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Research article

Roughness of soft sets and fuzzy sets in semigroups based on set-valued picture hesitant fuzzy relations

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Abstract: In the philosophy of rough set theory, the methodologies of rough soft sets and rough fuzzy sets have been being examined to be efficient mathematical tools to deal with unpredictability. The basic of approximations in rough set theory is based on equivalence relations. In the aftermath, such theory is extended by arbitrary binary relations and fuzzy relations for more wide approximation spaces. In recent years, the notion of picture hesitant fuzzy relations by Mathew et al. can be considered as a novel extension of fuzzy relations. Then this paper proposes extended approximations into rough soft sets and rough fuzzy sets from the viewpoint of its. We give corresponding examples to illustrate the correctness of such approximations. The relationships between the set-valued picture hesitant fuzzy relations with the upper (resp., lower) rough approximations of soft sets and fuzzy sets are investigated. Especially, it is shown that every non-rough soft set and non-rough fuzzy set can be induced by set-valued picture hesitant fuzzy reflexive relations and set-valued picture hesitant fuzzy antisymmetric relations. By processing the approximations and advantages in the new existing tools, some terms and products have been applied to semigroups. Then, we provide attractive results of upper (resp., lower) rough approximations of prime idealistic soft semigroups over semigroups and fuzzy prime ideals of semigroups induced by set-valued picture hesitant fuzzy relations on semigroups.

Keywords: rough set; rough soft set; rough fuzzy set; prime idealistic soft semigroup over semigroup; fuzzy prime ideal of semigroup; set-valued picture hesitant fuzzy relation
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1. Introduction

Most of the many concepts in our everyday life are vagueness than exact. In the notion of set theory, another topic discussed in association with the notion of a set is vagueness [1, 2]. Consequently, vagueness is the property of sets and can be viewed as a nettlesome problem for computer science, machine learning, artificial intelligence. Moreover, vagueness may be common sense reasoning based on natural language. Rough set theory can be viewed as a specific implementation of such an idea of vagueness. The basic of rough (inexact) sets and approximation spaces was proposed by Pawlak [3] in 1982. Based on an equivalence relation, rough set theory expresses vagueness on the assumption that any vague (imprecise) concept is replaced by a pair of crisp (precise) concepts so-called the upper and the lower approximation. The following model briefly describes basic concepts in rough set theory.

For a given non-empty universal set V and an equivalence relation E on V, (V, E) is denoted as a Pawlak's approximation space, and $[v]_E$ is denoted as an equivalence class of $v \in V$ induced by E. In the following, let (V, E) be a given Pawlak's approximation space and let X be a subset of V. Upon a collection of all equivalence classes generated by all elements in V, Pawlak suggests an approximation pattern as the following. The set

$$\lceil X \rceil_E := \bigcup_{v \in V} \{[v]_E : [v]_E \cap X \neq \emptyset\}$$

is said to be an upper approximation of X within (V, E). The set

$$\lfloor X \rfloor_E := \bigcup_{v \in V} \{[v]_E : [v]_E \subseteq X\}$$

is said to be a lower approximation of X within (V, E). A difference $\lceil X \rceil_E - \lfloor X \rfloor_E$ is said to be a boundary region of X within (V, E). As introduced above, such sets are obtained the following interpretation.

- The upper approximation $\lceil X \rceil_E$ of X contains all objects which possibly belong to X. In this way, a complement of $\lceil X \rceil_E$ is said to be a negative region of X within (V, E).
- The lower approximation $\lfloor X \rfloor_E$ of X consists of all objects which surely belong to X. In this way, such the set is said to be a positive region of X within (V, E).
- $\lceil X \rceil_E - \lfloor X \rfloor_E$ is a set of all objects, which can be classified neither as X nor as non-X using E.

In what follows, a pair $(\lceil X \rceil_E, \lfloor X \rfloor_E)$ is said to be a rough set of X within (V, E) if $\lceil X \rceil_E - \lfloor X \rfloor_E$ is a non-empty set. In this way, X is said to be a rough set. X is said to be a definable (or an exact) set within (V, E) if $\lceil X \rceil_E - \lfloor X \rfloor_E$ is an empty set.

Observe that rough set theory expresses vagueness by employing a boundary region of a set. If the boundary region of a set is empty it means that the set is crisp, otherwise, the set is rough. Moreover, a non-empty boundary region of a set means that our information (or knowledge) about the set is not adequate to define the set accurately.

Based on Zadeh's fuzzy set theory [4], one of the developments of Pawlak's rough set theory in terms of fuzzy set theory is the rough set approach provides tools for approximate construction of fuzzy membership functions. In particular, the notion of rough fuzzy sets was introduced by Dubois and Prade [5] in 1990. Observe that a rough fuzzy set is the approximation of a fuzzy set in a crisp approximation space.

Generally, to select the optimal objects for a decision problem of knowledge containing uncertainties, Molodtsov's soft set theory [6] is one of the powerful mathematical tools for dealing with such problems. A wide range of soft set theory based on Pawlak's rough set theory has been discovered in the notion of rough soft sets by Feng et al. [7]. In other words, it has been shown that the rough set approach can be used for the approximation of a set of approximate elements (or alternative objects) of a soft set.

The fundamental of a Pawlak's approximation space is induced by equivalence relations, but it has been extended to arbitrary binary relations and fuzzy binary relations (or fuzzy relations) (see e.g., [8–13]). Observe that the several definitions of relations between two objects of knowledge can be generated many extended approximation spaces. The concept of fuzzy relations was also proposed by Zadeh [14] as an extension of the classic relationship. In rough set theory, approximations are two basic operations in approximation spaces. Therefore, one of searching for approximations in extensions, it is better to define basic notions of rough set theory in terms of extended fuzzy relations.

Due to the extension of fuzzy set theory, in 1983, Atanassov [15] proposed a generalization of Zadeh's fuzzy sets so-called intuitionistic fuzzy sets. When fuzzy sets give the degree of membership of an element in a set, intuitionistic fuzzy sets give a degree of membership and a degree of non-membership. In addition, the sum of two memberships is less than or equal to 1. Next, the notion of intuitionistic fuzzy relations was given by Burillo and Bustince [16] in 1995. In development continuously, there are two interesting extensions of Atanassov's intuitionistic fuzzy sets as follows. Neutrosophic sets introduced by Smarandache [17] in 2005, which is a generalization of fuzzy sets and intuitionistic fuzzy sets. It is characterized by a truth membership function, an indeterminacy membership function, and a falsity membership function. Besides, all memberships are subsets of the nonstandard unit interval. Cuong [18] introduced an extension of intuitionistic fuzzy sets so-called picture fuzzy sets in 2014. The notion of picture fuzzy relations was also proposed. A picture fuzzy set expresses a degree of positive, neutral, negative memberships. Further, there is a restriction that the sum of these three grades is less than or equal to 1.

In 2010, Torra [19] introduced an extension of Zadeh's fuzzy sets so-called hesitant fuzzy sets. In the management of uncertain information, the hesitant fuzzy set is used to deal with group decision-making problems when experts have a hesitation among several possible memberships. Observe that Smarandache's Neutrosophic set is a generalization of hesitation fuzzy sets. In a different line, the concept of hesitant fuzzy preference relations was proposed by Zhu [20] in 2013. This is a powerful tool for group decision-making, and it is widely applied in many fields under evaluating problems (see e.g., [21–23]).

The combination of picture fuzzy sets and hesitant fuzzy sets was proposed by Wang and Li [24] in 2018. They introduced the notion of picture hesitant fuzzy sets based on properties of picture fuzzy sets and hesitant fuzzy sets. In recent years, Mathew et al. [25] introduced the notion of picture hesitant fuzzy relations in terms of picture hesitant fuzzy sets. This notion is an extension of fuzzy relations under the context of set-valued functions and fuzzy logic. Observe that the notion of picture hesitant fuzzy relations can be used for two extended approximation operations in the sense of rough set theory.

In applicability, rough set theory can solve uncertainty problems in information systems (see e.g., [2, 26, 27]), and it can be regarded in mathematical systems (see e.g., [10–13, 28–45]). Especially, in semigroups, completely prime ideals, fuzzy prime ideals, and prime idealistic soft semigroups were approximated in rough set theory. In 2006, Xiao and Zhang [28] introduced the notion of upper and lower rough approximations of completely prime ideals and fuzzy prime ideals of semigroups induced by congruence relations. In 2016, Wang and Zhan [35] proposed the notion of upper and lower rough approximations of completely prime ideals and fuzzy prime ideals of semigroups based on special congruence relations. In 2017, Wang and Zhan [36] proposed the notion of upper and lower rough approximations of prime idealistic soft semigroups based on special congruence relations. In 2018 and 2019, Prasertpong and Siripitukdet [10–13] introduced the concept of upper and lower rough approximations of completely prime ideals of semigroups based on arbitrary binary relations and fuzzy relations.

According to literature, based on picture hesitant fuzzy relations, this paper first constructs two approximation operations to rough soft sets and rough fuzzy sets together with a corresponding example in Section 3. Next, the relationships of the upper (resp., lower) rough approximations with set-valued picture hesitant fuzzy relations are provided. In Section 4, by making use of the novel models above, outcomes develop to semigroups. Results of upper (resp., lower) rough approximations of prime idealistic soft semigroups over semigroups and fuzzy prime ideals of semigroups induced by set-valued picture hesitant fuzzy relations on semigroups are verified. Besides, rough approximations models in the view of soft semigroup homomorphism problems are discussed in detail. In the end, some discussion and conclusions are summarized in Section 5.

2. Preliminaries

In this section, let us first recall some basic notions and properties which will be necessary for subsequent sections.

Throughout this paper, K, V and W denote non-empty sets, and $\mathcal{P}(V)$ represents a collection of all subsets of V.

2.1. Some essential attributes in semigroups

Definition 2.1. *[46] Let $*$ be a given binary operation on V. Recall that a semigroup is denoted by an algebraic system $(V, *)$, where $*$ is associative. For simplicity, we shall write V instead of $(V, *)$. In the following, if $(V, *)$ is a semigroup, then $\acute{v} * \grave{v}$ is denoted by $\acute{v}\grave{v}$ for all $\acute{v}, \grave{v} \in V$. Given two non-empty subsets X and Y of a semigroup V, the product $X * Y$ (simply XY) is defined by*

$$XY = \{\acute{v}\grave{v} : \acute{v} \in X \text{ and } \grave{v} \in Y\}.$$

Definition 2.2. *[47] Let V be a semigroup, and let X be a non-empty subset of V.*

(i) X is said to be a subsemigroup of V if $XX \subseteq X$.

(ii) X is said to be a left ideal of V if $VX \subseteq X$.

(iii) X is said to be a right ideal of V if $XV \subseteq X$.

(iv) X is said to be an ideal of V if it is a left ideal and a right ideal of V.

Definition 2.3. *[48] Let V be a semigroup. An ideal X of V is said to be a completely prime ideal of V if it satisfies the property that for all $\acute{v}, \grave{v} \in V$, if $\acute{v}\grave{v} \in X$, then $\acute{v} \in X$ or $\grave{v} \in X$.*

2.2. Some properties of fuzzy sets

Definition 2.4. *[4] f is said to be a fuzzy subset of V if it is a function from V to the closed unit interval $[0, 1]$. Specifically, 1_V is denoted as a fuzzy subset of V defined by $1_V(v) = 1$ for all $v \in V$, and 0_V is denoted as a fuzzy subset of V defined by $0_V(v) = 0$ for all $v \in V$.*

Definition 2.5. *[4] Let f and g be fuzzy subsets of V.*

(i) $f \subseteq g$ *is denoted by meaning* $f(v) \leq g(v)$ *for all* $v \in V$.

(ii) A fuzzy set intersection of f *and* g *is denoted by* $f \cap g$, *where* $(f \cap g)(v)$ *is a minimum value of* $f(v)$ *and* $g(v)$ *(simply* $f(v) \wedge g(v)$*) for all* $v \in V$.

(iii) A fuzzy set union of f *and* g *is denoted by* $f \cup g$, *where* $(f \cup g)(v)$ *is a maximum value of* $f(v)$ *and* $g(v)$ *(simply* $f(v) \vee g(v)$*) for all* $v \in V$.

(iv) A fuzzy set complement of f *is denoted by* f', *where* f' *is a function from* V *to* $[0,1]$ *defined by* $f'(v) = 1 - f(v)$ *for all* $v \in V$.

Definition 2.6. *[26] Let* f *be a fuzzy subset of* V *and* $\iota \in [0,1]$. *The set*

$$V^{(f,\iota,>)} := \{v \in V : f(v) > \iota\}$$

is said to be an ι*-strong level set of* f.

Definition 2.7. *[49] Let* f *be a fuzzy subset of a semigroup* V. f *is said to be a fuzzy ideal of* V *if* $f(\acute{v}\grave{v}) \geq f(\acute{v}) \vee f(\grave{v})$ *for all* $\acute{v}, \grave{v} \in V$.

Definition 2.8. *[28] Let* f *be a fuzzy subset of a semigroup* V. *A fuzzy ideal* f *of* V *is said to be a fuzzy prime ideal of* V *if* $f(\acute{v}\grave{v}) = f(\acute{v})$ *or* $f(\acute{v}\grave{v}) = f(\grave{v})$ *for all* $\acute{v}, \grave{v} \in V$.

Proposition 2.1. *[28] Let* f *be a fuzzy subset of a semigroup* V. *Then* f *is a fuzzy ideal (resp., a fuzzy prime ideal) of* V *if and only if for all* $\iota \in [0,1]$*; if* $V^{(f,\iota,>)}$ *is non-empty, then* $V^{(f,\iota,>)}$ *is an ideal (resp., a completely prime ideal) of* V.

Definition 2.9. *[14] Based on Definition 2.4, an element in a collection of all fuzzy subsets of* $V \times W$ *is said to be a fuzzy relation from* V *to* W. *Given a fuzzy relation* R *from* V *to* W *and elements* $v \in V$, $w \in W$, *the value* $R(v,w)$ *in* $[0,1]$ *is a membership grade of the relation between* v *and* w *based on* R.

Definition 2.10. *[50, 51] Let* R *be a fuzzy relation from* V *to* V.

(i) R *is said to be a classical fuzzy reflexive relation if* $R(v,v) = 1$ *for all* $v \in V$.

(ii) R *is said to be a classical fuzzy symmetric relation if* $R(\acute{v},\grave{v}) = R(\grave{v},\acute{v})$ *for all* $\acute{v}, \grave{v} \in V$.

(iii) R *is said to be a classical fuzzy transitive relation if it satisfies*

$$R(\acute{v},\grave{v}) \geq \sup_{v \in V}\{\min\{R(\acute{v},v), R(v,\grave{v})\}\}$$

for all $\acute{v}, \grave{v} \in V$.

(iv) R *is said to be a fuzzy equivalence relation if it is a classical fuzzy reflexive relation, a classical fuzzy symmetric relation and a classical fuzzy transitive relation.*

Definition 2.11. *[52] Let* R *be a fuzzy relation from* V *to* V. R *is said to be a classical fuzzy perfect antisymmetric relation if for all* $\acute{v}, \grave{v} \in V$, $R(\acute{v},\grave{v}) > 0$ *and* $R(\grave{v},\acute{v}) > 0$ *imply* $\acute{v} = \grave{v}$.

Definition 2.12. *[49] Let* V *be a semigroup, and let* R *be a fuzzy relation from* V *to* V. R *is said to be a classical fuzzy compatible relation if for all* $v, \acute{v}, \grave{v} \in V$, $R(\acute{v}v, \grave{v}v) \geq R(\acute{v},\grave{v})$ *and* $R(v\acute{v}, v\grave{v}) \geq R(\acute{v},\grave{v})$.

Throughout this paper, $\mathcal{P}([0,1])$ represents a collection of all subsets of $[0,1]$.

Definition 2.13. *[24] Let V be a finite set. $f := (f^-, f^\pm, f^+)$ is said to be a picture hesitant fuzzy set on V if f^-, $f^\pm$ and f^+ are functions from V to $\mathcal{P}([0,1])$ together with the property that*

$$0 \leq \sup\{f^-(v)\} + \sup\{f^\pm(v)\} + \sup\{f^+(v)\} \leq 1 \tag{2.1}$$

for all $v \in V$. For $v \in V$, $f^-(v)$, $f^\pm(v)$ and $f^+(v)$ are three sets of several values in $[0,1]$, representing the potential negative, neutral, and positive membership degrees, respectively.

Definition 2.14. *[25] Let V and W be finite sets. Based on Definition 2.13, an element in a collection of all picture hesitant fuzzy sets on $V \times W$ is said to be a picture hesitant fuzzy relation from V to W.*

2.3. Some essential definitions of soft sets

Definition 2.15. *[6] Let A be a non-empty subset of K. If F is a mapping from A to $\mathcal{P}(V)$, then (F, A) is said to be a soft set over V concerning for A. As the understanding of the soft set, V is said to be a universe of all alternative objects of (F, A), and K is said to be a set of all parameters of (F, A), where parameters are attributes, characteristics or statements of alternative objects in V. For any element $a \in A$, $F(a)$ is considered as a set of a-approximate elements (or a-alternative objects) of (F, A).*

Definition 2.16. *[53] Let A be a non-empty subset of K.*

(i) A relative null soft set over V with respect to A is denoted by $\mathfrak{N}_{\emptyset_A} := (\emptyset_A, A)$, where $\emptyset_A$ is a set valued-mapping given by $\emptyset_A(a) = \emptyset$ for all $a \in A$.

(ii) For a soft set $\mathfrak{F} := (F, A)$ over V with respect to A, a support of $\mathfrak{F}$ is denoted by $Supp(\mathfrak{F})$, where

$$Supp\mathfrak{F} := \{a \in A : F(a) \neq \emptyset\}.$$

(iii) A relative whole soft set over V with respect to A is denoted by $\mathfrak{W}_{V_A} := (V_A, A)$, where V_A is a set valued-mapping given by $V_A(a) = V$ for all $a \in A$.

(iv) If $\mathfrak{F} := (F, A)$ is a given soft set over V, then a relative complement of $\mathfrak{F}$ is denoted by $C(\mathfrak{F}) := (F^c, A)$, which is a soft set defined by $F^c(a) = V - F(a)$ for all $a \in A$.

Definition 2.17. *[53] Let $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ be two soft sets over a common alternative universe with respect to non-empty subsets A and B of K, respectively.*

(i) $\mathfrak{F}$ is a soft subset of $\mathfrak{G}$, denoted by $\mathfrak{F} \Subset \mathfrak{G}$, if $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$.

(ii) $\mathfrak{F}$ is equal to $\mathfrak{G}$ if $\mathfrak{F} \Subset \mathfrak{G}$ and $\mathfrak{G} \Subset \mathfrak{F}$.

Definition 2.18. *[54] Let $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ be two soft sets over a common alternative universe with respect to non-empty subsets A and B of K, respectively.*

(i) A restricted intersection of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F} \Cap \mathfrak{G}$, is defined as a soft set (H, C), where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$.

(ii) A restricted union of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F} \Cup \mathfrak{G}$, is defined as a soft set (H, C), where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

(iii) *An extended intersection of* $\mathfrak{F}$ *and* $\mathfrak{G}$, *denoted by* $\mathfrak{F} \sqcap \mathfrak{G}$, *is defined as a soft set* (H, C), *where* $C = A \cup B$ *and*

$$H(c) = \begin{cases} F(c) & \text{if } c \in A - B, \\ G(c) & \text{if } c \in B - A, \\ F(c) \cap G(c) & \text{if } c \in A \cap B \end{cases}$$

for all $c \in C$.

(iv) *An extended union of* $\mathfrak{F}$ *and* $\mathfrak{G}$, *denoted by* $\mathfrak{F} \sqcup \mathfrak{G}$, *is defined as a soft set* (H, C), *where* $C = A \cup B$ *and*

$$H(c) = \begin{cases} F(c) & \text{if } c \in A - B, \\ G(c) & \text{if } c \in B - A, \\ F(c) \cup G(c) & \text{if } c \in A \cap B \end{cases}$$

for all $c \in C$.

(v) *A restricted difference of* $\mathfrak{F}$ *and* $\mathfrak{G}$, *denoted by* $\mathfrak{F} \ominus \mathfrak{G}$, *is defined as a soft set* (H, C), *where* $C = A \cap B \neq \emptyset$ *and* $H(c) = F(c) - G(c)$ *for all* $c \in C$.

Definition 2.19. *[55] Let* $\mathfrak{F} := (F, A)$ *and* $\mathfrak{G} := (G, B)$ *be two soft sets over a semigroup* V *with respect to non-empty subsets* A *and* B *of* K, *respectively. A restricted product of* $\mathfrak{F}$ *and* $\mathfrak{G}$, *denoted by* $\mathfrak{F} \circledcirc \mathfrak{G}$, *is defined as a soft set* (H, C), *where* $C = A \cap B$ *and* $H(c) = (F(c))(G(c))$ *for all* $c \in C$.

Definition 2.20. *[36] Let* $\mathfrak{F} := (F, A)$ *be a soft set over a semigroup* V *with respect to a non-empty subset* A *of* K.

(i) $\mathfrak{F}$ *is said to be an idealistic soft semigroup if* $F(a)$ *is an ideal of* V *for all* $a \in Supp\mathfrak{F}$.

(ii) $\mathfrak{F}$ *is said to be a prime idealistic soft semigroup if* $F(a)$ *is a completely prime ideal of* V *for all* $a \in Supp\mathfrak{F}$.

Definition 2.21. *[56] Let* $\mathfrak{F} := (F, A)$ *be a soft set over a semigroup* V *with respect to a non-empty subset* A *of* K. $\mathfrak{F}$ *is said to be a soft semigroup if* $F(a)$ *is, if it is non-empty, a subsemigroup of* V *for all* $a \in A$.

Definition 2.22. *[56] Let* $\mathfrak{F} := (F, A)$ *be a soft semigroup over a semigroup* V *with respect to a non-empty subset* A *of* K, *and let* $\mathfrak{G} := (G, B)$ *be a soft semigroup over a semigroup* W *with respect to a non-empty subset* B *of* K. *If* $\Theta : V \to W$ *is an epimorphism and* $\Xi : A \to B$ *is a surjective function such that* $\Theta(F(a)) = G(\Xi(a))$ *for all* $a \in A$, *then* $(\Theta, \Xi)_h$ *is said to be a soft homomorphism from* $\mathfrak{F}$ *to* $\mathfrak{G}$.

2.4. Variations of rough sets

Definition 2.23. *[5] Let* (V, E) *be a Pawlak's approximation space, and let* f *be a fuzzy subset of* V. *An upper rough approximation of* f *within* (V, E) *is defined by the fuzzy subset* $\ulcorner f \urcorner_E$ *of* V, *where*

$$\ulcorner f \urcorner_E(\acute{v}) = \sup\{f(\grave{v}) : \grave{v} \in [\acute{v}]_E\}$$

for all $\acute{v} \in V$. *A lower rough approximation of* f *within* (V, E) *is defined by the fuzzy subset* $\llcorner f \lrcorner_E$ *of* V, *where*

$$\llcorner f \lrcorner_E(\acute{v}) = \inf\{f(\grave{v}) : \grave{v} \in [\acute{v}]_E\}$$

for all $\acute{v} \in V$. f is said to be a definable fuzzy set within (V, E) if $\ulcorner f \urcorner_E = \llcorner f \lrcorner_E$; otherwise f is said to be a rough fuzzy set within (V, E).

Definition 2.24. *[7] Let (V, E) be a Pawlak's approximation space, and let $\mathfrak{F} := (F, A)$ be a soft set over V. An upper rough approximation of $\mathfrak{F}$ within (V, E) is denoted by $\mathfrak{F}\rceil_E := (F\rceil_E, A)$, where*

$$F\rceil_E(a) = \lceil F(a) \rceil_E$$

for all $a \in A$. A lower rough approximation of $\mathfrak{F}$ within (V, E) is denoted by $\mathfrak{F}\rfloor_E := (F\rfloor_E, A)$, where

$$F\rfloor_E(a) = \lfloor F(a) \rfloor_E$$

for all $a \in A$. $\mathfrak{F}$ is said to be a definable soft set within (V, E) if $\mathfrak{F}\rceil_E = \mathfrak{F}\rfloor_E$; otherwise $\mathfrak{F}$ is said to be a rough soft set within (V, E).

Definition 2.25. *[10] Let R be a fuzzy relation from V to W, and let $\alpha \in [0, 1]$. For an element $v \in V$,*

$$[v]^s_{R,\alpha} := \{w \in W : R(v, w) \geq \alpha\}$$

is said to be a successor class of v with respect to α-level based on R.

Definition 2.26. *[10] Let R be a fuzzy relation from V to W, and let $\alpha \in [0, 1]$. For an element $v \in V$,*

$$[v]^{cs}_{R,\alpha} := \{\dot{v} \in V : [v]^s_{R,\alpha} = (\dot{v})^s_{R,\alpha}\}$$

is said to be a core of the successor class of v with respect to α-level based on R. $[V]^{cs}_{R,\alpha}$ is denoted as a collection of $[v]^{cs}_{R,\alpha}$ for all $v \in V$.

Definition 2.27. *[10] If $\alpha \in [0, 1]$ and R is a fuzzy relation from V to W related to $[V]^{cs}_{R,\alpha}$, then $(V, W, [V]^{cs}_{R,\alpha})$ is said to be an approximation space based on $[V]^{cs}_{R,\alpha}$.*

Definition 2.28. *[10] Let $(V, W, [V]^{cs}_{R,\alpha})$ be an approximation space based on $[V]^{cs}_{R,\alpha}$, and let X be a non-empty subset of V. An upper approximation of X within $(V, W, [V]^{cs}_{R,\alpha})$ is denoted by $\lceil X \rceil^{cs}_{R,\alpha}$, where*

$$\lceil X \rceil^{cs}_{R,\alpha} := \bigcup_{v \in V} \{[v]^{cs}_{R,\alpha} : [v]^{cs}_{R,\alpha} \cap X \neq \emptyset\}.$$

A lower approximation of X within $(V, W, [V]^{cs}_{R,\alpha})$ is denoted by $\lfloor X \rfloor^{cs}_{R,\alpha}$, where

$$\lfloor X \rfloor^{cs}_{R,\alpha} := \bigcup_{v \in V} \{[v]^{cs}_{R,\alpha} : [v]^{cs}_{R,\alpha} \subseteq X\}.$$

A boundary region of X within $(V, W, [V]^{cs}_{R,\alpha})$ is defined by $\lceil X \rceil^{cs}_{R,\alpha} - \lfloor X \rfloor^{cs}_{R,\alpha}$. We say that $(\lceil X \rceil^{cs}_{R,\alpha}, \lfloor X \rfloor^{cs}_{R,\alpha})$ is a rough set of X within $(V, W, [V]^{cs}_{R,\alpha})$ if $\lceil X \rceil^{cs}_{R,\alpha} - \lfloor X \rfloor^{cs}_{R,\alpha}$ is a non-empty set. X is said to be a definable set within $(V, W, [V]^{cs}_{R,\alpha})$ if $\lceil X \rceil^{cs}_{R,\alpha} - \lfloor X \rfloor^{cs}_{R,\alpha}$ is an empty set.

3. Rough soft sets and rough fuzzy sets induced by set-valued picture hesitant fuzzy relations

In this section, we first develop the character of the picture hesitant fuzzy set under infinite sets. We can use it to build some properties of a picture hesitant fuzzy relation. We construct rough approximation models for soft sets and fuzzy sets induced by picture hesitant fuzzy relations. Then we can also use it to give some properties related to upper (resp., lower) rough approximations of soft sets and fuzzy sets.

Throughout the entire remainder, A and B are two non-empty subsets of K such that $A \cap B$ is non-empty.

According to Definition 2.13, a universal set is defined as an infinite set in this work. For picture hesitant fuzzy sets $f := (f^-, f^\pm, f^+)$ and $g := (g^-, g^\pm, g^+)$ on V, an inclusion relation of f and g is defined as follows:

$$f \subseteq_{ir} g \text{ if } f^-(v) \supseteq g^-(v), f^\pm(v) = g^\pm(v)(\text{iff } f^\pm(v) \supseteq g^\pm(v) \text{ and } f^\pm(v) \subseteq g^\pm(v)) \text{ and } f^+(v) \subseteq g^+(v)$$

for all $v \in V$. Furthermore, given two elements $(\alpha,\beta), (\gamma,\delta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$, we define set-valued relations of (α,β) and (γ,δ) as follows:

(i) $(\alpha,\beta) = (\gamma,\delta)$ if $\alpha = \gamma$ and $\beta = \delta$;

(ii) $(\alpha,\beta) \subseteq_{sr} (\gamma,\delta)$ if $\alpha \supseteq \gamma$ and $\beta \subseteq \delta$.

Applying Definitions 2.10 and 2.11 to progress, we shall introduce some types of a new picture hesitant fuzzy relation on a single universe as Definition 3.1 below. In order to find the maximum value in $[0,1]$ of relationships between two elements based on Definitions 2.13 and 2.14, inequality (2.1) is redefined that for all $v \in V$, there exists $w \in W$ such that

$$\max\{\sup\{R^-(v,w)\}, \sup\{R^\pm(v,w)\}, \sup\{R^+(v,w)\} = 1. \tag{3.1}$$

Then $(R^-, R^\pm, R^+)$ is called a set-valued picture hesitant fuzzy relation from V to W. Observe that there exist functions $R^-, R^\pm$ and R^+ from $V \times W$ to $\mathcal{P}([0,1])$ such that the supremum of membership degrees is 1.

Definition 3.1. *Let $R := (R^-, R^\pm, R^+)$ be a set-valued picture hesitant fuzzy relation from V to V.*

(i) R is called a set-valued picture hesitant fuzzy reflexive relation if

- $R^+(v,v) = [0,1]$ *for all* $v \in V$;
- $R^\pm(v,v)$ *is either* $(0,1]$ *or the others in* $\{(0,1),[0,1)\}$ *for all* $v \in V$;
- $R^-(v,v) = \emptyset$ *for all* $v \in V$.

(ii) R is called a set-valued picture hesitant fuzzy symmetric relation if

- $R^+(\acute{v},\grave{v}) = R^+(\grave{v},\acute{v})$ *for all* $\acute{v},\grave{v} \in V$;
- $R^\pm(\acute{v},\grave{v}) = R^\pm(\grave{v},\acute{v})$ *for all* $\acute{v},\grave{v} \in V$;
- $R^-(\acute{v},\grave{v}) = R^-(\grave{v},\acute{v})$ *for all* $\acute{v},\grave{v} \in V$.

(iii) R is called a set-valued picture hesitant fuzzy transitive relation if

- $\bigcup_{v\in V}(R^{+}(\acute{v},v)\cap R^{+}(v,\grave{v}))\subseteq R^{+}(\acute{v},\grave{v})$ *for all* $\acute{v},\grave{v}\in V$;
- $\bigcup_{v\in V}(R^{\pm}(\acute{v},v)\cap R^{\pm}(v,\grave{v}))\subseteq R^{\pm}(\acute{v},\grave{v})\subseteq \bigcap_{v\in V}(R^{\pm}(\acute{v},v)\cup R^{\pm}(v,\grave{v}))$ *for all* $\acute{v},\grave{v}\in V$;
- $\bigcap_{v\in V}(R^{-}(\acute{v},v)\cup R^{-}(v,\grave{v}))\supseteq R^{-}(\acute{v},\grave{v})$ *for all* $\acute{v},\grave{v}\in V$.

(iv) R is called a set-valued picture hesitant fuzzy antisymmetric relation if

- *for all* $\acute{v},\grave{v}\in V$, *if* $R^{+}(\acute{v},\grave{v})\neq\emptyset$ *and* $R^{+}(\grave{v},\acute{v})\neq\emptyset$, *then* $\acute{v}=\grave{v}$;
- *for all* $\acute{v},\grave{v}\in V$, *if* $R^{\pm}(\acute{v},\grave{v})$ *and* $R^{\pm}(\grave{v},\acute{v})$ *is neither* $\emptyset$ *nor* $[0,1]$, *then* $\acute{v}=\grave{v}$;
- *for all* $\acute{v},\grave{v}\in V$, *if* $R^{-}(\acute{v},\grave{v})\neq[0,1]$ *and* $R^{-}(\grave{v},\acute{v})\neq[0,1]$, *then* $\acute{v}=\grave{v}$.

(v) R is called a set-valued picture hesitant fuzzy equivalence relation if it is a set-valued picture hesitant fuzzy reflexive relation, a set-valued picture hesitant fuzzy symmetric relation and a set-valued picture hesitant fuzzy transitive relation.

According to Definition 3.1, we assume that a set-valued picture hesitant fuzzy relation $R:=(R^{-},R^{\pm},R^{+})$ from V to V is defined by square matrix representations $R^{-}_{M},R^{\pm}_{M},R^{+}_{M}$ as follows:

$$R^{-}_{M}:=\left[\dot{v}_{ij}:=R^{-}(v_i,v_j)\right],R^{\pm}_{M}:=\left[\ddot{v}_{ij}:=R^{\pm}(v_i,v_j)\right],R^{+}_{M}:=\left[\dddot{v}_{ij}:=R^{+}(v_i,v_j)\right]\in M_n(\mathcal{P}([0,1])),$$

where

$$\dot{v}_{ij}=\begin{cases}\emptyset & \text{if } i\geq j,\\ [0,1] & \text{if } i<j,\end{cases}$$

$$\ddot{v}_{ij}=\begin{cases}(0,1] & \text{if } i\geq j,\\ \emptyset & \text{if } i<j\end{cases}$$

and

$$\dddot{v}_{ij}=\begin{cases}[0,1] & \text{if } i\geq j,\\ \emptyset & \text{if } i<j.\end{cases}$$

Therefore, it is easy to verify that R is a set-valued picture hesitant fuzzy reflexive relation, a set-valued picture hesitant fuzzy transitive relation and a set-valued picture hesitant fuzzy antisymmetric relation.

As expressed above, observe that during the evaluating process of each relationship between two elements of V in this simple example, however, these possible memberships maybe not only crisp values in $[0,1]$, but also interval values (or subsets of $[0,1]$). In addition, we see that there exist approximation functions $R^{-},R^{\pm}$ and R^{+} such that relationships between two elements have the maximum value belongs to $[0,1]$, which it satisfies Eq (3.1).

Definition 3.2. *Let* $R:=(R^{-},R^{\pm},R^{+})$ *be a set-valued picture hesitant fuzzy relation from V to W, and let* $(\alpha,\beta)\in\mathcal{P}([0,1])\times\mathcal{P}([0,1])$. *For an element* $v\in V$, *we call*

$$[v]^{s}_{R,(\alpha,\beta)}:=\{w\in W:R^{-}(v,w)\subseteq\alpha,\emptyset\subset R^{\pm}(v,w)\subset[0,1] \text{ and } R^{+}(v,w)\supseteq\beta,\}$$

a successor class of v with respect to (α,β)*-inclusion based on R. We generally denote by* $[V]^{s}_{R,(\alpha,\beta)}$ *a collection of* $[v]^{s}_{R,(\alpha,\beta)}$ *for all* $v\in V$.

Proposition 3.1. *If* $R:=(R^{-},R^{\pm},R^{+})$ *is a set-valued picture hesitant fuzzy reflexive relation from V to V and* $(\alpha,\beta)\in\mathcal{P}([0,1])\times\mathcal{P}([0,1])$, *then* $v\in[v]^{s}_{R,(\alpha,\beta)}$ *for all* $v\in V$.

Proof. Suppose that R is a set-valued picture hesitant fuzzy reflexive relation from V to V and $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$. Let $v \in V$ be given, then

$$R^{-}(v,v) = \emptyset \subseteq \alpha \text{ and } R^{+}(v,v) = [0,1] \supseteq \beta.$$

Next, we consider the following three cases.

Case 1. If $R^{\pm}(v,v) = (0,1]$, then $\emptyset \subset R^{\pm}(v,v) \subset [0,1]$.

Case 2. If $R^{\pm}(v,v) = (0,1)$, then $\emptyset \subset R^{\pm}(v,v) \subset [0,1]$.

Case 3. If $R^{\pm}(v,v) = [0,1)$, then $\emptyset \subset R^{\pm}(v,v) \subset [0,1]$.

This implies that $v \in [v]^{s}_{R,(\alpha,\beta)}$. □

Definition 3.3. *Let $R := (R^{-}, R^{\pm}, R^{+})$ be a set-valued picture hesitant fuzzy relation from V to W, and let $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$. For an element $v \in V$, we call*

$$[v]^{cs}_{R,(\alpha,\beta)} := \{\dot{v} \in V : [v]^{s}_{R,(\alpha,\beta)} = [\dot{v}]^{s}_{R,(\alpha,\beta)}\}$$

a core of the successor class of v with respect to (α,β)-inclusion based on R. We shall denote by $[V]^{cs}_{R,(\alpha,\beta)}$ a collection of $[v]^{cs}_{R,(\alpha,\beta)}$ for all $v \in V$.

Due to Definition 3.3, the following two statements hold.

Proposition 3.2. *If $R := (R^{-}, R^{\pm}, R^{+})$ is a set-valued picture hesitant fuzzy relation from V to W and $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$, then $v \in [v]^{cs}_{R,(\alpha,\beta)}$ for all $v \in V$.*

Proposition 3.3. *If $R := (R^{-}, R^{\pm}, R^{+})$ is a set-valued picture hesitant fuzzy relation from V to W and $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$, then the following two arguments are equivalent.*

(i) $\acute{v} \in [\grave{v}]^{cs}_{R,(\alpha,\beta)}$ *for all* $\acute{v}, \grave{v} \in V$.

(ii) $[\acute{v}]^{cs}_{R,(\alpha,\beta)} = [\grave{v}]^{cs}_{R,(\alpha,\beta)}$ *for all* $\acute{v}, \grave{v} \in V$.

Remark 3.1. *Propositions 3.2 and 3.3 indicate that if $R := (R^{-}, R^{\pm}, R^{+})$ is a set-valued picture hesitant fuzzy relation from V to W and $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$, then $[V]^{cs}_{R,(\alpha,\beta)}$ is the partition of V.*

For Propositions 3.4 and 3.5 in the following, proofs are straightforward.

Proposition 3.4. *If $R := (R^{-}, R^{\pm}, R^{+})$ is a set-valued picture hesitant fuzzy reflexive relation from V to V and $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$, then $[v]^{cs}_{R,(\alpha,\beta)} \subseteq [v]^{s}_{R,(\alpha,\beta)}$ for all $v \in V$.*

Proposition 3.5. *If $R := (R^{-}, R^{\pm}, R^{+})$ is a set-valued picture hesitant fuzzy symmetric relation and a set-valued picture hesitant fuzzy transitive relation from V to V and $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$, then $[v]^{s}_{R,(\alpha,\beta)} \subseteq [v]^{cs}_{R,(\alpha,\beta)}$ for all $v \in V$.*

According to Remark 3.1, Propositions 3.4 and 3.5, we have immediately the following propositions.

Proposition 3.6. *If $R := (R^{-}, R^{\pm}, R^{+})$ is a set-valued picture hesitant fuzzy equivalence relation from V to V and $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$, then $[V]^{s}_{R,(\alpha,\beta)}$ is the partition of V.*

Proposition 3.7. *If $R := (R^{-}, R^{\pm}, R^{+})$ is a set-valued picture hesitant fuzzy reflexive relation and a set-valued picture hesitant fuzzy antisymmetric relation from V to V and $(\alpha,\beta) \in \mathcal{P}([0,1])\backslash\{[0,1]\} \times \mathcal{P}([0,1])\backslash\{\emptyset\}$, then the following statements are equivalent.*

(i) $\acute{v} = \grave{v}$ *for all* $\acute{v}, \grave{v} \in V$.

(ii) $[\acute{v}]^{cs}_{R,(\alpha,\beta)} = [\grave{v}]^{cs}_{R,(\alpha,\beta)}$ *for all* $\acute{v}, \grave{v} \in V$.

(iii) $\acute{v} \in [\grave{v}]^{cs}_{R,(\alpha,\beta)}$ *for all* $\acute{v}, \grave{v} \in V$.

Proof. It is clear that (i) implies (ii). Due to Proposition 3.3, we obtain that (ii) implies (iii). In order to prove that (iii) implies (i), let $v_1, v_2 \in V$. Suppose $v_1 \in [v_2]^{cs}_{R,(\alpha,\beta)}$. Then $[v_1]^{s}_{R,(\alpha,\beta)} = [v_2]^{s}_{R,(\alpha,\beta)}$. Since R is a set-valued picture hesitant fuzzy reflexive relation, we have $v_1 \in [v_1]^{s}_{R,(\alpha,\beta)}$ and $v_2 \in [v_2]^{s}_{R,(\alpha,\beta)}$ due to Proposition 3.1. We see that $v_1 \in [v_2]^{s}_{R,(\alpha,\beta)}$ and $v_2 \in [v_1]^{s}_{R,(\alpha,\beta)}$. Thus,

$$R^{-}(v_2, v_1) \subseteq \alpha \neq [0,1], \emptyset \subset R^{\pm}(v_2, v_1) \subset [0,1] \text{ and } R^{+}(v_2, v_1) \supseteq \beta \neq \emptyset$$

and

$$R^{-}(v_1, v_2) \subseteq \alpha \neq [0,1], \emptyset \subset R^{\pm}(v_1, v_2) \subset [0,1] \text{ and } R^{+}(v_1, v_2) \supseteq \beta \neq \emptyset.$$

Since R is a set-valued picture hesitant fuzzy antisymmetric relation, we obtain that $v_1 = v_2$ as required. □

In the following, upper and lower rough approximations of soft sets and fuzzy sets are being considered under set-valued picture hesitant fuzzy relations.

Definition 3.4. *If* $(\alpha,\beta) \in \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$ *and* $R := (R^{-}, R^{\pm}, R^{+})$ *is a set-valued picture hesitant fuzzy relation from V to W related to* $[V]^{cs}_{R,(\alpha,\beta)}$, *then* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ *is called an approximation space based on* $[V]^{cs}_{R,(\alpha,\beta)}$. *In this way, we call* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ *an approximation space type I.*

According to Definition 3.4, observe that $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ can be considered as an extended approximation space of the approximation space in Definition 2.27 under the property of set-valued picture hesitant fuzzy relations.

Definition 3.5. *Let* $(V, W, [V]^{cs}_{R:=(R^{-},R^{\pm},R^{+}),(\alpha,\beta)})$ *be an approximation space type I, and let* $\mathfrak{F} := (F, A)$ *be a soft set over V. An upper rough approximation of* $\mathfrak{F}$ *within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ *is denoted by* $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} := (F\rceil^{cs}_{R,(\alpha,\beta)}, A)$, *where*

$$F\rceil^{cs}_{R,(\alpha,\beta)}(a) = \bigcup_{v \in V} \{[v]^{cs}_{R,(\alpha,\beta)} : [v]^{cs}_{R,(\alpha,\beta)} \cap F(a) \neq \emptyset\} \tag{3.2}$$

for all $a \in A$. *A lower rough approximation of* $\mathfrak{F}$ *within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ *is denoted by* $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} := (F\rfloor^{cs}_{R,(\alpha,\beta)}, A)$, *where*

$$F\rfloor^{cs}_{R,(\alpha,\beta)}(a) = \bigcup_{v \in V} \{[v]^{cs}_{R,(\alpha,\beta)} : [v]^{cs}_{R,(\alpha,\beta)} \subseteq F(a)\} \tag{3.3}$$

for all $a \in A$. *A boundary region of* $\mathfrak{F}$ *within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ *is denoted by* $\mathfrak{F}]^{cs}_{R,(\alpha,\beta)} := (F]^{cs}_{R,(\alpha,\beta)}, A)$, *where*

$$(F]^{cs}_{R,(\alpha,\beta)}, A) = \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \ominus \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}.$$

As introduced above, such sets are obtained the following interpretations.

(i) $F\rceil^{cs}_{R,(\alpha,\beta)}(a)$ *contains all objects which possibly belong to* $F(a)$ *for all* $a \in A$. *In this way, a complement of* $F\rceil^{cs}_{R,(\alpha,\beta)}(a)$ *is said to be a negative region of* $F(a)$ *within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ *for all* $a \in A$.

(ii) $F\rfloor^{cs}_{R,(\alpha,\beta)}(a)$ *consists of all objects which surely belong to* $F(a)$ *for all* $a \in A$*. In this way, such the set is said to be a positive region of* $F(a)$ *within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ *for all* $a \in A$*.*

(iii) $F\rceil^{cs}_{R,(\alpha,\beta)}(a)$ *is a set of all objects, which can be classified neither as* $F(a)$ *nor as non* $F(a)$ *using* R *for all* $a \in A$*.*

In what follows, for all $a \in A$*, if* $F\rceil^{cs}_{R,(\alpha,\beta)}(a) \neq \emptyset$*, then* $(F\rceil^{cs}_{R,(\alpha,\beta)}(a), F\rfloor^{cs}_{R,(\alpha,\beta)}(a))$ *is called a rough (or an inexact) set of* $F(a)$ *within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$*, and we call* $F(a)$ *a rough set. For all* $a \in A$*, if* $F\rceil^{cs}_{R,(\alpha,\beta)}(a) = \emptyset$*, then* $F(a)$ *is called a definable (or an exact) set within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$*. The soft set* $\mathfrak{F}$ *is called a definable soft set within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$ *if* $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} = \mathfrak{N}_{\emptyset_A}$*; otherwise* $\mathfrak{F}$ *is called a rough soft set within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$*.*

In view of Definition 3.5, we consider the following example under crisp sets.

Example 3.1. *Let* $(V, W, [V]^{cs}_{R:=(R^-,R^+),(\emptyset,[0,1])})$ *be a given approximation space type I, where* $V = \{v_n := n : n \text{ is a natural number}\}$*,* $W = \{w_n := n : n \text{ is an integer}\}$ *and* R *is a set-valued picture hesitant fuzzy relation from* V *to* W *defined by*

$$R^+(v,w) = \begin{cases} [0,1] & \text{if } 5|v-w, \\ \emptyset & \text{if } 5 \nmid v-w, \end{cases}$$

$$R^\pm(v,w) = \begin{cases} [0,1) & \text{if } 5|v-w, \\ \emptyset & \text{if } 5 \nmid v-w \end{cases}$$

and

$$R^-(v,w) = \begin{cases} \emptyset & \text{if } 5|v-w, \\ [0,1] & \text{if } 5 \nmid v-w \end{cases}$$

for all $(v,w) \in V \times W$*. Observe that if* n *is a natural number, then*

$[v_{5n-4}]^{s}_{R,(\emptyset,[0,1])} = \{w_{5i-4} : i \text{ is an integer}\}$,
$[v_{5n-3}]^{s}_{R,(\emptyset,[0,1])} = \{w_{5i-3} : i \text{ is an integer}\}$,
$[v_{5n-2}]^{s}_{R,(\emptyset,[0,1])} = \{w_{5i-2} : i \text{ is an integer}\}$,
$[v_{5n-1}]^{s}_{R,(\emptyset,[0,1])} = \{w_{5i-1} : i \text{ is an integer}\}$ *and*
$[v_{5n}]^{s}_{R,(\emptyset,[0,1])} = \{w_{5i} : i \text{ is an integer}\}$,

which yields

$[v_{5n-4}]^{cs}_{R,(\emptyset,[0,1])} = \{v_{5i-4} : i \text{ is a natural number}\}$,
$[v_{5n-3}]^{cs}_{R,(\emptyset,[0,1])} = \{v_{5i-3} : i \text{ is a natural number}\}$,
$[v_{5n-2}]^{cs}_{R,(\emptyset,[0,1])} = \{v_{5i-2} : i \text{ is a natural number}\}$,
$[v_{5n-1}]^{cs}_{R,(\emptyset,[0,1])} = \{v_{5i-1} : i \text{ is a natural number}\}$ *and*
$[v_{5n}]^{cs}_{R,(\emptyset,[0,1])} = \{v_{5i} : i \text{ is a natural number}\}$.

If $\mathfrak{F} := (F, A)$ *is a soft set over* V *defined by*

$$F(a) = \{v_{5i-2} : i \text{ is a natural number}\} \cup \{v_{5i} : i \text{ is a natural number with } 20 \leq i \leq 100\}$$

for all $a \in A$*, then we observe that*

$F\rceil^{cs}_{R,(\emptyset,[0,1])}(a) = \{v_{5i-2} : i \text{ is a natural number}\} \cup \{v_{5i} : i \text{ is a natural number}\}$,
$F\rfloor^{cs}_{R,(\emptyset,[0,1])}(a) = \{v_{5i-2} : i \text{ is a natural number}\}$ *and*
$F\rceil^{cs}_{R,(\emptyset,[0,1])}(a) = \{v_{5i} : i \text{ is a natural number}\}$

for all $a \in A$*. This implies that* $\mathfrak{F}$ *is a rough soft set within* $(V, W, [V]^{cs}_{R,(\emptyset,[0,1])})$*.*

Remark 3.2. *If* $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *is an approximation space type I and* $\mathfrak{F} := (F, A)$ *is a soft set over V, then it is easy to see that* $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{F} \Subset \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$.

The terminology of Definition 3.5 has some basic properties that we collect in the next four propositions.

Proposition 3.8. *Let* $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be an approximation space type I, and let* $\mathfrak{F} := (F, A)$ *be a given soft set over V. According to Eqs* (3.2) *and* (3.3) *in Definition 3.5, we have the following statements:*

(i) $F\rceil^{cs}_{R,(\alpha,\beta)}(a) = \{v \in V : [v]^{cs}_{R,(\alpha,\beta)} \cap F(a) \neq \emptyset\}$ *for all* $a \in A$.

(ii) $F\rfloor^{cs}_{R,(\alpha,\beta)}(a) = \{v \in V : [v]^{cs}_{R,(\alpha,\beta)} \subseteq F(a)\}$ *for all* $a \in A$.

Proposition 3.9. *Let* $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be an approximation space type I, and let* $\mathfrak{F} := (F, A)$ *be a soft set over V. Then we have the following statements:*

(i) If $\mathfrak{F} = \mathfrak{W}_{V_A}$, *then* $\mathfrak{F}$ *is equal to* $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ *and* $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$. *Moreover,* $\mathfrak{F}$ *is a definable soft set within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$.

(ii) If $\mathfrak{F} = \mathfrak{N}_{\emptyset_A}$, *then* $\mathfrak{F}$ *is equal to* $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ *and* $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$. *Moreover,* $\mathfrak{F}$ *is a definable soft set within* $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$.

(iii) $(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})\rceil^{cs}_{R,(\alpha,\beta)} = \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$.

(iv) $(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})\rfloor^{cs}_{R,(\alpha,\beta)} = \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$.

(v) $(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})\rfloor^{cs}_{R,(\alpha,\beta)} = \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$.

(vi) $(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})\rceil^{cs}_{R,(\alpha,\beta)} = \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$.

(vii) $C(\mathfrak{F})\rfloor^{cs}_{R,(\alpha,\beta)} = C(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$.

(viii) $C(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})\rceil^{cs}_{R,(\alpha,\beta)} = C(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$.

(ix) $C(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})\rfloor^{cs}_{R,(\alpha,\beta)} = C(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})$.

(x) $C(C(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})) = \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$.

(xi) $C(C(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})) = \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$.

Proposition 3.10. *Let* $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be an approximation space type I, and let* $\mathfrak{F} := (F, A)$ *and* $\mathfrak{G} := (G, B)$ *be soft sets over V. Then we have the following statements:*

(i) $(\mathfrak{F} \Cap \mathfrak{G})\rceil^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \Cap \mathfrak{G}\rceil^{cs}_{R,(\alpha,\beta)}$.

(ii) $(\mathfrak{F} \Cap \mathfrak{G})\rfloor^{cs}_{R,(\alpha,\beta)} = \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} \Cap \mathfrak{G}\rfloor^{cs}_{R,(\alpha,\beta)}$.

(iii) $(\mathfrak{F} \sqcap \mathfrak{G})\rceil^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \sqcap \mathfrak{G}\rceil^{cs}_{R,(\alpha,\beta)}$.

(iv) $(\mathfrak{F} \sqcap \mathfrak{G})\rfloor^{cs}_{R,(\alpha,\beta)} = \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} \sqcap \mathfrak{G}\rfloor^{cs}_{R,(\alpha,\beta)}$.

(v) $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \Cup \mathfrak{G}\rceil^{cs}_{R,(\alpha,\beta)} = (\mathfrak{F} \Cup \mathfrak{G})\rceil^{cs}_{R,(\alpha,\beta)}$.

(vi) $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} \Cup \mathfrak{G}\rfloor^{cs}_{R,(\alpha,\beta)} \Subset (\mathfrak{F} \Cup \mathfrak{G})\rfloor^{cs}_{R,(\alpha,\beta)}$.

(vii) $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \sqcup \mathfrak{G}\rceil^{cs}_{R,(\alpha,\beta)} = (\mathfrak{F} \sqcup \mathfrak{G})\rceil^{cs}_{R,(\alpha,\beta)}$.

(viii) $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} \sqcup \mathfrak{G}\rfloor^{cs}_{R,(\alpha,\beta)} \Subset (\mathfrak{F} \sqcup \mathfrak{G})\rfloor^{cs}_{R,(\alpha,\beta)}$.

Proposition 3.11. *Let* $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be an approximation space type I, and let* $\mathfrak{F} := (F, A)$ *and* $\mathfrak{G} := (G, B)$ *be given soft sets over V. If* $\mathfrak{F} \Subset \mathfrak{G}$*, then* $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{G}\rceil^{cs}_{R,(\alpha,\beta)}$ *and* $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{G}\rfloor^{cs}_{R,(\alpha,\beta)}$.

Proposition 3.12. *Let* $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *and* $(V, V, [V]^{cs}_{S:=(S^-,S^\pm,S^+),(\gamma,\delta)})$ *be two approximation spaces type I with the property that the inclusion relation of the set-valued picture hesitant fuzzy reflexive relation R and the set-valued picture hesitant fuzzy transitive relation S is* $R \subseteq_{ir} S$*, and* $(\gamma, \delta) \subseteq_{sr} (\alpha, \beta)$*. If* $\mathfrak{F} := (F, A)$ *is a soft set over V, then* $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{F}\rceil^{cs}_{S,(\gamma,\delta)}$ *and* $\mathfrak{F}\rfloor^{cs}_{S,(\gamma,\delta)} \Subset \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$.

Proof. Assume that $\mathfrak{F}$ is a soft set over V and $a \in A$. Let $v_1 \in F\rceil^{cs}_{R,(\alpha,\beta)}(a)$, then $[v_1]^{cs}_{R,(\alpha,\beta)} \cap F(a) \neq \emptyset$. Let $v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)} \cap F(a)$, then $[v_1]^{s}_{R,(\alpha,\beta)} = [v_2]^{s}_{R,(\alpha,\beta)}$. Since R is a set-valued picture hesitant fuzzy reflexive relation, we obtain that $v_1 \in [v_1]^{s}_{R,(\alpha,\beta)}$ and $v_2 \in [v_2]^{s}_{R,(\alpha,\beta)}$ due to Proposition 3.1. Observe that $v_1 \in [v_2]^{s}_{R,(\alpha,\beta)}$ and $v_2 \in [v_1]^{s}_{R,(\alpha,\beta)}$. Whence

$$S^-(v_2, v_1) \subseteq R^-(v_2, v_1) \subseteq \alpha \subseteq \gamma, S^-(v_1, v_2) \subseteq R^-(v_1, v_2) \subseteq \alpha \subseteq \gamma,$$

$$\emptyset \subset S^\pm(v_2, v_1) = R^\pm(v_2, v_1) \subset [0, 1], \emptyset \subset S^\pm(v_1, v_2) = R^\pm(v_1, v_2) \subset [0, 1]$$

and

$$S^+(v_2, v_1) \supseteq R^+(v_2, v_1) \supseteq \beta \supseteq \delta, S^+(v_1, v_2) \supseteq R^+(v_1, v_2) \supseteq \beta \supseteq \delta.$$

We shall prove that $[v_1]^{s}_{S,(\gamma,\delta)} = [v_2]^{s}_{S,(\gamma,\delta)}$. Let $v_3 \in [v_2]^{s}_{S,(\gamma,\delta)}$, then

$$S^-(v_2, v_3) \subseteq \gamma, \emptyset \subset S^\pm(v_2, v_3) \subset [0, 1] \text{ and } S^+(v_2, v_3) \supseteq \delta.$$

Since S is a set-valued picture hesitant fuzzy transitive relation, we see that

$$\begin{aligned} S^-(v_1, v_3) &\subseteq \bigcap_{v \in V}(S^-(v_1, v) \cup S^-(v, v_3)) \\ &\subseteq S^-(v_1, v_2) \cup S^-(v_2, v_3) \\ &\subseteq \gamma \cup \gamma \\ &= \gamma, \end{aligned}$$

$$\begin{aligned} S^\pm(v_1, v_3) &\subseteq \bigcap_{v \in V}(S^\pm(v_1, v) \cup S^\pm(v, v_3)) \\ &\subseteq S^\pm(v_1, v_2) \cup S^\pm(v_2, v_3) \\ &\subset [0, 1] \cup [0, 1] \\ &= [0, 1], \end{aligned}$$

$$\begin{aligned} S^{\pm}(v_1, v_3) &\supseteq \bigcup_{v\in V}(S^{\pm}(v_1, v) \cap S^{\pm}(v, v_3)) \\ &\supseteq S^{\pm}(v_1, v_2) \cap S^{\pm}(v_2, v_3) \\ &\supset \emptyset \cap \emptyset \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} S^{+}(v_1, v_3) &\supseteq \bigcup_{v\in V}(S^{+}(v_1, v) \cap S^{+}(v, v_3)) \\ &\supseteq S^{+}(v_1, v_2) \cap S^{+}(v_2, v_3) \\ &\supseteq \delta \cap \delta \\ &= \delta. \end{aligned}$$

Whence $v_3 \in [v_1]^s_{S,(\gamma,\delta)}$, which yields $[v_2]^s_{S,(\gamma,\delta)} \subseteq [v_1]^s_{S,(\gamma,\delta)}$. On the other hand, we can verify that $[v_1]^s_{S,(\gamma,\delta)} \subseteq [v_2]^s_{S,(\gamma,\delta)}$. Thus, $[v_1]^s_{S,(\gamma,\delta)} = [v_2]^s_{S,(\gamma,\delta)}$. Whence $v_2 \in [v_1]^{cs}_{S,(\gamma,\delta)}$. Observe that $v_2 \in [v_1]^{cs}_{S,(\gamma,\delta)} \cap F(a)$. Hence $[v_1]^{cs}_{S,(\gamma,\delta)} \cap F(a) \neq \emptyset$, which yields $v_1 \in F\rceil^{cs}_{S,(\gamma,\delta)}(a)$. Therefore, $F\rceil^{cs}_{R,(\alpha,\beta)}(a) \subseteq F\rceil^{cs}_{S,(\gamma,\delta)}(a)$. This means that $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{F}\rceil^{cs}_{S,(\gamma,\delta)}$.

To prove that $\mathfrak{F}\rfloor^{cs}_{S,(\gamma,\delta)} \Subset \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$, let $v_4 \in F\rfloor^{cs}_{S,(\gamma,\delta)}(a)$, then $[v_4]^{cs}_{S,(\gamma,\delta)} \subseteq F(a)$. It suffices to prove that $[v_4]^{cs}_{R,(\alpha,\beta)} \subseteq [v_4]^{cs}_{S,(\gamma,\delta)}$. Suppose $v_5 \in [v_4]^{cs}_{R,(\alpha,\beta)}$, then $[v_4]^s_{R,(\alpha,\beta)} = [v_5]^s_{R,(\alpha,\beta)}$. Since R is a set-valued picture hesitant fuzzy reflexive relation, we have $v_4 \in [v_4]^s_{R,(\alpha,\beta)}$ and $v_5 \in [v_5]^s_{R,(\alpha,\beta)}$ due to Proposition 3.1. We observe that $v_4 \in [v_5]^s_{R,(\alpha,\beta)}$ and $v_5 \in [v_4]^s_{R,(\alpha,\beta)}$. Thus,

$$S^{-}(v_5, v_4) \subseteq R^{-}(v_5, v_4) \subseteq \alpha \subseteq \gamma, S^{-}(v_4, v_5) \subseteq R^{-}(v_4, v_5) \subseteq \alpha \subseteq \gamma,$$

$$\emptyset \subset S^{\pm}(v_5, v_4) = R^{\pm}(v_5, v_4) \subset [0, 1], \emptyset \subset S^{\pm}(v_4, v_5) = R^{\pm}(v_4, v_5) \subset [0, 1]$$

and

$$S^{+}(v_5, v_4) \supseteq R^{+}(v_5, v_4) \supseteq \beta \supseteq \delta, S^{+}(v_4, v_5) \supseteq R^{+}(v_4, v_5) \supseteq \beta \supseteq \delta.$$

We shall show that $[v_4]^s_{S,(\gamma,\delta)} = [v_5]^s_{S,(\gamma,\delta)}$. Assume $v_6 \in [v_5]^s_{S,(\gamma,\delta)}$, then

$$S^{-}(v_5, v_6) \subseteq \gamma, \emptyset \subset S^{\pm}(v_5, v_6) \subset [0, 1] \text{ and } S^{+}(v_5, v_6) \supseteq \delta.$$

Since S is a set-valued picture hesitant fuzzy transitive relation, we observe that

$$\begin{aligned} S^{-}(v_4, v_6) &\subseteq \bigcap_{v\in V}(S^{-}(v_4, v) \cup S^{-}(v, v_6)) \\ &\subseteq S^{-}(v_4, v_5) \cup S^{-}(v_5, v_6) \\ &\subseteq \gamma \cup \gamma \\ &= \gamma, \end{aligned}$$

$$\begin{aligned} S^{\pm}(v_4, v_6) &\subseteq \bigcap_{v\in V}(S^{\pm}(v_4, v) \cup S^{\pm}(v, v_6)) \\ &\subseteq S^{\pm}(v_4, v_5) \cup S^{\pm}(v_5, v_6) \\ &\subset [0,1] \cup [0,1] \\ &= [0,1], \end{aligned}$$

$$\begin{aligned} S^{\pm}(v_4, v_6) &\supseteq \bigcup_{v\in V}(S^{\pm}(v_4, v) \cap S^{\pm}(v, v_6)) \\ &\supseteq S^{\pm}(v_4, v_5) \cap S^{\pm}(v_5, v_6) \\ &\supset \emptyset \cap \emptyset \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} S^{+}(v_4, v_6) &\supseteq \bigcup_{v\in V}(S^{+}(v_4, v) \cap S^{+}(v, v_6)) \\ &\supseteq S^{+}(v_4, v_5) \cap S^{+}(v_5, v_6) \\ &\supseteq \delta \cap \delta \\ &= \delta. \end{aligned}$$

We get $v_6 \in [v_4]^s_{S,(\gamma,\delta)}$. Hence $[v_5]^s_{S,(\gamma,\delta)} \subseteq [v_4]^s_{S,(\gamma,\delta)}$. Conversely, we can find that $[v_4]^s_{S,(\gamma,\delta)} \subseteq [v_5]^s_{S,(\gamma,\delta)}$. Thus, $[v_4]^s_{S,(\gamma,\delta)} = [v_5]^s_{S,(\gamma,\delta)}$. Whence $v_5 \in [v_4]^{cs}_{S,(\gamma,\delta)}$. Hence $[v_4]^{cs}_{R,(\alpha,\beta)} \subseteq [v_4]^{cs}_{S,(\gamma,\delta)} \subseteq F(a)$. Therefore $v_4 \in F\rfloor^{cs}_{R,(\alpha,\beta)}(a)$, which yields $F\rfloor^{cs}_{S,(\gamma,\delta)}(a) \subseteq F\rfloor^{cs}_{R,(\alpha,\beta)}(a)$. This implies that $\mathfrak{F}\rfloor^{cs}_{S,(\gamma,\delta)} \Subset \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$. The proof is complete. □

Proposition 3.13. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be an approximation space type I with the property that R is a set-valued picture hesitant fuzzy reflexive relation and a set-valued picture hesitant fuzzy antisymmetric relation and $(\alpha,\beta) \in \mathcal{P}([0,1])\backslash\{[0,1]\} \times \mathcal{P}([0,1])\backslash\{\emptyset\}$. If $\mathfrak{F} := (F, A)$ is a soft set over V, then $\mathfrak{F}$ is a definable soft set within $(V, V, [V]^{cs}_{R,(\alpha,\beta)})$.*

Proof. By Remark 3.2, it is true that $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$. Let $a \in A$ and $v_1 \in F\rceil^{cs}_{R,(\alpha,\beta)}(a)$, then $[v_1]^{cs}_{R,(\alpha,\beta)} \cap F(a) \neq \emptyset$. Let $v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)} \cap F(a)$. By Proposition 3.7, we get that $v_1 = v_2$. We must prove that $[v_1]^{cs}_{R,(\alpha,\beta)} \subseteq F(a)$. Let $v_3 \in [v_1]^{cs}_{R,(\alpha,\beta)}$, then $v_1 = v_3$ due to Proposition 3.7. Hence $v_3 = v_2 \in F(a)$, which implies that $[v_1]^{cs}_{R,(\alpha,\beta)} \subseteq F(a)$. Thus $v_1 \in F\rfloor^{cs}_{R,(\alpha,\beta)}(a)$. Whence $F\rceil^{cs}_{R,(\alpha,\beta)}(a) \subseteq F\rfloor^{cs}_{R,(\alpha,\beta)}(a)$. Therefore $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \Subset \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$. Thus, $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is equal to $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$. As a consequence, $\mathfrak{F}$ is a definable soft set within $(V, V, [V]^{cs}_{R,(\alpha,\beta)})$. □

We consider the following example.

Example 3.2. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^+),(\emptyset,[0,1])})$ be a given approximation space type I, where $V = \{v_n := 2^{n-1} : n \text{ is a natural number}\}$ and R is a set-valued picture hesitant fuzzy reflexive relation and a set-valued picture hesitant fuzzy antisymmetric relation from V to V defined by*

$$R^{+}(\acute{v}, \grave{v}) = \begin{cases} [0,1] & \text{if } \acute{v} \geq \grave{v}, \\ \emptyset & \text{if } \acute{v} < \grave{v}, \end{cases}$$

$$R^{\pm}(\acute{v},\grave{v}) = \begin{cases}(0,1] & \text{if } \acute{v} \geq \grave{v},\\ [0,1] & \text{if } \acute{v} < \grave{v}\end{cases}$$

and

$$R^{-}(\acute{v},\grave{v}) = \begin{cases}\emptyset & \text{if } \acute{v} \geq \grave{v},\\ [0,1] & \text{if } \acute{v} < \grave{v}\end{cases}$$

for all $\acute{v},\grave{v} \in V$. *Observe that if* n *is a natural number, then* $[v_n]^{cs}_{R,(\emptyset,[0,1])} = \{v_n\}$. *This implies that if* $\mathfrak{F} := (F,A)$ *is a soft set over* V, *then it is easy to see that* $\mathfrak{F}\rfloor^{cs}_{R,(\emptyset,[0,1])}$, $\mathfrak{F}$ *and* $\mathfrak{F}\rceil^{cs}_{R,(\emptyset,[0,1])}$ *are identical. It follows that* $\mathfrak{F}$ *is a definable soft set within* $(V,V,[V]^{cs}_{R,(\emptyset,[0,1])})$.

Definition 3.6. *Let* $(V,W,[V]^{cs}_{R:=(R^-,R^{\pm},R^+),(\alpha,\beta)})$ *be a given approximation space type I. Let* f *be a fuzzy subset of* V. *An upper rough approximation of* f *within* $(V,W,[V]^{cs}_{R,(\alpha,\beta)})$ *is defined by the fuzzy subset* $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ *of* V, *where*

$$\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}(\acute{v}) = \sup\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{R,(\alpha,\beta)}\}$$

for all $\acute{v} \in V$. *A lower rough approximation of* f *within* $(V,W,[V]^{cs}_{R,(\alpha,\beta)})$ *is defined by the fuzzy subset* $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$ *of* V, *where*

$$\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}(\acute{v}) = \inf\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{R,(\alpha,\beta)}\}$$

for all $\acute{v} \in V$. f *is called a definable fuzzy set within* $(V,W,[V]^{cs}_{R,(\alpha,\beta)})$ *if* $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} = \llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$; *otherwise* f *is called a rough fuzzy set within* $(V,W,[V]^{cs}_{R,(\alpha,\beta)})$.

Example 3.3 below illustrates Definition 3.6.

Example 3.3. *Based on* $(V,W,[V]^{cs}_{R:=(R^-,R^{\pm},R^+),(\emptyset,[0,1])})$ *in Example 3.1, suppose that* f *is a fuzzy subset of* V *defined by*

$$f(v) = \frac{1}{v}$$

for all $v \in V$. *Observe that if* n *is a natural number, then*

$$\ulcorner f \urcorner^{cs}_{R,(\emptyset,[0,1])}(v_n) = \begin{cases}1 & \text{if } v_n \in [v_{5n-4}]^{cs}_{R,(\emptyset,[0,1])},\\ \frac{1}{2} & \text{if } v_n \in [v_{5n-3}]^{cs}_{R,(\emptyset,[0,1])},\\ \frac{1}{3} & \text{if } v_n \in [v_{5n-2}]^{cs}_{R,(\emptyset,[0,1])},\\ \frac{1}{4} & \text{if } v_n \in [v_{5n-1}]^{cs}_{R,(\emptyset,[0,1])},\\ \frac{1}{5} & \text{if } v_n \in [v_{5n}]^{cs}_{R,(\emptyset,[0,1])},\end{cases}$$

and we also have $\llcorner f \lrcorner^{cs}_{R,(\emptyset,[0,1])}(v_n) = 0$. *Therefore,* f *is a rough fuzzy set within* $(V,W,[V]^{cs}_{R,(\emptyset,[0,1])})$.

Remark 3.3. *Let* $(V,W,[V]^{cs}_{R:=(R^-,R^{\pm},R^+),(\alpha,\beta)})$ *be a given approximation space type I. If* f *is a fuzzy subset of* V, *then it is easily obtained that* $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)} \subseteq f \subseteq \ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$.

For three propositions below, the proofs of all the parts can be proved under Definition 3.6.

Proposition 3.14. *Let* $(V,W,[V]^{cs}_{R:=(R^-,R^{\pm},R^+),(\alpha,\beta)})$ *be a given approximation space type I. If* f *is a fuzzy subset of* V, *then we have the following statements:*

(i) If $f = 1_V$, *then* f *is equal to* $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ *and* $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$. *Moreover,* f *is a definable fuzzy set within* $(V,W,[V]^{cs}_{R,(\alpha,\beta)})$.

(ii) *If $f = 0_V$, then f is equal to $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ and $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$. Moreover, f is a definable fuzzy set within $(V, W, [V]^{cs}_{R,(\alpha,\beta)})$.*

(iii) $\ulcorner(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)})\urcorner^{cs}_{R,(\alpha,\beta)} \subseteq \ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$.

(iv) $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)} \subseteq \llcorner(\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)})\lrcorner^{cs}_{R,(\alpha,\beta)}$.

(v) $f \subseteq \llcorner(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)})\lrcorner^{cs}_{R,(\alpha,\beta)}$.

(vi) $\ulcorner(\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)})\urcorner^{cs}_{R,(\alpha,\beta)} \subseteq f$.

(vii) $\llcorner f' \lrcorner^{cs}_{R,(\alpha,\beta)} = (\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)})'$.

(viii) $\ulcorner f' \urcorner^{cs}_{R,(\alpha,\beta)} = (\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)})'$.

Proposition 3.15. *Let $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type I, and let f and g be fuzzy subsets of V, then we have the following statements:*

(i) $\ulcorner f \cap g \urcorner^{cs}_{R,(\alpha,\beta)} \subseteq \ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} \cap \ulcorner g \urcorner^{cs}_{R,(\alpha,\beta)}$.

(ii) $\llcorner f \cap g \lrcorner^{cs}_{R,(\alpha,\beta)} = \llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)} \cap \llcorner g \lrcorner^{cs}_{R,(\alpha,\beta)}$.

(iii) $\ulcorner f \cup g \urcorner^{cs}_{R,(\alpha,\beta)} = \ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} \cup \ulcorner g \urcorner^{cs}_{R,(\alpha,\beta)}$.

(iv) $\llcorner f \cup g \lrcorner^{cs}_{R,(\alpha,\beta)} \supseteq \llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)} \cup \llcorner g \lrcorner^{cs}_{R,(\alpha,\beta)}$.

Proposition 3.16. *Let $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be an approximation space type I, and let f and g be fuzzy subsets of V. If $f \subseteq g$, then $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} \subseteq \ulcorner g \urcorner^{cs}_{R,(\alpha,\beta)}$ and $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)} \subseteq \llcorner g \lrcorner^{cs}_{R,(\alpha,\beta)}$.*

Proposition 3.17. *Let $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ and $(V, W, [V]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ be approximation spaces type I, where $R \subseteq_{ir} S$ and $(\gamma,\delta) \subseteq_{sr} (\alpha,\beta)$. If f is a fuzzy subset of V, then $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} \subseteq \ulcorner f \urcorner^{cs}_{S,(\gamma,\delta)}$ and $\llcorner f \lrcorner^{cs}_{S,(\gamma,\delta)} \subseteq \llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$.*

Proof. Suppose f is a fuzzy subset over V and $\acute{v} \in V$. Then

$$\begin{aligned}
\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}(\acute{v}) &= \sup\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{R,(\alpha,\beta)}\} \\
&= \sup_{\grave{v}\in V}\{f(\grave{v}) : R^-(\acute{v},\grave{v}) \subseteq \alpha, \emptyset \subset R^\pm(\acute{v},\grave{v}) \subset [0,1] \text{ and } R^+(\acute{v},\grave{v}) \supseteq \beta\} \\
&\leq \sup_{\grave{v}\in V}\{f(\grave{v}) : S^-(\acute{v},\grave{v}) \subseteq \gamma, \emptyset \subset S^\pm(\acute{v},\grave{v}) \subset [0,1] \text{ and } S^+(\acute{v},\grave{v}) \supseteq \delta\} \\
&= \ulcorner f \urcorner^{cs}_{S,(\gamma,\delta)}(\acute{v}).
\end{aligned}$$

Therefore $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} \subseteq \ulcorner f \urcorner^{cs}_{S,(\alpha,\beta)}$. By the fact that

$$\begin{aligned}
\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}(\acute{v}) &= \inf\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{R,(\alpha,\beta)}\} \\
&= \inf_{\grave{v}\in V}\{f(\grave{v}) : R^-(\acute{v},\grave{v}) \subseteq \alpha, \emptyset \subset R^\pm(\acute{v},\grave{v}) \subset [0,1] \text{ and } R^+(\acute{v},\grave{v}) \supseteq \beta\} \\
&\geq \inf_{\grave{v}\in V}\{f(\grave{v}) : S^-(\acute{v},\grave{v}) \subseteq \gamma, \emptyset \subset S^\pm(\acute{v},\grave{v}) \subset [0,1] \text{ and } S^+(\acute{v},\grave{v}) \supseteq \delta\} \\
&= \inf\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{S,(\gamma,\delta)}\} \\
&= \llcorner f \lrcorner^{cs}_{S,(\gamma,\delta)}(\acute{v}),
\end{aligned}$$

it follows that $\llcorner f \lrcorner^{cs}_{S,(\alpha,\beta)} \subseteq \llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$. □

Proposition 3.18. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be an approximation space type I with the property that R is a set-valued picture hesitant fuzzy reflexive relation and a set-valued picture hesitant fuzzy antisymmetric relation and $(\alpha,\beta) \in \mathcal{P}([0,1])\backslash\{[0,1]\} \times \mathcal{P}([0,1])\backslash\{\emptyset\}$. If f is a fuzzy subset of V, then f is a definable fuzzy set within $(V, V, [V]^{cs}_{R,(\alpha,\beta)})$.*

Proof. Assume f is a fuzzy subset of V and $\acute{v} \in V$. By Proposition 3.7, we see that

$$\begin{aligned}
\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}(\acute{v}) &= \sup\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{R,(\alpha,\beta)}\} \\
&= \sup_{\grave{v}\in V}\{f(\grave{v}) : \acute{v} = \grave{v}\} \\
&= \sup\{f(\acute{v})\} \\
&= f(\acute{v}).
\end{aligned}$$

Similarly, we can prove that $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}(\acute{v}) = f(\acute{v})$. Hence $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}(\acute{v}) = \llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}(\acute{v})$. It follows that $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} = \llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$. We deduce that f is a definable fuzzy set within $(V, V, [V]^{cs}_{R,(\alpha,\beta)})$. □

Definition 3.7. *Let f be a fuzzy subset of V and $\iota \in [0,1]$. A $(f,\iota,>)$-relative whole soft set over V with respect to A is denoted by $(V_A^{(f,\iota,>)}, A)$, where $V_A^{(f,\iota,>)}$ is a set valued-mapping given by $V_A^{(f,\iota,>)}(a) = V^{(f,\iota,>)}$ for all $a \in A$.*

Proposition 3.19. *Let $(V, W, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be an approximation space type I, and let f be a fuzzy subset of V and $\iota \in [0,1]$. Then we have the following statements:*

(i) $(V_A^{(f,\iota,>)}, A)\urcorner^{cs}_{R,(\alpha,\beta)} = (V_A^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}, A)$.

(ii) $(V_A^{(f,\iota,>)}, A)\lrcorner^{cs}_{R,(\alpha,\beta)} = (V_A^{(\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}, A)$.

Proof. (i) Let $a \in A$, then

$$\begin{aligned}
v_1 \in V_A^{(f,\iota,>)}\urcorner^{cs}_{R,(\alpha,\beta)}(a) &\iff [v_1]^{cs}_{R,(\alpha,\beta)} \cap V_A^{(f,\iota,>)}(a) \neq \emptyset \\
&\iff [v_1]^{cs}_{R,(\alpha,\beta)} \cap V^{(f,\iota,>)} \neq \emptyset \\
&\iff f(v_2) > \iota \text{ for some } v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)} \\
&\iff \sup\{f(v_2) : v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)}\} > \iota \\
&\iff \ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}(v_1) > \iota \\
&\iff v_1 \in V^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,>)} \\
&\iff v_1 \in V_A^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}(a).
\end{aligned}$$

Hence $V_A^{(f,\iota,>)}\urcorner^{cs}_{R,(\alpha,\beta)}(a) = V_A^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}(a)$. This implies that $(V_A^{(f,\iota,>)}, A)\urcorner^{cs}_{R,(\alpha,\beta)} = (V_A^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}, A)$.
(ii) Let $a \in A$, then

$$\begin{aligned}
v_1 \in V_A^{(f,\iota,>)}\lrcorner^{cs}_{R,(\alpha,\beta)}(a) &\iff [v_1]^{cs}_{R,(\alpha,\beta)} \subseteq V_A^{(f,\iota,>)}(a) \\
&\iff [v_1]^{cs}_{R,(\alpha,\beta)} \subseteq V^{(f,\iota,>)} \\
&\iff f(v_2) > \iota \text{ for all } v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)} \\
&\iff \inf\{f(v_2) : v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)}\} > \iota
\end{aligned}$$

$$\begin{aligned} &\Longleftrightarrow \llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}(v_1) > \iota \\ &\Longleftrightarrow v_1 \in V^{(\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)},\iota,>)} \\ &\Longleftrightarrow v_1 \in V_A^{(\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}(a). \end{aligned}$$

Therefore $V_A^{(f,\iota,>)}\rfloor^{cs}_{R,(\alpha,\beta)}(a) = V_A^{(\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}(a)$. Consequently $(V_A^{(f,\iota,>)}, A)\rfloor^{cs}_{R,(\alpha,\beta)} = (V_A^{(\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}, A)$. □

4. Upper and lower rough approximations of prime idealistic soft semigroups and fuzzy prime ideals of semigroups

Based on extended approximation spaces, in this section, interesting properties in semigroups are verified. We use set-valued picture hesitant fuzzy relations on semigroups to investigate the upper (resp., lower) rough approximations of prime idealistic soft semigroups and fuzzy prime ideals of semigroups. Furthermore, soft semigroup homomorphism problems are examined.

Throughout the remaining of this section, V and W are referred to as a semigroup. Using Definition 2.12 to extend, we define a novel characteristic of a set-valued picture hesitant fuzzy relation from V to V as Definition 4.1 below.

Definition 4.1. *Let $R := (R^-, R^\pm, R^+)$ be a set-valued picture hesitant fuzzy relation from V to V. R is called a set-valued picture hesitant fuzzy compatible relation if the following conditions are satisfied:*

- *For all $v, \acute{v}, \grave{v} \in V$, $R^+(\acute{v}v, \grave{v}v) \supseteq R^+(\acute{v}, \grave{v})$ and $R^+(v\acute{v}, v\grave{v}) \supseteq R^+(\acute{v}, \grave{v})$;*
- *For all $v, \acute{v}, \grave{v} \in V$, $R^\pm(\acute{v}v, \grave{v}v) = R^\pm(\acute{v}, \grave{v})$ and $R^\pm(v\acute{v}, v\grave{v}) = R^\pm(\acute{v}, \grave{v})$;*
- *For all $v, \acute{v}, \grave{v} \in V$, $R^-(\acute{v}v, \grave{v}v) \subseteq R^-(\acute{v}, \grave{v})$ and $R^-(v\acute{v}, v\grave{v}) \subseteq R^-(\acute{v}, \grave{v})$.*

Definition 4.2. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type I. If R is a set-valued picture hesitant fuzzy reflexive relation, a set-valued picture hesitant fuzzy transitive relation and a set-valued picture hesitant fuzzy compatible relation, then $(V, V, [V]^{cs}_{R,(\alpha,\beta)})$ is called an approximation space type II.*

Proposition 4.1. *If the triple $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ is an approximation space type II, then $([\acute{v}]^{cs}_{R,(\alpha,\beta)})([\grave{v}]^{cs}_{R,(\alpha,\beta)}) \subseteq [\acute{v}\grave{v}]^{cs}_{R,(\alpha,\beta)}$ for all $\acute{v}, \grave{v} \in V$.*

Proof. Let $v_1, v_2 \in V$ be given. Suppose $v_3 \in ([v_1]^{cs}_{R,(\alpha,\beta)})([v_2]^{cs}_{R,(\alpha,\beta)})$, then there exist $v_4 \in [v_1]^{cs}_{R,(\alpha,\beta)}$ and $v_5 \in [v_2]^{cs}_{R,(\alpha,\beta)}$ such that $v_3 = v_4v_5$. Thus, we observe that $[v_1]^{s}_{R,(\alpha,\beta)} = [v_4]^{s}_{R,(\alpha,\beta)}$ and $[v_2]^{s}_{R,(\alpha,\beta)} = [v_5]^{s}_{R,(\alpha,\beta)}$. Therefore $[v_1v_2]^{s}_{R,(\alpha,\beta)} = [v_4v_5]^{s}_{R,(\alpha,\beta)}$. In fact, suppose that $v_6 \in [v_1v_2]^{s}_{R,(\alpha,\beta)}$, then

$$R^-(v_1v_2, v_6) \subseteq \alpha, \emptyset \subset R^\pm(v_1v_2, v_6) \subset [0,1] \text{ and } R^+(v_1v_2, v_6) \supseteq \beta.$$

Since R is a set-valued picture hesitant fuzzy reflexive relation, we have $v_1 \in [v_1]^{s}_{R,(\alpha,\beta)}$ and $v_2 \in [v_2]^{s}_{R,(\alpha,\beta)}$ due to Proposition 3.1. Whence $v_1 \in [v_4]^{s}_{R,(\alpha,\beta)}$ and $v_2 \in [v_5]^{s}_{R,(\alpha,\beta)}$. Thus

$$R^-(v_4, v_1) \subseteq \alpha, R^-(v_5, v_2) \subseteq \alpha,$$

$$\emptyset \subset R^\pm(v_4, v_1) \subset [0,1], \emptyset \subset R^\pm(v_5, v_2) \subset [0,1]$$

and

$$R^+(v_4, v_1) \supseteq \beta, R^+(v_5, v_2) \supseteq \beta.$$

Since R is a set-valued picture hesitant fuzzy transitive relation and a set-valued picture hesitant fuzzy compatible relation, we obtain that

$$\begin{aligned} R^-(v_4v_5, v_1v_2) &\subseteq \bigcap_{v\in V}(R^-(v_4v_5, v) \cup R^-(v, v_1v_2)) \\ &\subseteq R^-(v_4v_5, v_1v_5) \cup R^-(v_1v_5, v_1v_2) \\ &\subseteq R^-(v_4, v_1) \cup R^-(v_5, v_2) \\ &\subseteq \alpha \cup \alpha \\ &= \alpha, \end{aligned}$$

$$\begin{aligned} R^\pm(v_4v_5, v_1v_2) &\subseteq \bigcap_{v\in V}(R^\pm(v_4v_5, v) \cup R^\pm(v, v_1v_2)) \\ &\subseteq R^\pm(v_4v_5, v_1v_5) \cup R^\pm(v_1v_5, v_1v_2) \\ &= R^\pm(v_4, v_1) \cup R^\pm(v_5, v_2) \\ &\subset [0,1] \cup [0,1] \\ &= [0,1], \end{aligned}$$

$$\begin{aligned} R^\pm(v_4v_5, v_1v_2) &\supseteq \bigcup_{v\in V}(R^\pm(v_4v_5, v) \cap R^\pm(v, v_1v_2)) \\ &\supseteq R^\pm(v_4v_5, v_1v_5) \cap R^\pm(v_1v_5, v_1v_2) \\ &= R^\pm(v_4, v_1) \cap R^\pm(v_5, v_2) \\ &\supset \emptyset \cap \emptyset \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} R^+(v_4v_5, v_1v_2) &\supseteq \bigcup_{v\in V}(R^+(v_4v_5, v) \cap R^+(v, v_1v_2)) \\ &\supseteq R^+(v_4v_5, v_1v_5) \cap R^+(v_1v_5, v_1v_2) \\ &\supseteq R^+(v_4, v_1) \cap R^+(v_5, v_2) \\ &\supseteq \beta \cap \beta \\ &= \beta. \end{aligned}$$

Since R is a set-valued picture hesitant fuzzy transitive relation, we observe that

$$\begin{aligned} R^-(v_4v_5, v_6) &\subseteq \bigcap_{v\in V}(R^-(v_4v_5, v) \cup R^-(v, v_6)) \\ &\subseteq R^-(v_4v_5, v_1v_2) \cup R^-(v_1v_2, v_6) \\ &\subseteq \alpha \cup \alpha \\ &= \alpha, \end{aligned}$$

$$\begin{aligned} R^{\pm}(v_4v_5, v_6) &\subseteq \bigcap_{v\in V}(R^{\pm}(v_4v_5, v) \cup R^{\pm}(v, v_6)) \\ &\subseteq R^{\pm}(v_4v_5, v_1v_2) \cup R^{\pm}(v_1v_2, v_6) \\ &\subset [0,1] \cup [0,1] \\ &= [0,1], \end{aligned}$$

$$\begin{aligned} R^{\pm}(v_4v_5, v_6) &\supseteq \bigcup_{v\in V}(R^{\pm}(v_4v_5, v) \cap R^{\pm}(v, v_6)) \\ &\supseteq R^{\pm}(v_4v_5, v_1v_2) \cap R^{\pm}(v_1v_2, v_6) \\ &\supset \emptyset \cap \emptyset \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} R^{+}(v_4v_5, v_6) &\supseteq \bigcup_{v\in V}(R^{+}(v_4v_5, v) \cap R^{+}(v, v_6)) \\ &\supseteq R^{+}(v_4v_5, v_1v_2) \cap R^{+}(v_1v_2, v_6) \\ &\supseteq \beta \cap \beta \\ &= \beta. \end{aligned}$$

Whence $v_6 \in [v_4v_5]^s_{R,(\alpha,\beta)}$. We get that $[v_1v_2]^s_{R,(\alpha,\beta)} \subseteq [v_4v_5]^s_{R,(\alpha,\beta)}$. Conversely, we can show that $[v_4v_5]^s_{R,(\alpha,\beta)} \subseteq [v_1v_2]^s_{R,(\alpha,\beta)}$. It follows that $[v_1v_2]^s_{R,(\alpha,\beta)} = [v_4v_5]^s_{R,(\alpha,\beta)}$. Thus $v_3 = v_4v_5 \in [v_1v_2]^{cs}_{R,(\alpha,\beta)}$. This shows that $([v_1]^{cs}_{R,(\alpha,\beta)})([v_2]^{cs}_{R,(\alpha,\beta)}) \subseteq [v_1v_2]^{cs}_{R,(\alpha,\beta)}$. □

Generally, we know from Proposition 4.1 that it does not holds for an equality case. In what follows, we shall consider a specific example.

Let $(V, V, [V]^{cs}_{R:=(R^-,R^+),(\emptyset,[0,1])})$ be a given approximation space type II, where $V = \{v_n := 2n-1 \ : \ n$ is a natural number$\}$ is a semigroup under the usual multiplication, and R is a set-valued picture hesitant fuzzy reflexive relation, a set-valued picture hesitant fuzzy transitive relation and a set-valued picture hesitant fuzzy compatible relation from V to V defined by

$$R^{+}(\acute{v}, \grave{v}) = \begin{cases} [0,1] & \text{if } 2|(\acute{v}+\grave{v}), \\ (0,1] & \text{if } 2 \nmid (\acute{v}+\grave{v}), \end{cases}$$

$$R^{\pm}(\acute{v}, \grave{v}) = \begin{cases} [0,1) & \text{if } 2|(\acute{v}+\grave{v}), \\ [0,1] & \text{if } 2 \nmid (\acute{v}+\grave{v}) \end{cases}$$

and

$$R^{-}(\acute{v}, \grave{v}) = \begin{cases} \emptyset & \text{if } 2|(\acute{v}+\grave{v}), \\ (0,1) & \text{if } 2 \nmid (\acute{v}+\grave{v}) \end{cases}$$

for all $\acute{v}, \grave{v} \in V$. It is clear that if n is a natural number, then $[v_n]^{cs}_{R,(\emptyset,[0,1])}$ is a set of all positive odd integers. In addition, it is clear that $([\acute{v}]^{cs}_{R,(\emptyset,[0,1])})([\grave{v}]^{cs}_{R,(\emptyset,[0,1])})$ and $[\acute{v}\grave{v}]^{cs}_{R,(\emptyset,[0,1])}$ are two sets of all positive odd integers for all $\acute{v}, \grave{v} \in V$. This implies that $([\acute{v}]^{cs}_{R,(\emptyset,[0,1])})([\grave{v}]^{cs}_{R,(\emptyset,[0,1])}) = [\acute{v}\grave{v}]^{cs}_{R,(\emptyset,[0,1])}$ for all $\acute{v}, \grave{v} \in V$. Then we see that the property can be considered as a special case of Proposition 4.1. It leads to the following definition.

Definition 4.3. *Let the triple $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type II. If $([\acute{v}]^{cs}_{R,(\alpha,\beta)})([\grave{v}]^{cs}_{R,(\alpha,\beta)}) = [\acute{v}\grave{v}]^{cs}_{R,(\alpha,\beta)}$ for all $\acute{v}, \grave{v} \in V$, then the set $[V]^{cs}_{R,(\alpha,\beta)}$ is called a complete collection induced by R. In what follows, we call R a set-valued picture hesitant fuzzy complete relation. In addition, $(V, V, [V]^{cs}_{R,(\alpha,\beta)})$ is called an approximation space type III if R is a set-valued picture hesitant fuzzy complete relation.*

Proposition 4.2. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be an approximation space type II. If $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ are soft sets over V, then $\mathfrak{F}]^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{G}]^{cs}_{R,(\alpha,\beta)} \Subset (\mathfrak{F} \circledcirc \mathfrak{G})]^{cs}_{R,(\alpha,\beta)}$.*

Proof. Suppose that $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ are soft sets over V. Let $\mathfrak{H}_1 := (H_1, C_1)$ be a soft set $\mathfrak{F}]^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{G}]^{cs}_{R,(\alpha,\beta)}$, then $C_1 = A \cap B$ and $H_1(c) = (F]^{cs}_{R,(\alpha,\beta)}(c))(G]^{cs}_{R,(\alpha,\beta)}(c))$ for all $c \in C_1$. Let $\mathfrak{H}_2 := (H_2, C_2)$ be a soft set $\mathfrak{F} \circledcirc \mathfrak{G}$, then $C_2 = A \cap B$ and $H_2(c) = (F(c))(G(c))$ for all $c \in C_2$. Now, we shall verify that $\mathfrak{H}_1 \Subset \mathfrak{H}_2]^{cs}_{R,(\alpha,\beta)}$. Obviously $C_1 = C_2$. Let $\acute{c}$ be an element in C_1, and let $v_1 \in H_1(\acute{c})$. Then $v_1 \in (F]^{cs}_{R,(\alpha,\beta)}(\acute{c}))(G]^{cs}_{R,(\alpha,\beta)}(\acute{c}))$. There exist $v_2 \in F]^{cs}_{R,(\alpha,\beta)}(\acute{c})$ and $v_3 \in G]^{cs}_{R,(\alpha,\beta)}(\acute{c})$ such that $v_1 = v_2v_3$. Hence, we get that $[v_2]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{c}) \neq \emptyset$ and $[v_3]^{cs}_{R,(\alpha,\beta)} \cap G(\acute{c}) \neq \emptyset$. Thus, there exist $v_4, v_5 \in V$ such that $v_4 \in [v_2]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{c})$ and $v_5 \in [v_3]^{cs}_{R,(\alpha,\beta)} \cap G(\acute{c})$. Using Proposition 4.1, we obtain that

$$v_4v_5 \in ([v_2]^{cs}_{R,(\alpha,\beta)})([v_3]^{cs}_{R,(\alpha,\beta)}) \subseteq [v_2v_3]^{cs}_{R,(\alpha,\beta)}.$$

Note that $v_4v_5 \in (F(\acute{c}))(G(\acute{c}))$. Then

$$[v_1]^{cs}_{R,(\alpha,\beta)} \cap H_2(\acute{c}) = [v_2v_3]^{cs}_{R,(\alpha,\beta)} \cap (F(\acute{c}))(G(\acute{c})) \neq \emptyset.$$

Thus $v_1 \in H_2]^{cs}_{R,(\alpha,\beta)}(\acute{c})$. Whence $H_1(\acute{c}) \subseteq H_2]^{cs}_{R,(\alpha,\beta)}(\acute{c})$. Therefore $\mathfrak{H}_1 \Subset \mathfrak{H}_2]^{cs}_{R,(\alpha,\beta)}$, which yields $\mathfrak{F}]^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{G}]^{cs}_{R,(\alpha,\beta)} \Subset (\mathfrak{F} \circledcirc \mathfrak{G})]^{cs}_{R,(\alpha,\beta)}$ as desired. □

Proposition 4.3. *Let the triple $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be an approximation space type III. If $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ are soft sets over V, then $\mathfrak{F}]^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{G}]^{cs}_{R,(\alpha,\beta)} \Subset (\mathfrak{F} \circledcirc \mathfrak{G})]^{cs}_{R,(\alpha,\beta)}$.*

Proof. Suppose that $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ are soft sets over V. Let $\mathfrak{H}_1 := (H_1, C_1) = \mathfrak{F}]^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{G}]^{cs}_{R,(\alpha,\beta)}$, then $C_1 = A \cap B$ and $H_1(c) = (F]^{cs}_{R,(\alpha,\beta)}(c))(G]^{cs}_{R,(\alpha,\beta)}(c))$ for all $c \in C_1$. Let $\mathfrak{H}_2 := (H_2, C_2) = \mathfrak{F} \circledcirc \mathfrak{G}$, then $C_2 = A \cap B$ and $H_2(c) = (F(c))(G(c))$ for all $c \in C_2$. We shall show that $\mathfrak{H}_1 \Subset \mathfrak{H}_2]^{cs}_{R,(\alpha,\beta)}$. Clearly $C_1 = C_2$. Let $\acute{c} \in C_1$, and let $v_1 \in H_1(\acute{c})$, then $v_1 \in (F]^{cs}_{R,(\alpha,\beta)}(\acute{c}))(G]^{cs}_{R,(\alpha,\beta)}(\acute{c}))$. Thus, there exist $v_2 \in F]^{cs}_{R,(\alpha,\beta)}(\acute{c})$ and $v_3 \in G]^{cs}_{R,(\alpha,\beta)}(\acute{c})$ such that $v_1 = v_2v_3$. Thus, we obtain that $[v_2]^{cs}_{R,(\alpha,\beta)} \subseteq F(\acute{c})$ and $[v_3]^{cs}_{R,(\alpha,\beta)} \subseteq G(\acute{c})$. Now

$$[v_1]^{cs}_{R,(\alpha,\beta)} = [v_2v_3]^{cs}_{R,(\alpha,\beta)} = ([v_2]^{cs}_{R,(\alpha,\beta)})([v_3]^{cs}_{R,(\alpha,\beta)}) \subseteq (F(\acute{c}))(G(\acute{c})) = H_2(\acute{c}).$$

Thus $v_1 \in H_2]^{cs}_{R,(\alpha,\beta)}(\acute{c})$. Hence $H_1(\acute{c}) \subseteq H_2]^{cs}_{R,(\alpha,\beta)}(\acute{c})$. Therefore $\mathfrak{H}_1 \Subset \mathfrak{H}_2]^{cs}_{R,(\alpha,\beta)}$. As a consequence, $\mathfrak{F}]^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{G}]^{cs}_{R,(\alpha,\beta)} \Subset (\mathfrak{F} \circledcirc \mathfrak{G})]^{cs}_{R,(\alpha,\beta)}$. □

We can now state our main theorems.

Theorem 4.1. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type II. If $\mathfrak{F} := (F, A)$ is an idealistic soft semigroup over V, then $\mathfrak{F}]^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V.*

Proof. Suppose $\mathfrak{F}$ is an idealistic soft semigroup over V. Then $F(a)$ is an ideal of V for all $a \in Supp\mathfrak{F}$. Let $\acute{a} \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$, then by Remark 3.2, we see that

$$F(\acute{a}) \subseteq F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}) \neq \emptyset.$$

If $F(\acute{a}) = \emptyset$, then it is easy to see that $F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}) = \emptyset$ due to Proposition 3.9 (ii). This is a contradiction. It is true that $F(\acute{a}) \neq \emptyset$. Thus $\acute{a} \in Supp\mathfrak{F}$. Suppose $v_1 \in V(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}))$. By Proposition 3.9 (i), we obtain that $\mathfrak{W}_{V_A}\rceil^{cs}_{R,(\alpha,\beta)} = \mathfrak{W}_{V_A}$. Hence $V_A\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}) = V$. By Proposition 4.2, we have $\mathfrak{W}_{V_A}\rceil^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)} \Subset (\mathfrak{W}_{V_A} \circledcirc \mathfrak{F})\rceil^{cs}_{R,(\alpha,\beta)}$. Now, we define the soft set $\mathfrak{H}_1 := (H_1, C_1)$ as $\mathfrak{W}_{V_A}\rceil^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$. Then $C_1 = K \cap A$ and $H_1(c) = (V_A\rceil^{cs}_{R,(\alpha,\beta)}(c))(F\rceil^{cs}_{R,(\alpha,\beta)}(c))$ for all $c \in C_1$. Define $\mathfrak{H}_2 := (H_2, C_2)$ is a soft set $\mathfrak{W}_{V_A} \circledcirc \mathfrak{F}$. Then $C_2 = K \cap A$ and $H_2(c) = (V_A(c))(F(c))$ for all $c \in C_2$. Observe that $H_1(a) \subseteq H_2\rceil^{cs}_{R,(\alpha,\beta)}(a)$ for all $a \in A$. Now

$$V(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})) = (V\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}))(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})) = H_1(\acute{a}) \subseteq H_2\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}).$$

Thus, we see that $v_1 \in H_2\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$. By the assumption, we get that

$$\emptyset \neq [v_1]^{cs}_{R,(\alpha,\beta)} \cap H_2(\acute{a}) = [v_1]^{cs}_{R,(\alpha,\beta)} \cap (V_A(\acute{a}))(F(\acute{a})) \subseteq [v_1]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{a}).$$

Observe that $[v_1]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{a}) \neq \emptyset$. Hence $v_1 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$. Thus $V(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})) \subseteq F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$. It follows that $F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is a left ideal of V. Similarly, we can prove that $F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is a right ideal of V. Therefore $F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is an ideal of V. Consequently $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V. □

Theorem 4.2. *Let* $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be a given approximation space type III. If* $\mathfrak{F} := (F, A)$ *is an idealistic soft semigroup over* V*, then* $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$ *is an idealistic soft semigroup over* V*.*

Proof. Suppose $\mathfrak{F}$ is an idealistic soft semigroup over V. Then $F(a)$ is an ideal of V for all $a \in Supp\mathfrak{F}$. Let $\acute{a} \in Supp(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})$, then $F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a}) \neq \emptyset$. Using Remark 3.2, it is easy to verify that $\acute{a} \in Supp\mathfrak{F}$. Suppose $v_1 \in V(F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a}))$. Then, by Proposition 3.9 (i), we obtain that $\mathfrak{W}_{V_A}\rfloor^{cs}_{R,(\alpha,\beta)} = \mathfrak{W}_{V_A}$. Thus $V_A\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}) = V$. From Proposition 4.3, we have $\mathfrak{W}_{V_A}\rfloor^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)} \Subset (\mathfrak{W}_{V_A} \circledcirc \mathfrak{F})\rfloor^{cs}_{R,(\alpha,\beta)}$. Now, we define $\mathfrak{H}_1 := (H_1, C_1)$ is a soft set $\mathfrak{W}_{V_A}\rfloor^{cs}_{R,(\alpha,\beta)} \circledcirc \mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$, then $C_1 = K \cap A$ and $H_1(c) = (V_A\rfloor^{cs}_{R,(\alpha,\beta)}(c))(F\rfloor^{cs}_{R,(\alpha,\beta)}(c))$ for all $c \in C_1$. Define $\mathfrak{H}_2 := (H_2, C_2)$ is a soft set $\mathfrak{W}_{V_A} \circledcirc \mathfrak{F}$, then $C_2 = K \cap A$ and $H_2(c) = (V_A(c))(F(c))$ for all $c \in C_2$. Thus, we see that $H_1(a) \subseteq H_2\rfloor^{cs}_{R,(\alpha,\beta)}(a)$ for all $a \in A$. Consider

$$V(F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})) = (V\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a}))(F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})) = H_1(\acute{a}) \subseteq H_2\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a}).$$

Observe that $v_1 \in H_2\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$. Since $\mathfrak{F}$ is an idealistic soft semigroup over V, we get

$$[v_1]^{cs}_{R,(\alpha,\beta)} \subseteq H_2(\acute{a}) = (V_A(\acute{a}))(F(\acute{a})) \subseteq F(\acute{a}).$$

Whence $[v_1]^{cs}_{R,(\alpha,\beta)} \subseteq F(\acute{a})$. Hence $v_1 \in F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$. Thus $V(F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})) \subseteq F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$. It is true that $F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is a left ideal of V. Similarly, we can prove that $F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is a right ideal of V. Therefore $F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is an ideal of V. As a consequence, $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V. □

Theorem 4.3. *Let* $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be a given approximation space type III. If* $\mathfrak{F} := (F, A)$ *is a prime idealistic soft semigroup over* V*, then* $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ *is a prime idealistic soft semigroup over* V*.*

Proof. Suppose $\mathfrak{F}$ is a prime idealistic soft semigroup over V. Then $\mathfrak{F}$ is an idealistic soft semigroup over V. From Theorem 4.1, it follows that $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V. Thus, $F\rceil^{cs}_{R,(\alpha,\beta)}(a)$ is an ideal of V for all $a \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$. Now, we let $\acute{a} \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$ be given. Then, by Remark 3.2, we see that

$$F(\acute{a}) \subseteq F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}) \neq \emptyset.$$

If $F(\acute{a}) = \emptyset$, then it is easy to verify that $F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}) = \emptyset$ since Proposition 3.9 (ii). This is a contradiction, and hence $F(\acute{a}) \neq \emptyset$. Whence $\acute{a} \in Supp\mathfrak{F}$, and so $F(\acute{a})$ is a completely prime ideal of V. We shall prove that $F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is a completely prime ideal of V. Let $v_1, v_2 \in V$. Suppose that $v_1v_2 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$. Then $[v_1v_2]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{a}) \neq \emptyset$. Since R is a set-valued picture hesitant fuzzy complete relation, we have

$$([v_1]^{cs}_{R,(\alpha,\beta)})([v_2]^{cs}_{R,(\alpha,\beta)}) \cap F(\acute{a}) = [v_1v_2]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{a}) \neq \emptyset.$$

Then there exist $v_3 \in [v_1]^{cs}_{R,(\alpha,\beta)}$ and $v_4 \in [v_2]^{cs}_{R,(\alpha,\beta)}$ such that $v_3v_4 \in F(\acute{a})$. Since $F(\acute{a})$ is a completely prime ideal of V, we have $v_3 \in F(\acute{a})$ or $v_4 \in F(\acute{a})$. Observe that

$$[v_1]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{a}) \neq \emptyset \text{ or } [v_2]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{a}) \neq \emptyset.$$

It follows that $v_1 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$ or $v_2 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$. Therefore $F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is a completely prime ideal of V. We deduce that $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is a prime idealistic soft semigroup over V. □

Theorem 4.4. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type III. If $\mathfrak{F} := (F, A)$ is a prime idealistic soft semigroup over V, then $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$ is a prime idealistic soft semigroup over V.*

Proof. Suppose $\mathfrak{F}$ is a prime idealistic soft semigroup over V. Then $\mathfrak{F}$ is an idealistic soft semigroup over V. From Theorem 4.2, it follows that $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V. Thus $F\rfloor^{cs}_{R,(\alpha,\beta)}(a)$ is an ideal of V for all $a \in Supp(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})$. Now, we let $\acute{a} \in Supp(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})$, then $F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a}) \neq \emptyset$. By Remark 3.2, it is easy to prove that $\acute{a} \in Supp\mathfrak{F}$. Let $v_1, v_2 \in V$, assume that $v_1v_2 \in F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$, then $[v_1v_2]^{cs}_{R,(\alpha,\beta)} \subseteq F(\acute{a})$. Since R is a set-valued picture hesitant fuzzy complete relation, we have

$$([v_1]^{cs}_{R,(\alpha,\beta)})([v_2]^{cs}_{R,(\alpha,\beta)}) = [v_1v_2]^{cs}_{R,(\alpha,\beta)} \subseteq F(\acute{a}).$$

Assume that $v_1 \notin F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$, then $[v_1]^{cs}_{R,(\alpha,\beta)}$ is not a subset of $F(\acute{a})$. Thus, there exists $v_3 \in [v_1]^{cs}_{R,(\alpha,\beta)}$ but $v_3 \notin F(\acute{a})$. Observe that, if $v_4 \in [v_2]^{cs}_{R,(\alpha,\beta)}$, then

$$v_3v_4 \in ([v_1]^{cs}_{R,(\alpha,\beta)})([v_2]^{cs}_{R,(\alpha,\beta)}) \subseteq F(\acute{a}),$$

and so $v_4 \in F(\acute{a})$ since $F(\acute{a})$ is a completely prime ideal of V. Here, it is true that $[v_2]^{cs}_{R,(\alpha,\beta)} \subseteq F(\acute{a})$. Thus $v_2 \in F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$. Therefore $F\rfloor^{cs}_{R,(\alpha,\beta)}(\acute{a})$ is a completely prime ideal of V. As a consequence, $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$ is a prime idealistic soft semigroup over V. □

Proposition 4.4. *Let f be a fuzzy subset of V, then f is a fuzzy ideal (resp., a fuzzy prime ideal) of V if and only if $(V_A^{(f,\iota,>)}, A)$ is an idealistic soft semigroup (resp., a prime idealistic soft semigroup) over V for all $\iota \in [0, 1]$.*

Proof. Suppose that f is a fuzzy ideal of V. Let $\iota \in [0, 1]$, and let $a \in Supp(V_A^{(f,\iota,>)}, A)$. Then

$$V^{(f,\iota,>)} = V_A^{(f,\iota,>)}(a) \neq \emptyset.$$

Then, by Proposition 2.1, we obtain that $V^{(f,\iota,>)}$ is an ideal of V. Hence $V_A^{(f,\iota,>)}(a)$ is an ideal of V. It follows that $(V_A^{(f,\iota,>)}, A)$ is an idealistic soft semigroup over V. In this way, we can prove that if f is a fuzzy prime ideal, then $(V_A^{(f,\iota,>)}, A)$ is a prime idealistic soft semigroup over V. Conversely, assume that $(V_A^{(f,\iota,>)}, A)$ is an idealistic soft semigroup over V for all $\iota \in [0,1]$. Then we have $V_A^{(f,\iota,>)}(a)$ is an ideal of V for all $a \in Supp(V_A^{(f,\iota,>)}, A), \iota \in [0,1]$. Based on this point, we observe that for all $a \in A, \iota \in [0,1]$, if $V_A^{(f,\iota,>)}(a) \neq \emptyset$, then $V_A^{(f,\iota,>)}(a)$ is an ideal of V. From Proposition 2.1, once again, it follows that f is a fuzzy ideal of V. In the same way, we can verify that if $(V_A^{(f,\iota,>)}, A)$ is a prime idealistic soft semigroup over V for all $\iota \in [0,1]$, then f is a fuzzy prime ideal of V. □

Theorem 4.5. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type II. If f is a fuzzy ideal of V, then $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ is a fuzzy ideal of V.*

Proof. Suppose f is a fuzzy ideal of V. Then, by Proposition 4.4, we have $(V_A^{(f,\iota,>)}, A)$ is an idealistic soft semigroup over V for all $\iota \in [0,1]$. From Theorem 4.1, it follows that $(V_A^{(f,\iota,>)}, A)\urcorner^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V for all $\iota \in [0,1]$. By Proposition 3.19 (i), we get that $(V_A^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}, A)$ is an idealistic soft semigroup over V for all $\iota \in [0,1]$. Using Proposition 4.4, once again, we obtain that $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ is a fuzzy ideal of V. □

Theorem 4.6. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type III. If f is a fuzzy ideal of V, then $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$ is a fuzzy ideal of V.*

Proof. From Propositions 3.19 (ii), 4.4 and Theorem 4.2, the statement is easily provided. □

Theorem 4.7. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type III. If f is a fuzzy prime ideal of V, then $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ is a fuzzy prime ideal of V.*

Proof. Assume f is a fuzzy prime ideal of V. Then, by Proposition 4.4, we have $(V_A^{(f,\iota,>)}, A)$ is a prime idealistic soft semigroup over V for all $\iota \in [0,1]$. Using Theorem 4.3, it follows that $(V_A^{(f,\iota,>)}, A)\urcorner^{cs}_{R,(\alpha,\beta)}$ is a prime idealistic soft semigroup over V for all $\iota \in [0,1]$. By Proposition 3.19 (i), we get that $(V_A^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,>)}, A)$ is a prime idealistic soft semigroup over V for all $\iota \in [0,1]$. Using Proposition 4.4, once again, we obtain that $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ is a fuzzy prime ideal of V. □

Theorem 4.8. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be a given approximation space type III. If f is a fuzzy prime ideal of V, then $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$ is a fuzzy prime ideal of V.*

Proof. Using Propositions 3.19 (ii), 4.4 and Theorem 4.4, we can verify that the statement holds. □

Proposition 4.5. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ and $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ be approximation spaces type I, and let $(\Theta, \Xi)_h$ be a soft homomorphism from a soft semigroup $\mathfrak{F} := (F, A)$ over V to a soft semigroup $\mathfrak{G} := (G, B)$ over W, where*

$$R^-(\acute{v}, \grave{v}) = S^-(\Theta(\acute{v}), \Theta(\grave{v})), \tag{4.1}$$

$$R^\pm(\acute{v}, \grave{v}) = S^\pm(\Theta(\acute{v}), \Theta(\grave{v})), \tag{4.2}$$

$$R^+(\acute{v}, \grave{v}) = S^+(\Theta(\acute{v}), \Theta(\grave{v})) \tag{4.3}$$

for all $\acute{v}, \grave{v} \in V$. Then we have the following statements:

(i) *If* $\acute{v}, \grave{v} \in V$, *then* $\acute{v} \in [\grave{v}]^{cs}_{R,(\alpha,\beta)}$ *if and only if* $\Theta(\acute{v}) \in [\Theta(\grave{v})]^{cs}_{S,(\alpha,\beta)}$.

(ii) $\Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(a)) = G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(a))$ *for all* $a \in A$.

(iii) $\Theta(F\rfloor^{cs}_{R,(\alpha,\beta)}(a)) \subseteq G\rfloor^{cs}_{S,(\alpha,\beta)}(\Xi(a))$ *for all* $a \in A$.

(iv) *If* Θ *is injective, then* $\Theta(F\rfloor^{cs}_{R,(\alpha,\beta)}(a)) = G\rfloor^{cs}_{S,(\alpha,\beta)}(\Xi(a))$ *for all* $a \in A$.

(v) $\Xi(Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})) = Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$.

(vi) $\Xi(Supp(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})) \subseteq Supp(\mathfrak{G}\rfloor^{cs}_{S,(\alpha,\beta)})$.

(vii) *If* Θ *is injective, then* $\Xi(Supp(\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)})) = Supp(\mathfrak{G}\rfloor^{cs}_{S,(\alpha,\beta)})$.

(viii) *R is a set-valued picture hesitant fuzzy reflexive relation, a set-valued picture hesitant fuzzy symmetric relation, a set-valued picture hesitant fuzzy transitive relation and a set-valued picture hesitant fuzzy compatible relation if and only if S is a set-valued picture hesitant fuzzy reflexive relation, a set-valued picture hesitant fuzzy symmetric relation, a set-valued picture hesitant fuzzy transitive relation and a set-valued picture hesitant fuzzy compatible relation, respectively.*

(ix) *If R is a set-valued picture hesitant fuzzy antisymmetric relation and a set-valued picture hesitant fuzzy complete relation, then S is a set-valued picture hesitant fuzzy antisymmetric relation and a set-valued picture hesitant fuzzy complete relation, respectively.*

(x) *If Θ is injective, then R is a set-valued picture hesitant fuzzy antisymmetric relation and a set-valued picture hesitant fuzzy complete relation if and only if S is a set-valued picture hesitant fuzzy antisymmetric relation and a set-valued picture hesitant fuzzy complete relation, respectively.*

Proof. (i) Let $v_1, v_2 \in V$ be given. Suppose $v_1 \in [v_2]^{cs}_{R,(\alpha,\beta)}$. Then $[v_1]^{s}_{R,(\alpha,\beta)} = [v_2]^{s}_{R,(\alpha,\beta)}$. Note that $\Theta(v_1), \Theta(v_2) \in W$. Suppose $w_1 \in [\Theta(v_1)]^{s}_{S,(\alpha,\beta)}$. Then there exists $v_3 \in V$ such that $\Theta(v_3) = w_1$. We observe that

$$R^-(v_1, v_3) = S^-(\Theta(v_1), \Theta(v_3)) \subseteq \alpha,$$

$$\emptyset \subset R^\pm(v_1, v_3) = S^\pm(\Theta(v_1), \Theta(v_3)) \subset [0, 1]$$

and

$$R^+(v_1, v_3) = S^+(\Theta(v_1), \Theta(v_3)) \supseteq \beta.$$

Thus $v_3 \in [v_1]^{s}_{R,(\alpha,\beta)}$, and so $v_3 \in [v_2]^{s}_{R,(\alpha,\beta)}$. Now

$$S^-(\Theta(v_2), \Theta(v_3)) = R^-(v_2, v_3) \subseteq \alpha,$$

$$\emptyset \subset S^\pm(\Theta(v_2), \Theta(v_3)) = R^\pm(v_2, v_3) \subset [0, 1]$$

and

$$S^+(\Theta(v_2), \Theta(v_3)) = R^+(v_2, v_3) \supseteq \beta.$$

Whence $\Theta(v_3) \in [\Theta(v_2)]^{s}_{S,(\alpha,\beta)}$. Thus, we get $[\Theta(v_1)]^{s}_{S,(\alpha,\beta)} \subseteq [\Theta(v_2)]^{s}_{S,(\alpha,\beta)}$. Conversely, we can prove that $[\Theta(v_2)]^{s}_{S,(\alpha,\beta)} \subseteq [\Theta(v_1)]^{s}_{S,(\alpha,\beta)}$. Hence $[\Theta(v_1)]^{s}_{S,(\alpha,\beta)} = [\Theta(v_2)]^{s}_{S,(\alpha,\beta)}$. It follows that $\Theta(v_1) \in [\Theta(v_2)]^{cs}_{S,(\alpha,\beta)}$. On the other hand, suppose that $\Theta(v_1) \in [\Theta(v_2)]^{cs}_{S,(\alpha,\beta)}$, then $[\Theta(v_1)]^{s}_{S,(\alpha,\beta)} = [\Theta(v_2)]^{s}_{S,(\alpha,\beta)}$. Now, we assume $v_4 \in [v_1]^{s}_{R,(\alpha,\beta)}$, then

$$S^-(\Theta(v_1), \Theta(v_4)) = R^-(v_1, v_4) \subseteq \alpha,$$

$$\emptyset \subset S^{\pm}(\Theta(v_1), \Theta(v_4)) = R^{\pm}(v_1, v_4) \subset [0, 1]$$

and

$$S^{+}(\Theta(v_1), \Theta(v_4)) = R^{+}(v_1, v_4) \supseteq \beta.$$

Thus $\Theta(v_4) \in [\Theta(v_1)]^{s}_{S,(\alpha,\beta)}$. Hence $\Theta(v_4) \in [\Theta(v_2)]^{s}_{S,(\alpha,\beta)}$. Now

$$R^{-}(v_2, v_4) = S^{-}(\Theta(v_2), \Theta(v_4)) \subseteq \alpha,$$

$$\emptyset \subset R^{\pm}(v_2, v_4) = S^{\pm}(\Theta(v_2), \Theta(v_4)) \subset [0, 1]$$

and

$$R^{+}(v_2, v_4) = S^{+}(\Theta(v_2), \Theta(v_4)) \supseteq \beta.$$

Thus, we get that $v_4 \in [v_2]^{s}_{R,(\alpha,\beta)}$. It follows that $[v_1]^{s}_{R,(\alpha,\beta)} \subseteq [v_2]^{s}_{R,(\alpha,\beta)}$. Conversely, we can show that $[v_2]^{s}_{R,(\alpha,\beta)} \subseteq [v_1]^{s}_{R,(\alpha,\beta)}$, which yields $[v_1]^{s}_{R,(\alpha,\beta)} = [v_2]^{s}_{R,(\alpha,\beta)}$. This implies that $v_1 \in [v_2]^{cs}_{R,(\alpha,\beta)}$. The proof is complete.

(ii) Let $a \in A$ and $w_1 \in \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(a))$. Then there exists $v_1 \in F\rceil^{cs}_{R,(\alpha,\beta)}(a)$ such that $\Theta(v_1) = w_1$. We see that $[v_1]^{cs}_{R,(\alpha,\beta)} \cap F(a) \neq \emptyset$. Let $v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)} \cap F(a)$. By item (i), we have $\Theta(v_2) \in [\Theta(v_1)]^{cs}_{S,(\alpha,\beta)}$ and $\Theta(v_2) \in \Theta(F(a))$. Since $\Theta(F(a)) = G(\Xi(a))$, we have $\Theta(v_2) \in G(\Xi(a))$. Thus,

$$[w_1]^{cs}_{S,(\alpha,\beta)} \cap G(\Xi(a)) = [\Theta(v_1)]^{cs}_{S,(\alpha,\beta)} \cap G(\Xi(a)) \neq \emptyset.$$

Therefore, we get $w_1 \in G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(a))$, which yields $\Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(a)) \subseteq G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(a))$. Conversely, we let $w_2 \in G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(a))$, then $[w_2]^{cs}_{S,(\alpha,\beta)} \cap G(\Xi(a)) \neq \emptyset$. Let $w_3 \in [w_2]^{cs}_{S,(\alpha,\beta)} \cap G(\Xi(a))$, since $\Theta(F(a)) = G(\Xi(a))$, we have $w_3 \in \Theta(F(a))$. Thus, there exists $v_3 \in F(a)$ such that $\Theta(v_3) = w_3$. Since Θ is surjective, there exists $v_4 \in V$ such that $\Theta(v_4) = w_2$. We see that $\Theta(v_3) \in [\Theta(v_4)]^{cs}_{S,(\alpha,\beta)}$. By item (i), we get that $v_3 \in [v_4]^{cs}_{R,(\alpha,\beta)}$. Hence $[v_4]^{cs}_{R,(\alpha,\beta)} \cap F(a) \neq \emptyset$. Thus $v_4 \in F\rceil^{cs}_{R,(\alpha,\beta)}(a)$. Whence $w_2 = \Theta(v_4) \in \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(a))$. Therefore $G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(a)) \subseteq \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(a))$. This implies that $\Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(a)) = G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(a))$ as required.

(iii) Let $a \in A$ be given. Suppose $w_1 \in \Theta(F\rfloor^{cs}_{R,(\alpha,\beta)}(a))$. Then there exists $v_1 \in F\rfloor^{cs}_{R,(\alpha,\beta)}(a)$ such that $\Theta(v_1) = w_1$. Thus $[v_1]^{cs}_{R,(\alpha,\beta)} \subseteq F(a)$. We shall prove that $[w_1]^{cs}_{S,(\alpha,\beta)} \subseteq G(\Xi(a))$. Let $w_2 \in [w_1]^{cs}_{S,(\alpha,\beta)}$, then there exists $v_2 \in V$ such that $\Theta(v_2) = w_2$. Thus $\Theta(v_2) \in [\Theta(v_1)]^{cs}_{S,(\alpha,\beta)}$. By item (i), we get $v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)}$. We see that $v_2 \in F(a)$. Thus $\Theta(v_2) \in \Theta(F(a))$. Since $\Theta(F(a)) = G(\Xi(a))$, we have $w_2 = \Theta(v_2) \in G(\Xi(a))$. It follows that $w_1 \in [w_1]^{cs}_{S,(\alpha,\beta)} \subseteq G(\Xi(a))$. This implies that $\Theta(F\rfloor^{cs}_{R,(\alpha,\beta)}(a)) \subseteq G\rfloor^{cs}_{S,(\alpha,\beta)}(\Xi(a))$.

(iv) Let $a \in A$ and $w_1 \in G\rfloor^{cs}_{S,(\alpha,\beta)}(\Xi(a))$. Then $[w_1]^{cs}_{S,(\alpha,\beta)} \subseteq G(\Xi(a))$. Since $\Theta(F(a)) = G(\Xi(a))$, we have $[w_1]^{cs}_{S,(\alpha,\beta)} \subseteq \Theta(F(a))$. Since Θ is surjective, there exists $v_1 \in V$ such that $\Theta(v_1) = w_1$. Thus $[\Theta(v_1)]^{cs}_{S,(\alpha,\beta)} \subseteq \Theta(F(a))$. We shall prove that $[v_1]^{cs}_{R,(\alpha,\beta)} \subseteq F(a)$. Let $v_2 \in [v_1]^{cs}_{R,(\alpha,\beta)}$, then $\Theta(v_2) \in [\Theta(v_1)]^{cs}_{S,(\alpha,\beta)}$due to item (i). Thus $\Theta(v_2) \in \Theta(F(a))$. Then there exists $v_3 \in F(a)$ such that $\Theta(v_2) = \Theta(v_3)$. Since Θ is injective, we obtain that $v_2 = v_3$. Observe that $v_2 \in F(a)$. Therefore $[v_1]^{cs}_{R,(\alpha,\beta)} \subseteq F(a)$. Whence $v_1 \in F\rfloor^{cs}_{R,(\alpha,\beta)}(a)$. Thus $w_1 = \Theta(v_1) \in \Theta(F\rfloor^{cs}_{R,(\alpha,\beta)}(a))$. This means that $G\rfloor^{cs}_{S,(\alpha,\beta)}(\Xi(a)) \subseteq \Theta(F\rfloor^{cs}_{R,(\alpha,\beta)}(a))$. By item (iii), we obtain that $\Theta(F\rfloor^{cs}_{R,(\alpha,\beta)}(a)) = G\rfloor^{cs}_{S,(\alpha,\beta)}(\Xi(a))$.

(v) Assume $\acute{b} \in \Xi(Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}))$. Then there exists $\acute{a} \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$ such that $\acute{b} = \Xi(\acute{a})$. Observe that $F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}) \neq \emptyset$. Let $v_1 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$. By item (ii), we obtain that

$$\Theta(v_1) \in \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})) = G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\acute{a})) = G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b}).$$

Thus $G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b}) \neq \emptyset$. Therefore $\acute{b} \in Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$. Hence $\Xi(Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})) \subseteq Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$. Conversely, let $\grave{b} \in Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$, then $G\rceil^{cs}_{S,(\alpha,\beta)}(\grave{b}) \neq \emptyset$. Let $w \in G\rceil^{cs}_{S,(\alpha,\beta)}(\grave{b})$, since Ξ is surjective, there exists $\grave{a} \in A$ such that $\Xi(\grave{a}) = \grave{b}$. Using item (ii), we get that

$$w \in G\rceil^{cs}_{S,(\alpha,\beta)}(\grave{b}) = G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a})) = \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})).$$

Then there exists $v_2 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$ such that $\Theta(v_2) = w$. Observe that $F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a}) \neq \emptyset$. Thus $\grave{a} \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$. Hence $\grave{b} \in \Xi(Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}))$. Whence $Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}) \subseteq \Xi(Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}))$. This means that $\Xi(Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})) = Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$.

Items $(vi) - (x)$ are not hard to verify that arguments are true, so we omit it. □

Theorem 4.9. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ and $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ be two given approximation spaces type I, and let $(\Theta, \Xi)_h$ be a given soft homomorphism from a soft semigroup $\mathfrak{F} := (F, A)$ over V to a soft semigroup $\mathfrak{G} := (G, B)$ over W satisfying Eqs (4.1)–(4.3). Then $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V if and only if $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ is an idealistic soft semigroup over W.*

Proof. Suppose that $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V. Then $F\rceil^{cs}_{R,(\alpha,\beta)}(a)$ is an ideal of V for all $a \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$. Let $\acute{b} \in Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$, then $\acute{b} \in \Xi(Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}))$ due to Proposition 4.5 (v). Thus, there exists $\acute{a} \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$ such that $\Xi(\acute{a}) = \acute{b}$. We note that $\Theta(V) = W$. From the hypothesis and Proposition 4.5 (ii), we observe that

$$\begin{aligned}
W(G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b})) &= W(G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\acute{a})))\\
&= (\Theta(V))(\Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})))\\
&= \Theta(V(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})))\\
&\subseteq \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a}))\\
&= G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\acute{a}))\\
&= G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b}).
\end{aligned}$$

Hence $G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b})$ is a left ideal of W. Similarly, we can prove that $G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b})$ is a right ideal of W. Thus, $G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b})$ is an ideal of W. Therefore, $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ is an idealistic soft semigroup over W.

On the other hand, we suppose that $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ is an idealistic soft semigroup over W. Then $G\rceil^{cs}_{S,(\alpha,\beta)}(b)$ is an ideal of W for all $b \in Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$. Let $\grave{a} \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$ be given. Suppose that $v_1 \in V(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a}))$. Then, by Proposition 4.5 (ii),

$$\begin{aligned}
\Theta(v_1) &\in \Theta(V(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})))\\
&= (\Theta(V))(\Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})))\\
&= W(G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a})))\\
&\subseteq G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a}))\\
&= \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})).
\end{aligned}$$

Thus, there exists $v_2 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$ such that $\Theta(v_1) = \Theta(v_2)$. By Proposition 3.2, we obtain that $\Theta(v_1) \in [\Theta(v_2)]^{cs}_{S,(\alpha,\beta)}$. From Proposition 4.5 (i), we argue that $v_1 \in [v_2]^{cs}_{R,(\alpha,\beta)}$. From Proposition 3.3, we get $[v_1]^{cs}_{R,(\alpha,\beta)} = [v_2]^{cs}_{R,(\alpha,\beta)}$. Observe that $[v_2]^{cs}_{R,(\alpha,\beta)} \cap F(\grave{a}) \neq \emptyset$. Then $[v_1]^{cs}_{R,(\alpha,\beta)} \cap F(\grave{a}) \neq \emptyset$, and so $v_1 \in$

$F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$. Hence $V(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})) \subseteq F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$. Whence $F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$ is a left ideal of V. Similarly, we can show that $F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$ is a right ideal of V. It follows that $F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$ is an ideal of V. Consequently $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V. □

Theorem 4.10. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ and $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ be two approximation spaces type I, and let $(\Theta, \Xi)_h$ be a given soft homomorphism from a soft semigroup $\mathfrak{F} := (F, A)$ over V to a soft semigroup $\mathfrak{G} := (G, B)$ over W satisfying Eqs (4.1)–(4.3). If Θ is injective, then $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V if and only if $\mathfrak{G}\rfloor^{cs}_{S,(\alpha,\beta)}$ is an idealistic soft semigroup over W.*

Proof. According to Proposition 4.5 (iv), we can show that the statement is true. □

Theorem 4.11. *Let $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ be an approximation space type I, let the triple $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ be an approximation space type III, and Let $(\Theta, \Xi)_h$ be a soft homomorphism from a soft semigroup $\mathfrak{F} := (F, A)$ over V to a soft semigroup $\mathfrak{G} := (G, B)$ over W satisfying Eqs (4.1)–(4.3). Then $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is a prime idealistic soft semigroup over V if and only if $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ is a prime idealistic soft semigroup over W.*

Proof. Suppose that $\mathfrak{F}\rfloor^{cs}_{R,(\alpha,\beta)}$ is a prime idealistic soft semigroup over V. Then $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V. By Theorem 4.9, we obtain that $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ is an idealistic soft semigroup over W. Thus, $G\rceil^{cs}_{S,(\alpha,\beta)}(b)$ is an ideal of W for all $b \in Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$. Let $\acute{b} \in Supp(\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)})$, then, by Proposition 4.5 (v), we obtain that $\acute{b} \in \Xi(Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}))$. Thus, there exists $\acute{a} \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$ such that $\Xi(\acute{a}) = \acute{b}$. Next, we let $w_1, w_2 \in W$. Assume that $w_1w_2 \in G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b})$. Then $w_1w_2 \in G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\acute{a}))$. Since Θ is surjective, there exist $v_1, v_2 \in V$ such that $\Theta(v_1) = w_1$ and $\Theta(v_2) = w_2$. Since S is a set-valued picture hesitant fuzzy complete relation, we observe

$$
\begin{aligned}
\emptyset &\neq [w_1w_2]^{cs}_{S,(\alpha,\beta)} \cap G(\Xi(\acute{a})) \\
&= ([w_1]^{cs}_{S,(\alpha,\beta)})([w_2]^{cs}_{S,(\alpha,\beta)}) \cap \Theta(F(\acute{a})) \\
&= ([\Theta(v_1)]^{cs}_{S,(\alpha,\beta)})([\Theta(v_2)]^{cs}_{S,(\alpha,\beta)}) \cap \Theta(F(\acute{a})).
\end{aligned}
$$

Then there exist $\Theta(v_3) \in [\Theta(v_1)]^{cs}_{S,(\alpha,\beta)}$ and $\Theta(v_4) \in [\Theta(v_2)]^{cs}_{S,(\alpha,\beta)}$ such that $(\Theta(v_3))(\Theta(v_4)) \in \Theta(F(\acute{a}))$. By the property of Θ, we get that $\Theta(v_3v_4) \in \Theta(F(\acute{a}))$. Thus, there exists $v_5 \in F(\acute{a})$ such that $\Theta(v_3v_4) = \Theta(v_5)$. Since $\Theta(v_3) \in [\Theta(v_1)]^{cs}_{S,(\alpha,\beta)}$ and $\Theta(v_4) \in [\Theta(v_2)]^{cs}_{S,(\alpha,\beta)}$, by Proposition 4.5 (i), we get that $v_3 \in [v_1]^{cs}_{R,(\alpha,\beta)}$ and $v_4 \in [v_2]^{cs}_{R,(\alpha,\beta)}$, respectively. By Propositions 4.1 and 4.5 (viii), we get

$$v_3v_4 \in ([v_1]^{cs}_{R,(\alpha,\beta)})([v_2]^{cs}_{R,(\alpha,\beta)}) \subseteq [v_1v_2]^{cs}_{R,(\alpha,\beta)}.$$

By Proposition 3.3, we get $[v_1v_2]^{cs}_{R,(\alpha,\beta)} = [v_3v_4]^{cs}_{R,(\alpha,\beta)}$. From Proposition 3.2, we have $\Theta(v_3v_4) \in [\Theta(v_3v_4)]^{cs}_{S,(\alpha,\beta)}$. Thus $\Theta(v_5) \in [\Theta(v_3v_4)]^{cs}_{S,(\alpha,\beta)}$. By Proposition 4.5 (i), we obtain $v_5 \in [v_3v_4]^{cs}_{R,(\alpha,\beta)}$. Hence $v_5 \in [v_1v_2]^{cs}_{R,(\alpha,\beta)}$. Thus, we see that $[v_1v_2]^{cs}_{R,(\alpha,\beta)} \cap F(\acute{a}) \neq \emptyset$. Thus $v_1v_2 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$. By the assumption, we get that $v_1 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$ or $v_2 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})$. From Proposition 4.5 (ii), it follows that

$$w_1 = \Theta(v_1) \in \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})) = G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\acute{a})) = G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b})$$

or

$$w_2 = \Theta(v_2) \in \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\acute{a})) = G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\acute{a})) = G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b}).$$

Whence $G\rceil^{cs}_{S,(\alpha,\beta)}(\acute{b})$ is a completely prime ideal of W. Consequently $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ is a prime idealistic soft semigroup over W.

Conversely, assume that $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ is a prime idealistic soft semigroup over W. Then $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ is an idealistic soft semigroup over W. From Theorem 4.9, we get that $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is an idealistic soft semigroup over V, and hence $F\rceil^{cs}_{R,(\alpha,\beta)}(a)$ is an ideal of V for all $a \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$. Let $\grave{a} \in Supp(\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)})$ and $v_1, v_2 \in V$. Suppose that $v_1v_2 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$. Then $\Theta(v_1v_2) \in \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a}))$. Thus, by Proposition 4.5 (ii), we get

$$(\Theta(v_1))(\Theta(v_2)) = \Theta(v_1v_2) \in \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})) = G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a})).$$

By the fact that $G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a}))$ is a completely prime ideal of W, we get $\Theta(v_1) \in G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a}))$ or $\Theta(v_2) \in G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a}))$. If $\Theta(v_1) \in G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a}))$, then $\Theta(v_1) \in \Theta(F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a}))$ due to Proposition 4.5 (ii). Thus, there exists $v_3 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$ such that $\Theta(v_1) = \Theta(v_3)$. By Proposition 3.2, we get that $\Theta(v_3) \in [\Theta(v_3)]^{cs}_{S,(\alpha,\beta)}$, which yields $\Theta(v_1) \in [\Theta(v_3)]^{cs}_{S,(\alpha,\beta)}$. Then, by Proposition 4.5 (i), it follows that $v_1 \in [v_3]^{cs}_{R,(\alpha,\beta)}$. By Proposition 3.3, we have $[v_1]^{cs}_{R,(\alpha,\beta)} = [v_3]^{cs}_{R,(\alpha,\beta)}$. Observe that $[v_3]^{cs}_{R,(\alpha,\beta)} \cap F(\grave{a}) \neq \emptyset$. Thus $[v_1]^{cs}_{R,(\alpha,\beta)} \cap F(\grave{a}) \neq \emptyset$. Therefore $v_1 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$. Similarly, if $\Theta(v_2) \in G\rceil^{cs}_{S,(\alpha,\beta)}(\Xi(\grave{a}))$, then $v_2 \in F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$. Thus, we obtain that $F\rceil^{cs}_{R,(\alpha,\beta)}(\grave{a})$ is a completely prime ideal of V. This means that $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ is a prime idealistic soft semigroup over V. □

Theorem 4.12. *Let* $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be an approximation space type I, and let the triple* $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ *be an approximation space type III. Let* $(\Theta, \Xi)_h$ *be a soft homomorphism from a soft semigroup* $\mathfrak{F} := (F, A)$ *over V to a soft semigroup* $\mathfrak{G} := (G, B)$ *over W satisfying Eqs (4.1)–(4.3). If* Θ *is injective, then* $\mathfrak{F}\rceil^{cs}_{R,(\alpha,\beta)}$ *is a prime idealistic soft semigroup over V if and only if* $\mathfrak{G}\rceil^{cs}_{S,(\alpha,\beta)}$ *is a prime idealistic soft semigroup over W.*

Proof. By Proposition 4.5 (iv), we can verify that the statement holds. □

Theorem 4.13. *Let* $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *and* $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ *be two approximation spaces type I. Let f and g be fuzzy subsets of V and W, respectively, and let* $\iota, \kappa \in [0, 1]$. *Let* $(\Theta, \Xi)_h$ *be a soft homomorphism from a soft semigroup* $(V_A^{(f,\iota,>)}, A)$ *over V to a soft semigroup* $(W_B^{(g,\kappa,>)}, B)$ *over W satisfying Eqs (4.1)–(4.3). Then* $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ *is a fuzzy ideal of V if and only if* $\ulcorner g \urcorner^{cs}_{S,(\alpha,\beta)}$ *is a fuzzy ideal of W.*

Proof. Using Propositions 3.19 (i), 4.4 and Theorem 4.9, we see that

$$\begin{aligned}
\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} \text{ is a fuzzy ideal of } V &\iff (V_A^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,\geq)}, A) \text{ is an idealistic soft} \\
&\qquad \text{semigroup over } V \text{ for all } \iota \in [0,1] \\
&\iff (V_A^{(f,\iota,\geq)}, A)\rceil^{cs}_{R,(\alpha,\beta)} \text{ is an idealistic soft} \\
&\qquad \text{semigroup over } V \text{ for all } \iota \in [0,1] \\
&\iff (W_B^{(g,\kappa,\geq)}, B)\rceil^{cs}_{S,(\alpha,\beta)} \text{ is an idealistic soft} \\
&\qquad \text{semigroup over } W \text{ for all } \kappa \in [0,1] \\
&\iff (W_B^{(\ulcorner g \urcorner^{cs}_{S,(\alpha,\beta)},\kappa,\geq)}, B) \text{ is an idealistic soft} \\
&\qquad \text{semigroup over } W \text{ for all } \kappa \in [0,1] \\
&\iff \ulcorner g \urcorner^{cs}_{S,(\alpha,\beta)} \text{ is a fuzzy ideal of } W.
\end{aligned}$$

Therefore, $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ is a fuzzy ideal of V if and only if $\ulcorner g \urcorner^{cs}_{S,(\alpha,\beta)}$ is a fuzzy ideal of W. □

Theorem 4.14. *Let* $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *and* $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ *be two approximation spaces type I. Let f and g be fuzzy subsets of V and W, respectively, and let* $\iota, \kappa \in [0, 1]$. *Let* $(\Theta, \Xi)_h$ *be a soft homomorphism from a soft semigroup* $(V_A^{(f,\iota,>)}, A)$ *over V to a soft semigroup* $(W_B^{(g,\kappa,>)}, B)$ *over W satisfying Eqs (4.1)–(4.3). If* Θ *is injective, then* $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$ *is a fuzzy ideal of V if and only if* $\llcorner g \lrcorner^{cs}_{S,(\alpha,\beta)}$ *is a fuzzy ideal of W.*

Proof. By Propositions 3.19 (ii), 4.4 and Theorem 4.10, this argument is easily provided. □

Theorem 4.15. *Let* $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be an approximation space type I, and let the triple* $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ *be an approximation space type III. Let f and g be fuzzy subsets of V and W, respectively, and let* $\iota, \kappa \in [0, 1]$. *Let* $(\Theta, \Xi)_h$ *be a soft homomorphism from a soft semigroup* $(V_A^{(f,\iota,>)}, A)$ *over V to a soft semigroup* $(W_B^{(g,\kappa,>)}, B)$ *over W satisfying Eqs (4.1)–(4.3). Then* $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ *is a fuzzy prime ideal of V if and only if* $\ulcorner g \urcorner^{cs}_{S,(\alpha,\beta)}$ *is a fuzzy prime ideal of W.*

Proof. Using Propositions 3.19 (i), 4.4 and Theorem 4.11, we observe that

$$\begin{aligned}
\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)} \text{ is a fuzzy prime ideal of } V &\iff (V_A^{(\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)},\iota,\geq)}, A) \text{ is a prime idealistic} \\
&\qquad \text{soft semigroup over } V \text{ for all } \iota \in [0,1] \\
&\iff (V_A^{(f,\iota,\geq)}, A)\rceil^{cs}_{R,(\alpha,\beta)} \text{ is a prime idealistic} \\
&\qquad \text{soft semigroup over } V \text{ for all } \iota \in [0,1] \\
&\iff (W_B^{(g,\kappa,\geq)}, B)\rceil^{cs}_{S,(\alpha,\beta)} \text{ is a prime idealistic} \\
&\qquad \text{soft semigroup over } W \text{ for all } \kappa \in [0,1] \\
&\iff (W_B^{(\ulcorner g \urcorner^{cs}_{S,(\alpha,\beta)},\kappa,\geq)}, B) \text{ is a prime idealistic} \\
&\qquad \text{soft semigroup over } W \text{ for all } \kappa \in [0,1] \\
&\iff \ulcorner g \urcorner^{cs}_{S,(\alpha,\beta)} \text{ is a fuzzy prime ideal of } W.
\end{aligned}$$

It follows that $\ulcorner f \urcorner^{cs}_{R,(\alpha,\beta)}$ is a fuzzy prime ideal of V if and only if $\ulcorner g \urcorner^{cs}_{S,(\alpha,\beta)}$ is a fuzzy prime ideal of W. □

Theorem 4.16. *Let* $(V, V, [V]^{cs}_{R:=(R^-,R^\pm,R^+),(\alpha,\beta)})$ *be an approximation space type I, and let the triple* $(W, W, [W]^{cs}_{S:=(S^-,S^\pm,S^+),(\alpha,\beta)})$ *be an approximation space type III. Let f and g be fuzzy subsets of V and W, respectively, and let* $\iota, \kappa \in [0, 1]$. *Let* $(\Theta, \Xi)_h$ *be a given soft homomorphism from a soft semigroup* $(V_A^{(f,\iota,>)}, A)$ *over V to a soft semigroup* $(W_B^{(g,\kappa,>)}, B)$ *over W satisfying Eqs (4.1)–(4.3). If* Θ *is injective, then* $\llcorner f \lrcorner^{cs}_{R,(\alpha,\beta)}$ *is a fuzzy prime ideal of V if and only if* $\llcorner g \lrcorner^{cs}_{S,(\alpha,\beta)}$ *is a fuzzy prime ideal of W.*

Proof. According to Propositions 3.19 (ii), 4.4 and Theorem 4.12, we can prove that the statement holds. □

5. Conclusions and discussion

In this work, we have studied the concept of picture hesitant fuzzy relations in terms of picture hesitant fuzzy sets on infinite sets, which is a new extension of fuzzy relations given by Mathew et al. [25] in 2020. Then the notion of extended approximation spaces under set-valued picture hesitant fuzzy relations was proposed as the followings:

- The basic element of the rough approximation of soft sets constitute upper and lower rough approximations, boundary regions, definable soft sets and rough soft sets.
- The basic element of the rough approximation of fuzzy sets constitute upper and lower rough approximations, definable fuzzy sets and rough fuzzy sets.

As a consequence, we proved that the definable soft set and the definable fuzzy set are induced by a set-valued picture hesitant fuzzy reflexive relation and a set-valued picture hesitant fuzzy antisymmetric relation on a single universe.

In addition, we suggested studying the use of the theory of semigroups to advance in the investigation of rough soft sets and rough fuzzy sets. We used models in Section 3 to consider upper and lower rough approximations of prime idealistic soft semigroups over semigroups and fuzzy prime ideals of semigroups. Then we obtained statements as the followings:

- Every upper rough approximation of a prime idealistic soft semigroup (resp., a fuzzy prime ideal) is a prime idealistic soft semigroup (resp., a fuzzy prime ideal) based on a picture hesitant fuzzy reflexive relation, a picture hesitant fuzzy transitive relation, and a picture hesitant fuzzy complete relation.
- Every lower rough approximation of a prime idealistic soft semigroup (resp., a fuzzy prime ideal) is a prime idealistic soft semigroup (resp., a fuzzy prime ideal) based on a picture hesitant fuzzy reflexive relation, a picture hesitant fuzzy transitive relation, and a picture hesitant fuzzy complete relation.

Moreover, we observed that a set-valued picture hesitant fuzzy symmetric relation and a set-valued picture hesitant fuzzy antisymmetric relation on semigroups are not a sufficient condition for all results. We used soft homomorphisms to check upper and lower rough approximations of prime idealistic soft semigroups over semigroups and fuzzy prime ideals of semigroups. Furthermore, we got necessary and sufficient conditions for upper and lower rough approximations of prime idealistic soft semigroups over semigroups and fuzzy prime ideals of semigroups.

However, when we consider other types of algebraic structures, the corresponding issues need to be further proved. Increasingly, in future work, we will adapt the proposed rough approximations approach to deal with decision problems in semigroups. Furthermore, group decision making under rough approximation models received more and more attention as the existing literature [37,38]. Thus, the group decision-making approach based on extended models of this paper is a future research focus.

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Conflict of interest

The author declares no conflict of interest.

References

1. G. Cantor, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*, Leipzig: Teubner, 1883.
2. Z. Pawlak, A. Skowron, Rudiments of rough sets, *Inform. Sci.*, **177** (2007), 3–27. doi: 10.1016/j.ins.2006.06.003.
3. Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341–356. doi: 10.1007/BF01001956.
4. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353.
5. D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, *Int. J. Gen. Syst.*, **17** (1990), 191–209. doi: 10.1080/03081079008935107.
6. D. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19–31. doi: 10.1016/S0898-1221(99)00056-5.
7. F. Feng, C. X. Li, B. Davvaz, M. I. Ali, Soft sets combined with fuzzy sets and rough sets: A tentative approach, *Soft Comput.*, **14** (2010), 899–911. doi: 10.1007/s00500-009-0465-6.
8. Y. Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, *Inform. Sci.*, **111** (1998), 239–259. doi: 10.1016/S0020-0255(98)10006-3.
9. R. Mareay, Generalized rough sets based on neighborhood systems and topological spaces, *J. Egypt. Math. Soc.*, **24** (2016), 603–608. doi: 10.1016/j.joems.2016.02.002.
10. R. Prasertpong, M. Siripitukdet, On rough sets induced by fuzzy relations approach in semigroups, *Open Math.*, **16** (2018), 1634–1650. doi: 10.1515/math-2018-0136.
11. R. Prasertpong, M. Siripitukdet, Rough set models induced by serial fuzzy relations approach in semigroups, *Eng. Let.*, **27** (2019), 216–225.
12. R. Prasertpong, M. Siripitukdet, Generalizations of rough sets induced by binary relations approach in semigroups, *J. Intell. Fuzzy Syst.,* **36** (2019), 5583–5596. doi: 10.3233/JIFS-181435.
13. R. Prasertpong, M. Siripitukdet, Applying generalized rough set concepts to approximation spaces of semigroups, *IAENG Int. J. Appl. Math.*, **49** (2019), 51–60.
14. L. A. Zadeh, Similarity relations and fuzzy orderings, *Inform. Sci.*, **3** (1971), 117–200. doi: 10.1016/S0020-0255(71)80005-1.
15. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.,* **20** (1986), 87–96. doi: 10.1016/S0165-0114(86)80034-3.
16. P. J. Burillo, H. Bustince, Intuitionistic fuzzy relations (Part I), *Mathware Soft Comput.,* **2** (1995), 5–38.
17. F. Smarandache, *A unifying field in logics: Neutrosophic logic. Neutrosophy set, neutrosophic probability and statistics*, 4 Eds., Rehoboth: American Research Press, 2005.

18. B. C. Cuong, Picture fuzzy sets, *J. Comput. Sci. Cybern.*, **30** (2014), 409–420. doi: 10.15625/1813-9663/30/4/5032.

19. V. Torra, Hesitant fuzzy sets, *Int. J. Intell. Syst.*, **25** (2010), 529–539. doi:10.1002/int.20418.

20. B. Zhu, Studies on consistency measure of hesitant fuzzy preference relations, *Procedia Comput. Sci.*, **17** (2013), 457–464. doi: 10.1016/j.procs.2013.05.059.

21. R. Wang, B. Shuai, Z. S. Chen, K. S. Chin, J. H. Zhu, Revisiting the role of hesitant multiplicative preference relations in group decision making with novel consistency improving and consensus reaching processes, *Int. J. Comput. Intell. Syst.*, **12** (2019), 1029–1046. doi: 10.2991/ijcis.d.190823.001.

22. Z. S. Chen, X. Zhang, W. Pedrycz, X. J. Wang, K. S. Chin, L Martínezf, *K*-means clustering for the aggregation of HFLTS possibility distributions: *N*-two-stage algorithmic paradigm, *Knowl.-Based Syst.*, **227** (2021), 107230. doi: 10.1016/j.knosys.2021.107230.

23. Z. M. Zhang, S. M. Chen, Group decision making based on acceptable multiplicative consistency and consensus of hesitant fuzzy linguistic preference relations, *Inform. Sci.*, **541** (2020), 531–550. doi: 10.1016/j.ins.2020.07.024.

24. R. Wang, Y. L. Li, Picture hesitant fuzzy set and its application to multiple criteria decision-making, *Symmetry*, **10** (2018), 1–29. doi: 10.3390/sym10070295.

25. B. Mathew, S. J. John, J. C. R. Alcantud, Multi-granulation picture hesitant fuzzy rough sets, *Symmetry*, **12** (2020), 1–17. doi: 10.3390/sym12030362.

26. T. Y. Lin, N. Cercone, *Rough sets and data mining*, 1 Ed., Boston: Springer, 1997. doi: 10.1007/978-1-4613-1461-5.

27. Q. H. Zhang, Q. Xie, G. Y. Wang, A survey on rough set theory and its applications, *CAAI T. Intell. Techno.*, **1** (2016), 323–333. doi: 10.1016/j.trit.2016.11.001.

28. Q. M. Xiao, Z. L. Zhang, Rough prime ideals and rough fuzzy prime ideals in semigroups, *Inform. Sci.*, **176** (2006), 725–733. doi: 10.1016/j.ins.2004.12.010.

29. O. Kazancı, B. Davvaz, On the structure of rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings, *Inform. Sci.,* **178** (2008), 1343–1354. doi: 10.1016/j.ins.2007.10.005.

30. Q. J. Luo, G. J. Wang, Roughness and fuzziness in quantales, *Inform. Sci.*, **271** (2014), 14–30. doi: 10.1016/j.ins.2014.02.105.

31. J. M. Zhan, Q. Liu, B. Davvaz, A new rough set theory: Rough soft hemirings, *J. Intell. Fuzzy Syst.*, **28** (2015), 1687–1697. doi: 10.3233/IFS-141455.

32. S. K. Roy, S. Bera, Approximation of rough soft set and its application to lattice, *Fuzzy Inf. Eng.*, **7** (2015), 379–387. doi: 10.1016/j.fiae.2015.09.008.

33. W. J. Pan, J. M. Zhan, Rough fuzzy groups and rough soft groups, *Ital. J. Pure Appl. Math.*, **36** (2016), 617–628.

34. J. M. Zhan, B. Davvaz, A kind of new rough set: Rough soft sets and rough soft rings, *J. Intell. Fuzzy Syst.*, **30** (2016), 475–483. doi: 10.3233/IFS-151772.

35. Q. M. Wang, J. M. Zhan, Rough semigroups and rough fuzzy semigroups based on fuzzy ideals, *Open Math.*, **14** (2016), 1114–1121. doi: 10.1515/math-2016-0102.

36. Q. M. Wang, J. M. Zhan, A novel view of rough soft semigroups based on fuzzy ideals, *Ital. J. Pure Appl. Math.*, **37** (2017), 673–686.

37. J. M. Zhan, X. W. Zhou, D. J. Xiang, Rough soft n-ary semigroups based on a novel congruence relation and corresponding decision making, *J. Intell. Fuzzy Syst.*, **33** (2017), 693–703. doi: 10.3233/JIFS-161497.

38. J. M. Zhan, Q. Liu, W. Zhu, Another approach to rough soft hemirings and corresponding decision making, *Soft Comput.*, **21** (2017), 3769–3780. doi: 10.1007/s00500-016-2058-5.

39. S. M. Qurashi, M. Shabir, Generalized rough fuzzy ideals in quantales, *Discrete Dyn. Nat. Soc.*, **2018** (2018), 1–11. doi: 10.1155/2018/1085201.

40. J. C. R. Alcantud, F. Feng, R. R. Yager, An N-soft set approach to rough sets, *IEEE Trans. Fuzzy Syst.*, **28** (2019), 2996–3007. doi: 10.1109/TFUZZ.2019.2946526.

41. R. S. Kanwal, M. Shabir, Rough approximation of a fuzzy set in semigroups based on soft relations, *Comp. Appl. Math.*, **38** (2019), 1–23. doi: 10.1007/s40314-019-0851-3.

42. A. Hussain, T. Mahmood, M. I. Ali, Rough Pythagorean fuzzy ideals in semigroups, *Comp. Appl. Math.*, **38** (2019), 1–15. doi: 10.1007/s40314-019-0824-6.

43. R. Chinram, T. Panityakul, Rough Pythagorean fuzzy ideals in ternary semigroups, *J. Math. Comput. Sci.*, **20** (2020), 302–312. doi: 10.22436/jmcs.020.04.04.

44. A. Satirad, R. Chinram, A. Iampan, Pythagorean fuzzy sets in UP-algebras and approximations, *AIMS Math.*, **6** (2021), 6002–6032. doi: 10.3934/math.2021354.

45. A. Elmoasry, On rough fuzzy prime ideals in left almost semigroups, *Int. J. Anal. Appl.*, **19** (2021), 455–464. doi: 10.28924/2291-8639-19-2021-455.

46. A. H. Clifford, G. B. Preston, *The algebraic theory of semigroups*, Providence, RI: American Mathematical Society, 1961. doi: 10.1090/surv/007.1.

47. J. M. Howie, *Fundamentals of semigroup theory*, United States: Oxford University Press, 1995.

48. Š. Schwarz, Prime ideals and maximal ideals in semigroups, *Czechoslovak Math. J.*, **19** (1969), 72–79.

49. J. N. Mordeson, D. S. Malik, N. Kuroki, *Fuzzy semigroups*, Berlin, Heidelberg: Springer, 2003. doi: 10.1007/978-3-540-37125-0.

50. M. K. Chakraborty, M. Das, On fuzzy equivalence-I, *Fuzzy Sets Syst.*, **11** (1983), 185–193.

51. M. K. Chakraborty, M. Das, On fuzzy equivalence-II, *Fuzzy Sets Syst.*, **11** (1983), 299–307.

52. M. K. Chakraborty, S. Sarkar, Fuzzy antisymmetry and order, *Fuzzy Sets Syst.*, **21** (1987), 169–182. doi: 10.1016/0165-0114(87)90162-X.

53. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, *Comput. Math. Appl.*, **45** (2003), 555–562. doi: 10.1016/S0898-1221(03)00016-6.

54. M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M.Shabir, On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547–1553. doi: 10.1016/j.camwa.2008.11.009.

55. M. I. Ali, M. Shabir, K. P. Shum, On soft ideals over semigroups, *Southeast Asian Bull. Math.*, **34** (2010), 595–610.

56. F. Feng, M. I. Ali, M. Shabir, Soft relations applied to semigroups, *Filomat*, **27** (2013), 1183–1196. doi: 10.2298/FIL1307183F.

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