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Research article

On some types of functions and a form of compactness via ω_s-open sets

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Abstract: In this paper, ω_s-irresoluteness as a strong form of ω_s-continuity is introduced. It is proved that ω_s-irresoluteness is independent of each of continuity and irresoluteness. Also, ω_s-openness which lies strictly between openness and semi-openness is defined. Sufficient conditions for the equivalence between ω_s-openness and openness, and between ω_s-openness and semi-openness are given. Moreover, pre-ω_s-openness which is a strong form of ω_s-openness and independent of each of openness and pre-semi-openness is introduced. Furthermore, slight ω_s-continuity as a new class of functions which lies between slight continuity and slight semi-continuity is introduced. Several results related to slight ω_s-continuity are introduced, in particular, sufficient conditions for the equivalence between slight ω_s-continuity and slight continuity, and between slight ω_s-continuity and slight semi-continuity are given. In addition to these, ω_s-compactness as a new class of topological spaces that lies strictly between compactness and semi-compactness is introduced. It is proved that locally countable compact topological spaces are ω_s-compact. Also, it is proved that anti-locally countable ω_s-compact topological spaces are semi-compact. Several implications, examples, counter-examples, characterizations, and mapping theorems are introduced related to the above concepts are introduced.

Keywords: ω_s-open sets; semi-continuous function; ω_s-continuous function; irresolute function; semi-open function; pre-semi-open function; semi-compact
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1. Introduction

Let B be a subset of a topological space (Y, σ). The set of condensation points of B is denoted by $Cond(B)$ and defined by $Cond(B) = \{y \in Y$: for every $V \in \sigma$ with $y \in V$, $V \cap B$ is uncountable$\}$. For the purpose of characterizing Lindelöf topological spaces and improving some known mapping theorems, the author in [1] defined ω-closed sets as follows: B is called an ω-closed set in (Y, σ) if $Cond(B) \subseteq B$. B is called an ω-open set in (Y, σ) [1] if $Y - B$ is ω-closed. It is well known that B is

ω-open in (Y, σ) if and only if for every $b \in B$, there are $V \in \sigma$ and a countable subset $F \subseteq Y$ with $b \in V - F \subseteq B$. The family of all ω-open sets in (Y, σ) is denoted by σ_ω. It is well known that σ_ω is a topology on Y that is finer than σ. Many research papers related to ω-open sets have appeared in [2–8] and others. Authors in [9–11] included ω-openness in both soft and fuzzy topological spaces. In this paper, we will denote the closure of B in (Y, σ), the closure of B in (Y, σ_ω), the interior of B in (Y, σ), and the interior of B in (Y, σ_ω) by $\overline{B}$, $\overline{B}^{\omega}$, $Int(B)$, and $Int_\omega(B)$, respectively. B is called a semi-open set in (Y, σ) [12] if there exists $V \in \sigma$ such that $V \subseteq B \subseteq \overline{V}$. Complements of semi-open sets are called semi-closed sets. The family of all semi-open sets in (Y, σ) will be denoted by $SO(Y, \sigma)$. Authors in [12–15] have used semi-open sets to define semi-continuity, semi-openness, irresoluteness, pre-semi-openness, and slight semi-continuity. The area of research related to semi-open sets is still hot [16–28]. Authors in [29] have defined ω_s-open sets as a strong form of semi-open sets as follows: B is called an ω_s-open set in (Y, σ) if there exists $O \in \sigma$ such that $O \subseteq B \subseteq \overline{O}^{\omega}$. Complements of ω_s-open sets are called ω_s-closed sets. The family of all ω_s-open sets in (Y, σ) will be denoted by $\omega_s(Y, \sigma)$. The intersection of all ω_s-closed sets in (Y, σ) which contains B will be denoted by $\overline{B}^{\omega_s}$, and the union of all ω_s-open sets in (Y, σ) which contained in B will be denoted by $Int_{\omega_s}(B)$. Authors in [29] have defined and investigated the class of ω_s-continuity which lies strictly between the classes of continuity and semi-continuity.

In this paper, ω_s-irresoluteness as a strong form of ω_s-continuity is introduced. It is proved that ω_s-irresoluteness is independent of each of continuity and ω_s-irresoluteness. Also, ω_s-openness which lies strictly between openness and semi-openness is introduced and investigated, and pre-ω_s-openness which is a strong form of ω_s-openness and independent of openness is introduced and investigated. Moreover, slight ω_s-continuity as a new class of functions which lies between slight continuity and slightly semi-continuity is introduced and investigated. In addition to these, ω_s-compactness as a new class of topological spaces that lies strictly between compactness and semi-compactness is introduced. Several implications, examples, counter-examples, characterizations, and mapping theorems are introduced. In particular, several sufficient conditions for the equivalence between our new concepts and other related concepts are given. We hope that this will open the door for a number of future related studies such as ω_s-separation axioms and ω_s-connectedness.

Throughout this paper, the usual topology on $\mathbb{R}$ will be denoted by τ_u.

Recall that a topological space (Y, σ) is called locally countable [30] (resp. anti-locally countable [31]) if (Y, σ) has a base consisting of countable sets (all non-empty open sets are uncountable sets).

The following results will be used in the sequel:

Proposition 1.1. [29] Let (Y, σ) be a topological space. Then

(a) $\sigma \subseteq \omega_s(Y, \sigma) \subseteq SO(Y, \sigma)$.

(b) If (Y, σ) is locally countable, then $\sigma = \omega_s(Y, \sigma)$.

(c) If (Y, σ) is anti-locally countable, then $\omega_s(Y, \sigma) = SO(Y, \sigma)$.

Proposition 1.2. [29] Let (Y, σ) be a topological space and let $B, C \subseteq Y$. Then we have the following:

(a) If $B \subseteq C \subseteq \overline{B}^{\omega}$ and $B \in \omega_s(Y, \sigma)$, then $C \in \omega_s(Y, \sigma)$.

(b) If $B \in \sigma$ and $C \in \omega_s(Y, \sigma)$, then $B \cap C \in \omega_s(Y, \sigma)$.

Proposition 1.3. [31] Let (Y, σ) be anti-locally countable. Then for every $B \in \sigma_\omega$, we have $\overline{B}^{\omega} = \overline{B}$.

2. ω_s-Irresolute functions

Definition 2.1. A function $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ is called irresolute [14] (resp. ω_s-continuous [29]), if for every $V \in SO(Z, \gamma)$ (resp. $V \in \gamma$), $g^{-1}(V) \in SO(Y, \sigma)$ (resp. $g^{-1}(V) \in \omega_s(Y, \sigma)$).

Definition 2.2. A function $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ is called ω_s-irresolute, if for every $V \in \omega_s(Z, \gamma)$, $g^{-1}(V) \in \omega_s(Y, \sigma)$.

The following two examples will show that irresoluteness and ω_s-irresoluteness are independent concepts:

Example 2.3. Let $X = \mathbb{R}$, $Y = \{a, b\}$, $\tau = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$, and $\sigma = \{\emptyset, Y, \{a\}, \{b\}\}$. Define $f : (X, \tau) \longrightarrow (Y, \sigma)$ by

$$f(x) = \begin{cases} a & \text{if } x \in \mathbb{Q}^c \\ b & \text{if } x \in \mathbb{Q} \end{cases}.$$

Since $f^{-1}(\{a\}) = \mathbb{Q}^c \in \tau \subseteq SO(X, \tau)$ and $f^{-1}(\{b\}) = \mathbb{Q} \in SO(X, \tau) - \omega_s(X, \tau)$, then f is irresolute but not ω_s-irresolute.

Example 2.4. Consider the topology $\sigma = \{\emptyset, \mathbb{N}, \{1\}, \{2\}, \{1, 2\}\}$ on $\mathbb{N}$. Define $g : (\mathbb{N}, \sigma) \longrightarrow (\mathbb{N}, \sigma)$ by

$$g(t) = \begin{cases} 1 & \text{if } t = 1 \\ 1 & \text{if } t = 2 \\ t & \text{if } t \in \mathbb{N} - \{1, 2\} \end{cases}.$$

Since $(\mathbb{N}, \sigma)$ is locally countable, then by Proposition 1.1 (b) we have $\omega_s(\mathbb{N}, \sigma) = \sigma$. Since $f^{-1}(\{1\}) = \{1, 2\} \in \sigma$ and $f^{-1}(\{2\}) = \emptyset \in \sigma$, then f is ω_s-irresolute. However, f is not irresolute because there is $\overline{\{2\}} = \mathbb{N} - \{1\} \in SO(\mathbb{N}, \sigma)$ such that $f^{-1}(\mathbb{N} - \{1\}) = \mathbb{N} - \{1, 2\} \notin SO(\mathbb{N}, \sigma)$.

Theorem 2.5. Let (Y, σ) and (Z, γ) be two anti-locally countable topological spaces. Then for any function $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ the followings are equivalent:

(a) g is irresolute.

(b) g is ω_s-irresolute.

Proof. Follows from the definitions and Proposition 1.1 (c).

The following two examples will show that continuity and ω_s-irresoluteness are independent concepts:

Example 2.6. Let $X = \mathbb{R}$ and $\tau = \{\emptyset, \mathbb{R}, (3, \infty), \{2\}, \{2\} \cup (3, \infty)\}$. Define $f : (X, \tau) \longrightarrow (X, \tau)$ by

$$f(x) = \begin{cases} 2 & \text{if } x \in \{2\} \cup (3, \infty) \\ x & \text{if } x \in \mathbb{R} - (\{2\} \cup (3, \infty)) \end{cases}.$$

Since $f^{-1}((3, \infty)) = \emptyset \in \tau$ and $f^{-1}(\{2\}) = \{2\} \cup (3, \infty) \in \tau$, then f is continuous. Since $\overline{(3, \infty)}^{\omega} = \mathbb{R} - \{2\}$, then $\mathbb{R} - \{2\} \in \omega_s(X, \tau)$. Since $f^{-1}(\mathbb{R} - \{2\}) = \mathbb{R} - (\{2\} \cup (3, \infty)) \notin \omega_s(X, \tau)$, then f is not ω_s-irresolute.

Example 2.7. Let τ_{disc} be the discrete topology on $\{a, b\}$. Define $f : (\mathbb{R}, \tau_u) \longrightarrow (\{a, b\}, \tau_{disc})$ by

$$f(x) = \begin{cases} a & \text{if } x \in (-\infty, 0) \\ b & \text{if } x \in [0, \infty) \end{cases}.$$

Clearly that $\omega_s(\{a,b\},\tau_{disc}) = \tau_{disc}$. Since $f^{-1}(\{a\}) = (-\infty,0) \in \tau_u \subseteq \omega_s(\mathbb{R},\tau_u)$ and $f^{-1}(\{b\}) = [0,\infty) \in \omega_s(\mathbb{R},\tau_u)$, then f is ω_s-irresolute. However, f is not continuous because there is $\{b\} \in \tau_{disc}$ such that $f^{-1}(\{b\}) = [0,\infty) \notin \tau_u$.

Theorem 2.8. Let (Y,σ) and (Z,γ) be two locally countable topological spaces. Then for any function $g : (Y,\sigma) \longrightarrow (Z,\gamma)$ the followings are equivalent:

(a) g is continuous.

(b) g is ω_s-irresolute.

Proof. Follows from the definitions and Proposition 1.1 (b).

Theorem 2.9. If $g : (Y,\sigma) \longrightarrow (Z,\gamma)$ is a continuous function such that $g : (Y,\sigma_\omega) \longrightarrow (Z,\gamma_\omega)$ is an open function, then g is ω_s-irresolute.

Proof. Let $H \in \omega_s(Z,\gamma)$. Then there exists $W \in \gamma$ such that $W \subseteq H \subseteq \overline{W}^{\omega}$ and so $g^{-1}(W) \subseteq g^{-1}(H) \subseteq g^{-1}\left(\overline{W}^{\omega}\right)$. Since $g : (Y,\sigma) \longrightarrow (Z,\gamma)$ is continuous, then $g^{-1}(W) \in \sigma$. Since $g : (Y,\sigma_\omega) \longrightarrow (Z,\gamma_\omega)$ is open, then $g^{-1}\left(\overline{W}^{\omega}\right) \subseteq \overline{g^{-1}(W)}^{\omega}$. Thus, $g^{-1}(W) \subseteq g^{-1}(H) \subseteq \overline{g^{-1}(W)}^{\omega}$, and hence $g^{-1}(H) \in \omega_s(Y,\sigma)$. Therefore, g is ω_s-irresolute.

Theorem 2.10. Every ω_s-irresolute function is ω_s-continuous.

Proof. Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be ω_s-irresolute. Let $V \in \sigma$. Then by Proposition 1.1 (a), $V \in \omega_s(Y,\sigma)$. Since f is ω_s-irresolute, then $f^{-1}(V) \in \omega_s(X,\tau)$. Therefore, f is ω_s-irresolute.

The function in Example 2.6 is continuous and hence ω_s-continuous. Therefore, the converse of Theorem 2.10 is not true, in general.

Theorem 2.11. A function $g : (Y,\sigma) \longrightarrow (Z,\gamma)$ is ω_s-irresolute if and only if for every ω_s-closed subset B of (Z,γ), $g^{-1}(B)$ is ω_s-closed in (Y,σ).

Proof. Necessity. Assume that g is ω_s-irresolute. Let B be an ω_s-closed set in (Z,γ). Then $Z - B \in \omega_s(Z,\gamma)$. Since g is ω_s-irresolute, then $g^{-1}(Z - B) = Y - g^{-1}(B) \in \omega_s(Y,\sigma)$. Hence, $g^{-1}(B)$ is ω_s-closed in (Y,σ).

Sufficiency. Suppose that for every ω_s-closed subset B of (Z,γ), $g^{-1}(B)$ is ω_s-closed in (Y,σ). Let $W \in \omega_s(Z,\gamma)$. Then $Z - W$ is ω_s-closed in (Z,γ). By assumption, $g^{-1}(Z - W) = Y - g^{-1}(W)$ is ω_s-closed in (Y,σ), and so $g^{-1}(W) \in \omega_s(Y,\sigma)$. Therefore, g is ω_s-irresolute.

Theorem 2.12. A function $g : (Y,\sigma) \longrightarrow (Z,\gamma)$ is ω_s-irresolute if and only if for every $H \subseteq Y$, $g\left(\overline{H}^{\omega_s}\right) \subseteq \overline{g(H)}^{\omega_s}$.

Proof. Necessity. Suppose that g is ω_s-irresolute and let $H \subseteq Y$. Then $\overline{g(H)}^{\omega_s}$ is ω_s-closed in (Z,γ), and by Theorem 2.11, $g^{-1}\left(\overline{g(H)}^{\omega_s}\right)$ is ω_s-closed in (Y,σ). Since $H \subseteq g^{-1}\left(\overline{g(H)}^{\omega_s}\right)$, then $\overline{H}^{\omega_s} \subseteq g^{-1}\left(\overline{g(A)}^{\omega_s}\right)$. Thus,

$$
\begin{aligned}
g\left(\overline{H}^{\omega_s}\right) &\subseteq g\left(g^{-1}\left(\overline{g(H)}^{\omega_s}\right)\right) \\
&\subseteq \overline{g(H)}^{\omega_s}.
\end{aligned}
$$

Sufficiency. Suppose that for every subset $H \subseteq Y$, $g\left(\overline{H}^{\omega_s}\right) \subseteq \overline{g(H)}^{\omega_s}$. We will apply Theorem 2.11 to show that g is ω_s-irresolute. Let B be an ω_s-closed subset of (Z,γ). Then by assumption we have $g\left(\overline{g^{-1}(B)}^{\omega_s}\right) \subseteq \overline{g(g^{-1}(B))}^{\omega_s} \subseteq \overline{B}^{\omega_s} = B$, and so

$$\overline{g^{-1}(B)}^{\omega_s} \subseteq g^{-1}\left(g\left(\overline{g^{-1}(B)}^{\omega_s}\right)\right) \subseteq g^{-1}(B).$$

Therefore, $\overline{g^{-1}(B)}^{\omega_s} = g^{-1}(B)$, and hence $g^{-1}(B)$ is ω_s-closed in (Y,σ). This shows that g is ω_s-irresolute.

Theorem 2.13. A function $g : (Y,\sigma) \longrightarrow (Z,\gamma)$ is ω_s-irresolute if and only if for every $H \subseteq Z$, $\overline{g^{-1}(H)}^{\omega_s} \subseteq g^{-1}\left(\overline{H}^{\omega_s}\right)$.

Proof. Necessity. Suppose that g is ω_s-irresolute and let $H \subseteq Z$. Then by Theorem 2.11, $g^{-1}\left(\overline{H}^{\omega_s}\right)$ is ω_s-closed in (Y,σ). Since $g^{-1}(H) \subseteq g^{-1}\left(\overline{H}^{\omega_s}\right)$, then $\overline{g^{-1}(H)}^{\omega_s} \subseteq g^{-1}\left(\overline{H}^{\omega_s}\right)$.

Sufficiency. Suppose that for every $H \subseteq Z$, $\overline{g^{-1}(H)}^{\omega_s} \subseteq g^{-1}\left(\overline{H}^{\omega_s}\right)$. We will apply Theorem 2.11 to show that g is ω_s-irresolute. Let B be an ω_s-closed subset of (Z,γ). Then $\overline{B}^{\omega_s} = B$. So by assumption, $\overline{g^{-1}(B)}^{\omega_s} \subseteq g^{-1}(B)$, and so $\overline{g^{-1}(B)}^{\omega_s} = g^{-1}(B)$. Therefore, $g^{-1}(B)$ is ω_s-closed in (Y,σ).

Theorem 2.14. The composition of two ω_s-irresolute functions is ω_s-irresolute.

Proof. Let $g : (Y,\sigma) \longrightarrow (Z,\lambda)$ and $h : (Z,\lambda) \longrightarrow (M,\gamma)$ be ω_s-irresolute functions. Let $C \in \omega_s(M,\gamma)$. Since h is ω_s-irresolute, then $h^{-1}(C) \in \omega_s(Z,\lambda)$. Since g is ω_s-irresolute, then $(h \circ g)^{-1}(C) = g^{-1}\left(h^{-1}(C)\right) \in \omega_s(Y,\sigma)$. Therefore, $h \circ g$ is ω_s-irresolute.

3. ω_s-open and pre-ω_s-open functions

Definition 3.1. A function $g : (Y,\sigma) \longrightarrow (Z,\lambda)$ is called

(a) ω_s-open (resp. semi-open [13]) if for each $U \in \sigma$, $f(U) \in \omega_s(Z,\lambda)$ (resp. $f(U) \in SO(Z,\lambda)$).

(b) pre-semi-open [14] if for each $U \in SO(Y,\sigma)$, $f(U) \in SO(Z,\lambda)$.

Theorem 3.2. Let $g : (Y,\sigma) \longrightarrow (Z,\lambda)$ be a function. If for a base $\mathcal{B}$ of (Y,σ), $g(B) \in \lambda$ for all $B \in \mathcal{B}$, then g is ω_s-open.

Proof. Suppose that for a base $\mathcal{B}$ of (Y,σ), $g(B) \in \omega_s(Z,\lambda)$ for all $B \in \mathcal{B}$. Let $V \in \sigma - \{\emptyset\}$. Choose $\mathcal{B}_1 \subseteq \mathcal{B}$ such that $V = \cup\{B : B \in \mathcal{B}_1\}$. Then

$$\begin{aligned} g(V) &= g(\cup\{B : B \in \mathcal{B}_1\}) \\ &= \cup\{g(B) : B \in \mathcal{B}_1\}. \end{aligned}$$

Since by assumption, $g(B) \in \omega_s(Z,\lambda)$ for all $B \in \mathcal{B}_1$, then $g(V) \in \omega_s(Z,\lambda)$.

Theorem 3.3. Every open function is ω_s-open.

Proof. Let $g : (Y,\sigma) \longrightarrow (Z,\gamma)$ be an open function and let $V \in \sigma$. Since g is open, then $g(V) \in \gamma \subseteq \omega_s(Z,\gamma)$.

The converse of Theorem 3.3 is not true as shown in the next example:

Example 3.4. Consider the function $g : (\mathbb{R},\tau_u) \longrightarrow (\mathbb{R},\tau_u)$ defined by $g(y) = y^2$. We apply Theorem 3.2 to show that g is ω_s-open. Consider the base $\{(c,d) : c,d \in \mathbb{R} \text{ and } c < d\}$ for $(\mathbb{R},\tau_u)$. Then for all $c,d \in \mathbb{R}$ with $c < d$ we have

$$g((c,d)) = \begin{cases} \left(d^2, c^2\right) & \text{if } c < d \leq 0 \\ \left(c^2, d^2\right) & \text{if } 0 \leq c < d \\ \left[0, \max\left\{c^2, d^2\right\}\right) & \text{if } c < 0 < d \end{cases},$$

and so $g((c,d)) \in \omega_s(\mathbb{R}, \tau_u)$. Therefore, g is ω_s-open. On the other hand, since $\mathbb{R} \in \tau_u$ but $g(\mathbb{R}) = [0, \infty) \notin \tau_u$, then g is not open.

Theorem 3.5. If $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ is an ω_s-open function such that (Z, γ) is locally countable, then g is open.

Proof. Follows from the definitions and Proposition 1.1 (b).

Theorem 3.6. Every ω_s-open function is semi-open.

Proof. Let $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ be an ω_s-open function and let $V \in \sigma$. Since g is ω_s-open, then $g(V) \in \omega_s(Z, \gamma) \subseteq SO(Z, \gamma)$.

The converse of Theorem 3.6 is not true as shown in the next example:

Example 3.7. Consider $(\mathbb{R}, \sigma)$ where $\sigma = \{\emptyset, \mathbb{N}, \mathbb{R}\}$. Define $g : (\mathbb{R}, \sigma) \longrightarrow (\mathbb{R}, \sigma)$ by $g(y) = y - 1$. Since $g(\mathbb{N}) = \{0\} \cup \mathbb{N} \subseteq \overline{\mathbb{N}} = \mathbb{R}$, then, $g(\mathbb{N}) \in SO(\mathbb{R}, \sigma)$. Also, $g(\emptyset) = \emptyset \in SO(\mathbb{R}, \sigma)$ and $g(\mathbb{R}) = \mathbb{R} \in SO(\mathbb{R}, \sigma)$. Therefore, g is semi-open. Conversely, g is not ω_s-open since there is $\mathbb{N} \in \sigma$ such that $g(\mathbb{N}) = \{0\} \cup \mathbb{N} \notin \omega_s(\mathbb{R}, \sigma)$.

Theorem 3.8. If $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ is a semi-open function such that (Z, γ) is anti-locally countable, then g is ω_s-open.

Proof. Follows from the definitions and Proposition 1.1 (c).

Definition 3.9. A function $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ is called pre-ω_s-open, if for every $A \in \omega_s(Y, \sigma)$, $g(A) \in \omega_s(Z, \gamma)$.

The following two examples will show that openness and pre-ω_s-openness are independent concepts:

Example 3.10. Let $X = \mathbb{R}$, $Y = \{a, b, c\}$, $\tau = \{\emptyset, (-\infty, 1), X\}$, and $\sigma = \{\emptyset, \{a\}, Y\}$. Define $f : (X, \tau) \longrightarrow (Y, \sigma)$ by

$$f(x) = \begin{cases} a & \text{if } x \in (-\infty, 1) \\ b & \text{if } x = 1 \\ c & \text{if } x \in (1, \infty) \end{cases}.$$

Since $f(\emptyset) = \emptyset \in \sigma$, $f((-\infty, 1)) = \{a\} \in \sigma$, and $f(X) = Y \in \sigma$, then f is open. Since (X, τ) is anti-locally countable, then by Proposition 1.3, $\overline{(-\infty, 1)}^{\omega} = \overline{(-\infty, 1)} = \mathbb{R}$. So, we have $(-\infty, 1] \in \omega_s(X, \tau)$ but $f((-\infty, 1]) = \{a, b\} \notin \omega_s(Y, \sigma) = \sigma$. This shows that f is not pre-ω_s-open.

Example 3.11. Consider $(\mathbb{R}, \tau)$ where $\tau = \{\emptyset, \mathbb{R}, [1, \infty)\}$. Define $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \tau)$ by $f(x) = x - 1$. Since $(\mathbb{R}, \tau)$ is anti-locally countable, then by Proposition 1.1 (c), $\omega_s(\mathbb{R}, \tau) = SO(\mathbb{R}, \tau) = \{\emptyset\} \cup \{H : [1, \infty) \subseteq H\}$. To see that f is pre-ω_s-open, let $H \in \omega_s(\mathbb{R}, \tau) - \{\emptyset\}$. Then $[0, \infty) = f([1, \infty)) \subseteq f(H)$, and thus $[1, \infty) \subseteq f(H)$. Hence, $f(H) \in \omega_s(\mathbb{R}, \tau)$. Therefore, f is pre-ω_s-open. Conversely, f is not open since there is $[1, \infty) \in \tau$ such that $f([1, \infty)) = [0, \infty) \notin \tau$.

Theorem 3.12. Let (Y, σ) and (Z, γ) be locally countable, and let $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ be a function. Then the followings are equivalent:

(a) g is open.

(b) g is pre-ω_s-open.

Proof. Follows from the definitions and Proposition 1.1 (b).

The following two examples will show that pre-semi-openness and pre-ω_s-openness are independent concepts:

Example 3.13. Consider the function g as in Example 3.7. It is not difficult to see that $SO(\mathbb{R},\sigma) = \{\emptyset\} \cup \{H : \mathbb{N} \subseteq H\}$ and $\omega_s(\mathbb{R},\sigma) = \sigma$. To see that g is pre-semi-open, let $H \in SO(\mathbb{R},\sigma) - \{\emptyset\}$. Then $\mathbb{N} \subseteq H$ and so $\mathbb{N} \subseteq \mathbb{N} \cup \{0\} = g(\mathbb{N})$. Thus, $g(\mathbb{N}) \in SO(\mathbb{R},\sigma)$. This shows that g is pre-semi-open. Conversely, g is not pre-ω_s-open since there is $\mathbb{N} \in \omega_s(\mathbb{R},\sigma)$ such that $g(\mathbb{N}) = \{0\} \cup \mathbb{N} \notin \omega_s(\mathbb{R},\sigma)$.

Example 3.14. Let $X = Y = \{1,2,3,4\}$, $\tau = \{\emptyset, X, \{1,2\}, \{1\}, \{2\}\}$, and $\sigma = \{\emptyset, Y, \{2,3,4\}, \{1,2\}, \{1\}, \{2\}\}$. Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be the identity function. It is clear that f is open. Since (X,τ) and (Y,σ) are locally countable, then by Theorem 3.12, f is pre-ω_s-open. Conversely, f is not pre-semi-open because there is $\{1,3\} \in SO(X,\tau)$ such that $f(\{1,3\}) = \{1,3\} \notin SO(Y,\sigma)$.

Theorem 3.15. Let (X,τ) and (Y,σ) be anti-locally countable topological spaces and let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a function. Then f is pre-semi-open if and only if f is pre-ω_s-open.

Proof. Follows from definitions and Proposition 1.1 (c).

Theorem 3.16. Every pre-ω_s-open function is ω_s-open. *Proof.* Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a pre-ω_s-open function and let $U \in \tau \subseteq \omega_s(X,\tau)$. Since f is pre-ω_s-open, then $f(U) \in \omega_s(Y,\sigma)$.

The function f in Example 3.10 is open but not pre-ω_s-open, and by Theorem 3.3, f is ω_s-open. Therefore, the converse of the implication in Theorem 3.16 is not true, in general.

Theorem 3.17. For a function $g : (Y,\sigma) \longrightarrow (Z,\lambda)$, the followings are equivalent:

(a) g is ω_s-open.

(b) $g^{-1}\left(\overline{B}^{\omega_s}\right) \subseteq \overline{g^{-1}(B)}$ for every $B \subseteq Z$.

(c) $Int\left(g^{-1}(B)\right) \subseteq g^{-1}(Int_{\omega_s}(B))$ for every $B \subseteq Z$.

Proof. (a) $\Longrightarrow$ (b): Suppose that g is ω_s-open and let $B \subseteq Z$. Let $y \in g^{-1}\left(\overline{B}^{\omega_s}\right)$. To show that $y \in \overline{g^{-1}(B)}$, let $V \in \sigma$ such that $y \in V$. Then $g(y) \in g(V)$. Since g is ω_s-open, then $g(V) \in \omega_s(Z,\lambda)$. Since $g(y) \in g(V) \cap \overline{B}^{\omega_s}$, then $g(V) \cap B \neq \emptyset$. Choose $t \in V$ such that $g(t) \in B$. Then $t \in V \cap g^{-1}(B)$, and hence $V \cap g^{-1}(B) \neq \emptyset$. It follows that $y \in \overline{g^{-1}(B)}$.

(b) $\Longrightarrow$ (a): Suppose that $g^{-1}\left(\overline{B}^{\omega_s}\right) \subseteq \overline{g^{-1}(B)}$ for every $B \subseteq Z$, and suppose to the contrary that g is not ω_s-open. Then there exists $V \in \sigma$ such that $g(V) \notin \omega_s(Z,\lambda)$ and so, $Z - g(V)$ is not ω_s-closed. Thus, there exists $z \in g(V) \cap \overline{Z - g(V)}^{\omega_s}$. Choose $y \in V$ such that $z = g(y)$. Then $y \in g^{-1}\left(\overline{Z - g(V)}^{\omega_s}\right)$. By assumption, we have

$$\begin{aligned} g^{-1}\left(\overline{Z - g(V)}^{\omega_s}\right) &\subseteq \overline{g^{-1}(Z - g(V))} \\ &= \overline{Y - g^{-1}(g(V))} \\ &\subseteq \overline{Y - V} \\ &= Y - V. \end{aligned}$$

Therefore, $y \in Y - V$. But $y \in V$, a contradiction.

(b) $\Longrightarrow$ (c): Suppose that $g^{-1}\left(\overline{B}^{\omega_s}\right) \subseteq \overline{g^{-1}(B)}$ for every $B \subseteq Z$. Let $B \subseteq Z$. Then by (b),

$$g^{-1}(Int_{\omega_s}(B)) = g^{-1}\left(Z - \overline{Z - B}^{\omega_s}\right)$$

$$
\begin{aligned}
&= Y - g^{-1}\left(\overline{Z-B}^{\omega_s}\right) \\
&\supseteq Y - \overline{g^{-1}(Z-B)} \\
&= Y - \overline{Y - g^{-1}(B)} \\
&= Int\left(g^{-1}(B)\right).
\end{aligned}
$$

(c) $\Longrightarrow$ (b): Suppose that $Int\left(g^{-1}(B)\right) \subseteq g^{-1}(Int_{\omega_s}(B))$ for every $B \subseteq Z$. Let $B \subseteq Z$. Then by (c),

$$
\begin{aligned}
g^{-1}\left(\overline{B}^{\omega_s}\right) &= g^{-1}(Z - Int_{\omega_s}(Z-B)) \\
&= Y - g^{-1}(Int_{\omega_s}(Z-B)) \\
&\subseteq Y - Int\left(g^{-1}(Z-B)\right) \\
&= Y - Int\left(Y - g^{-1}(B)\right) \\
&= \overline{g^{-1}(B)}.
\end{aligned}
$$

Theorem 3.18. For a function $g : (Y, \sigma) \longrightarrow (Z, \lambda)$, the followings are equivalent:

(a) g is pre-ω_s-open.

(b) $g^{-1}\left(\overline{B}^{\omega_s}\right) \subseteq \overline{g^{-1}(B)}^{\omega_s}$ for every $B \subseteq Z$.

(c) $Int_{\omega_s}\left(g^{-1}(B)\right) \subseteq g^{-1}(Int_{\omega_s}(B))$ for every $B \subseteq Z$.

Proof. (a) $\Longrightarrow$ (b): Suppose that g is pre-ω_s-open and let $B \subseteq Z$. Let $y \in g^{-1}\left(\overline{B}^{\omega_s}\right)$. To show that $y \in \overline{g^{-1}(B)}^{\omega_s}$, let $C \in \omega_s(Y, \sigma)$ such that $y \in C$. Then $g(y) \in g(C)$. Since g is pre-ω_s-open, then $g(C) \in \omega_s(Z, \lambda)$. Since $g(y) \in g(C) \cap \overline{B}^{\omega_s}$, then $g(C) \cap B \neq \emptyset$. Choose $t \in C$ such that $g(t) \in B$. Then $t \in C \cap g^{-1}(B)$ and hence $C \cap g^{-1}(B) \neq \emptyset$. It follows that $y \in \overline{g^{-1}(B)}^{\omega_s}$.

(b) $\Longrightarrow$ (a): Suppose that $g^{-1}\left(\overline{B}^{\omega_s}\right) \subseteq \overline{g^{-1}(B)}^{\omega_s}$ for every $B \subseteq Z$, and suppose to the contrary that g is not pre-ω_s-open. Then there is $C \in \omega_s(Y, \sigma)$ such that $g(C) \notin \omega_s(Z, \lambda)$ and so $Z - g(C)$ is not ω_s-closed. And so there exists $z \in g(C) \cap \overline{Z - g(C)}^{\omega_s}$. Choose $y \in C$ such that $z = g(y)$. Then $y \in g^{-1}\left(\overline{Z - g(C)}^{\omega_s}\right)$. By assumption we have

$$
\begin{aligned}
g^{-1}\left(\overline{Z - g(C)}^{\omega_s}\right) &\subseteq \overline{g^{-1}(Z - g(C))}^{\omega_s} \\
&= \overline{Y - g^{-1}(g(C))}^{\omega_s} \\
&\subseteq \overline{Y - C}^{\omega_s} \\
&= Y - C.
\end{aligned}
$$

Therefore, $y \in Y - C$ but $y \in C$, a contradiction.

(b) $\Longrightarrow$ (c): Suppose that $g^{-1}\left(\overline{B}^{\omega_s}\right) \subseteq \overline{g^{-1}(B)}^{\omega_s}$ for every $B \subseteq Z$. Let $B \subseteq Z$. Then by (b),

$$
\begin{aligned}
g^{-1}(Int_{\omega_s}(B)) &= g^{-1}\left(Z - \overline{Z-B}^{\omega_s}\right) \\
&= Y - g^{-1}\left(\overline{Z-B}^{\omega_s}\right) \\
&\supseteq Y - \overline{g^{-1}(Z-B)}^{\omega_s} \\
&= Y - \overline{Y - g^{-1}(B)}^{\omega_s}
\end{aligned}
$$

$$= \quad Int_{\omega_s}\left(g^{-1}(B)\right).$$

(c) $\Longrightarrow$ (b): Suppose that $Int_{\omega_s}\left(g^{-1}(B)\right) \subseteq g^{-1}\left(Int_{\omega_s}(B)\right)$ for every $B \subseteq Z$. Let $B \subseteq Z$. Then by (c),

$$\begin{aligned} g^{-1}\left(\overline{B}^{\omega_s}\right) &= g^{-1}\left(Z - Int_{\omega_s}(Z - B)\right) \\ &= Y - g^{-1}\left(Int_{\omega_s}(Z - B)\right) \\ &\subseteq Y - Int_{\omega_s}\left(g^{-1}(Z - B)\right) \\ &= Y - Int_{\omega_s}\left(Y - g^{-1}(B)\right) \\ &= \overline{g^{-1}(B)}^{\omega_s}. \end{aligned}$$

Theorem 3.19. If $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω_s-continuous such that $g : (Y, \sigma_\omega) \longrightarrow (Z, \lambda_\omega)$ is open, then $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω_s-irresolute.

Proof. Suppose that $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω_s-continuous with $g : (Y, \sigma_\omega) \longrightarrow (Z, \lambda_\omega)$ is open. Let $G \in \omega_s(Z, \lambda)$, choose $W \in \lambda$ such that $W \subseteq G \subseteq \overline{W}^{\omega}$. Thus, we have $g^{-1}(W) \subseteq g^{-1}(G) \subseteq g^{-1}\left(\overline{W}^{\omega}\right)$. Since $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω_s-continuous, then $g^{-1}(W) \in \omega_s(Y, \sigma)$. Since $g : (Y, \sigma_\omega) \longrightarrow (Z, \lambda_\omega)$ is open, then $g^{-1}\left(\overline{W}^{\omega}\right) \subseteq \overline{g^{-1}(W)}^{\omega}$. Since we have $g^{-1}(W) \subseteq g^{-1}(G) \subseteq \overline{g^{-1}(W)}^{\omega}$ with $g^{-1}(W) \in \omega_s(Y, \sigma)$, then by Proposition 1.2 (a), $g^{-1}(G) \in \omega_s(Y, \sigma)$. This ends the proof.

Theorem 3.20. If $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is an ω_s-open function such that $g : (Y, \sigma_\omega) \longrightarrow (Z, \lambda_\omega)$ is a continuous function, then $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is pre-ω_s-open.

Proof. Suppose that $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω_s-open with $g : (Y, \sigma_\omega) \longrightarrow (Z, \lambda_\omega)$ is continuous. Let $H \in \omega_s(Y, \sigma)$, then we find $V \in \sigma$ such that $V \subseteq H \subseteq \overline{V}^{\omega}$. Thus, we have $g(V) \subseteq g(H) \subseteq g\left(\overline{V}^{\omega}\right)$. Since $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω_s-open, then $g(V) \in \omega_s(Z, \lambda)$. By continuity of $g : (Y, \sigma_\omega) \longrightarrow (Z, \lambda_\omega)$, $g\left(\overline{V}^{\omega}\right) \subseteq \overline{g(V)}^{\omega}$. Since we have $g(V) \subseteq g(H) \subseteq \overline{g(V)}^{\omega}$ with $g(V) \in \omega_s(Z, \lambda)$, then by Proposition 1.2 (a), $g(V) \in \omega_s(Z, \lambda)$. This ends the proof.

As defined in [32], a function $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω-continuous if for each $W \in \lambda$, $g^{-1}(W) \in \sigma_\omega$.

Theorem 3.21. Let $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ be pre-ω_s-open and ω_s-irresolute such that (Z, λ) is semi-regular and dense in itself, then g is ω-continuous.

Proof. Suppose to the contrary that g is not ω-continuous. Then there is $W \in \lambda$ such that $g^{-1}(W) \notin \sigma_\omega$. So, there exists $y \in g^{-1}(W) - Int_\omega\left(g^{-1}(W)\right)$. Since $g(y) \in W$ and (Z, λ) is semi-regular, then there is a regular open set M of (Z, λ) such that $g(y) \in M \subseteq W$. Since $Int\left(\overline{M}^{\omega}\right) \subseteq Int\left(\overline{M}\right) = M$, then by Theorem 2.16 of [29], M is ω_s-closed. Since g is ω_s-irresolute, then by Theorem 2.11, $g^{-1}(M)$ is ω_s-closed, and so $Y - g^{-1}(M) \in \omega_s(Y, \sigma)$. Since $g^{-1}(M) \subseteq g^{-1}(W)$, then $Int_\omega\left(g^{-1}(M)\right) \subseteq Int_\omega\left(g^{-1}(W)\right)$. Since $y \notin Int_\omega\left(g^{-1}(W)\right)$, then $y \notin Int_\omega\left(g^{-1}(M)\right)$, and so

$$y \in Y - Int_\omega\left(g^{-1}(M)\right) = \overline{Y - g^{-1}(M)}^{\omega}.$$

Thus, by Proposition 1.2 (a), we have $\left(Y - g^{-1}(M)\right) \cup \{y\} \in \omega_s(Y, \sigma)$ with $y \notin Y - g^{-1}(M)$. Put $S = g\left(\left(Y - g^{-1}(M)\right) \cup \{y\}\right) = g\left(Y - g^{-1}(M)\right) \cup \{g(y)\}$. Since g is pre-ω_s-open, then $S \in \omega_s(Z, \lambda)$. So by Proposition 1.2 (b), we have $S \cap M \in \omega_s(Z, \lambda)$. Since $g\left(Y - g^{-1}(M)\right) \subseteq Z - M$, then we have

$$g(y) \quad \in \quad S \cap M \subseteq ((Z - M) \cup \{g(y)\}) \cap M$$

$$= \{g(y)\},$$

and thus $S \cap M = \{g(y)\}$. Therefore, $\{g(y)\} \in \omega_s(Z, \lambda)$. Thus, there exists $O \in \lambda$ such that $O \subseteq \{g(y)\} \subseteq \overline{O}^{\omega}$, and hence $O = \{g(y)\}$. This implies that $\{g(y)\} \in \lambda$. But by assumption (Z, λ) is dense in itself, a contradiction.

The condition '(Z, λ) is dense in itself' in Theorem 3.21 cannot be dropped as our next example shows:

Example 3.22. Take f as in Example 2.7. Then f is ω_s-irresolute and pre-ω_s-open. Also, $(\{a, b\}, \tau_{disc})$ is semi-regular. On the other hand, since $\{b\} \in \tau_{disc}$ but $f^{-1}(\{b\}) = [0, \infty) \notin (\tau_u)_{\omega}$, then f is not ω-continuous.

Theorem 3.23. Let $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ be injective, pre-ω_s-open, and ω_s-irresolute such that (Z, λ) is semi-regular, then g is ω-continuous.

Proof. Suppose to the contrary that g is not ω-continuous. Then there is $W \in \lambda$ such that $g^{-1}(W) \notin \sigma_{\omega}$. So, there exists $y \in g^{-1}(W) - Int_{\omega}\left(g^{-1}(W)\right)$. Since $g(y) \in W$ and (Z, λ) is semi-regular, then there is a regular open set M of (Z, λ) such that $g(y) \in M \subseteq W$. Since $Int\left(\overline{M}^{\omega}\right) \subseteq Int\left(\overline{M}\right) = M$, then by Theorem 2.16 of [29], M is ω_s-closed. Since g is ω_s-irresolute, then by Theorem 2.11, $g^{-1}(M)$ is ω_s-closed, and so $Y - g^{-1}(M) \in \omega_s(Y, \sigma)$. Since $g^{-1}(M) \subseteq g^{-1}(W)$, then $Int_{\omega}\left(g^{-1}(M)\right) \subseteq Int_{\omega}\left(g^{-1}(W)\right)$. Since $y \notin Int_{\omega}\left(g^{-1}(W)\right)$, then $y \notin Int_{\omega}\left(g^{-1}(M)\right)$, and so

$$y \in Y - Int_{\omega}\left(g^{-1}(M)\right) = \overline{Y - g^{-1}(M)}^{\omega}.$$

Thus, by Proposition 1.2 (a), we have $\left(Y - g^{-1}(M)\right) \cup \{y\} \in \omega_s(Y, \sigma)$ with $y \notin Y - g^{-1}(M)$. Put $S = g\left(\left(Y - g^{-1}(M)\right) \cup \{y\}\right) = g\left(Y - g^{-1}(M)\right) \cup \{g(y)\}$. Since g is pre-ω_s-open, then $S \in \omega_s(Z, \lambda)$, and by Proposition 1.2 (b) we have $S \cap M \in \omega_s(Z, \lambda)$. Since $g\left(Y - g^{-1}(M)\right) \subseteq Z - M$, then we have

$$\begin{aligned} g(y) &\in S \cap M \subseteq ((Z - M) \cup \{g(y)\}) \cap M \\ &= \{g(y)\}, \end{aligned}$$

and thus $S \cap M = \{g(y)\}$. Therefore, $\{g(y)\} \in \omega_s(Z, \lambda)$. Since g is ω_s-irresolute and injective, then $g^{-1}(\{g(y)\}) = \{y\} \in \omega_s(Y, \sigma)$. So, there exists $V \in \sigma$ such that $V \subseteq \{y\} \subseteq \overline{V}^{\omega}$, and hence $\{y\} \in \sigma$. Since $y \in g^{-1}(W)$ and $\{y\} \in \sigma \subseteq \sigma_{\omega}$, then $y \in Int_{\omega}\left(g^{-1}(W)\right)$, a contradiction.

Example 3.23 shows that the condition 'injective' in Theorem 3.23 cannot be dropped.

Definition 3.24. A function $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is called ω_s-closed if for each ω_s-closed set C of (Y, σ), $g(C)$ is ω_s-closed set in (Z, λ).

As defined in [33] a topological space (X, τ) is called ω-regular if for each closed set F in (X, τ) and $x \in X - F$, there exist $U \in \tau$ and $V \in \tau_{\omega}$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. As defined in [31], a function $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω-open if for each $V \in \sigma$, $g(V) \in \lambda_{\omega}$.

Theorem 3.25. If $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is ω_s-closed, pre-ω_s-open, and ω_s-irresolute such that (Y, σ) is ω-regular, then g is ω-open.

Proof. Suppose to the contrary that there exists $V \in \sigma$ such that $g(V) \notin \lambda_{\omega}$. Then we find $y \in V$ such that $g(y) \in g(V) - Int_{\omega}(g(V))$. By ω-regularity of (Y, σ), we find $M \in \sigma$ such that $y \in M \subseteq \overline{M}^{\omega} \subseteq V$. Since $\overline{M}^{\omega}$ is ω_s-closed and g is ω_s-closed, then $Z - g\left(\overline{M}^{\omega}\right) \in \omega_s(Z, \lambda)$ with $g(y) \notin Z - g\left(\overline{M}^{\omega}\right)$. Since $g(y) \notin Int_{\omega}(g(V))$, then $g(y) \notin Int_{\omega}\left(g\left(\overline{M}^{\omega}\right)\right)$. Thus,

$$g(y) \in Z - Int_\omega\left(g\left(\overline{M}^\omega\right)\right) = \overline{Z - g\left(\overline{M}^\omega\right)}^\omega.$$

So by Proposition 1.2 (a), $\left(Z - g\left(\overline{M}^\omega\right)\right) \cup \{g(y)\} \in \omega_s(Z, \lambda)$. Set $B = g^{-1}\left(\left(Z - g\left(\overline{M}^\omega\right)\right) \cup \{g(y)\}\right)$. Since g is ω_s-irresolute, then $B \in \omega_s(Y, \sigma)$ and by Proposition 1.2 (b), $M \cap B \in \omega_s(Y, \sigma)$. Since g is pre-ω_s-open, then $g(M \cap B) \in \omega_s(Z, \lambda)$. Since

$$\begin{aligned} g(y) &\in g(M \cap B) \\ &\subseteq g(M) \cap g(B) \\ &\subseteq g(M) \cap \left(\left(Z - g\left(\overline{M}^\omega\right)\right) \cup \{g(y)\}\right) \\ &= \{g(y)\}, \end{aligned}$$

then $\{g(y)\} \in \omega_s(Z, \lambda)$. Thus, there is $K \in \lambda$ such that $K \subseteq \{g(y)\} \subseteq \overline{K}^\omega$. Hence, $\{g(y)\} \in \lambda$. Since $g(y) \in g(V)$, then $g(y) \in Int(g(V)) \subseteq Int_\omega(g(V))$, a contradiction.

4. Slightly ω_s-continuous functions

Let (Y, σ) be a topological space and let B be a subset of Y. Then B is called clopen (resp. semi-clopen, ω_s-clopen) in (Y, σ) if B both open and closed (resp. semi-open and semi-closed, ω_s-open and ω_s-closed) in (Y, σ). Throughout this section, the family of all clopen (resp., semi-clopen, ω_s-clopen) subsets of the topological space (Y, σ) will be denoted by $CO(Y, \sigma)$ (resp. $SCO(Y, \sigma)$, $\omega_s CO(Y, \sigma)$).

Definition 4.1. A function $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is called slightly continuous [34] (resp. slightly semi-continuous [15], slightly ω_s-continuous), if for every $y \in Y$ and every $W \in CO(Z, \lambda)$ with $g(y) \in W$, there exists $V \in \sigma$ (resp. $V \in SO(Y, \sigma)$, $V \in \omega_s(Y, \sigma)$) such that $y \in V$ and $g(V) \subseteq W$.

As an example of a slightly continuous function that is not continuous, take the function $g : (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau_u)$ defined by $g(y) = [y]$, where $[y]$ is the greatest integer of y.

Theorem 4.2. For a function $g : (Y, \sigma) \longrightarrow (Z, \lambda)$, the followings are equivalent:

(a) g is slightly ω_s-continuous.

(b) For all $W \in CO(Z, \lambda)$, $g^{-1}(W) \in \omega_s(Y, \sigma)$.

(c) For all $W \in CO(Z, \lambda)$, $g^{-1}(W) \in \omega_s CO(Y, \sigma)$.

Proof. (a) $\Longrightarrow$ (b): Let $W \in CO(Z, \lambda)$. Then for each $y \in g^{-1}(W)$, $g(y) \in W$ and by (a), there exists $V_y \in \omega_s(Y, \sigma)$ such that $y \in V_y$ and $g(V_y) \subseteq W$. Thus,

$$g^{-1}(W) = \cup\{V_y : y \in g^{-1}(W)\}.$$

Therefore, $g^{-1}(W)$ is a union of ω_s-open sets, and hence $g^{-1}(W)$ is ω_s-open.

(b) $\Longrightarrow$ (c): Let $W \in CO(Z, \lambda)$. Then $Z - W \in CO(Z, \lambda)$. Thus, by (b), $g^{-1}(W) \in \omega_s(Y, \sigma)$ and $g^{-1}(Z - W) = Y - g^{-1}(W) \in \omega_s(Y, \sigma)$. Therefore, $g^{-1}(W) \in \omega_s CO(Y, \sigma)$.

(c) $\Longrightarrow$ (a): Let $y \in Y$ and $W \in CO(Z, \lambda)$ with $g(y) \in W$. By (c), $g^{-1}(W) \in \omega_s CO(Y, \sigma) \subseteq \omega_s(Y, \sigma)$. Put $V = g^{-1}(W)$. Then $V \in \omega_s(Y, \sigma)$, $y \in V$, and $g(V) = g(g^{-1}(W)) \subseteq W$. This shows that g is slightly ω_s-continuous.

Theorem 4.3. Every slightly continuous function is slightly ω_s-continuous.

Proof. Assume that $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is slightly continuous. Let $W \in CO(Z, \lambda)$. Since g is slightly continuous, then $g^{-1}(W) \in \sigma$. So by Proposition 1.1 (a), $g^{-1}(W) \in \omega_s(Y, \sigma)$. Therefore, by Theorem 4.2, it follows that g is slightly ω_s-continuous.

Our next example shows that the converse of Theorem 4.3 is not true, in general:

Example 4.4. Let $X = Y = \mathbb{R}$, $\tau = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$, and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{R} - \mathbb{N}\}$. Define $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ by $f(x) = x$. Note that $CO(\mathbb{R}, \sigma) = \sigma$. Since $f^{-1}(\mathbb{N}) = \mathbb{N} \in \tau \subseteq \omega_s(X, \tau)$ and $f^{-1}(\mathbb{R} - \mathbb{N}) = \mathbb{R} - \mathbb{N} \in \omega_s(X, \tau)$, then f is slightly ω_s-continuous. On the other hand, since $\mathbb{R} - \mathbb{N} \in CO(\mathbb{R}, \sigma)$ but $f^{-1}(\mathbb{R} - \mathbb{N}) = \mathbb{R} - \mathbb{N} \notin \tau$, then f is not slightly continuous.

Theorem 4.5. Every slightly ω_s-continuous function is slightly semi-continuous.

Proof. Assume that $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is slightly ω_s-continuous. Let $W \in CO(Z, \lambda)$. Since g is slightly ω_s-continuous, then $g^{-1}(W) \in \omega_s(Y, \sigma)$. So by Proposition 1.1 (a), $g^{-1}(W) \in SO(Y, \sigma)$. Therefore, g is slightly semi-continuous.

The converse of Theorem 4.5 is not true in general as the following example shows:

Example 4.6. Let $X = Y = \mathbb{R}$, $\tau = \{\emptyset, \mathbb{R}, \mathbb{N}, \mathbb{Q}^c, \mathbb{N} \cup \mathbb{Q}^c\}$, and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}\}$. Define $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ by $f(x) = x$. Note that $CO(\mathbb{R}, \sigma) = \sigma$. Since $f^{-1}(\mathbb{Q}) = \mathbb{Q} \in SO(X, \tau)$ and $f^{-1}(\mathbb{R} - \mathbb{Q}) = \mathbb{R} - \mathbb{Q} \in \tau \subseteq SO(X, \tau)$, then f is slightly semi-continuous. On the other hand, since $\mathbb{Q} \in CO(\mathbb{R}, \sigma)$ but $f^{-1}(\mathbb{Q}) = \mathbb{Q} \notin \omega_s(\mathbb{R}, \tau)$, then f is not slightly ω_s-continuous.

Theorem 4.7. (a) If $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is slightly ω_s-continuous such that (Y, σ) is locally countable, then g is continuous.

(b) If $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is slightly semi-continuous with (Y, σ) is anti-locally countable, then g is ω_s-continuous.

Proof. (a) Follows from the definitions and Proposition 1.1 (c).

(b) Follows from the definitions and Proposition 1.1 (b).

Theorem 4.8. Let $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ be a function and let γ be the product topology of (Y, σ) and (Z, λ). Let $h : (Y, \sigma) \longrightarrow (Y \times Z, \gamma)$, where $h(y) = (y, g(y))$ be the graph of g. Then g is slightly ω_s-continuous if and only if h is slightly ω_s-continuous.

Proof. Let $y \in Y$ and let $M \in CO(Y \times Z, \gamma)$ such that $h(y) = (y, g(y)) \in M$. Then $M \cap (\{y\} \times Z)$ is a clopen set in $\{y\} \times Z$ which contains $h(y) = (y, g(y))$. Since $\{y\} \times Z$ is homeomorphic to Z, then $\{z \in Z : (y, z) \in M\} \in CO(Z, \lambda)$. Since g is slightly ω_s-continuous, then $\cup\{g^{-1}(z) : (y, z) \in M\} \in \omega_s(Y, \sigma)$. Moreover, $y \in \cup \{g^{-1}(z) : (y, z) \in M\} \subseteq h^{-1}(M)$. Hence, $h^{-1}(M) \in \omega_s(Y, \sigma)$. It follows that h is slightly ω_s-continuous.

Conversely, let $H \in CO(Z, \lambda)$. Then $Y \times H \in CO(Y \times Z, \gamma)$. By slight ω_s-continuity of h, $h^{-1}(Y \times H) \in \omega_s CO(Y, \sigma)$. Also, $h^{-1}(Y \times H) = g^{-1}(H)$. It follows that g is slightly ω_s-continuous.

Theorem 4.9. If $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ is slightly ω_s-continuous and $h : (Z, \lambda) \longrightarrow (W, \delta)$ is slightly continuous, then $h \circ g : (Y, \sigma) \longrightarrow (W, \delta)$ is slightly ω_s-continuous.

Proof. Let $M \in CO(W, \delta)$. By slight continuity of h, $g^{-1}(M) \in CO(Z, \lambda)$. By slight ω_s-continuity of g, $g^{-1}(h^{-1}(M)) = (h \circ g)^{-1}(M) \in \omega_s CO(Y, \sigma)$. Hence, $h \circ g$ is slightly ω_s-continuous.

As defined a topological space (Y, σ) is called semi-connected if $SCO(Y, \sigma) = \{\emptyset, Y\}$.

Definition 4.10. A topological space (Y, σ) is called ω_s-connected if $\omega_s CO(Y, \sigma) = \{\emptyset, Y\}$.

Theorem 4.11. Every ω_s-connected topological space is connected.

Proof. Let (Y, σ) be ω_s-connected. Then $\omega_s CO(Y, \sigma) = \{\emptyset, Y\}$. Thus, by Proposition 1.1 (a), we have $\{\emptyset, Y\} \subseteq CO(Y, \sigma) \subseteq \omega_s CO(Y, \sigma) = \{\emptyset, Y\}$. Hence, $CO(Y, \sigma) = \{\emptyset, Y\}$. Therefore, (Y, σ) is connected.

The following example will show that the converse of Theorem 4.11 is not true, in general:

Example 4.12. Consider the topological space $(\mathbb{R}, \tau_u)$. To see that $(\mathbb{R}, \tau_u)$ is not ω_s-connected, let $M = (-\infty, 0)$, then $M \in \tau_u \subseteq \omega_s(\mathbb{R}, \tau_u)$. Since $(0, \infty) \in \tau_u$ and $\overline{(0, \infty)}^{\omega} = \overline{(0, \infty)} = [0, \infty) = \mathbb{R} - M$, then $\mathbb{R} - M \in \omega_s(\mathbb{R}, \tau_u)$. Therefore, $M \in \omega_s CO(\mathbb{R}, \tau_u) - \{\emptyset, \mathbb{R}\}$, and hence $(\mathbb{R}, \tau_u)$ is not ω_s-connected. On the other hand, $(\mathbb{R}, \tau_u)$ is connected.

Theorem 4.13. Every connected locally countable topological space is ω_s-connected.

Proof. Follows from the definitions and Proposition 1.1 (b).

Theorem 4.14. Every semi-connected topological space is ω_s-connected.

Proof. Let (Y, σ) be semi-connected. Then $SCO(Y, \sigma) = \{\emptyset, Y\}$. Thus, by Proposition 1.1 (a), we have $\{\emptyset, Y\} \subseteq \omega_s CO(Y, \sigma) \subseteq SCO(Y, \sigma) = \{\emptyset, Y\}$. Hence, $\omega_s CO(Y, \sigma) = \{\emptyset, Y\}$. Therefore, (Y, σ) is ω_s-connected.

Question 4.15. Is it true that ω_s-connected topological spaces are semi-connected?

The following theorem answers Question 4.15 partially:

Theorem 4.16. Every anti-locally countable ω_s-connected topological space is semi-connected.

Proof. Follows from the definitions and Proposition 1.1 (c).

Theorem 4.17. A slightly ω_s-continuous image of an ω_s-connected space is connected.

Proof . Let $g : (Y, \sigma) \longrightarrow (Z, \lambda)$ be surjective and slightly ω_s-continuous, where (Y, σ) is ω_s-connected. Suppose that (Z, λ) is not connected. Then there exists $M \in CO(Z, \lambda) - \{\emptyset, Z\}$. By Theorem 4.2, $g^{-1}(M) \in \omega_s CO(Y, \sigma)$. Since $\emptyset \neq M \neq Z$ and g is surjective, then $\emptyset \neq g^{-1}(M) \neq Z$. Therefore, $g^{-1}(M) \in \omega_s CO(Y, \sigma) - \{\emptyset, Y\}$ which contradicts the assumption that (Y, σ) is ω_s-connected.

5. ω_s-Compact topological spaces

Definition 5.1. A topological space (Y, σ) is called ω_s-compact (resp. semi-compact [36]) if for any cover $\mathcal{A}$ of Y with $\mathcal{A} \subseteq \omega_s(Y, \sigma)$ (resp. $\mathcal{A} \subseteq SO(Y, \sigma)$), there is a finite subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}$ is also a cover of Y.

Theorem 5.2. Every ω_s-compact topological space is compact.

Proof. Let (Y, σ) be ω_s-compact and let $\mathcal{A}$ be a cover of Y with $\mathcal{A} \subseteq \sigma$. Then by Proposition 1.1 (a), $\mathcal{A} \subseteq \omega_s(Y, \sigma)$. Since (Y, σ) is ω_s-compact, then there exists a finite subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}$ is also a cover of Y. This shows that (Y, σ) is compact.

The following example will show that the converse of Theorem 5.2 is not true, in general:

Example 5.3. Consider $([0, \infty), \sigma)$, where $\sigma = \{\emptyset, [0, \infty)\} \cup \{(a, \infty) : a \geq 0\}$. To see that $([0, \infty), \sigma)$ is compact, let $\mathcal{A}$ be a cover of $[0, \infty)$ with $\mathcal{A} \subseteq \sigma$. Then $[0, \infty) \in \mathcal{A}$. Choose $\mathcal{B} = \{[0, \infty)\}$. Then $\mathcal{B}$ is a finite subfamily of $\mathcal{A}$ such that $\mathcal{B}$ is also a cover of $[0, \infty)$. This shows that $([0, \infty), \sigma)$ is compact. Let $\mathcal{A} = \{(1, \infty) \cup \{x\} : x \in [0, 1]\}$. Then $\mathcal{A}$ is a cover of $[0, \infty)$. Since $(1, \infty) \in \sigma \subseteq \omega_s([0, \infty), \sigma)$ and $\overline{(1, \infty)}^{\omega} = \overline{(1, \infty)} = [0, \infty)$, then by Proposition 1.2 (a), we have $\mathcal{A} \subseteq \omega_s([0, \infty), \sigma)$. On the other hand, if $\mathcal{B}$ is a finite subfamily of $\mathcal{A}$, then $\mathcal{B}$ is not a cover of $[0, \infty)$. This shows that $([0, \infty), \sigma)$ is not ω_s-compact.

Theorem 5.4. Let (Y, σ) be a locally countable topological space. Then (Y, σ) is ω_s-compact if and only if (Y, σ) is compact.

Proof. Necessity. Follows from Theorem 5.2.

Sufficiency. Suppose that (Y, σ) is compact and let $\mathcal{A}$ be a cover of Y such that $\mathcal{A} \subseteq \omega_s(Y, \sigma)$. Since (Y, σ) is locally countable, then by Proposition 1.1 (b), $\mathcal{A} \subseteq \sigma$. Since (Y, σ) is compact, then there exists a finite subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}$ is also a cover of Y. This shows that (Y, σ) is ω_s-compact.

Theorem 5.5. Every semi-compact topological space is ω_s-compact.

Proof. Let (Y,σ) be semi-compact and let $\mathcal{A}$ be a cover of Y such that $\mathcal{A} \subseteq \omega_s(Y,\sigma)$. Then by Proposition 1.1 (a), $\mathcal{A} \subseteq SO(Y,\sigma)$. Since (Y,σ) is semi-compact, then there exists a finite subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}$ is also a cover of Y. This shows that (Y,σ) is ω_s-compact.

The following example will show that the converse of Theorem 5.5 is not true, in general:

Example 5.6. Consider $(\mathbb{N},\sigma)$, where $\sigma = \{\emptyset,\mathbb{N},\{1\},\{2\},\{1,2\}\}$. Since $\{1\}$,$\{2\}$, and $\{1,2\}$ are countable sets, then $\overline{\{1\}}^{\omega} = \{1\}$,$\overline{\{2\}}^{\omega}$, and $\overline{\{1,2\}}^{\omega} = \{1,2\}$. Thus, $\sigma = \omega_s(Y,\sigma)$. This shows that $(\mathbb{N},\sigma)$ is ω_s-compact. Let $\mathcal{A} = \{\{2\}\} \cup \{\{1,x\} : x \in \mathbb{N} - \{1,2\}\}$. Then $\mathcal{A}$ is a cover of $\mathbb{N}$. Since $\overline{\{1\}} = \mathbb{N} - \{2\}$, then $\mathcal{A} \subseteq SO(Y,\sigma)$. If $\mathcal{B}$ is a finite subfamily of $\mathcal{A}$, then $\bigcup\mathcal{B}$ is a finite subset of $\mathbb{N}$. This shows that $(\mathbb{N},\sigma)$ is not semi-compact.

Theorem 5.7. Let (Y,σ) be an anti-locally countable topological space. Then (Y,σ) is semi-compact if and only if (Y,σ) is ω_s-compact.

Proof. Necessity. Follows from Theorem 2.5.

Sufficiency. Suppose that (Y,σ) is ω_s-compact and let $\mathcal{A}$ be a cover of Y such that $\mathcal{A} \subseteq SO(Y,\sigma)$. Since (Y,σ) is anti-locally countable, then by Proposition 1.1 (c), $\mathcal{A} \subseteq \omega_s(Y,\sigma)$. Since (Y,σ) is ω_s-compact, then there is a finite subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}$ is also a cover of Y. This shows that (Y,σ) is semi-compact.

Theorem 5.8. A topological space (Y,σ) is ω_s-compact if and only if every family of ω_s-closed sets which has the finite intersection property must have non-empty intersection.

Proof. Necessity. Suppose that (Y,σ) is ω_s-compact, and suppose to the contrary that there exists a family $\mathcal{H}$ of ω_s-closed such that $\mathcal{H}$ has the finite intersection property and $\bigcap\mathcal{H} = \emptyset$. Let $\mathcal{A} = \{Y - H : H \in \mathcal{H}\}$. Then $\mathcal{A}$ is a cover of Y and $\mathcal{A} \subseteq \omega_s(Y,\sigma)$. Since (Y,σ) is ω_s-compact, then there is a finite subfamily $\mathcal{A}_1 \subseteq \mathcal{A}$ such that $\mathcal{A}_1$ is also a cover of Y. Let $\mathcal{H}_1 = \{Y - A : A \in \mathcal{A}_1\}$. Then $\mathcal{H}_1$ is a finite subcollection of $\mathcal{H}$ such that

$$\begin{aligned}
\bigcap\mathcal{H}_1 &= \bigcap_{A\in\mathcal{A}_1}(Y - A)\\
&= Y - \bigcup_{A\in\mathcal{A}_1} A\\
&= Y - Y\\
&= \emptyset.
\end{aligned}$$

This contradicts the assumption that $\mathcal{H}$ has the finite intersection property.

Sufficiency. Suppose that every family of ω_s-closed sets which has the finite intersection property must have non-empty intersection, and suppose to the contrary that (Y,σ) is not ω_s-compact. Then there is a cover $\mathcal{A}$ of Y such that $\mathcal{A} \subseteq \omega_s(Y,\sigma)$ and any finite subcollection of $\mathcal{A}$ is not a cover of Y. Let $\mathcal{H} = \{Y - A : A \in \mathcal{A}\}$. Then $\mathcal{H}$ is a family of ω_s-closed sets and $\mathcal{H}$ has the finite intersection property. So, by assumption $\bigcap\mathcal{H} \neq \emptyset$, and thus $Y - \bigcap\mathcal{H} \neq Y$. But

$$\begin{aligned}
Y - \bigcap\mathcal{H} &= \bigcup_{A\in\mathcal{A}} A\\
&\neq Y,
\end{aligned}$$

a contradiction.

Definition 5.9. Let (Y, σ) be a topological space and let $(x_d)_{d \in D}$ be a net in (Y, σ). A point $y \in Y$ is called an ω_s-cluster point of $(y_d)_{d \in D}$ in (Y, σ) if for every $V \in \omega_s(Y, \sigma)$ with $y \in V$ and every $d \in D$, there is $d_0 \in D$ such that $d \leq d_0$ and $y_{d_0} \in V$.

Theorem 5.10. A topological space (Y, σ) is ω_s-compact if and only if every net in (Y, σ) has an ω_s-cluster point.

Proof. Necessity. Suppose that (Y, σ) is ω_s-compact and let $(y_d)_{d \in D}$ be a net in (Y, σ). For each $d \in D$, let $T_d = \{y_{d'} : d' \in D \text{ and } d \leq d'\}$. Let $\mathcal{A} = \left\{\overline{T_d}^{\omega_s} : d \in D\right\}$. Then $\mathcal{A}$ is a family of ω_s-closed sets.

Claim 1. $\mathcal{A}$ has the finite intersection property.

Proof of Claim 1. Let $d_1, d_2, ..., d_n \in D$. Choose $d_0 \in D$ such that $d_i \leq d_0$ for all $i = 1, 2, ..., n$. Then $y_{d_0} \in \bigcap_{i=1}^{n} T_{d_i} \subseteq \bigcap_{i=1}^{n} \overline{T_{d_i}}^{\omega_s}$. This ends the proof that $\mathcal{A}$ has the finite intersection property.

Since (Y, σ) is ω_s-compact, then by Claim 1 and Theorem 5.8, there exists $y \in \bigcap_{d \in D} \overline{T_d}^{\omega_s}$.

Claim 2. y is an ω_s-cluster point of $(y_d)_{d \in D}$ in (Y, σ).

Proof of Claim 2. Let $V \in \omega_s(Y, \sigma)$ such that $y \in V$, and let $d \in D$. Since $y \in V \cap \overline{T_d}^{\omega_s}$, then $V \cap T_d \neq \emptyset$, and so there exists $d' \in D$ such that $d \leq d'$ and $x_{d'} \in V$. This shows that y is an ω_s-cluster point of $(y_d)_{d \in D}$ in (Y, σ).

Sufficiency. Suppose that every net in (Y, σ) has an ω_s-cluster point. We will apply Theorem 5.8. Let $\mathcal{A}$ be a family of ω_s-closed sets which has the finite intersection property. Let D be the family of all finite intersections of members of $\mathcal{A}$. Define the relation $\leq$ on D as follows:

For every $d_1, d_2 \in D$, $d_1 \leq d_2$ if and only if $d_2 \subseteq d_1$.

Then $(D, \leq)$ is a directed set. For every $d \in D$, choose $y_d \in d$. By assumption, there is an ω_s-cluster point y of $(y_d)_{d \in D}$.

Claim 3. $y \in \overline{A}^{\omega_s}$ for all $A \in \mathcal{A}$, and hence $y \in \bigcap_{A \in \mathcal{A}} \overline{A}^{\omega_s} = \bigcap_{A \in \mathcal{A}} A$.

Proof of Claim 3. Let $A \in \mathcal{A}$ and $V \in \omega_s(Y, \sigma)$ with $y \in V$. Let $d = A$, then $d \in D$. Since y is an ω_s-cluster point of $(y_d)_{d \in D}$, then there is $d' \in D$ such that $d \leq d'$ and $y_{d'} \in V$, say $d' = F$. Then $F \subseteq A$, and hence $y_{d'} \in V \cap A$. Therefore, $y \in \overline{A}^{\omega_s}$.

Theorem 5.11. Let $g : (Y, \sigma) \longrightarrow (Z, \gamma)$ be ω_s-irresolute and surjective. If (Y, σ) is ω_s-compact, then (Z, γ) is ω_s-compact.

Proof. Suppose that (Y, σ) is ω_s-compact and let $\mathcal{H}$ be a cover of Z such that $\mathcal{H} \subseteq \omega_s(Z, \gamma)$. Let $\mathcal{M} = \left\{g^{-1}(H) : H \in \mathcal{H}\right\}$. Then $\mathcal{M}$ is a cover of Y. Also, by ω_s-irresoluteness of g, we have $\mathcal{M} \subseteq \omega_s(Y, \sigma)$. Since (Y, σ) is ω_s-compact, then there exist $H_1, H_2, ..., H_n \in \mathcal{H}$ such that $\bigcup_{i=1}^{n} g^{-1}(H_i) = Y$, and so $\bigcup_{i=1}^{n} H_i \subseteq g\left(g^{-1}\left(\bigcup_{i=1}^{n} H_i\right)\right) = g(Y)$. Since g is surjective, then $g(Y) = Z$. Therefore, $Z = \bigcup_{i=1}^{n} H_i$. Hence, (Z, γ) is ω_s-compact.

6. Conclusions

In this paper, we introduce ω_s-irresoluteness, ω_s-openness, pre-ω_s-openness, and slight ω_s-continuity as new classes of functions. And, we define ω_s-compactness as a new class of topological spaces which lies between the classes compactness and semi-compactness. Several implications, examples, counter-examples, characterizations, and mapping theorems are introduced. The following topics could be considered in future studies: (1) To define ω_s-open separation axioms; (2) To define ω_s-connectedness; (3) To improve some known topological results.

Conflict of interest

We declare no conflicts of interest in this paper.

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