Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(2): 2237–2254.
DOI:10.3934/math.2022127
Received: 23 August 2021
Accepted: 07 November 2021
Published: 10 November 2021

Research article

Extremal orders and races between palindromes in different bases

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Abstract: Let $b \geq 2$ and $n \geq 1$ be integers. Then n is said to be a palindrome in base b (or b-adic palindrome) if the representation of n in base b reads the same backward as forward. Let $A_b(n)$ be the number of b-adic palindromes less than or equal to n. In this article, we obtain extremal orders of $A_b(n)$. We also study the comparison between the number of palindromes in different bases and prove that if $b \neq b_1$, then $A_b(n) - A_{b_1}(n)$ changes signs infinitely often as $n \to \infty$.

Keywords: palindrome; palindromic number; extremal order; b-adic expansion; minimal order; maximal order
Mathematics Subject Classification: 11A63, 11N37, 11N56

1. Introduction

Let $b \geq 2$ and $n \geq 1$ be integers. We call n a *palindrome in base b* (or *b-adic palindrome*) if the b-adic expansion of $n = (a_k a_{k-1} \cdots a_0)_b$ with $a_k \neq 0$ has the symmetric property $a_{k-i} = a_i$ for $0 \leq i \leq k$. As usual, if we write a number without specifying the base, then it is always in base 10, and if we write $n = (a_k a_{k-1} \cdots a_0)_b$, then it means that $n = \sum_{i=0}^{k} a_i b^i$, $a_k \neq 0$, and $0 \leq a_i < b$ for all $i = 0, 1, \ldots, k$. So, for example, $9 = (1001)_2 = (100)_3$ is a palindrome in bases 2 and 10 but not in base 3.

For a long time ago, palindromes were considered only as a part of recreational mathematics, but in recent years, there has been an increasing interest in the importance of palindromes in mathematics [1–3,16,25], theoretical computer science [4,15,17,18], and theoretical physics [5,13,21]. For example, in 2016, Banks [6] showed that every positive integer can be written as the sum of at most 49 palindromes in base 10. Two years later Cilleruelo, Luca, and Baxter [11] improved it by showing that if $b \geq 5$ is fixed, then every positive integer is the sum of at most three b-adic palindromes. Then Rajasekaran, Shallit, and Smith [31] completed the study by proving that the theorem of Cilleruelo, Luca, and Baxter [11] also holds when $b \in \{3, 4\}$, and if $b = 2$, then we need four summands to write every positive integer as a sum of b-adic palindromes. Nevertheless, we still have many other interesting open problems concerning palindromes. In particular, on Mathematics Stack Exchange, Vepir asked: which number base contains the most palindromic numbers?

Pongsriiam and Subwattanachai [30] started the investigation by deriving an exact formula for the number of b-adic palindromes not exceeding n, denoted by $A_b(n)$, but the formula is difficult to analyze, and so it does not give an answer to Vepir's question. Then Phunphayap and Pongsriiam [28] calculated the reciprocal sum of all b-adic palindromes implying that if $b > b_1$ and if we use the logarithmic measure, then there are more b-adic palindromes than b_1-adic palindromes. Nevertheless, this does not answer Vepir's question according to the counting measure.

In this article, we obtain extremal orders of $A_b(n)$ and show that $A_b(n) - A_{b_1}(n)$ has infinitely many sign changes. We also obtain other related results and use them to solve Vepir's problem.

For more information on the palindromes, we refer the reader to Banks, Hart, and Sakata [7] and Banks and Shparlinski [8] for some multiplicative properties of palindromes, Bašić [9, 10], Di Scala and Sombra [14], Goins [19], Luca and Togbé [26] for the study of palindromes in different bases, Cilleruelo, Luca, and Tesoro [12] for palindromes in linear recurrence sequences, Harminc and Soták [20] for b-adic palindromes in arithmetic progressions, Korec [23] for nonpalindromic numbers having palindromic squares, and Pongsriiam [29] for the longest arithmetic progressions of palindromes.

2. Preliminaries and lemmas

In this section, we provide some definitions and lemmas which are needed in the proof of the main theorems. Recall that for a real number x, $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\lceil x \rceil$ is the smallest integer greater than or equal to x, and $\{x\}$ is the *fractional part* of x given by $\{x\} = x - \lfloor x \rfloor$. Furthermore, for a mathematical statement P, the *Iverson notation* $[P]$ is defined by

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

In the proof of our main results, we often use Pongsriiam and Subwattanachai's formula [30]. So it is convenient to define $C_b(n)$ as follows.

Definition 1. Let $b \geq 2$ and $n \geq 1$ be integers, and $n = (a_k a_{k-1} \cdots a_1 a_0)_b$. Define $C_b(n) = (c_k c_{k-1} \cdots c_1 c_0)_b$ to be the b-adic palindrome satisfying $c_i = a_i$ for $k - \lfloor k/2 \rfloor \leq i \leq k$. In other words, $C_b(n)$ is the b-adic palindrome having $k+1$ digits, the first half of which are the same as those of n in its b-adic expansion, that is, $C_b(n) = (a_k a_{k-1} \cdots a_{k - \lfloor \frac{k}{2} \rfloor} \cdots a_{k-1} a_k)_b$.

Example 2. If $m = (247853)_9$ and $n = (1327021)_8$, then $C_9(m) = (247742)_9$ and $C_8(n) = (1327231)_8$.

Note that our definition of $C_b(n)$ is slightly different from that in Pongsriiam and Subwattanachai's formula [30]. In addition, while we focus only on positive integers, they [30] also count zero. After a slight modification, their formula is as follows.

Lemma 3 (Pongsriiam and Subwattanachai [30]). *Let $b \geq 2$ and $n \geq 1$ be integers, and $n = (a_k a_{k-1} \cdots a_1 a_0)_b$. Then the number of b-adic palindromes less than or equal to n is given by*

$$A_b(n) = b^{\lceil \frac{k}{2} \rceil} + \sum_{0 \leq i \leq \lfloor \frac{k}{2} \rfloor} a_{k-i} b^{\lfloor \frac{k}{2} \rfloor - i} + [n \geq C_b(n)] - 2,$$

where $[n \geq C_b(n)]$ is the Iverson notation.

The next lemma gives an upper bound for $A_b(n)$. By Theorem 9, we will see later that the inequality in Lemma 4 is sharp.

Lemma 4. *Let $b \geq 2$ and $n \geq 1$ be integers. Then*

$$A_b(n) + 1 \leq \left(\sqrt{b} + \frac{1}{\sqrt{b}}\right)\sqrt{n}.$$

Proof. We write $n = (a_k a_{k-1} \cdots a_1 a_0)_b$ and let

$$y = \sum_{0 \leq i \leq \lfloor k/2 \rfloor} a_{k-i} b^{k-i} \ \text{ and } \ z = \sum_{0 \leq i \leq \lfloor k/2 \rfloor} a_{k-i} b^{-i}.$$

We see that $y \leq n$, $y = b^k z$, and $1 \leq z \leq (b-1)\sum_{i=0}^{\infty} b^{-i} = b$. By Lemma 3, we obtain

$$A_b(n) + 1 \leq b^{\lceil \frac{k}{2} \rceil} + \sum_{0 \leq i \leq \lfloor \frac{k}{2} \rfloor} a_{k-i} b^{\lfloor \frac{k}{2} \rfloor - i} = b^{\lceil \frac{k}{2} \rceil} + b^{\lfloor \frac{k}{2} \rfloor} z.$$

Therefore,

$$\frac{A_b(n) + 1}{\sqrt{n}} \leq \frac{A_b(n) + 1}{\sqrt{y}} \leq \frac{b^{\lceil \frac{k}{2} \rceil} + b^{\lfloor \frac{k}{2} \rfloor} z}{b^{\frac{k}{2}} \sqrt{z}}.$$

We divide the consideration into two cases according to the parity of k.
Case 1. k is even. Then

$$\frac{A_b(n) + 1}{\sqrt{n}} \leq \frac{b^{\frac{k}{2}} + b^{\frac{k}{2}} z}{b^{\frac{k}{2}} \sqrt{z}} = \sqrt{z} + \frac{1}{\sqrt{z}}.$$

Since the function $x \mapsto \sqrt{x} + \frac{1}{\sqrt{x}}$ is increasing on $[1, \infty]$ and $1 \leq z \leq b$, we have

$$\frac{A_b(n) + 1}{\sqrt{n}} \leq \sqrt{z} + \frac{1}{\sqrt{z}} \leq \sqrt{b} + \frac{1}{\sqrt{b}}.$$

Case 2. k is odd. This case is similar to Case 1. We have

$$\frac{A_b(n) + 1}{\sqrt{n}} \leq \frac{b^{\frac{k+1}{2}} + b^{\frac{k-1}{2}} z}{b^{\frac{k}{2}} \sqrt{z}} = \sqrt{\frac{b}{z}} + \sqrt{\frac{z}{b}} = \sqrt{\frac{b}{z}} + \frac{1}{\sqrt{b/z}}.$$

Since $1 \leq z \leq b$, we obtain $1 \leq b/z \leq b$ and therefore

$$\frac{A_b(n) + 1}{\sqrt{n}} \leq \sqrt{\frac{b}{z}} + \sqrt{\frac{z}{b}} \leq \sqrt{b} + \frac{1}{\sqrt{b}}.$$

In any case, we obtain the desired result. □

Recall that a sequence $(x_n)_{n \geq 1}$ of real numbers is said to be *uniformly distributed modulo 1* if for any $0 \leq a < b \leq 1$,

$$\lim_{x \to \infty} \left(\frac{1}{x} \sum_{\substack{n \leq x \\ a \leq \{x_n\} < b}} 1 \right) = b - a.$$

A well-known criterion for uniform distribution modulo 1 is as follows [24, page 7].

Lemma 5. (Weyl's Criterion) *The sequence* $(x_n)_{n\geq 1}$ *is uniformly distributed modulo 1 if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \quad \text{for all integers } h \neq 0.$$

Lemma 6. *Let* $(x_n)_{n\geq 1}$ *be an arithmetic progression of real numbers,* d *the common difference,* $d \in \mathbb{Z} \setminus \{0\}$*, and* $\alpha \in \mathbb{R}$ *an irrational number. Then* $(\alpha x_n)_{n\geq 1}$ *is uniformly distributed modulo 1.*

Proof. Since α is irrational, $d \neq 0$, and $d \in \mathbb{Z}$, we have $\left|e^{2\pi i h\alpha d} - 1\right| \neq 0$ for all integers $h \neq 0$. Recall that $x_n = x_1 + (n-1)d$ for all $n \geq 1$. So if h is a nonzero integer, then

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h\alpha x_n}\right| = \frac{\left|e^{2\pi i h\alpha x_1}\right|\left|e^{2\pi i h\alpha N d} - 1\right|}{N\left|e^{2\pi i h\alpha d} - 1\right|} = \frac{\left|e^{2\pi i h\alpha N d} - 1\right|}{N\left|e^{2\pi i h\alpha d} - 1\right|} \leq \frac{2}{N\left|e^{2\pi i h\alpha d} - 1\right|},$$

which converges to 0 as $N \to \infty$. By Weyl's criterion, the desired result is verified. □

The next result is an important tool for obtaining Theorem 11.

Lemma 7. *Let* $(x_n)_{n\geq 1}$*,* $d \geq 1$*, and* α *satisfy the same assumption as in Lemma 6,* $x_n \in \mathbb{Z}$ *for every* n*, and let* $c \in (0, 1)$*. Then there are infinitely many integers* $k \geq 1$ *such that*

$$\{\alpha x_k\} < c \text{ and } \lfloor \alpha x_k \rfloor \equiv 0 \pmod 2. \tag{2.1}$$

Proof. Suppose for a contradiction that there is only a finite number of integers $k \geq 1$ such that x_k satisfies (2.1). Let k be the largest such an integer if it exists and let $k = 1$ otherwise. Since $d \geq 1$, we see that $(x_n)_{n\geq 1}$ is strictly increasing, and so if $n > k$, then x_n does not satisfy (2.1). We will get a contradiction by constructing an integer $x = x_n$ satisfying (2.1) and $n > k$. By Lemma 6, $(\alpha x_n)_{n\geq 1}$ is uniformly distributed modulo 1. From this point on, we apply the above fact repeatedly without reference. Let $k_1, k_2, \ldots, k_d$ be integers satisfying

$$k_d > k_{d-1} > \cdots > k_1 > k \ \text{ and } \ \{\alpha x_{k_i}\} < \frac{c}{d+1} \ \text{ for all } i = 1, 2, \ldots, d.$$

Then $\lfloor \alpha x_{k_i} \rfloor \equiv 1 \pmod 2$ for all $i = 1, 2, \ldots, d$. We divide the consideration into two cases.
Case 1. d is even. Let k_{d+1} be an integer such that

$$k_{d+1} > k_d \ \text{ and } \ \{\alpha x_{k_{d+1}}\} > 1 - \sum_{i=1}^{d} \{\alpha x_{k_i}\}. \tag{2.2}$$

Case 1.1 $\lfloor \alpha x_{k_{d+1}} \rfloor \equiv 1 \pmod 2$. We see that

$$\alpha \sum_{i=1}^{d+1} x_{k_i} = \sum_{i=1}^{d+1} \lfloor \alpha x_{k_i} \rfloor + \sum_{i=1}^{d+1} \{\alpha x_{k_i}\}, \ \text{ and}$$
$$1 < \sum_{i=1}^{d+1} \{\alpha x_{k_i}\} < 1 + \sum_{i=1}^{d} \{\alpha x_{k_i}\} < 1 + c.$$

This implies that

$$\left\{\alpha \sum_{i=1}^{d+1} x_{k_i}\right\} < c \ \text{ and } \ \left\lfloor \alpha \sum_{i=1}^{d+1} x_{k_i} \right\rfloor = 1 + \sum_{i=1}^{d+1} \lfloor \alpha x_{k_i} \rfloor \equiv 1 + d + 1 \equiv 0 \pmod 2. \tag{2.3}$$

Let $A = \sum_{i=1}^{d+1} x_{k_i}$. Then we have

$$A > x_k \ \text{ and } \ A \equiv \sum_{i=1}^{d+1} (x_1 + (k_i - 1)d) \equiv (d+1)x_1 \equiv x_1 \pmod d.$$

Since $A > x_k$ and $A \equiv x_1 \pmod d$, we see that $A = x_n$ for some $n > k$. But by (2.3), $A = x_n$ satisfies (2.1), contradicting the fact that k is the largest integer such that x_k satisfies (2.1).

Case 1.2 $\lfloor \alpha x_{k_{d+1}} \rfloor \equiv 0 \pmod 2$. Let M be the maximum value of $\{\alpha x_{k_i}\}$ for $i = 1, 2, \ldots, d$. Since d is even, we must have $d \geq 2$. Then

$$M < \frac{c}{d+1} \ \text{ and } \ 0 < \frac{1}{d} < \frac{1}{d} + \frac{c}{d} - M < 1.$$

Therefore there are infinitely many $t \in \mathbb{N}$ such that

$$\frac{1}{d} < \{\alpha x_t\} < \frac{1}{d} + \frac{c}{d} - M. \tag{2.4}$$

Then one of the two sets

$$X = \{t \in \mathbb{N} \mid x_t \text{ satisfies (2.4) and } \lfloor \alpha x_t \rfloor \equiv 0 \pmod 2\} \ \text{ and}$$
$$Y = \{t \in \mathbb{N} \mid x_t \text{ satisfies (2.4) and } \lfloor \alpha x_t \rfloor \equiv 1 \pmod 2\}$$

is infinite, say Y. Then we can choose integers $t_d > t_{d-1} > \cdots > t_1 > k$ in Y so that

$$\frac{1}{d} < \{\alpha x_{t_i}\} < \frac{1}{d} + \frac{c}{d} - M \ \text{ for all } i = 1, 2, \ldots, d, \tag{2.5}$$

and

$$\sum_{i=1}^{d} \lfloor \alpha x_{t_i} \rfloor \equiv 1 + 1 + \cdots + 1 \equiv d \equiv 0 \pmod 2.$$

(If X is infinite, then we can choose such $t_1, t_2, \ldots, t_d \in X$ too.) Let

$$B = \sum_{i=1}^{d+1} x_{k_i} + \sum_{i=1}^{d} x_{t_i}.$$

Then

$$\alpha B = \sum_{i=1}^{d+1} \lfloor \alpha x_{k_i} \rfloor + \sum_{i=1}^{d} \lfloor \alpha x_{t_i} \rfloor + \sum_{i=1}^{d+1} \{\alpha x_{k_i}\} + \sum_{i=1}^{d} \{\alpha x_{t_i}\}.$$

By (2.2) and (2.5), we obtain

$$2 = 1 + \sum_{i=1}^{d} \frac{1}{d} < \sum_{i=1}^{d+1} \{\alpha x_{k_i}\} + \sum_{i=1}^{d} \{\alpha x_{t_i}\}$$

$$\begin{aligned} &< \{\alpha x_{k_{d+1}}\} + \sum_{i=1}^{d} (\{\alpha x_{k_i}\} - M) + \sum_{i=1}^{d} \left(\frac{1}{d} + \frac{c}{d}\right) \\ &\leq \{\alpha x_{k_{d+1}}\} + 1 + c < 2 + c, \end{aligned}$$

which implies that

$$\{\alpha B\} < c \ \text{ and } \ \lfloor \alpha B \rfloor = 2 + \sum_{i=1}^{d+1} \lfloor \alpha x_{k_i} \rfloor + \sum_{i=1}^{d} \lfloor \alpha x_{t_i} \rfloor \equiv 0 \ (\text{mod } 2).$$

Similar to Case 1.1, we have

$$B > x_k \ \text{ and } \ B \equiv x_1 \ (\text{mod } d).$$

Therefore $B = x_n$ for some $n > k$ and x_n satisfies (2.1), a contradiction.
Case 2. d is odd. Let k_{d+1} be an integer satisfying

$$k_{d+1} > k_d \ \text{ and } \ \{\alpha x_{k_{d+1}}\} < \frac{c}{d+1}.$$

This case is similar to Case 1. Let $D = \sum_{i=1}^{d+1} x_{k_i}$. Then $D > x_k$, $D \equiv x_1$ (mod d), and

$$\alpha D = \sum_{i=1}^{d+1} \alpha x_{k_i} = \sum_{i=1}^{d+1} \lfloor \alpha x_{k_i} \rfloor + \sum_{i=1}^{d+1} \{\alpha x_{k_i}\}. \tag{2.6}$$

Since $k_{d+1} > k$ and $\{\alpha x_{k_{d+1}}\} < c$, we see that $\lfloor \alpha x_{k_{d+1}} \rfloor \equiv 1$ (mod 2). In addition, we have

$$\sum_{i=1}^{d+1} \{\alpha x_{k_i}\} < \sum_{i=1}^{d+1} \frac{c}{d+1} = c, \tag{2.7}$$

$$\sum_{i=1}^{d+1} \lfloor \alpha x_{k_i} \rfloor \equiv 1 + 1 + \cdots + 1 \equiv d + 1 \equiv 0 \ (\text{mod } 2). \tag{2.8}$$

From (2.6), (2.7), and (2.8), we obtain

$$\{\alpha D\} = \sum_{i=1}^{d+1} \{\alpha x_{k_i}\} < c \ \text{ and } \ \lfloor \alpha D \rfloor = \sum_{i=1}^{d+1} \lfloor \alpha x_{k_i} \rfloor \equiv 0 \ (\text{mod } 2).$$

Therefore $D = x_n$ for some $n > k$ and x_n satisfies (2.1), a contradiction.

In any case, we have a contradiction. So the proof is complete. □

Lemma 8. *Suppose a and b are positive rational numbers and $a, b \neq 1$. Then $\log_b a$ is rational if and only if there exist integers m and n such that $a^m = b^n$.*

Proof. This is well-known. For more details, see for example, pages 24–25, Chapter 2 in the book of Niven [27]. □

3. Main results

We are now ready to prove our main theorems. We begin with maximal and minimal orders of $A_b(n)$.

Theorem 9. *Let $b \geq 2$ be an integer. Then*

$$\limsup_{n\to\infty} \frac{A_b(n)}{\sqrt{n}} = \sqrt{b} + \frac{1}{\sqrt{b}}.$$

In particular, a maximal order of $A_b(n)$ is $\left(\sqrt{b} + \frac{1}{\sqrt{b}}\right)\sqrt{n}$.

Proof. Let $\varepsilon > 0$. By Lemma 4, it remains to show that $A_b(n)/\sqrt{n} \geq \sqrt{b} + \frac{1}{\sqrt{b}} - \varepsilon$ for infinitely many $n \in \mathbb{N}$. To prove this, we construct a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that

$$\frac{A_b(n_k)}{\sqrt{n_k}} \to \sqrt{b} + \frac{1}{\sqrt{b}} \text{ as } k \to \infty.$$

For each $k \in \mathbb{N}$, let $n_k = b^{2k+1} + 1$. By Lemma 3, we obtain that $A_b(n_k) = b^{k+1} + b^k - 1$. Therefore,

$$\frac{A_b(n_k)}{\sqrt{n_k}} = \frac{b^{k+1} + b^k - 1}{b^{k+\frac{1}{2}}\sqrt{1 + b^{-2k-1}}} = \left(\sqrt{b} + \frac{1}{\sqrt{b}} - \frac{1}{b^{k+\frac{1}{2}}}\right)\frac{1}{\sqrt{1 + b^{-2k-1}}} \to \sqrt{b} + \frac{1}{\sqrt{b}} \text{ as } k \to \infty,$$

as desired. □

Theorem 10. *Let $b \geq 2$ be an integer. Then*

$$\liminf_{n\to\infty} \frac{A_b(n)}{\sqrt{n}} = 2.$$

In particular, a minimal order of $A_b(n)$ is $2\sqrt{n}$.

Proof. Let $\varepsilon > 0$. We first show that $A_b(n)/\sqrt{n} \geq 2 - \varepsilon$ for all large n. Let N be a large positive integer to be determined later and let $n \geq b^N$. Since $[b^N, \infty) = \bigcup_{k=N}^{\infty}[b^k, b^{k+1})$, we see that $b^k \leq n < b^{k+1}$ for some $k \geq N$. Then the number of digits of n in its b-adic expansion is $k + 1$. Let

$$n = (a_k a_{k-1} \cdots a_1 a_0)_b, \quad y = \sum_{0\leq i\leq \lfloor k/2\rfloor} a_{k-i}b^{k-i}, \quad \text{and} \quad z = \sum_{0\leq i\leq \lfloor k/2\rfloor} a_{k-i}b^{-i}.$$

So $y = (a_k a_{k-1} \cdots a_{k-\lfloor \frac{k}{2}\rfloor} 00 \cdots 0)_b$, $n \geq y$, and $A_b(n) \geq A_b(y)$. We divide the calculation into two cases according to the parity of k.

Case 1. k is even. Then by Lemma 3,

$$A_b(y) = b^{\frac{k}{2}} + \sum_{0\leq i\leq \frac{k}{2}} a_{k-i}b^{\frac{k}{2}-i} - 2 = b^{\frac{k}{2}}(1 + z) - 2.$$

Observe that

$$n \leq \sum_{0\leq i\leq \frac{k}{2}} a_{k-i}b^{k-i} + (b-1)\sum_{\frac{k}{2}<i\leq k} b^{k-i} = \sum_{0\leq i\leq \frac{k}{2}} a_{k-i}b^{k-i} + b^{\frac{k}{2}} - 1 = b^k z + b^{\frac{k}{2}} - 1.$$

Then $\sqrt{n} \le b^{k/2}\sqrt{z+\delta}$, where $\delta = b^{-k/2} - b^{-k} > 0$. Therefore

$$\begin{aligned}\frac{A_b(n)}{\sqrt{n}} \ge \frac{A_b(y)}{\sqrt{n}} \ge \frac{b^{\frac{k}{2}}(1+z)}{b^{\frac{k}{2}}\sqrt{z+\delta}} - \frac{2}{\sqrt{n}} &= \frac{1+z}{\sqrt{z+\delta}} - 2 - \frac{2}{\sqrt{n}} + 2\\ &= \frac{\left(\sqrt{z+\delta}-1\right)^2}{\sqrt{z+\delta}} - \frac{\delta}{\sqrt{z+\delta}} - \frac{2}{\sqrt{n}} + 2\\ &\ge 2 - \frac{2}{\sqrt{n}} - \frac{\delta}{\sqrt{z+\delta}}.\end{aligned}$$

As $N \to \infty$, we see that $n \to \infty$, $k \to \infty$, and $\delta/\sqrt{z+\delta} \le \sqrt{\delta} = \sqrt{b^{-\frac{k}{2}} - b^{-k}} \to 0$. Therefore we can choose N large enough so that

$$\frac{A_b(n)}{\sqrt{n}} \ge 2 - \frac{2}{\sqrt{n}} - \frac{\delta}{\sqrt{z+\delta}} \ge 2 - \varepsilon, \quad \text{for all } n \ge b^N.$$

Case 2. k is odd. Some calculations in this part are similar to those in Case 1, so we skip some details. Let $\delta = b^{-(k-1)/2} - b^{-k} > 0$. By Lemma 3,

$$A_b(y) = b^{\frac{k+1}{2}} + \sum_{0\le i\le \frac{k-1}{2}} a_{k-i} b^{\frac{k-1}{2}-i} - 2 = b^{\frac{k}{2}}\left(\sqrt{b} + \frac{z}{\sqrt{b}}\right) - 2.$$

In addition,

$$\sqrt{n} \le \sqrt{b^k z + b^{\frac{k+1}{2}} - 1} = b^{\frac{k}{2}}\sqrt{z+\delta}, \quad \text{and}$$

$$\begin{aligned}\frac{A_b(n)}{\sqrt{n}} \ge \frac{A_b(y)}{\sqrt{n}} &\ge \frac{b^{\frac{k}{2}}\left(\sqrt{b} + \frac{z}{\sqrt{b}}\right)}{b^{\frac{k}{2}}\sqrt{z+\delta}} - \frac{2}{\sqrt{n}}\\ &= \frac{b+z}{\sqrt{b(z+\delta)}} - 2 - \frac{2}{\sqrt{n}} + 2\\ &= \frac{\left(\sqrt{z+\delta} - \sqrt{b}\right)^2}{\sqrt{b(z+\delta)}} - \frac{\delta}{\sqrt{b(z+\delta)}} - \frac{2}{\sqrt{n}} + 2\\ &\ge 2 - \frac{2}{\sqrt{n}} - \frac{\delta}{\sqrt{b(z+\delta)}} \ge 2 - \varepsilon \ \text{ when } N \text{ is large enough.}\end{aligned}$$

In any case, we see that if N is large and $n \ge b^N$, then $A_b(n)/\sqrt{n} \ge 2 - \varepsilon$. So it remains to show that $A_b(n)/\sqrt{n} \le 2 + \varepsilon$ for infinitely many $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $n = n_k = b^{2k} - 2$. Then $n = (a_{2k-1}a_{2k-2}\cdots a_1 a_0)_b$ where $a_0 = b - 2$ and $a_i = b - 1$ for all $i = 1, 2, \cdots, 2k-1$, and $n < b^{2k} - 1 = C_b(n)$. By Lemma 3, we have

$$A_b(n) = b^k + (b-1)\sum_{0\le i\le k-1} b^{k-1-i} - 2 = 2b^k - 3.$$

This implies

$$\frac{A_b(n)}{\sqrt{n}} = \frac{2b^k}{\sqrt{b^{2k}-2}} - \frac{3}{\sqrt{b^{2k}-2}} \to 2 \ \text{ as } k \to \infty.$$

Since k is arbitrary, there are infinitely many $n \in \mathbb{N}$ such that $A_b(n)/\sqrt{n} \le 2 + \varepsilon$. This completes the proof. □

We used a computer to compare the values of $A_b(n)$ when $b = 2, 3, 5, 10$ and $n = 10^k$ for $k = 1, 2, \ldots, 20$. The data are shown in Table 1 at the end of this article. We see that for distinct $b, b_1 \in \{2, 3, 5, 10\}$, there is an integer $n \geq 1$ such that $A_b(n) > A_{b_1}(n)$ and there is an integer $m \geq 1$ such that $A_b(m) < A_{b_1}(m)$. For example, $A_2(10^{20}) > A_{10}(10^{20})$ while $A_2(10^{19}) < A_{10}(10^{19})$. In general, we have the following theorem, which is the main result of this paper.

Theorem 11. *Let $b > b_1 \geq 2$ be integers. Then $A_b(n) - A_{b_1}(n)$ has infinitely many sign changes as $n \to \infty$. More precisely, the following statements hold.*

(i) *There are infinitely many $n \in \mathbb{N}$ such that $A_{b_1}(n) > A_b(n)$.*
(ii) *There are infinitely many $n \in \mathbb{N}$ such that $A_b(n) > A_{b_1}(n)$.*

Proof. Throughout the proof, we apply Lemma 3 repeatedly without reference and separate the proof into two parts. In the first part, we show that (i) holds.

Case 1(i). b is not a rational power of b_1. By Lemma 8, we see that $\log_b b_1$ is irrational. Applying Lemma 7 to the sequence of odd positive integers $(1, 3, 5, 7, \ldots)$ with $d = 2$, $\alpha = \log_b b_1$, and $c = \log_b\left(1 + \frac{1}{b^2}\right)$, we obtain that there are infinitely many integers k such that

$$k \equiv 1 \pmod 2, \quad k \geq 5\log_{b_1} b, \quad \{k \log_b b_1\} < \log_b\left(1 + \frac{1}{b^2}\right), \text{ and } \lfloor k \log_b b_1 \rfloor \equiv 0 \pmod 2. \tag{3.1}$$

Let k be one of those integers. Since $k \equiv 1 \pmod 2$ and $b_1^k < b_1^k + 1 = C_{b_1}(b_1^k)$, we obtain

$$A_{b_1}(b_1^k) = b_1^{\lceil \frac{k}{2} \rceil} + b_1^{\lfloor \frac{k}{2} \rfloor} - 2 = b_1^{\frac{k+1}{2}} + b_1^{\frac{k-1}{2}} - 2 = \left(\sqrt{b_1} + \frac{1}{\sqrt{b_1}}\right) b_1^{\frac{k}{2}} - 2. \tag{3.2}$$

Next, we write $b_1^k = (a_r a_{r-1} \cdots a_1 a_0)_b$. Since $b^r \leq a_r b^r \leq b_1^k < b^{r+1}$, we see that r is the largest integer such that $b^r \leq b_1^k$. Therefore $r = \lfloor k \log_b b_1 \rfloor \geq \left\lfloor 5 \log_{b_1} b \cdot \log_b b_1 \right\rfloor = 5$. In addition, $a_r b^r \leq b_1^k < (a_r + 1) b^r$, so a_r is the largest integer such that $a_r b^r \leq b_1^k$. Therefore

$$a_r = \left\lfloor \frac{b_1^k}{b^r} \right\rfloor = \left\lfloor \frac{b_1^k}{b^{k \log_b b_1 - \{k \log_b b_1\}}} \right\rfloor = \left\lfloor b^{\{k \log_b b_1\}} \right\rfloor.$$

Since $\{k \log_b b_1\} < \log_b(1 + 1/b^2) < \log_b 2$, we see that $b^{\{k \log_b b_1\}} < b^{\log_b 2} = 2$. Thus $\left\lfloor b^{\{k \log_b b_1\}} \right\rfloor < 2$, which implies $a_r < 2$. So $a_r = 1$. Next, we calculate a_{r-1}. We see that

$$a_{r-1} = \left\lfloor \frac{b_1^k - b^r}{b^{r-1}} \right\rfloor = \left\lfloor b^{1 + \{k \log_b b_1\}} \right\rfloor - b. \tag{3.3}$$

Since $\{k \log_b b_1\} < \log_b\left(1 + 1/b^2\right) \leq \log_b(1 + 1/b)$, we have

$$b^{1 + \{k \log_b b_1\}} < b^{1 + \log_b\left(1 + \frac{1}{b}\right)} = b\left(1 + \frac{1}{b}\right) = b + 1.$$

We obtain from (3.3) that $a_{r-1} < b + 1 - b = 1$, which implies $a_{r-1} = 0$. Similarly, we have

$$a_{r-2} = \left\lfloor \frac{b_1^k - b^r}{b^{r-2}} \right\rfloor = \left\lfloor b^{2 + \{k \log_b b_1\}} \right\rfloor - b^2 < b^2\left(1 + \frac{1}{b^2}\right) - b^2 = 1.$$

This implies $a_{r-2} = 0$. So we have $a_r = 1$ and $a_{r-1} = a_{r-2} = 0$, and thus $b_1^k \le b^r + b^{r-2}$. Recall that $r = \lfloor k \log_b b_1 \rfloor \equiv 0 \pmod 2$, $r \ge 5$, and $b^r \le b_1^k$. Since $b^r + b^{r-2} < b^r + b^{r-2} + b^2 + 1 = C_b(b^r + b^{r-2})$, we obtain

$$A_b(b_1^k) \le A_b\left(b^r + b^{r-2}\right) = b^{\lceil \frac{r}{2} \rceil} + b^{\lfloor \frac{r}{2} \rfloor} + b^{\lfloor \frac{r}{2} \rfloor - 2} - 2$$
$$= 2b^{\frac{r}{2}} + b^{\frac{r}{2}-2} - 2 \le 2b_1^{\frac{k}{2}} + b_1^{\frac{k}{2}} b^{-2} - 2 = \left(2 + b^{-2}\right) b_1^{\frac{k}{2}} - 2.$$

From this and (3.2), we obtain

$$A_{b_1}(b_1^k) - A_b(b_1^k) \ge \left(\sqrt{b_1} + \frac{1}{\sqrt{b_1}}\right) b_1^{\frac{k}{2}} - \left(2 + \frac{1}{b^2}\right) b_1^{\frac{k}{2}} = \left(\sqrt{b_1} + \frac{1}{\sqrt{b_1}} - 2 - \frac{1}{b^2}\right) b_1^{\frac{k}{2}}. \tag{3.4}$$

Since $b > b_1 \ge 2$ and the function $x \mapsto \sqrt{x} + \frac{1}{\sqrt{x}}$ is increasing on $[1, \infty)$,

$$\sqrt{b_1} + \frac{1}{\sqrt{b_1}} - 2 - \frac{1}{b^2} \ge \sqrt{2} + \frac{1}{\sqrt{2}} - 2 - \frac{1}{3^2} > 0.$$

Therefore $A_{b_1}(b_1^k) - A_b(b_1^k) > 0$. Since $A_{b_1}(b_1^k) - A_b(b_1^k) > 0$ holds for any k satisfying (3.1), we can choose $n = b_1^k$ and obtain that $A_{b_1}(n) - A_b(n) > 0$ for infinitely many n, as required.

Case 2(i). b is a rational power of b_1. Let $b^s = b_1^t$ for some $s, t \in \mathbb{N}$ with $\gcd(s, t) = 1$. Let $m \in \mathbb{N}$ and $k = 2mt + 1$. Then k is odd. Since $b_1^k + 1 = C_{b_1}(b_1^k + 1)$, we obtain

$$A_{b_1}(b_1^k + 1) = b_1^{\lceil \frac{k}{2} \rceil} + b_1^{\lfloor \frac{k}{2} \rfloor} - 1 = b_1^{\frac{k+1}{2}} + b_1^{\frac{k-1}{2}} - 1 = (b_1 + 1) b_1^{\frac{k-1}{2}} - 1. \tag{3.5}$$

Since $b^s = b_1^t$ and $k = 2mt + 1$, we obtain $b_1^k = b_1^{2mt+1} = b_1 \cdot b^{2ms}$. Therefore $b_1^k + 1 = (a_r a_{r-1} \cdots a_0)_b$ where $r = 2ms$, $a_r = b_1$, $a_0 = 1$, and $a_i = 0$ for $i = 1, 2, \ldots, r-1$. So

$$A_b\left(b_1^k + 1\right) = b^{\lceil \frac{r}{2} \rceil} + a_r b^{\lfloor \frac{r}{2} \rfloor} - 2 = (b_1 + 1) b^{ms} - 2 = (b_1 + 1) b_1^{mt} - 2 = (b_1 + 1) b_1^{\frac{k-1}{2}} - 2.$$

From this and (3.5), we obtain $A_{b_1}(b_1^k + 1) - A_b(b_1^k + 1) = 1$. Since m is arbitrary, we can choose $k = 2mt + 1$ and $n = b_1^k + 1$ so that $A_{b_1}(n) - A_b(n) > 0$ for infinitely many n.

Case 1(i) and Case 2(i) give a proof of (i). The proof of (ii) is quite similar to that of (i). So we omit some details. We divide the consideration into two cases.

Case 1(ii). b is not a rational power of b_1. Then $\log_{b_1} b$ is irrational. Similar to Case 1(i), we apply Lemma 7 and let k be an integer such that

$$k \equiv 1 \pmod 2, \quad k \ge 5, \quad \left\{k \log_{b_1} b\right\} < \log_{b_1}\left(1 + \frac{1}{b_1^2}\right), \quad \text{and} \quad \left\lfloor k \log_{b_1} b \right\rfloor \equiv 0 \pmod 2. \tag{3.6}$$

Then

$$A_b(b^k) = b^{\frac{k+1}{2}} + b^{\frac{k-1}{2}} - 2 = \left(\sqrt{b} + \frac{1}{\sqrt{b}}\right) b^{\frac{k}{2}} - 2. \tag{3.7}$$

We write $b^k = (a_r a_{r-1} \cdots a_1 a_0)_{b_1}$. Then $r = \left\lfloor k \log_{b_1} b \right\rfloor \ge k \ge 5$ and

$$a_r = \left\lfloor \frac{b^k}{b_1^r} \right\rfloor = \left\lfloor \frac{b^k}{b_1^{k \log_{b_1} b - \left\{k \log_{b_1} b\right\}}} \right\rfloor = \left\lfloor b_1^{\left\{k \log_{b_1} b\right\}} \right\rfloor < 1 + \frac{1}{b_1^2} < 2,$$

which implies $a_r = 1$. Similarly,

$$a_{r-1} = \left\lfloor \frac{b^k - b_1^r}{b_1^{r-1}} \right\rfloor = \left\lfloor b_1^{1+\{k\log_{b_1} b\}} \right\rfloor - b_1 < b_1\left(1 + \frac{1}{b_1^2}\right) - b_1 < 1,$$

which implies $a_{r-1} = 0$. Then

$$a_{r-2} = \left\lfloor \frac{b^k - b_1^r}{b_1^{r-2}} \right\rfloor = \left\lfloor b_1^{2+\{k\log_{b_1} b\}} \right\rfloor - b_1^2 < b_1^2\left(1 + \frac{1}{b_1^2}\right) - b_1^2 = 1,$$

which implies $a_{r-2} = 0$. So $b^k \le b_1^r + b_1^{r-2} < C_{b_1}\left(b_1^r + b_1^{r-2}\right)$. Recall that $r = \left\lfloor k\log_{b_1} b\right\rfloor \equiv 0 \pmod 2$ and $r \ge 5$. Then

$$\begin{aligned}A_{b_1}(b^k) \le A_{b_1}\left(b_1^r + b_1^{r-2}\right) &= b_1^{\lceil \frac{r}{2}\rceil} + b_1^{\lfloor \frac{r}{2}\rfloor} + b_1^{\lfloor \frac{r}{2}\rfloor-2} - 2\\ &= 2b_1^{\frac{r}{2}} + b_1^{\frac{r}{2}}b_1^{-2} - 2 \le 2b^{\frac{k}{2}} + b^{\frac{k}{2}}b_1^{-2} - 2 = \left(2 + b_1^{-2}\right)b^{\frac{k}{2}} - 2.\end{aligned}$$

From this and (3.7), we obtain

$$A_b(b^k) - A_{b_1}(b^k) \ge \left(\sqrt{b} + \frac{1}{\sqrt{b}} - 2 - \frac{1}{b_1^2}\right)b^{\frac{k}{2}}. \tag{3.8}$$

Since $b > b_1 \ge 2$ and the function $x \mapsto \sqrt{x} + \frac{1}{\sqrt{x}}$ is increasing on $[1, \infty)$, we have

$$\sqrt{b} + \frac{1}{\sqrt{b}} - 2 - \frac{1}{b_1^2} \ge \sqrt{3} + \frac{1}{\sqrt{3}} - 2 - \frac{1}{2^2} > 0.$$

Therefore $A_b(b^k) - A_{b_1}(b^k) > 0$. By Lemma 7, there are infinitely many integers k satisfying (3.6). So we can choose $n = b^k$ so that $A_b(n) - A_{b_1}(n) > 0$ for infinitely many n.

Case 2(ii). $b^s = b_1^t$ for some $s, t \in \mathbb{N}$ with $\gcd(s, t) = 1$. Let $m \in \mathbb{N}$ and $k = 2ms + 1$. Then k is odd and

$$A_b(b^k) = b^{\frac{k+1}{2}} + b^{\frac{k-1}{2}} - 2 = \left(\sqrt{b} + \frac{1}{\sqrt{b}}\right)b^{\frac{k}{2}} - 2. \tag{3.9}$$

Since $b^s = b_1^t$, we obtain $b^k = b^{2ms+1} = b_1^{2mt}b_1^{\frac{t}{s}} = b_1^{\{\frac{t}{s}\}} \cdot b_1^{2mt+\lfloor\frac{t}{s}\rfloor}$. Since $\left\{\frac{t}{s}\right\} = \frac{j}{s}$ for some $j = 0, 1, \ldots, s-1$, we obtain $b_1^{\{\frac{t}{s}\}} = b_1^{\frac{j}{s}}$. Recall that for any positive integers x and y, $x^{1/y}$ is either an irrational number or an integer. Then $b_1^{\{\frac{t}{s}\}} = \left(b_1^j\right)^{\frac{1}{s}}$ is either an irrational number or an integer. If $b_1^{\{\frac{t}{s}\}}$ is irrational, then $b = b_1^{\frac{t}{s}} = b_1^{\{\frac{t}{s}\}} \cdot b_1^{\lfloor\frac{t}{s}\rfloor}$ is irrational, a contradiction. So $b_1^{\{\frac{t}{s}\}} \in \mathbb{N}$ and $b^k = (a_r a_{r-1} \cdots a_0)_{b_1}$ where $r = 2mt + \left\lfloor\frac{t}{s}\right\rfloor$, $a_r = b_1^{\{\frac{t}{s}\}} = b_1^{\frac{j}{s}}$, and $a_i = 0$ for all $i = 0, 1, \ldots, r-1$. Therefore

$$A_{b_1}\left(b^k\right) = b_1^{\lceil\frac{r}{2}\rceil} + a_r b_1^{\lfloor\frac{r}{2}\rfloor} - 2.$$

In addition, $b^{\frac{k}{2}} = b^{ms+\frac{1}{2}} = b_1^{mt+\frac{t}{2s}}$. Suppose first that $\left\lfloor\frac{t}{s}\right\rfloor$ is even. Then

$$A_{b_1}\left(b^k\right) = b_1^{mt+\frac{1}{2}\lfloor\frac{t}{s}\rfloor} + b_1^{\frac{j}{s}}b_1^{mt+\frac{1}{2}\lfloor\frac{t}{s}\rfloor} - 2$$

$$= \left(1 + b_1^{\frac{j}{s}}\right) b_1^{mt+\frac{t}{2s}} b_1^{-\frac{1}{2}\left\{\frac{t}{s}\right\}} - 2$$
$$= \left(1 + b_1^{\frac{j}{s}}\right) b^{\frac{k}{2}} b_1^{-\frac{j}{2s}} - 2$$
$$= \left(\sqrt{b_1^{\frac{j}{s}}} + \frac{1}{\sqrt{b^{\frac{j}{s}}}}\right) b^{\frac{k}{2}} - 2. \tag{3.10}$$

Since $b > b_1^{\frac{j}{s}} \geq 1$ and the function $x \mapsto \sqrt{x} + \frac{1}{\sqrt{x}}$ is strictly increasing on $[1, \infty)$, we obtain from (3.9) and (3.10) that

$$A_b\left(b^k\right) - A_{b_1}\left(b^k\right) = \left(\sqrt{b} + \frac{1}{\sqrt{b}} - \sqrt{b_1^{\frac{j}{s}}} - \frac{1}{\sqrt{b_1^{\frac{j}{s}}}}\right) b^{\frac{k}{2}} > 0. \tag{3.11}$$

Suppose $\left\lfloor \frac{t}{s} \right\rfloor$ is odd. Then similar to (3.10), we obtain

$$A_{b_1}\left(b^k\right) = b_1^{mt+\frac{1}{2}\left\lfloor\frac{t}{s}\right\rfloor+\frac{1}{2}} + b_1^{\left\{\frac{t}{s}\right\}} b_1^{mt+\frac{1}{2}\left\lfloor\frac{t}{s}\right\rfloor-\frac{1}{2}} - 2$$
$$= b_1^{mt+\frac{t}{2s}+\frac{1}{2}-\frac{1}{2}\left\{\frac{t}{s}\right\}} + b_1^{mt+\frac{t}{2s}-\frac{1}{2}+\frac{1}{2}\left\{\frac{t}{s}\right\}} - 2$$
$$= b^{\frac{k}{2}}\left(b_1^{\frac{1}{2}\left(1-\left\{\frac{t}{s}\right\}\right)} + b_1^{-\frac{1}{2}\left(1-\left\{\frac{t}{s}\right\}\right)}\right) - 2$$
$$= \left(\sqrt{b_1^{\ell}} + \frac{1}{\sqrt{b_1^{\ell}}}\right) b^{\frac{k}{2}} - 2,$$

where $\ell = 1 - \left\{\frac{t}{s}\right\}$. From this and (3.9), we obtain

$$A_b\left(b^k\right) - A_{b_1}\left(b^k\right) = \left(\sqrt{b} + \frac{1}{\sqrt{b}} - \sqrt{b_1^{\ell}} - \frac{1}{\sqrt{b_1^{\ell}}}\right) b^{\frac{k}{2}} > 0. \tag{3.12}$$

Since m is arbitrary, we can choose $k = 2ms + 1$ and $n = b^k$ so that $A_b(n) - A_{b_1}(n) > 0$ for infinitely many n.

Case 1(ii) and Case 2(ii) give a proof of (ii). Therefore the proof of this theorem is complete. □

Observing the proof of Theorem 11 carefully, we can state some parts of Theorem 11 in a stronger form as follows.

Theorem 12. *Let $b > b_1 \geq 2$ be integers. Then the following statements hold.*

(i) *There are infinitely many $k \in \mathbb{N}$ and a constant $c > 0$ depending at most on b and b_1 and not on k such that*

$$A_b\left(b^k\right) - A_{b_1}\left(b^k\right) \geq c b^{\frac{k}{2}}.$$

Consequently, $\limsup_{n\to\infty}\left(A_b(n) - A_{b_1}(n)\right) = +\infty$.

(ii) *Suppose b is not a rational power of b_1. Then there are infinitely many $k \in \mathbb{N}$ and a constant $d > 0$ which depends at most on b and b_1 and not on k such that*

$$A_{b_1}\left(b_1^k\right) - A_b\left(b_1^k\right) \geq d b_1^{\frac{k}{2}}.$$

Consequently, $\liminf_{n\to\infty}\left(A_b(n) - A_{b_1}(n)\right) = -\infty$

Proof. For (i), we consider Case 1(ii) and Case 2(ii) in the proof of Theorem 11. In Case 1(ii), we see from (3.8) that we can take $c = \sqrt{b} + \frac{1}{\sqrt{b}} - 2 - \frac{1}{b_1^2}$ so that $A_b(b^k) - A_{b_1}(b^k) \geq cb^{\frac{k}{2}}$. Next, we consider (3.11) and (3.12) in Case 2(ii). Let $0 < \alpha < 1$. Since $b_1 > b_1^\alpha \geq 1$ and the function $x \mapsto \sqrt{x} + \frac{1}{\sqrt{x}}$ is increasing on $[1, \infty)$, we obtain

$$\sqrt{b} + \frac{1}{\sqrt{b}} - \sqrt{b_1^\alpha} - \frac{1}{\sqrt{b_1^\alpha}} \geq \sqrt{b} + \frac{1}{\sqrt{b}} - \sqrt{b_1} - \frac{1}{\sqrt{b_1}} > 0.$$

Setting $\alpha = \frac{j}{s}$ in (3.11) and let $\alpha = \ell$ in (3.12), we see that we can take

$$c = \sqrt{b} + \frac{1}{\sqrt{b}} - \sqrt{b_1} - \frac{1}{\sqrt{b_1}} > 0$$

so that

$$A_b\left(b^k\right) - A_{b_1}\left(b^k\right) \geq cb^{\frac{k}{2}}.$$

If we would like to obtain c that works in (3.8), (3.11), and (3.12), then we can choose

$$c = \min\left\{\sqrt{b} + \frac{1}{\sqrt{b}} - 2 - \frac{1}{b_1^2},\ \sqrt{b} + \frac{1}{\sqrt{b}} - \sqrt{b_1} - \frac{1}{\sqrt{b_1}}\right\}.$$

This proves (i). For (ii), we consider (3.4) in Case 1(i), take $d = \sqrt{b_1} + \frac{1}{\sqrt{b_1}} - 2 - \frac{1}{b^2}$, and obtain that

$$A_{b_1}\left(b_1^k\right) - A_b\left(b_1^k\right) \geq d b_1^{\frac{k}{2}}.$$

This proves (ii). So the proof is complete. □

We are now ready to give a complete answer to Vepir's question [32] posted on Mathematics Stack Exchange. The title of Vepir's post is as follows:

which number base contains the most palindromic numbers?

In the comment, Vepir also says that he is only interested in the palindromes having more than one digit. Therefore for each integers $b \geq 2$ and $n \geq 1$, we let

$$f_b(n) = A_b(n) - (b-1).$$

So $f_b(n)$ is the number of b-adic palindromes which have more than one digit and are less than or equal to n. We have the following corollary.

Corollary 13. *Let $b > b_1 \geq 2$ be integers. Then $f_b(n) - f_{b_1}(n)$ changes sign infinitely often as $n \to \infty$. In other words, if we use counting measure, then the races between palindromes in any two different bases have infinitely many wins and infinitely many losses.*

Proof. By Theorem 11, there are infinitely many $n \in \mathbb{N}$ such that $A_{b_1}(n) > A_b(n)$. So if n is such an integer, then

$$f_{b_1}(n) - f_b(n) = A_{b_1}(n) - A_b(n) + b - b_1 > 0.$$

By Theorem 12, there are infinitely many $m \in \mathbb{N}$ such that $A_b(m) - A_{b_1}(m) \geq b$. So if m is such an integer, then

$$f_b(m) - f_{b_1}(m) = A_b(m) - A_{b_1}(m) - b + b_1 \geq b_1 > 0.$$

This completes the proof. □

Corollary 14. *Let $b > b_1 \geq 2$ be integers. Then $A_b(n) - A_{b_1}(n) = 0$ for infinitely many $n \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}$, let $g(n) = A_b(n) - A_{b_1}(n)$. We know that, for any $n \in \mathbb{N}$, both $A_b(n+1) - A_b(n)$ and $A_{b_1}(n+1) - A_{b_1}(n)$ are either 0 or 1. So $g(n+1) - g(n)$ is $-1, 0$, or 1, that is, the difference of any two consecutive terms of the sequence $(A_b(n) - A_{b_1}(n))_{n \geq 1}$ is one of -1, 0, or 1. Therefore if $A_b(r) - A_{b_1}(r) < 0$ and $A_b(m) - A_{b_1}(m) > 0$, then there exists an integer n lying between r and m such that $A_b(n) - A_{b_1}(n) = 0$. By Theorem 11, there are infinitely many $n \in \mathbb{N}$ such that $A_b(n) - A_{b_1}(n) = 0$, as required. □

4. Conclusion and some future projects

We obtain extremal orders of the palindromes counting function $A_b(n)$ and show that if $b > b_1 \geq 2$, then $A_b(n) - A_{b_1}(n)$ has infinitely many sign changes as $n \to \infty$. Moreover, we obtain that

$$\limsup_{n \to \infty} (A_b(n) - A_{b_1}(n)) = +\infty,$$

and if b is not a rational power of b_1, then

$$\liminf_{n \to \infty} (A_b(n) - A_{b_1}(n)) = -\infty.$$

Problem 1. Suppose $b > b_1$ and b is a rational power of b_1. Then, perhaps, $\liminf_{n \to \infty} (A_b(n) - A_{b_1}(n))$ is either $-\infty$ or -1. More precisely, it is -1 if and only if b is an integral power of b_1, and it is $-\infty$ if and only if $b \neq b_1^m$ for any $m \in \mathbb{N}$. We do not have a proof of this yet but we plan to do it in the future.

Problem 2. Suppose $b_1, b_2, \ldots, b_k$ are distinct integers larger than 1. We believe that our results can be extended to the string of inequalities

$$A_{b_1}(n) < A_{b_2}(n) < \cdots < A_{b_k}(n)$$

for infinitely many $n \in \mathbb{N}$. Maybe, if $\delta_i \in \{0, 1, -1\}$ for every i, $\delta_1 + \delta_2 + \cdots + \delta_k = 0$, and $b_1, b_2, \ldots, b_k$ satisfy some natural conditions such as $\log b_i / \log b_j$ is not rational for any $i \neq j$, then

$$\limsup_{n \to \infty} \left(\sum_{i=1}^{k} \delta_i A_{b_i}(n) \right) = +\infty \text{ and } \liminf_{n \to \infty} \left(\sum_{i=1}^{k} \delta_i A_{b_i}(n) \right) = -\infty.$$

Problem 3. For positive integers $b \geq 2$, $n \geq 1$, $q \geq 2$, and $1 \leq a \leq q$, let $A_b(n, q, a)$ be the number of b-adic palindromes which are less than or equal to n and are congruent to a modulo q. Perhaps, we can find an asymptotic formula for $A_b(n, q, a)$. Then the study of the race between palindromes in different congruence classes (in the same or different bases) may be interesting. For example, under some natural conditions on b and q, are there infinitely many sign changes in $A_b(n, q, a_1) - A_b(n, q, a_2)$ for distinct a_1, a_2? If $k \in \mathbb{N}$ is fixed, are there b, q, a_1, a_2 such that the number of sign changes in $A_b(n, q, a_1) - A_b(n, q, a_2)$ is exactly k?

Problem 4. Can Lemma 7 be extended to any congruence classes? For example, suppose (x_n) is an arithmetic progression, $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, $d = x_2 - x_1 \geq 1$, α is an irrational number, and $0 \leq c_1 < c_2 \leq 1$. Then for any $q \geq 2$ and $0 \leq a < q$, we may be able to prove that there are infinitely many $k \in \mathbb{N}$ such that $c_1 < \{\alpha x_k\} < c_2$ and $\lfloor \alpha x_k \rfloor \equiv a \pmod{q}$. Furthermore, we may be able to replace the assumption that $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$ by a weaker condition.

Problem 5. Considering Remark 3.19 in the article by Kawsumarng et al. [22], we see that there exists a palindromic pattern in the sumset $B(\alpha^2) + B(\alpha^2)$ with respect to the Fibonacci numbers, where $B(\alpha^2)$ is the upper Wythoff sequence. It may be interesting to see whether or not these kinds of palindromic patterns occur in the h-fold sumset $hB(x)$ with respect to the members of linear recurrence sequence (a_n), where x is a particular root of the characteristic polynomial of the sequence (a_n) and $h + 1$ is the smallest positive integer such that $(h + 1)B(x)$ is cofinite.

We plan to solve some of these problems in the future but we do not mind if the readers solve them before us.

Acknowledgments

We are grateful to the editor and reviewers for their comments and suggestions which improve the quality of this article. Prapanpong Pongsriiam's research project is supported jointly by the Faculty of Science Silpakorn University and the National Research Council of Thailand (NRCT), grant number NRCT5-RSA63021-02.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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Appendix

Table 1. The value of $A_b(n)$ when $b = 2, 3, 5, 10$.

n	$A_2(n)$	$A_3(n)$	$A_5(n)$	$A_{10}(n)$
10	5	5	5	9
10^2	19	19	23	18
10^3	61	62	63	108
10^4	204	202	203	198
10^5	644	652	783	1,098
10^6	1,999	2,099	2,223	1,998
10^7	6,535	6,758	6,323	10,998
10^8	20,397	21,801	22,023	19,998
10^9	63,283	70,487	79,623	109,998
10^{10}	207,364	228,398	206,123	199,998
10^{11}	643,612	719,607	646,623	1,099,998
10^{12}	2,002,248	2,221,547	2,465,123	1,999,998
10^{13}	6,578,488	6,873,719	7,073,123	10,999,998
10^{14}	20,309,535	21,318,077	20,005,623	19,999,998
10^{15}	63,356,753	66,277,292	69,308,123	109,999,998
10^{16}	208,723,532	206,575,404	253,628,123	199,999,998
10^{17}	640,964,484	645,537,966	653,740,623	1,099,999,998
10^{18}	2,005,064,397	2,022,653,063	2,039,903,123	1,999,999,998
10^{19}	6,623,273,731	6,354,756,390	7,741,915,623	10,999,999,998
10^{20}	20,231,466,772	20,020,259,837	22,487,515,623	19,999,999,998