^g \|\phi\|_{\infty} \right) \|\phi\|_{\infty} \\
 & \times \int_t^{t+h} e^{-a^m(t-s)} ds + \left( \int_t^{t+h} e^{-qa^m(t-s)} ds \right)^{\frac{1}{q}} \left( \int_t^{t+h} \|I_{ij}(s)\|_{\mathbb{H}}^p ds \right)^{\frac{1}{p}} \\
 \leq & \frac{a^M}{a^m} \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} L_{ij}^f \|\phi\|_{\infty} + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} L_{ij}^g \|\phi\|_{\infty} \right) \|\phi\|_{\infty} h + a^M h \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \|I_{ij}(s)\|_{\mathbb{H}} ds \\
 & + e^{a^m h} \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} L_{ij}^f \|\phi\|_{\infty} + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} L_{ij}^g \|\phi\|_{\infty} \right) \|\phi\|_{\infty} h + e^{a^m h} \|I_{ij}\|_{W^p} h,
 \end{aligned}$$

where  $ij \in \Lambda$ . Hence, letting  $h \rightarrow 0^+$ , by (3.1), we have

$$\|(T_{ij}\phi)(t+h) - (T_{ij}\phi)(t)\|_{\mathbb{H}} \rightarrow 0,$$

which means that  $(T_{ij}\phi)$  is uniformly continuous on  $\mathbb{R}$ ,  $ij \in \Lambda$ . Therefore,  $T\phi \in BUC(\mathbb{R}, \mathbb{H}^{m \times n})$ .

*Step 2*, we will prove that  $T$  is a self-mapping from  $\Omega$  to  $\Omega$ .

Actually, for arbitrary  $\phi \in \Omega$ , from  $(H_2)$ – $(H_3)$ , we have

$$\|T\phi - \phi^0\|_{\infty} \leq \sup_{t \in \mathbb{R}} \left\{ \max_{ij \in \Lambda} \left[ \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v)dv} \left\| \check{a}_{ij}(s)\phi_{ij}(s) + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)(f_{ij}(\phi_{kl}(s)) - f(\mathbf{0}))\phi_{ij}(s) \right. \right. \right.$$

$$\begin{aligned}
& + \left. \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s) (g_{ij}(\phi_{kl}(s - \tau_{kl}(s))) - g_{ij}(\mathbf{0})) \phi_{ij}(s) \right\|_{\mathbb{H}} ds \Big\} \\
& \leq \max_{ij \in \Lambda} \left\{ \frac{1}{a^m} \left[ \check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right] \right\} \|\phi\|_{\infty} \\
& \leq \kappa \alpha \leq \alpha,
\end{aligned}$$

which implies that  $T\phi \in \Omega$ . Consequently,  $T$  is a self-mapping from  $\Omega$  to  $\Omega$ .

*Step 3*, we will prove  $T$  is a contraction mapping.

As a matter of fact, in view of  $(H_1)$ – $(H_2)$ , for any  $\phi, \nu \in \Omega$ , we can get

$$\begin{aligned}
& \|T\phi - T\nu\|_{\infty} \\
& \leq \sup_{t \in \mathbb{R}} \left\{ \max_{ij \in \Lambda} \left[ \int_{-\infty}^t e^{-a^m(t-s)} \left( \check{\alpha}_{ij}^M \|\phi_{ij}(s) - \nu_{ij}(s)\|_{\mathbb{H}} + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} \|f_{ij}(\phi_{kl}(s))(\phi_{ij}(s) - \nu_{ij}(s)) \right. \right. \right. \\
& \quad + (f_{ij}(\phi_{kl}(s)) - f_{ij}(\nu_{kl}(s))) \nu_{ij}(s)\|_{\mathbb{H}} + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} \|g_{ij}(\phi_{kl}(s - \tau_{kl}(s))) (\phi_{ij}(s) - \nu_{ij}(s)) \\
& \quad \left. \left. \left. + (g_{ij}(\phi_{kl}(s - \tau_{kl}(s))) - g_{ij}(\nu_{kl}(s - \tau_{kl}(s)))) \nu_{ij}(s)\|_{\mathbb{H}} \right) ds \right] \right\} \\
& \leq \sup_{t \in \mathbb{R}} \left\{ \max_{ij \in \Lambda} \left[ \int_{-\infty}^t e^{-a^m(t-s)} \left[ \check{\alpha}_{ij}^M \|\phi_{ij}(s) - \nu_{ij}(s)\|_{\mathbb{H}} + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{klM} (2L_{ij}^f \alpha \|\phi_{ij}(s) - \nu_{ij}(s)\|_{\mathbb{H}} \right. \right. \right. \\
& \quad + 2L_{ij}^f \alpha \|\phi_{kl}(s) - \nu_{kl}(s)\|_{\mathbb{H}}) + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} (2L_{ij}^g \alpha \|\phi_{ij}(s) - \nu_{ij}(s)\|_{\mathbb{H}} \\
& \quad \left. \left. \left. 2L_{ij}^g \alpha \|\phi_{kl}(s - \tau_{kl}(s)) - \nu_{kl}(s - \tau_{kl}(s))\|_{\mathbb{H}}) \right] ds \right] \right\} \\
& \leq \max_{ij \in \Lambda} \left\{ \frac{1}{a^m} \left[ \check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha \right] \right\} \|\phi - \nu\|_{\infty}.
\end{aligned}$$

From this and  $(H_3)$ , one has

$$\|T\phi - T\nu\|_{\infty} \leq \kappa \|\phi - \nu\|_{\infty}.$$

Noticing that  $\kappa < 1$ ,  $T$  is a contraction mapping. Consequently, system (1.1) has a unique solution  $x$  in  $\Omega$ .

*Step 4*, we will prove that the unique solution  $x \in \Omega$  is  $W^p$ -almost periodic.

Indeed, since  $x = (x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn})^T \in \Omega$ ,  $x$  is bounded and uniformly continuous. Hence, for every  $\epsilon > 0$ , there exists a  $\delta \in (0, \epsilon)$  such that for any  $t_1, t_2 \in \mathbb{R}$  with  $|t_1 - t_2| < \delta$  and  $ij \in \Lambda$ , we have

$$\|x_{ij}(t_1) - x_{ij}(t_2)\|_{\mathbb{H}} < \epsilon. \quad (3.4)$$

Also, for this  $\delta$ , in view of  $(H_1)$  and Lemma 2.2, we see that there exists a common  $\delta$ -translation number  $\sigma$  such that

$$\limsup_{r \rightarrow +\infty} \sup_{\beta \in \mathbb{R}} \left( \frac{1}{r} \int_{\beta}^{\beta+r} \|I_{ij}(t + \sigma) - I_{ij}(t)\|_{\mathbb{H}}^p dt \right)^{\frac{1}{p}} < \delta < \epsilon, \quad (3.5)$$



$$|B_{ij}^{kl}(t + \sigma) - B_{ij}^{kl}(t)| < \epsilon, \quad (3.6)$$

$$|C_{ij}^{kl}(t + \sigma) - C_{ij}^{kl}(t)| < \epsilon, \quad (3.7)$$

$$|a_{ij}^R(t + \sigma) - a_{ij}^R(t)| < \epsilon, \|\check{a}_{ij}(t + \sigma) - \check{a}_{ij}(t)\|_{\mathbb{H}} < \epsilon \quad (3.8)$$

and

$$|\tau_{ij}(t + \sigma) - \tau_{ij}(t)| < \delta, \quad (3.9)$$

where  $ij, kl \in \Lambda$ . Consequently, from (3.4) and (3.9), we get

$$\|x_{ij}(t - \tau_{ij}(t + \sigma)) - x_{ij}(t - \tau_{ij}(t))\|_{\mathbb{H}} < \epsilon. \quad (3.10)$$

Since  $x$  is a solution of system (1.1), by (3.2), for  $ij \in \Lambda$ , we have

$$\begin{aligned} & \|x_{ij}(t + \sigma) - x_{ij}(t)\|_{\mathbb{H}} \\ & \leq \left\| \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v+\sigma)dv} (\check{a}_{ij}(s + \sigma)x_{ij}(s + \sigma) - \check{a}_{ij}(s)x_{ij}(s)) ds \right\|_{\mathbb{H}} \\ & \quad + \left\| \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v+\sigma)dv} \sum_{C_{kl} \in N_r(i,j)} [B_{ij}^{kl}(s + \sigma)f_{ij}(x_{kl}(s + \sigma))x_{ij}(s + \sigma) \right. \\ & \quad \left. - B_{ij}^{kl}(s)f_{ij}(x_{kl}(s))x_{ij}(s)] ds \right\|_{\mathbb{H}} + \left\| \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v+\sigma)dv} \sum_{C_{kl} \in N_s(i,j)} [C_{ij}^{kl}(s + \sigma) \right. \\ & \quad \left. \times g_{ij}(x_{kl}(s + \sigma - \tau_{kl}(s + \sigma)))x_{ij}(s + \sigma) - C_{ij}^{kl}(s)g_{ij}(x_{kl}(s - \tau_{kl}(s)))x_{ij}(s)] ds \right\|_{\mathbb{H}} \\ & \quad + \left\| \int_{-\infty}^t e^{-\int_s^t a_{ij}^R(v+\sigma)dv} [I_{ij}(s + \sigma) - I_{ij}(s)] ds \right\|_{\mathbb{H}} \\ & \quad + \left\| \int_{-\infty}^t (e^{-\int_s^t a_{ij}^R(v+\sigma)dv} - e^{-\int_s^t a_{ij}^R(v)dv}) \check{a}_{ij}(s)x_{ij}(s) ds \right\|_{\mathbb{H}} \\ & \quad + \left\| \int_{-\infty}^t (e^{-\int_s^t a_{ij}^R(v+\sigma)dv} - e^{-\int_s^t a_{ij}^R(v)dv}) \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)f_{ij}(x_{kl}(s))x_{ij}(s) ds \right\|_{\mathbb{H}} \\ & \quad + \left\| \int_{-\infty}^t (e^{-\int_s^t a_{ij}^R(v+\sigma)dv} - e^{-\int_s^t a_{ij}^R(v)dv}) \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(s)g_{ij}(x_{kl}(s - \tau_{kl}(s)))x_{ij}(s) ds \right\|_{\mathbb{H}} \\ & \quad + \left\| \int_{-\infty}^t (e^{-\int_s^t a_{ij}^R(v+\sigma)dv} - e^{-\int_s^t a_{ij}^R(v)dv}) I_{ij}(s) ds \right\|_{\mathbb{H}} \\ & := \sum_{l=1}^8 B_{lij}(t). \end{aligned} \quad (3.11)$$

When  $p > 2$ , it follows from Hölder's inequality  $(\frac{2}{p}, \frac{p-2}{p})$ , Hölder's inequality  $(\frac{1}{2}, \frac{1}{2})$  and  $(H_2)$  that

$$B_{2ij}(t) \leq \int_{-\infty}^t e^{-a^m(t-s)} \sum_{C_{kl} \in N_r(i,j)} \|B_{ij}^{kl}(s + \sigma)f_{ij}(x_{kl}(s + \sigma))(x_{ij}(s + \sigma) - x_{ij}(s))\|_{\mathbb{H}} ds$$

$$\begin{aligned}
& + \int_{-\infty}^t e^{-a^m(t-s)} \sum_{C_{kl} \in N_r(i,j)} \left\| B_{ij}^{kl}(s+\sigma)(f_{ij}(x_{kl}(s+\sigma)) - f_{ij}(x_{kl}(s)))x_{ij}(s) \right\|_{\mathbb{H}} ds \\
& + \int_{-\infty}^t e^{-a^m(t-s)} \sum_{C_{kl} \in N_r(i,j)} \left\| (B_{ij}^{kl}(s+\sigma) - B_{ij}^{kl}(s))f_{ij}(x_{kl}(s))x_{ij}(s) \right\|_{\mathbb{H}} ds \\
\leq & \left( \int_{-\infty}^t e^{-\frac{p}{2p-4}a^m(t-s)} ds \right)^{\frac{p-2}{p}} \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^{\frac{p}{2}} ds \right]^{\frac{2}{p}} \\
& + \left( \int_{-\infty}^t e^{-\frac{p}{2p-4}a^m(t-s)} ds \right)^{\frac{p-2}{p}} \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{kl}(s+\sigma) - x_{kl}(s)\|_{\mathbb{H}} \right)^{\frac{p}{2}} ds \right]^{\frac{2}{p}} \\
& + \left( \int_{-\infty}^t e^{-\frac{p}{2p-4}a^m(t-s)} ds \right)^{\frac{p-2}{p}} \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 4L_{ij}^f \alpha^2 |B_{ij}^{kl}(s+\sigma) - B_{ij}^{kl}(s)| ds \right)^{\frac{p}{2}} ds \right]^{\frac{2}{p}} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left\{ \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^{\frac{p}{2}} ds \right]^{\frac{2}{p}} \right. \\
& + \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{kl}(s+\sigma) - x_{kl}(s)\|_{\mathbb{H}} \right)^{\frac{p}{2}} ds \right]^{\frac{2}{p}} \\
& \left. + \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 4L_{ij}^f \alpha^2 |B_{ij}^{kl}(s+\sigma) - B_{ij}^{kl}(s)| ds \right)^{\frac{p}{2}} ds \right]^{\frac{2}{p}} \right\} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left\{ \left[ \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{ij}(s+\sigma) \right. \right. \right. \right. \\
& \left. \left. \left. - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds \right)^{\frac{1}{2}} \right]^{\frac{2}{p}} + \left[ \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right. \right. \right. \right. \\
& \left. \left. \left. \times \|x_{kl}(s+\sigma) - x_{kl}(s)\|_{\mathbb{H}} \right)^p ds \right)^{\frac{1}{2}} \right]^{\frac{2}{p}} + \left[ \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \right. \right. \\
& \left. \left. \times \left( \sum_{C_{kl} \in N_r(i,j)} 4L_{ij}^f \alpha^2 |B_{ij}^{kl}(s+\sigma) - B_{ij}^{kl}(s)| \right)^p ds \right)^{\frac{1}{2}} \right]^{\frac{2}{p}} \right\} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left( \frac{4}{a^m p} \right)^{\frac{1}{p}} \left\{ \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds \right]^{\frac{1}{p}} \right. \\
& + \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{kl}(s+\sigma) - x_{kl}(s)\|_{\mathbb{H}} \right)^p ds \right]^{\frac{1}{p}} \\
& \left. + \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 4L_{ij}^f \alpha^2 |B_{ij}^{kl}(s+\sigma) - B_{ij}^{kl}(s)| \right)^p ds \right]^{\frac{1}{p}} \right\}, \tag{3.12}
\end{aligned}$$

for  $ij \in \Lambda$ . Similarly, we have

$$B_{1ij}(t) \leq \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left( \frac{4}{a^m p} \right)^{\frac{1}{p}} \left\{ \check{a}_{ij}^M \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}}^p ds \right]^{\frac{1}{p}} \right.$$

$$+ 2\alpha \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \|\check{\alpha}_{ij}(s+\sigma) - \check{\alpha}_{ij}(s)\|_{\mathbb{H}}^p ds \right]^{\frac{1}{p}}, \quad (3.13)$$

$$\begin{aligned} B_{3ij}(t) &\leq \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left( \frac{4}{a^m p} \right)^{\frac{1}{p}} \left\{ \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds \right]^{\frac{1}{p}} \right. \\ &+ \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{kl}(s+\sigma - \tau_{kl}(s+\sigma)) - x_{kl}(s - \tau_{kl}(s))\|_{\mathbb{H}} \right)^p ds \right]^{\frac{1}{p}} \\ &\left. + \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 4L_{ij}^g \alpha^2 |C_{ij}^{kl}(s+\sigma) - C_{ij}^{kl}(s)| \right)^p ds \right]^{\frac{1}{p}} \right\} \end{aligned} \quad (3.14)$$

and

$$B_{4ij}(t) \leq \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left( \frac{4}{a^m p} \right)^{\frac{1}{p}} \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \|I_{ij}(s+\sigma) - I_{ij}(s)\|_{\mathbb{H}}^p ds \right)^{\frac{1}{p}}, \quad (3.15)$$

for  $ij \in \Lambda$ .

Besides, combining with Hölder's inequality  $(\frac{2}{p}, \frac{p-2}{p})$ , Hölder's inequality  $(\frac{1}{2}, \frac{1}{2})$  and  $(H_2)$  that

$$\begin{aligned} B_{5ij}(t) &\leq 2\check{\alpha}_{ij}^M \alpha \int_{-\infty}^t e^{-a^m(t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right) ds \\ &\leq 2\check{\alpha}_{ij}^M \alpha \left( \int_{-\infty}^t e^{-\frac{p}{2p-4}a^m(t-s)} ds \right)^{\frac{p-2}{p}} \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^{\frac{p}{2}} ds \right]^{\frac{2}{p}} \\ &\leq \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} 2\check{\alpha}_{ij}^M \alpha \left[ \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) \right. \right. \right. \\ &\quad \left. \left. \left. - a_{ij}^R(v)| dv \right)^p ds \right)^{\frac{1}{2}} \right]^{\frac{2}{p}} \\ &\leq \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left( \frac{4}{a^m p} \right)^{\frac{1}{p}} 2\check{\alpha}_{ij}^M \alpha \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^p ds \right]^{\frac{1}{p}}, \end{aligned} \quad (3.16)$$

for  $ij \in \Lambda$ . In a similar way, one can get that

$$B_{6ij}(t) \leq \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left( \frac{4}{a^m p} \right)^{\frac{1}{p}} \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha^2 \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^p ds \right]^{\frac{1}{p}}, \quad (3.17)$$

$$B_{7ij}(t) \leq \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left( \frac{4}{a^m p} \right)^{\frac{1}{p}} \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha^2 \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^p ds \right]^{\frac{1}{p}} \quad (3.18)$$

and

$$B_{8ij}(t) \leq \left( \frac{2p-4}{a^m p} \right)^{\frac{p-2}{p}} \left( \frac{4}{a^m p} \right)^{\frac{1}{p}} \left[ \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^p \|I_{ij}(s)\|_{\mathbb{H}}^p ds \right]^{\frac{1}{p}}, \quad (3.19)$$

for  $ij \in \Lambda$ . Hence, together with a change of variables, Fubini's theorem, Hölder's inequality, (3.8) and (3.13), we derive that

$$\begin{aligned}
& \frac{1}{r} \int_{\beta}^{\beta+r} B_{1ij}^p(t) dt \\
& \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left\{ \frac{1}{r} \int_{\beta}^{\beta+r} \left[ \left( \check{\alpha}_{ij}^M \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}}^p ds \right)^{\frac{1}{p}} \right. \right. \\
& \quad \left. \left. + 2\alpha \left[ \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \|\check{\alpha}_{ij}(s+\sigma) - \check{\alpha}_{ij}(s)\|_{\mathbb{H}}^p ds \right]^{\frac{1}{p}} \right]^p dt \right\} \\
& \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{8}{a^m p} \left\{ (\check{\alpha}_{ij}^M)^p \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}}^p ds dt \right. \\
& \quad \left. + (2\alpha)^p \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \|\check{\alpha}_{ij}(s+\sigma) - \check{\alpha}_{ij}(s)\|_{\mathbb{H}}^p ds dt \right\} \\
& \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{8}{a^m p} \left\{ (\check{\alpha}_{ij}^M)^p \int_{-\infty}^{\beta} e^{-\frac{p}{4} a^m (\beta-s)} \left( \frac{1}{r} \int_s^{s+r} \|x_{ij}(t+\sigma) - x_{ij}(t)\|_{\mathbb{H}}^p dt \right) ds \right. \\
& \quad \left. + (2\alpha)^p \int_{-\infty}^{\beta} e^{-\frac{p}{4} a^m (\beta-s)} \left( \frac{1}{r} \int_s^{s+r} \|\check{\alpha}_{ij}(t+\sigma) - \check{\alpha}_{ij}(t)\|_{\mathbb{H}}^p dt \right) ds \right\} \\
& = (\check{\alpha}_{ij}^M)^p \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{8}{a^m p} \int_{-\infty}^{\beta} e^{-\frac{p}{4} a^m (\beta-s)} \Theta^{\sigma,r}(s) ds + \rho_{1ij},
\end{aligned}$$

where

$$\Theta^{\sigma,r}(s) := \frac{1}{r} \int_s^{s+r} \|x(t+\sigma) - x(t)\|_{\mathbb{H}^m}^p dt$$

and

$$\rho_{1ij} := 2 \left( \frac{2p-4}{a^m p} \right)^{p-2} \left( \frac{4}{a^m p} \right)^2 (2\alpha\epsilon)^p, \quad (3.20)$$

and, together with a change of variables, Fubini's theorem, Hölder's inequality, (3.6) and (3.12), we derive that

$$\begin{aligned}
& \frac{1}{r} \int_{\beta}^{\beta+r} B_{2ij}^p(t) dt \\
& \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left\{ \frac{1}{r} \int_{\beta}^{\beta+r} \left[ \left( \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds \right)^{\frac{1}{p}} \right. \right. \\
& \quad \left. \left. + \left( \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{kl}(s+\sigma) - x_{kl}(s)\|_{\mathbb{H}} \right)^p ds \right)^{\frac{1}{p}} \right. \right. \\
& \quad \left. \left. + \left( \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 4L_{ij}^f \alpha^2 |B_{ij}^{kl}(s+\sigma) - B_{ij}^{kl}(s)| \right)^p ds \right)^{\frac{1}{p}} \right]^p dt \right\} \\
& \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left\{ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds dt \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \|x_{kl}(s+\sigma) - x_{kl}(s)\|_{\mathbb{H}} \right)^p ds dt \\
& + \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_r(i,j)} 4L_{ij}^f \alpha^2 |B_{ij}^{kl}(s+\sigma) - B_{ij}^{kl}(s)| \right)^p ds dt \Big\} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left[ \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^p \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \|x(s+\sigma) - x(s)\|_{\mathbb{H}^{m \times n}}^p ds dt \right. \\
& + \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^p \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \|x(s+\sigma) - x(s)\|_{\mathbb{H}^{m \times n}} ds dt \\
& \left. + \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} (4mnL_{ij}^f \alpha^2 \epsilon)^p ds dt \right] \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left[ \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^p \int_{-\infty}^{\beta} e^{-\frac{p}{4}a^m(\beta-s)} \left( \frac{1}{r} \int_s^{s+r} \|x(t+\sigma) - x(t)\|_{\mathbb{H}^{m \times n}}^p dt \right) ds \right. \\
& \left. + \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^p \int_{-\infty}^{\beta} e^{-\frac{p}{4}a^m(\beta-s)} \left( \frac{1}{r} \int_s^{s+r} \|x(t+\sigma) - x(t)\|_{\mathbb{H}^{m \times n}} dt \right) ds + \frac{4}{a^m p} (4mnL_{ij}^f \alpha^2 \epsilon)^p \right] \\
= & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{24}{a^m p} \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^p \int_{-\infty}^{\beta} e^{-\frac{p}{4}a^m(\beta-s)} \Theta^{\sigma,r}(s) ds + \rho_{2ij},
\end{aligned}$$

where

$$\rho_{2ij} := 3 \left( \frac{2p-4}{a^m p} \right)^{p-2} \left( \frac{4}{a^m p} \right)^2 (4mnL_{ij}^f \alpha^2 \epsilon)^p. \quad (3.21)$$

Moreover, based on a change of variables, Fubini's theorem, Hölder's inequality, (3.7), (3.10) and (3.14), we deduce that

$$\begin{aligned}
& \frac{1}{r} \int_{\beta}^{\beta+r} B_{3ij}^p(t) dt \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left\{ \frac{1}{r} \int_{\beta}^{\beta+r} \left[ \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds \right)^{\frac{1}{p}} \right. \right. \\
& + \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{kl}(s+\sigma - \tau_{kl}(s+\sigma)) - x_{kl}(s - \tau_{kl}(s))\|_{\mathbb{H}} \right)^p ds \right)^{\frac{1}{p}} \\
& \left. \left. + \left( \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 4L_{ij}^g \alpha^2 \|C_{ij}^{kl}(s+\sigma) - C_{ij}^{kl}(s)\|_{\mathbb{H}} \right)^p ds \right)^{\frac{1}{p}} \right]^p dt \right\} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left\{ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds dt \right. \\
& \left. + \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{kl}(s+\sigma - \tau_{kl}(s+\sigma)) - x_{kl}(s - \tau_{kl}(s))\|_{\mathbb{H}} \right)^p ds dt \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 4L_{ij}^g \alpha^2 |C_{ij}^{kl}(s+\sigma) - C_{ij}^{kl}(s)| \right)^p ds dt \Big\} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left\{ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds dt \right. \\
& + \frac{1}{r} \int_{\beta}^{\beta+r} \left[ 2 \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{kl}(s+\sigma - \tau_{kl}(s+\sigma)) - x_{kl}(s - \tau_{kl}(s+\sigma))\|_{\mathbb{H}} \right)^p ds \right. \\
& + 2 \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha^2 \|x_{kl}(s - \tau_{kl}(s+\sigma)) - x_{kl}(s - \tau_{kl}(s))\|_{\mathbb{H}} \right)^p ds \Big] dt \\
& + \left. \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} (4mnL_{ij}^g \alpha^2 \epsilon)^p ds dt \right\} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left\{ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds dt \right. \\
& + \frac{1}{r} \int_{\beta}^{\beta+r} \left[ \frac{2}{1-\tau'} \int_{-\infty}^{t-\tau_{kl}(t+\sigma)} e^{-\frac{p}{4}a^m(t-u-\tau)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{kl}(u+\sigma) - x_{kl}(u)\|_{\mathbb{H}} \right)^p du \right. \\
& + 2 \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \epsilon \right)^p \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} ds \Big] dt + \frac{4}{a^m p} (4mnL_{ij}^g \alpha^2 \epsilon)^p \Big\} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left\{ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{ij}(s+\sigma) - x_{ij}(s)\|_{\mathbb{H}} \right)^p ds dt \right. \\
& + \frac{1}{r} \int_{\beta}^{\beta+r} \left[ \frac{2e^{\frac{p}{4}a^m \tau}}{1-\tau'} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \|x_{kl}(s+\sigma) - x_{kl}(s)\|_{\mathbb{H}} \right)^p ds \right] dt \\
& + \left. \frac{8}{a^m p} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \epsilon \right)^p + \frac{4}{a^m p} (4mnL_{ij}^g \alpha^2 \epsilon)^p \right\} \\
\leq & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left[ \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^p \int_{-\infty}^{\beta} e^{-\frac{p}{4}a^m(\beta-s)} \left( \frac{1}{r} \int_s^{s+r} \|x(t+\sigma) - x(t)\|_{\mathbb{H}^{m \times n}} dt \right) ds \right. \\
& + \frac{2e^{\frac{p}{4}a^m \tau}}{1-\tau'} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^p \int_{-\infty}^{\beta} e^{-\frac{p}{4}a^m(\beta-s)} \left( \frac{1}{r} \int_s^{s+r} \|x(t+\sigma) - x(t)\|_{\mathbb{H}^{m \times n}}^p dt \right) ds \\
& + \left. \frac{8}{a^m p} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \epsilon \right)^p + \frac{4}{a^m p} (4mnL_{ij}^g \alpha^2 \epsilon)^p \right] \\
= & \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{12}{a^m p} \left( 1 + \frac{2e^{\frac{p}{4}a^m \tau}}{1-\tau'} \right) \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^p \int_{-\infty}^{\beta} e^{-\frac{p}{4}a^m(\beta-s)} \Theta^{\sigma,r}(s) ds + \rho_{3ij},
\end{aligned}$$

where

$$\rho_{3ij} := 3 \left( \frac{2p-4}{a^m p} \right)^{p-2} \left( \frac{4}{a^m p} \right)^2 \left[ 2 \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^p + (4mnL_{ij}^g \alpha^2 \epsilon)^p \right] \epsilon^p. \quad (3.22)$$

In view of (3.5), (3.8) and (3.15)–(3.19), we can easily obtain that

$$\begin{aligned} & \frac{1}{r} \int_{\beta}^{\beta+r} B_{4ij}^p(t) dt \\ & \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \int_{-\infty}^{\beta} e^{-\frac{p}{4} a^m (\beta-s)} \left( \frac{1}{r} \int_s^{\beta+r} \|I_{ij}(t+\sigma) - I_{ij}(t)\|_{\mathbb{H}}^p dt \right) ds \\ & \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \left( \frac{4}{a^m p} \right)^2 \epsilon^p := \rho_{4ij}, \end{aligned} \quad (3.23)$$

$$\begin{aligned} & \frac{1}{r} \int_{\beta}^{\beta+r} B_{5ij}^p(t) dt \\ & \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left( 2\check{\alpha}_{ij}^M \alpha \right)^p \left[ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^p ds dt \right] \\ & \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left( 2\check{\alpha}_{ij}^M \alpha \epsilon \right)^p \int_0^{\infty} e^{-\frac{p}{4} a^m s} s^p ds := \rho_{5ij}, \end{aligned} \quad (3.24)$$

$$\begin{aligned} & \frac{1}{r} \int_{\beta}^{\beta+r} B_{6ij}^p(t) dt \\ & \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left( \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha^2 \right)^p \left[ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \right. \\ & \quad \left. \times \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^p ds dt \right] \\ & \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left( \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha^2 \epsilon \right)^p \int_0^{\infty} e^{-\frac{p}{4} a^m s} s^p ds := \rho_{6ij}, \end{aligned} \quad (3.25)$$

$$\begin{aligned} & \frac{1}{r} \int_{\beta}^{\beta+r} B_{7ij}^p(t) dt \\ & \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left( \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha^2 \right)^p \left[ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4} a^m (t-s)} \right. \\ & \quad \left. \times \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^p ds dt \right] \\ & \leq \left( \frac{2p-4}{a^m p} \right)^{p-2} \frac{4}{a^m p} \left( \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha^2 \epsilon \right)^p \int_0^{\infty} e^{-\frac{p}{4} a^m s} s^p ds := \rho_{7ij} \end{aligned} \quad (3.26)$$

and

$$\frac{1}{r} \int_{\beta}^{\beta+r} B_{8ij}^p(t) dt$$

$$\begin{aligned}
&\leq \left(\frac{2p-4}{a^m p}\right)^{p-2} \frac{4}{a^m p} \left[ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} \left( \int_s^t |a_{ij}^R(v+\sigma) - a_{ij}^R(v)| dv \right)^p \|I_{ij}(s)\|_{\mathbb{H}}^p ds dt \right] \\
&\leq \epsilon^p \left(\frac{2p-4}{a^m p}\right)^{p-2} \frac{4}{a^m p} \left[ \frac{1}{r} \int_{\beta}^{\beta+r} \int_{-\infty}^t e^{-\frac{p}{4}a^m(t-s)} (t-s)^p \|I_{ij}(s)\|_{\mathbb{H}}^p ds dt \right] \\
&\leq \left(\frac{2p-4}{a^m p}\right)^{p-2} \frac{4}{a^m p} \|I_{ij}\|_{WP}^p \epsilon^p \int_0^{\infty} e^{-\frac{p}{4}a^m s} s^p ds := \rho_{8ij},
\end{aligned} \tag{3.27}$$

for  $ij \in \Lambda$ . Consequently, combining with (3.11) and (3.20)–(3.27), we obtain

$$\begin{aligned}
\Theta^{\sigma,r}(\beta) &\leq \max_{ij \in \Lambda} \left\{ 8 \sum_{l=1}^8 \frac{1}{r} \int_{\beta}^{\beta+r} B_{lij}^p(t) dt \right\} \\
&\leq \rho + \gamma \int_{-\infty}^{\beta} e^{-\eta(\beta-s)} \Theta^{\sigma,r}(s) ds,
\end{aligned}$$

where  $\eta = \frac{p}{4}a^m$ ,

$$\begin{aligned}
\rho &= 8 \sum_{l=1}^6 \max_{ij \in \Lambda} \{\rho_{lij}\} \\
&= 8 \max_{ij \in \Lambda} \left\{ \left(\frac{2p-4}{a^m p}\right)^{p-2} \frac{4}{a^m p} \left[ \frac{8}{a^m p} (2\alpha)^p + \frac{12}{a^m p} (4mnL_{ij}^f \alpha^2)^p \right. \right. \\
&\quad + \frac{24}{a^m p} \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^p + \frac{12}{a^m p} (4mnL_{ij}^g \alpha^2)^p + \frac{4}{a^m p} + \left[ (2\check{\alpha}_{ij}^M \alpha)^p \right. \\
&\quad + \left( \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha^2 \right)^p + \left. \left. \left( \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha^2 \right)^p \right. \right. \\
&\quad \left. \left. + \|I_{ij}\|_{WP}^p \int_0^{\infty} e^{-\frac{p}{4}a^m s} s^p ds \right] \right\} \epsilon^p
\end{aligned}$$

and

$$\begin{aligned}
\gamma &= 8 \max_{ij \in \Lambda} \left\{ \left(\frac{2p-4}{a^m p}\right)^{p-2} \frac{12}{a^m p} \left[ \frac{2}{3} (\check{\alpha}_{ij}^M)^p + 2 \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^p \right. \right. \\
&\quad \left. \left. + \left( 1 + \frac{2e^{\frac{p}{4}a^m \tau}}{1-\tau'} \right) \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^p \right] \right\}.
\end{aligned}$$

By  $(H_4)$ , we have  $\gamma < \eta$ . Thus, it follows from Lemma 2.3 that

$$\frac{1}{r} \int_{\beta}^{\beta+r} \|x(t+\sigma) - x(t)\|_{\mathbb{H}^{m \times n}}^p dt \leq \rho \frac{\eta}{\eta - \gamma}.$$

Hence,  $x \in APW^p(\mathbb{R}, \mathbb{H}^{m \times n})$ .

When  $p = 2$ , similar to the proof of the case of  $p > 2$ , one can obtain

$$\Theta^{\sigma,r}(\beta) \leq \tilde{\rho} + \tilde{\gamma} \int_{-\infty}^{\beta} e^{-\tilde{\eta}(\beta-s)} \Theta^{\sigma,r}(s) ds,$$



where  $\tilde{\eta} = \frac{a^m}{2}$ ,

$$\begin{aligned} \tilde{\rho} &= 8 \sum_{l=1}^8 \max_{ij \in \Lambda} \{\tilde{\rho}_{lij}\} \\ &= \max_{ij \in \Lambda} \left\{ \frac{16}{a^m} \left[ \frac{4}{a^m} (2\alpha)^2 + \frac{6}{a^m} (4mnL_{ij}^f \alpha^2 \epsilon)^2 + \frac{12}{a^m} \left( 2 \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{klM} L_{ij}^g \alpha \right)^2 \right. \right. \\ &\quad \left. \left. + \frac{6}{a^m} (4mnL_{ij}^g \alpha^2)^2 + \frac{2}{a^m} + \left[ (2\check{\alpha}_{ij}^M \alpha)^2 + \left( \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha^2 \right)^2 \right. \right. \right. \\ &\quad \left. \left. \left. + \left( \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha^2 \right)^2 + \|I_{ij}\|_{W^2}^2 \right] \int_0^\infty e^{-\frac{1}{2}a^m s} s^2 ds \right] \right\} \epsilon^2 \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma} &= \max_{ij \in \Lambda} \left\{ \frac{48}{a^m} \left[ \frac{2}{3} (\check{\alpha}_{ij}^M)^2 + 2 \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^2 \right. \right. \\ &\quad \left. \left. + \left( 1 + \frac{2e^{\frac{1}{2}a^m \tau}}{1 - \tau'} \right) \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^2 \right] \right\}. \end{aligned}$$

By  $(H_4)$ , we have  $\tilde{\gamma} < \tilde{\eta}$ . Thus, it follows from Lemma 2.3 that

$$\frac{1}{r} \int_\beta^{\beta+r} \|x(t+\sigma) - x(t)\|_{\mathbb{H}^{m \times n}}^p dt \leq \tilde{\rho} \frac{\tilde{\eta}}{\tilde{\eta} - \tilde{\gamma}},$$

which means that  $x \in APW^2(\mathbb{R}, \mathbb{H}^{m \times n})$ . The proof is complete.  $\square$

**Definition 3.1.** [14] Let  $x$  be a solution of system (1.1) with the initial value  $\varphi$  and  $y$  be an arbitrary solution of system (1.1) with the initial value  $\psi$ , respectively. If there exist positive constants  $\lambda$  and  $M$  such that

$$\|x(t) - y(t)\|_{\mathbb{H}^{m \times n}} \leq M e^{-\lambda t} \|\varphi - \psi\|_\tau, \quad t \in \mathbb{R}^+,$$

where  $\|\varphi - \psi\|_\tau = \sup_{t \in [-\tau, 0]} \|\varphi(t) - \psi(t)\|_{\mathbb{H}^{m \times n}}$ . Then the solution  $x$  of system (1.1) is said to be globally exponentially stable.

**Theorem 3.2.** Assume that  $(H_1)$ – $(H_3)$  hold, then system (1.1) has a unique  $W^p$ -almost periodic solution that is globally exponentially stable.

*Proof.* Let  $x(t)$  be the  $W^p$ -almost periodic solution with the initial value  $\varphi(t)$  and  $y(t)$  be an arbitrary solution with the initial value  $\psi(t)$ . Taking

$$z_{ij}(t) = x_{ij}(t) - y_{ij}(t), \quad \phi_{ij}(t) = \varphi_{ij}(t) - \psi_{ij}(t), \quad ij \in \Lambda,$$

we have

$$\dot{z}_{ij}(t) = -a_{ij}(t)z_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(t)(f_{ij}(x_{kl}(t))x_{ij}(t)$$

$$\begin{aligned}
& - f_{ij}(y_{kl}(t))y_{ij}(t) - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(t)(g_{ij}(x_{kl}(t - \tau_{kl}(t)))x_{ij}(t) \\
& - g_{ij}(y_{kl}(t - \tau_{kl}(t)))y_{ij}(t)), \quad ij \in \Lambda.
\end{aligned} \tag{3.28}$$

For  $ij \in \Lambda$ , we define the following functions:

$$\Pi_{ij}(u) = a^m - u - \left( \check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha e^{u\tau_{ij}^M} + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha \right).$$

From  $(H_3)$ , we get

$$\Pi_{ij}(0) = a^m - \left( \check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha \right) > 0, \quad ij \in \Lambda.$$

Since  $\Pi_{ij}(u)$  is continuous on  $[0, +\infty)$  and  $\Pi_{ij}(u) \rightarrow -\infty$  as  $u \rightarrow \infty$ , there exists  $\zeta_{ij} > 0$  such that  $\Pi_{ij}(\zeta_{ij}) = 0$  and  $\Pi_{ij}(u) > 0$ , for  $u \in (0, \zeta_{ij})$ ,  $ij \in \Lambda$ . Let  $\varsigma = \min_{ij \in \Lambda} \{\zeta_{ij}\}$ , then we have  $\Pi_{ij}(\varsigma) \geq 0$ ,  $ij \in \Lambda$ .

Hence, we can choose a positive constant  $\lambda$  such that  $0 < \lambda < \min\{\varsigma, a^m\}$  and  $\Pi_{ij}(\lambda) > 0$ . Thus, one has

$$\begin{aligned}
& \frac{1}{a^m - \lambda} \left( \check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha e^{\lambda\tau_{ij}^M} \right. \\
& \left. + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha \right) < 1,
\end{aligned} \tag{3.29}$$

where  $ij \in \Lambda$ . Take a constant  $M = \max_{ij \in \Lambda} \left\{ \frac{a^m}{\check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 4C_{ij}^{klM} L_{ij}^g \alpha} \right\}$ , then by  $(H_3)$ , we have

$M > 1$ . Thus,

$$\begin{aligned}
& \frac{1}{M} - \frac{1}{a^m - \lambda} \left( \check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha e^{\lambda\tau_{ij}^M} \right. \\
& \left. + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha \right) < 0, \quad ij \in \Lambda.
\end{aligned} \tag{3.30}$$

From (3.28), we have

$$\begin{aligned}
\dot{z}_{ij}(t) + a_{ij}^R(t)z_{ij}(t) & = -\check{\alpha}_{ij}(t)z_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(t)(f_{ij}(x_{kl}(t))x_{ij}(t) - f_{ij}(y_{kl}(t))y_{ij}(t)) \\
& - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(t)(g_{ij}(x_{kl}(t - \tau_{kl}(t)))x_{ij}(t) \\
& - g_{ij}(y_{kl}(t - \tau_{kl}(t)))y_{ij}(t)), \quad ij \in \Lambda.
\end{aligned} \tag{3.31}$$

Multiplying both sides of (3.31) by  $e^{\int_0^t a_{ij}^R(v)dv}$  and integrating over  $[0, t]$ , we have

$$z_{ij}(t) = \phi_{ij}(0)e^{-\int_0^t a_{ij}^R(v)dv} + \int_0^t e^{-\int_s^t a_{ij}^R(v)dv} \left[ -\check{\alpha}_{ij}(s)z_{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)(f_{ij}(x_{kl}(s))x_{ij}(s) \right.$$

$$\begin{aligned} & - f_{ij}(y_{kl}(t))y_{ij}(t) - \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(t) (g_{ij}(x_{kl}(t - \tau_{kl}(t)))x_{ij}(t) \\ & - g_{ij}(y_{kl}(t - \tau_{kl}(t)))y_{ij}(t)) \Big] ds, \quad ij \in \Lambda. \end{aligned}$$

Hence, for any  $\epsilon > 0$ , it is easy to see that

$$\|z(t)\|_{\mathbb{H}^{m \times n}} < (\|\phi\|_{\tau} + \epsilon)e^{-\lambda t} < M(\|\phi\|_{\tau} + \epsilon)e^{-\lambda t}, \quad \forall t \in (-\tau, 0].$$

We claim that

$$\|z(t)\|_{\mathbb{H}^{m \times n}} < M(\|\phi\|_{\tau} + \epsilon)e^{-\lambda t}, \quad \forall t \in [0, +\infty). \quad (3.32)$$

Otherwise, there exists  $t^* > 0$  such that

$$\|z(t^*)\|_{\mathbb{H}^{m \times n}} = M(\|\phi\|_{\tau} + \epsilon)e^{-\lambda t^*} \quad (3.33)$$

and

$$\|z(t)\|_{\mathbb{H}^{m \times n}} < M(\|\phi\|_{\tau} + \epsilon)e^{-\lambda t}, \quad t < t^*. \quad (3.34)$$

From (3.29), (3.30) and (3.34), we get

$$\begin{aligned} \|z_{ij}(t^*)\|_{\mathbb{H}} & \leq \|\phi_{ij}(0)\|_{\mathbb{H}} e^{-t^* a^m} + \int_0^{t^*} e^{-t^* a^m} \left[ \check{a}_{ij}^M \|z_{ij}(s)\|_{\mathbb{H}} + \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(t) \right. \\ & \quad \times \left( \|f_{ij}(x_{kl}(t))(x_{ij}(t) - y_{ij}(t))\|_{\mathbb{H}} + \|(f_{ij}(x_{kl}(t)) - f_{ij}(y_{kl}(t)))y_{ij}(t)\|_{\mathbb{H}} \right) \\ & \quad + \sum_{C_{kl} \in N_s(i,j)} C_{ij}^{kl}(t) \left( \|g_{ij}(x_{kl}(t - \tau_{kl}(t)))(x_{ij}(t) - y_{ij}(t))\|_{\mathbb{H}} \right. \\ & \quad \left. \left. + \|(g_{ij}(x_{kl}(t - \tau_{kl}(t))) - g_{ij}(y_{kl}(t - \tau_{kl}(t)))y_{ij}(t)\|_{\mathbb{H}} \right) \right] ds \\ & \leq (\|\phi\|_{\tau} + \epsilon)e^{-a^m t^*} + M(\|\phi\|_{\tau} + \epsilon) \int_0^{t^*} e^{-(t^*-s)a^m} \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha \right. \\ & \quad \left. + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha e^{\lambda \tau_{ij}^M} + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha \right) e^{-\lambda s} ds \\ & \leq (\|\phi\|_{\tau} + \epsilon)e^{-\lambda t^*} e^{(\lambda - a^m)t^*} + M(\|\phi\|_{\tau} + \epsilon) \frac{e^{-\lambda t^*} (1 - e^{(\lambda - a^m)t^*})}{a^m - \lambda} \left( \check{a}_{ij}^M \right. \\ & \quad \left. + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha e^{\lambda \tau_{ij}^M} + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha \right) \\ & = M(\|\phi\|_{\tau} + \epsilon)e^{-\lambda t^*} \left[ \frac{e^{(\lambda - a^m)t^*}}{M} + \frac{1 - e^{(\lambda - a^m)t^*}}{a^m - \lambda} \left( \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha \right. \right. \\ & \quad \left. \left. + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha e^{\lambda \tau_{ij}^M} + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha \right) \right] \end{aligned}$$

$$\begin{aligned}
&= M(\|\phi\|_\tau + \epsilon)e^{-\lambda t^*} \left[ \left( \frac{1}{M} - \frac{1}{a^m - \lambda} \left( \check{\alpha}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha \right) \right. \right. \\
&\quad + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha e^{\lambda \tau_{ij}^M} + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha \left. \right) e^{(\lambda - a^m)t^*} + \frac{1}{a^m - \lambda} \left( \check{\alpha}_{ij}^M \right. \\
&\quad \left. \left. + \sum_{C_{kl} \in N_r(i,j)} 4B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha e^{\lambda \tau_{ij}^M} + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{kl} L_{ij}^g \alpha \right) \right] \\
&< M(\|\phi\|_\tau + \epsilon)e^{-\lambda t^*}, \quad ij \in \Lambda,
\end{aligned}$$

which contradicts the Eq (3.33). Hence, (3.32) holds. Letting  $\epsilon \rightarrow 0^+$ , from (3.32), we have

$$\|z(t)\|_{\mathbb{H}^{m \times n}} \leq M\|\phi\|_\tau e^{-\lambda t}, \quad \forall t > 0.$$

Therefore, the  $W^p$ -almost periodic solution of system (1.1) is globally exponentially stable. This completes the proof.  $\square$

#### 4. A numerical example

**Example 4.1.** In system (1.1), let  $i, j = 1, 2$ ,  $r = s = 1$  and take

$$\begin{aligned}
x_{ij}(t) &= x_{ij}^R(t) + ix_{ij}^I(t) + jx_{ij}^J(t) + kx_{ij}^K(t) \in \mathbb{H}, \\
a_{11}(t) &= 32 \sin^2(\sqrt{3}t) + i \frac{\sqrt{2}}{5} \cos(2t) - j \frac{2}{5} \sin(\sqrt{5}t) + k \frac{1}{100} \cos^2(3t), \\
a_{21}(t) &= 39 \left| \cos(\sqrt{2}t) \right| - i \frac{2}{10} \sin(2t) + j \frac{3}{10} \sin^2(\sqrt{7}t) + k \frac{\sqrt{3}}{100} \cos(t), \\
a_{12}(t) &= 33 \cos^2(t) - i \frac{\sqrt{3}}{6} \sin(5t) - j \frac{1}{4} \cos^3(\sqrt{5}t) + k \frac{1}{6} \cos^2(3t) \\
a_{22}(t) &= 37 \sin^4(\sqrt{3}t) + i \frac{1}{4} \cos(7t) - j \frac{\sqrt{29}}{16} \left| \sin(\sqrt{11}t) \right| + k \frac{1}{8} \cos^2(5t), \\
\tau_{11}(t) &= \frac{\sqrt{2}}{32} \sin^8(\sqrt{2}t), \tau_{12}(t) = \frac{1}{16} \cos^2(\sqrt{3}t), \tau_{21}(t) = \frac{1}{41} \sin^2\left(\frac{\sqrt{2}}{5}t\right), \tau_{22}(t) = \frac{1}{29} \sin^4\left(\frac{\sqrt{3}}{7}t\right), \\
B_{11}(t) &= \frac{1}{3} \left| \cos(\sqrt{2}t) \right|, B_{12}(t) = \frac{1}{6} \sin(4t) + \frac{1}{2} B_{21}(t) = \frac{3}{4} \sin(2t) + 1, B_{22}(t) = \frac{1}{4} \sin^2(\sqrt{5}t), \\
C_{11}(t) &= \cos(\sqrt{3}t) + 3, C_{12}(t) = \left| \sin(\sqrt{5}t) \right|, C_{21}(t) = 2 \sin^2(\sqrt{7}t), C_{22}(t) = \frac{1}{4} \sin(2t) + \frac{7}{4}, \\
f_{11}(x) &= f_{12}(x) = \frac{1}{5} \sin\left(\frac{1}{4}x^R + \frac{\sqrt{3}}{6}x^K\right) - i \frac{1}{5} \left| \frac{3\sqrt{2}}{4}x^I + \frac{\sqrt{3}}{5}x^J \right| + k \frac{1}{4} \sin\left(\frac{1}{5}x^R\right), \\
f_{21}(x) &= f_{22}(x) = \frac{1}{4} \left| \frac{\sqrt{2}}{3}x^I + \frac{\sqrt{3}}{7}x^K \right| - i \frac{\sqrt{2}}{8} \sin\left(\frac{1}{5}x^R + \frac{\sqrt{2}}{5}x^K\right) + k \frac{\sqrt{2}}{3} \sin\left(\frac{1}{5}x^I\right), \\
g_{11}(x) &= g_{12}(x) = \frac{1}{20} \sin\left(\frac{9\sqrt{2}}{2}x^R\right) - i \frac{\sqrt{3}}{40} \sin\left(\frac{7}{5}x^K\right) + j \frac{\sqrt{2}}{20} \sin\left(\frac{1}{3}x^I + \frac{3}{4}x^I\right),
\end{aligned}$$

$$g_{21}(x) = g_{22}(x) = \frac{\sqrt{2}}{25} \left| \frac{1}{3}x^J + \frac{2}{3}x^J \right| - i \frac{\sqrt{3}}{40} \sin\left(\frac{7}{5}x^K + x^R\right) + j \frac{1}{21} \sin(4\sqrt{2}x^R),$$

$$I_{11}(t) = I_{21}(t) = \sqrt{2} \sin t + i \frac{4}{3} e^{-|t|} + k \frac{\sqrt{2}}{3} \cos\left(\frac{1}{2}t\right),$$

$$I_{12}(t) = I_{22}(t) = i \frac{\sqrt{2}}{3} \sin \sqrt{2}t - j \frac{1}{5} \cos 2t + k e^{-|t|}.$$

By computing, we obtain

$$L_{11}^f = L_{12}^f = \frac{3}{10}, L_{21}^f = L_{22}^f = \frac{1}{6}, L_{11}^g = L_{12}^g = \frac{9}{20}, L_{21}^g = L_{22}^g = \frac{8}{21},$$

$$a^m = 32, \check{a}_{11}^M = \frac{1}{2}, \check{a}_{12}^M = \frac{2}{5}, \check{a}_{21}^M = \frac{5}{12}, \check{a}_{22}^M = \frac{7}{16}, \tau = \frac{1}{16}, \tau' = \frac{1}{2},$$

$$\sum_{C_{kl} \in N_1(1,1)} B_{11}^{klM} = \sum_{C_{kl} \in N_1(1,2)} B_{12}^{klM} = \sum_{C_{kl} \in N_1(2,1)} B_{21}^{klM} = \sum_{C_{kl} \in N_1(2,2)} B_{22}^{klM} = 4,$$

$$\sum_{C_{kl} \in N_1(1,1)} C_{11}^{klM} = \sum_{C_{kl} \in N_1(1,2)} C_{12}^{klM} = \sum_{C_{kl} \in N_1(2,1)} C_{21}^{klM} = \sum_{C_{kl} \in N_1(2,2)} C_{22}^{klM} = 8.$$

Choose a constant  $\alpha = \frac{1}{16} \geq \|\phi^0\|_\infty$ , for  $i, j = 1, 2$ , we obtain that

$$\kappa = \max_{ij \in \Lambda} \left\{ \frac{2}{a^m} \left[ \check{a}_{ij}^M + \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha + \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right] \right\} = 0.06875 < 1.$$

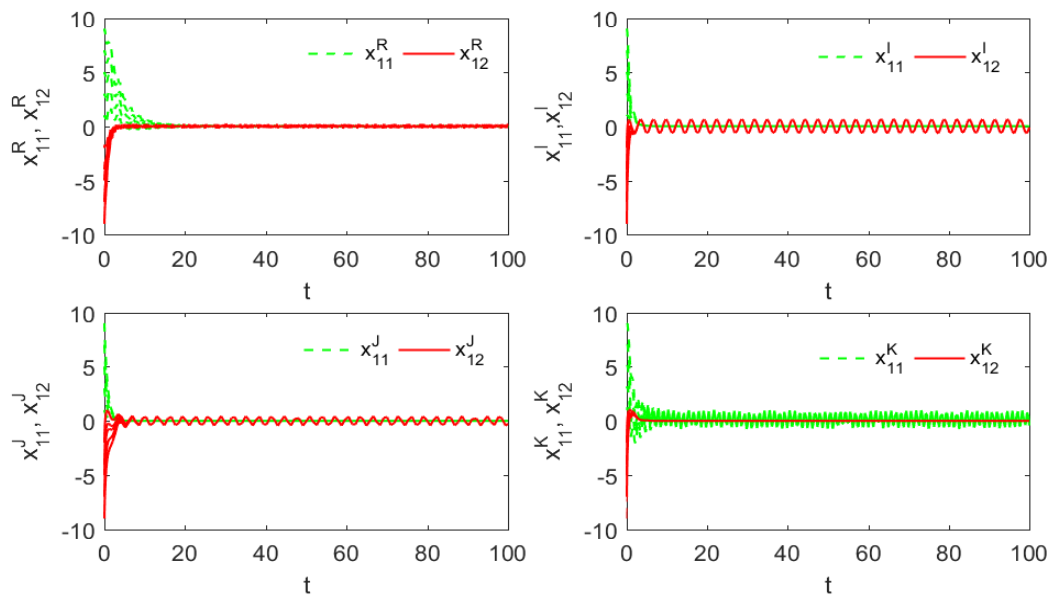
For  $i, j = 1, 2$ , take  $p = 3$ , it is easy to obtain that

$$\begin{aligned} & \max_{ij \in \Lambda} \left\{ 96 \left( \frac{2}{3a^m} \right)^3 \left[ \frac{2}{3} (\check{a}_{ij}^M)^3 + 2 \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^3 \right. \right. \\ & \left. \left. + \left( 1 + \frac{2e^{\frac{3}{4}a^m\tau}}{1-\tau'} \right) \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^3 \right] \right\} = 0.0016 < 1. \end{aligned}$$

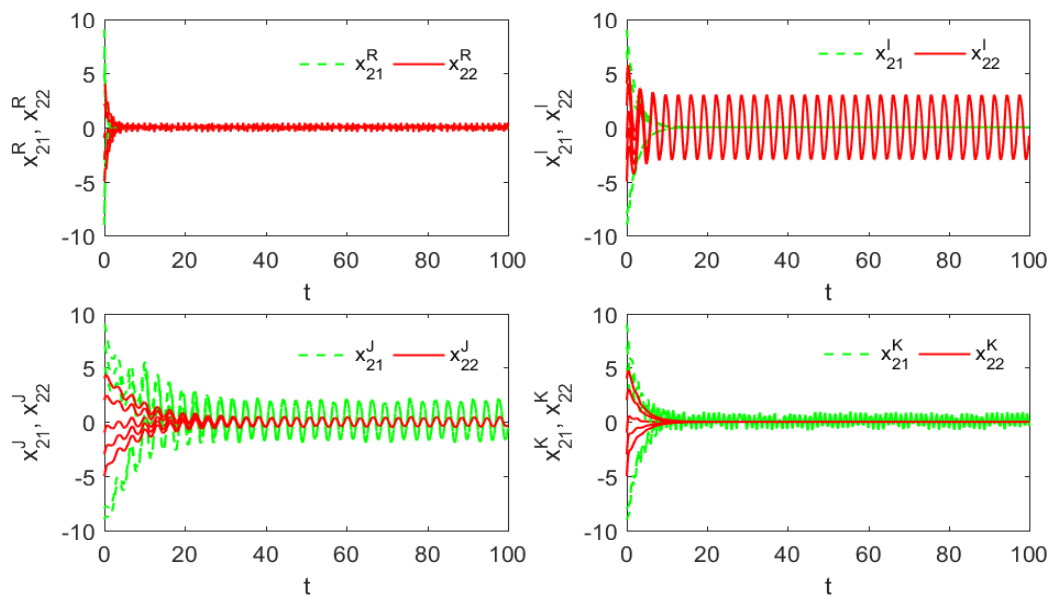
Take  $p = 2$ , we have

$$\begin{aligned} & \max_{ij \in \Lambda} \left\{ 24 \left( \frac{2}{a^m} \right)^2 \left[ \frac{2}{3} (\check{a}_{ij}^M)^2 + 2 \left( \sum_{C_{kl} \in N_r(i,j)} 2B_{ij}^{klM} L_{ij}^f \alpha \right)^2 \right. \right. \\ & \left. \left. + \left( 1 + \frac{2e^{\frac{1}{2}a^m\tau}}{1-\tau'} \right) \left( \sum_{C_{kl} \in N_s(i,j)} 2C_{ij}^{klM} L_{ij}^g \alpha \right)^2 \right] \right\} = 0.2832 < 1. \end{aligned}$$

Thus, conditions  $(H_1)$ – $(H_4)$  of Theorems 3.1 and Theorems 3.2 are satisfied. Hence, system (1.1) has a unique  $W^p$ -almost periodic solution that is globally exponentially stable (see Figures 1, 2).



**Figure 1.** Curves of  $(x_{11}^l(t), x_{12}^l(t))^T$  of system (1.1) with the initial values  $(x_{11}^l(0), x_{12}^l(0))^T = (1, -2)^T, (3, -4)^T, (5, -5)^T, (7, -7)^T, (9, -9)^T, l = R, I, J, K$ .



**Figure 2.** Curves of  $(x_{21}^l(t), x_{22}^l(t))^T$  of system (1.1) with the initial values  $(x_{21}^l(0), x_{22}^l(0))^T = (-8, -1)^T, (-9, -3)^T, (5, -5)^T, (7, -2)^T, (9, 4)^T, l = R, I, J, K$ .

**Remark 4.1.** No known results are available to give the results of Example 4.1.

## 5. Conclusions

In this paper, the existence and global exponential stability of Weyl almost periodic solutions for a class of quaternion-valued neural networks with time-varying delays are established. Even when the system we consider is a real-valued system, our results are brand-new. In addition, the method in this paper can be used to study the existence of Weyl almost periodic solutions for other types of neural networks.

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## Conflict of interest

The authors declare no conflict of interest in this paper.

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