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Research article

Common fixed points of fuzzy set-valued contractive mappings on metric spaces with a directed graph

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Abstract: We introduce a new class of generalized graphic fuzzy F- contractive mappings on metric spaces and establish the existence of common fuzzy coincidence and fixed point results for such contractions. It is significant to note that we do not use any form of continuity of mappings to prove these results. Some examples are provided to verify our proven results. Various developments in the existing literature are generalized and extended by our results. It is aimed that the initiated concepts in this work will encourage new research aspects in fixed point theory and related hybrid models in the literature of fuzzy mathematics.

Keywords: fuzzy set; fuzzy set-valued mapping; fuzzy fixed point; directed graph
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The most celebrated fixed point theorem familiar as Banach contraction principle (BCP) (see [13]), is largely used to obtain the existence of a solution of linear and nonlinear functional equations. Given an initial guess of the solution, BCP provides sufficient conditions to guarantee the convergence of successive approximations to actual solution of the problem. BCP [13] has been modified and applied in different directions for instance [19, 26, 33] and the references therein. Existence of fixed points of certain mappings established on partially ordered metric spaces has been considered by Ran and Reurings [34]. For further consequences in this direction [21, 35]. Jachymski and Jozwik [23] used graph structure on metric fixed point theory instead of the order structure and proved fixed point results. In this fashion the consequences proven in ordered structured upgraded and generalized (see also [24] and the reference therein); In 2009, Gwozdz-CLukawska and Jachymski [36] incorporated graph theory in metric fixed point theory and flourished the results of the Hutchinson-Barnsley theory for specific families of mappings on a metric space. Bojor [16] amalgamated fixed point theory on metric space with graph theory and established fixed point results for Reich type contractions on metric spaces. Abbas and Nazir [3] used graphic structure and proved fixed points results of power graph contraction pair on a metric space. This attracted the attention of many authors and various interesting results have been obtained in this direction (see, for example, [7,8,15,17,18]). Wardowski [37] introduced F-contraction and achieved an interesting fixed point consequence as an extension of BCP. On the domain of sets equipped with directed graph, latterly, Abbas et al. [1] established some fixed point results of set-valued mappings fulfilling certain graphic contraction conditions (see also, [2]). On the other hand, one of the obstacles in mathematical modeling of real circumstances is the indefiniteness persuaded by our inabilities to classify events with ample precision. The crisp set theory cannot cope effectively with imprecisions. As an attempt to deal with the problems of inadequate data, crisp sets were replaced with fuzzy sets [38] which gave a birth to Fuzzy set theory. It provides appropriate mathematical tools for handling information with non statistical uncertainty. As a result, fuzzy set theory has gained much recognition because of its utilization in several domains such as management sciences, engineering, environmental sciences, medical sciences and in other emerging fields. The fundamental notions of fuzzy sets have been modified and polished up in different fashions; for example, see [4, 9, 20, 27, 28]. In 1981, Heilpern [22] initiated the study of fuzzy set-valued maps and obtained a fuzzy replica of Nadler's fixed point results [31]. Afterwards, many authors worked on the existence of fixed points of fuzzy set-valued maps, for example, Al-Mazrooei et al. [5, 6], Azam et al. [10–12], Bose and Sahani [14], Mohammed [29], Mohammed and Azam [30], Qiu and Shu [32], and so on.

In this paper, we develop a generalized graphic fuzzy F- contractive mappings on metric spaces and obtain the existence of common fuzzy coincidence and fixed point results for such contractions. We present some examples to endorse the results established herein. Our results extend and unify comparable results in the present literature.

Persistent with Jacehymski [24], let (Ψ, φ) be a metric space and the diagonal of $\Psi \times \Psi$ is denoted by Δ. $V(G)$ denotes the set of vertices coincides with Ψ of a directed graph G and $E(G)$ represents the set of edges of the graph containing all loops, that is, $\Delta \subseteq E(G)$. Also it is assumed that the graph G has no multiple edges and hence, one can recognize G with the pair $(V(G), E(G))$. Moreover, the number $\varphi(\xi, \zeta)$ is interpreted as the weight of the edge (ξ, ζ) of G.

2. Prelimnaries

Definition 2.1. *In a metric space (Ψ, φ), a mapping $h : \Psi \to \Psi$ is defined a G-contraction if*

- *for each $\xi, \zeta \in \Psi$ with $(\xi, \zeta) \in E(G)$, we possess $(h(\xi), h(\zeta)) \in E(G)$. Viz, h preserves edges of the given graph G;*
- *h decreases weights of edges of G; there exits $\eta \in (0, 1)$ such that for all $\xi, \zeta \in \Psi$ with $(\xi, \zeta) \in G$,*

we have $\varphi(h(\xi), h(\zeta)) \leq \eta\varphi(\xi, \eta)$.

A directed path between ξ and ζ of length $\ell \in \mathbb{N}$ in graph G is a finite sequence $\{\xi_n\}(n \in \{0, 1, 2, ..., \ell\})$ of vertices such that $\xi_0 = \xi$, $\xi_\ell = \zeta$ and $(\xi_{i-1}, \xi_i) \in E(G)$ for $i \in \{1, 2, ..., \ell\}$. Remember that a graph G is said to be connected if there is a directed path between every pair of vertex that is from every vertex to any other vertex whereas it is said to be weakly connected if $\tilde{G}$ is connected, where $\tilde{G}$ represents the undirected graph acquired from G by neglecting the direction of edges.

The graph obtained by reversing the direction of edges is denoted by G^{-1}, furthermore for the sake of convenience we treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric.

In $V(G)$ we define the relation R in the following way. For $\xi, \zeta \in V(G)$, $\xi R \zeta$ if and only if, there is a path in G from ξ to ζ. Let $h : \Psi \to \Psi$ be an operator, then by F_h we represent the set of all fixed points of h. Set

$$\Psi_h := \{\xi \in \Psi : (\xi, h(\xi)) \in E(G)\}.$$

A metric space (Ψ, φ) equipped with a directed graph G is said to possess the property (P) [23]:

(P) if for any sequence $\{\xi_n\} \in \Psi$ satisfying $\xi_n \to \xi$ as $n \to \infty$ and $(\xi_n, \xi_{n+1}) \in E(G)$, we have $(\xi_n, \xi) \in E(G)$.

Theorem 2.2. *[23] Let* (Ψ, φ) *be a complete metric space and G a directed graph such that* $V(G) = \Psi$ *and* $h : \Psi \to \Psi$ *a G- contraction. Assume that* $E(G)$ *and the triplet* (Ψ, φ, G) *possess property* (P)*. Then the following statements hold.*

- $F_h \neq \emptyset$ *if and only if* $\Psi_h \neq \emptyset$;
- *if* $\Psi_h \neq \emptyset$ *and G is weakly connected, then h is a Picard operator, that is* $F_h = \{\xi^*\}$ *and sequence* $\{h^n(\xi)\} \to \xi^*$ *as* $n \to \infty$ *for all* $\xi \in \Psi$*;*
- *for any* $\xi \in \Psi_h$, $h|[\xi]_{\tilde{G}}$ *is picard operator;*
- *if* $\Psi_h \subseteq E(G)$ *then h is weakly picard operator, that is* $F_h \neq \emptyset$ *and for each* $\xi \in \Psi$*, we have sequence* $\{h^n(\xi)\} \to \xi^* \in F_h$ *as* $n \to \infty$.

Recall that a crisp set Λ in Ψ is determined by its characteristic function $\chi_\Lambda : \Lambda \longrightarrow \{0, 1\}$ interpreted as

$$\chi_\Lambda(x) = \begin{cases} 1, & \text{if } \xi \in \Lambda \\ 0, & \text{if } x \notin \Lambda. \end{cases}$$

The value of χ_Λ at ξ indicates whether an element ξ belongs to Λ or not. A fuzzy set is illustrated by allowing a mapping χ_Λ to assume any possible value in the interval $[0, 1]$. Thus, a fuzzy set Λ in Ψ is characterized by the function Λ with domain Ψ and values in $[0, 1] = I$. The collection of all fuzzy sets in Ψ is denoted by I^Ψ. If Λ is a fuzzy set in Ψ, then $\Lambda(\xi)$ is called the grade or degree of membership of an element ξ in Λ. The α-level set of a fuzzy set Λ is represented by $[\Lambda]_\alpha$ and is explained as follows:

$$[\Lambda]_\alpha = \begin{cases} \overline{\{\xi \in \Lambda : \Lambda(\xi) > 0\}}, & \text{if } \alpha = 0, \\ \{\xi \in \Psi : \Lambda(\xi) \geq \alpha\}, & \text{if } \alpha \in (0, 1]. \end{cases}$$

Where $\bar{N}$ represents the closure of the crisp set N.

Example 2.3. *Let Ψ be the set of all individuals in a certain town, and*

$$\Lambda = \{\xi \in \Psi | \ \xi \text{ is an old person}\}.$$

Then, it is more appropriate to identify an individual be an old person by membership function Λ on Ψ because the term "old" is not well defined.

Example 2.4. *Let $\Psi = \{1, 2, 3, 4\}$ be endowed with the usual metric. Let $\daleth : \Psi \longrightarrow I^{\Psi}$ be a fuzzy set-valued map, that is, for each $\xi \in \Psi$, $\daleth(\xi) : \Psi \longrightarrow [0, 1]$ is a fuzzy set. For instance, for some $\alpha \in (0, 1]$, we may define one of the fuzzy set $\daleth(1)$ by*

$$\daleth(1)(t) = \begin{cases} \alpha, & \text{if } t = 1 \\ \frac{\alpha}{3}, & \text{if } t = 2 \\ \frac{\alpha}{7}, & \text{if } t = 3 \\ \frac{\alpha}{9}, & \text{if } t = 4. \end{cases}$$

In a metric space (Ψ, φ) , $CB(\Psi)$ represents a class of all non-empty closed and bounded subsets of Ψ. For $\Lambda, \Upsilon \in CB(\Psi)$, the Pompeiu-Hausdorff metric induced by metric φ is defined as

$$H(\Lambda, \Upsilon) = \max\{\sup_{\varrho \in \Upsilon} \varphi(\varrho, \Lambda), \sup_{\kappa \in \Lambda} \varphi(\kappa, \Upsilon)\},$$

where the distance of a point ξ to the set Υ is defined as

$$\varphi(\xi, \Upsilon) = \inf\{\varphi(\xi, \varrho) : \varrho \in \Upsilon\}.$$

Consistent with Abbas et al. [2], let $(CB(\Psi), \varphi)$ the Pompeiu-Hausdorff metric induced by φ and Δ represents the diagonal of $CB(\Psi) \times CB(\Psi)$. Throughout this work, we assume that for any $\Lambda, \Upsilon \in CB(\Psi)$, there is an edge between Λ and Υ, which aims that there is an edge between some $\kappa \in \Lambda$ and $\varrho \in \Upsilon$ which we represent by $(\Lambda, \Upsilon) \subset E(G)$. We now identify the directed graph G, called a directed set graph with the pair $(V(G), E(G))$ if the set $V(G)$ of its vertices coincides with $CB(\Psi)$ and $E(G)$ the set of edges of the graph containing all loops, that is, $\Delta \subseteq E(G)$. In addition, suppose that the graph G has no multiple edges. Moreover, for each $\Theta, \Phi \in CB(\Psi)$, the number $H(\Theta, \Phi)$ is interpreted as the weight of the edge (Θ, Φ) of a directed set graph G.

We suppose that a directed set graph G has no multiple edge and G is a weighted graph in the meaning that each vertex Θ is given the weight $H(\Theta, \Theta) = 0$ and each edge (Θ, Φ) is given the weight $H(\Theta, \Phi)$. Abbas et al. [2] introduced the following (P^*) property. A directed set graph G is said to possess property

P^* : If for any sequence of sets $\{\Psi_n\}$ in $CB(\Psi)$ with $\Psi_n \to \Psi$ as $n \to \infty$, there exists an edge between Ψ_{n+1} and Ψ_n for $n \in \mathbb{N}$, and it further implies that, there is a subsequence Ψ_{n_k} of Ψ_n with an edge between Ψ and Ψ_{n_k} for $n \in \mathbb{N}$.

Definition 2.5. *Let $\Lambda, \Upsilon \in I^{\Psi}$. Then by definition $[\Lambda]_\alpha, [\Upsilon]_\alpha \subseteq \Psi$.*

1) *There is an edge between $[\Lambda]_\alpha, [\Upsilon]_\alpha \subseteq \Psi$ for some $\alpha \in (0, 1]$, we aim that there is an edge between some $\xi \in [\Lambda]_\alpha$ and $\zeta \in [\Upsilon]_\alpha$ which we denote by $([\Lambda]_\alpha, [\Upsilon]_\alpha) \subset E(G)$.*
2) *There is path between $[\Lambda]_\alpha$ and $[\Upsilon]_\alpha$ we aim that there is a path between some $\xi \in [\Lambda]_\alpha$ and $\zeta \in [\Upsilon]_\alpha$.*

Definition 2.6. *Define the set $I_{Fc}(\Psi)$ by*

$$I_{Fc}(\Psi) = \{\Lambda \in I^{\Psi} : [\Lambda]_{\alpha} \in CB(\Psi)\}. \tag{2.1}$$

A relation R on $I_{Fc}(\Psi)$ is interpreted as follows: For $\Lambda, \Upsilon \in I_{Fc}(\Psi)$, $[\Lambda]_{\alpha}R[\Upsilon]_{\alpha}$ if there is a path between $[\Lambda]_{\alpha}$ and $[\Upsilon]_{\alpha}$ for some $\alpha \in (0, 1]$. The relation R on $I_{Fc}(\Psi)$ is said to be transitive. if for some $\alpha \in (0, 1]$ there is path between $[\Lambda]_{\alpha}$ and $[\Upsilon]_{\alpha}$ and there is a path between $[\Upsilon]_{\alpha}$ and $[\Omega]_{\alpha}$ imply that there is a path between $[\Lambda]_{\alpha}$ and $[\Omega]_{\alpha}$.

Definition 2.7. *Consider the fuzzy set-valued mapping $\gimel : CB(\Psi) \to I_{Fc}(\Psi)$, the set $\Psi_{\gimel}$ is explained as*

$$\Psi_{\gimel} = \{\Theta \in CB(\Psi) : (\Theta, [\gimel(\Theta)]_{\alpha}) \subseteq E(G) \text{ for some } \alpha \in (0, 1]\}.$$

Definition 2.8. *Let $\daleth, \gimel : CB(\Psi) \to I_{Fc}(\Psi)$ be two fuzzy set-valued mappings. Then $\Theta \in CB(\Psi)$ is said to be a fuzzy coincidence point of $\daleth$ and $\gimel$ if $[\daleth(\Theta)]_{\alpha} = [\gimel(\Theta)]_{\alpha}$ for some $\alpha \in (0, 1]$. Also a set $A \in CB(\Psi)$ is said to be a fuzzy fixed point of $\daleth$ if there exists $\alpha \in (0, 1]$ such that $[\daleth(\Lambda)]_{\alpha} = \Lambda$.*

Note that in our work the set of all fuzzy coincidence points of $\daleth$ and $\gimel$ is represented by $C_F(\daleth, \gimel)$ and the set of all fuzzy fixed points of $\daleth$ is represented by $Fuz(\daleth)$.

Definition 2.9. *Two fuzzy set-valued maps $\daleth, \gimel : CB(\Psi) \to I_{Fc}(\Psi)$ are called weakly compatible if they commute at their coincidence point.*

Definition 2.10. *A subset Γ of $CB(\Psi)$ is said to be complete if for any two fuzzy sets $\Upsilon, \Omega \in I_{Fc(X)}$ such that $[\Upsilon]_{\alpha}, [\Omega]_{\alpha} \subseteq \Gamma$ and there is an edge between $[\Upsilon]_{\alpha}$ and $[\Omega]_{\alpha}$.*

Let F be the collection of all continuous mappings $F : \mathbb{R}^{+} \to \mathbb{R}$ that satisfies the following requirements [37]:

F_1) F is strictly increasing, that is, for all $\kappa, \varrho \in \mathbb{R}^{+}$ with $\kappa < \varrho$ gives that $F(\kappa) < F(\varrho)$.

F_2) for every sequence $\{\kappa_n\}$ of positive real numbers,

$$\lim_{n\to\infty} \kappa_n = 0$$

is equivalent to

$$\lim_{n\to\infty} F(\kappa_n) = -\infty,$$

F_3) there exists $k \in (0, 1)$ such that

$$\lim_{\kappa\to 0^{+}} \kappa^{k} F(\kappa) = 0.$$

Definition 2.11. *For some $\varepsilon > 0$, a metric space (Ψ, φ) is said to be ε-chainable if for given $\xi, \zeta \in \Psi$, there is $n \in \mathbb{N}$ and a finite sequence $\{\xi_n\}$ in Ψ such that*

$$\xi_0 = \xi,\ \xi_n = \zeta \text{ and } \varphi(\xi_{i-1}, \xi_i) < \varepsilon \text{ for } i = 1, 2, ..., n.$$

Lemma 2.12. *Let (Ψ, φ) be a metric space if $H(\Lambda, \Upsilon) < \varepsilon$ for $\Lambda, \Phi \in CB(\Psi)$ then for every $\kappa \in \Lambda$ we have an element $\varrho \in \Upsilon$ such that $\varphi(\kappa, \varrho) < \varepsilon$.*

Motivated by the work in [25], we introduce the following definition.

Definition 2.13. *Let* $\daleth, \gimel : CB(\Psi) \to I_{Fc}(\Psi)$ *be fuzzy set-valued maps. The pair* $(\daleth, \gimel)$ *is said to be a generalized graphic fuzzy* F*-contractive mappings if the following statements are satisfied for some* $\alpha \in (0, 1]$.

1) *For any* Θ *in* $CB(\Psi)$, $([\daleth(\Theta)]_\alpha, \Theta) \subseteq E(G)$ *and* $(\Theta, \gimel[\Theta]_\alpha) \subseteq E(G)$.
2) *There is a function* $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\liminf_{\eta \to t^+} \tau(\eta) > 0$ *for all* $t \geq 0$ *such that for* $F \in F$ *there is an edge between* Λ *and* Υ *with* $[\daleth(\Lambda)_\alpha] \neq [\daleth(\Upsilon)]_\alpha$ *such that*

$$\tau(M(\Lambda, \Upsilon)) + F(H([\daleth(\Lambda)]_\alpha, [\daleth(\Upsilon)]_\alpha)) \leq F(M(\Lambda, \Upsilon)) \tag{2.2}$$

holds, where

$$M(\Lambda, \Upsilon) = \max\{H([\gimel(\Lambda)]_\alpha, [\gimel(\Upsilon)]_\alpha), H([\daleth(\Lambda)]_\alpha, [\gimel(\Lambda)]_\alpha), H([\daleth(\Upsilon)]_\alpha, [\gimel(\Upsilon)]_\alpha), \frac{H([\daleth(\Lambda)]_\alpha, [\gimel(\Upsilon)]_\alpha) + H([\daleth(\Upsilon)]_\alpha, [\gimel(\Lambda)]_\alpha)}{2}\}.$$

It is important to note that if any pair $(\daleth, \gimel)$ of set-valued mappings from $CB(\Psi)$ to $I_{Fc}(\Psi)$ is a generalized graphic fuzzy F-contractive mappings for a graph G, then the pair $(\daleth, \gimel)$ is also generalized graphic fuzzy F-contractive mappings for a graph G^{-1} and $\tilde{G}$. In addition, a pair $(\daleth, \gimel)$ of generalized graphic fuzzy F-contractive mappings for graph G is also generalized graphic fuzzy F-contractive mappings for the graph G_0 where G_0 is graph with $E(G_0) = \Psi \times \Psi$.

3. Main results

Here, we establish some common fuzzy coincidence and fixed point results for fuzzy set-valued maps on $I_{Fc}(\Psi)$ fulfilling generalized graphic fuzzy F-contractive mappings conditions.

Theorem 3.1. *Let* (Ψ, φ) *be a metric space equipped with a directed graph G with* $V(G) = \Psi$, $E(G) \supseteq \Delta$ *and the relation R on* $I_{Fc}(\Psi)$ *is transitive. Suppose that* $\daleth, \gimel : CB(\Psi) \to I_{Fc}(\Psi)$ *is a generalized graphic fuzzy* F*-contractive mappings pair such that the range of* $\gimel$ *contains the range of* $\daleth$*, then the following statements are satisfied.*

1) $C_F(\daleth, \gimel) \neq \emptyset$ *given that* G *is weakly connected which holds the property* (P^*) *and there exists* $\alpha \in (0, 1]$ *such that* $[\gimel(\Psi)]_\alpha$ *is a complete subspace of* $I_{Fc}(\Psi)$.
2) *If* $C_F(\daleth, \gimel)$ *is complete, then the Hausdorff weight assigned to* $[\daleth(\Theta)]_\alpha$ *and* $[\daleth(\Phi)]_\alpha$ *is* 0 *for some* $\alpha \in (0, 1]$ *and for all* $\Theta, \Phi \in C_F(\daleth, \gimel)$.
3) *If* $C_F(\daleth, \gimel)$ *is complete and* $\daleth$, $\gimel$ *are weakly compatible, then* $Fuz(\daleth) \bigcap Fuz(\gimel)$ *is singleton.*
4) $Fuz(\daleth) \bigcap Fuz(\gimel)$ *is complete if and only if* $Fuz(\daleth) \bigcap Fuz(\gimel)$ *is singleton.*

Proof. To verify (1): Let Λ_0 be an arbitrary element in $CB(\Psi)$. As the range of $\gimel$ contains the range of $\daleth$, choose $\Lambda_1 \in CB(\Psi)$ such that $[\daleth(\Lambda_0)]_\alpha = [\gimel(\Lambda_1)]_\alpha$. and for $\Lambda_2 \in CB(\Psi)$ such that $[S(\Lambda_1)]_\alpha = [T(\Lambda_2)]_\alpha$. Carrying on this procedure, for $\Lambda_n \in CB(X)$ we get an Λ_{n+1} in $CB(\Psi)$ such that $[\daleth(\Lambda_n)]_\alpha = [\gimel(\Lambda_{n+1})]_\alpha$ for all $n \in \mathbb{N}$.

By the hypothesis of the theorem it is given that the pair $(\daleth, \gimel)$ is generalized graphic fuzzy F-contractive mappings, therefore for $\Lambda_{n+1}, \Lambda_n \in CB(\Psi)$ we have $(\Lambda_{n+1}, [\gimel(\Lambda_{n+1})]_\alpha) \subseteq E(G)$ and $([\daleth(\Lambda_n)]_\alpha, \Lambda_n) \subseteq E(G)$, as $[\daleth(\Lambda_n)]_\alpha = [\gimel(\Lambda_{n+1})]_\alpha$ so we have $([\daleth(\Lambda_n)]_\alpha, \Lambda_n) = ([\gimel(\Lambda_{n+1})]_\alpha, \Lambda_n) \subseteq E(G)$. By using transitivity we have $(\Lambda_{n+1}, \Lambda_n) \subseteq E(G)$.

Let us suppose that for $\Lambda_n, \Lambda_{n+1} \in CB(\Psi)$ where $n \in \mathbb{N}$ we have $[\daleth(\Lambda_n)]_\alpha \neq [\daleth(\Lambda_{n+1})]_\alpha$ with $\alpha \in (0, 1]$, otherwise, for some $k \in \mathbb{N}$ we have $[\daleth(\Lambda_{2k})]_\alpha = [\daleth(\Lambda_{2k+1})]_\alpha$, Also as $[\daleth(\Lambda_{2k})]_\alpha = [\gimel(\Lambda_{2k+1})]_\alpha$ therefore we can write $[\gimel(\Lambda_{2k+1})]_\alpha = [\daleth(\Lambda_{2k+1})]_\alpha$ and hence $\Lambda_{2k+1} \in C_F(\daleth, \gimel)$. Since $(\Lambda_{n+1}, \Lambda_n) \subseteq E(G)$ for all $n \in \mathbb{N}$, by (2.2) we get

$$\begin{aligned} & \tau(M(\Lambda_n, \Lambda_{n+1})) + F(H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)) \\ = \ & \tau(M(\Lambda_n, \Lambda_{n+1})) + F(H([\daleth(\Lambda_n)]_\alpha, [\daleth(\Lambda_{n+1})]_\alpha)) \\ \leq \ & F(M(\Lambda_n, \Lambda_{n+1})), \end{aligned}$$

where

$$\begin{aligned} & M(\Lambda_n, \Lambda_{n+1}) \\ = \ & \max\{H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha), H([\daleth(\Lambda_n)]_\alpha, [\gimel(\Lambda_n)]_\alpha), \\ & H([\daleth(\Lambda_{n+1})]_\alpha, \gimel([\Lambda_{n+1}]_\alpha)), \frac{H([\daleth(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha) + H([\daleth(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_n)]_\alpha)}{2}\} \\ = \ & \max\{H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha), H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_n)]_\alpha), \\ & H([\gimel(\Lambda_{n+2})]_\alpha, \gimel([\Lambda_{n+1}]_\alpha)), \frac{H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha) + H([\gimel(\Lambda_{n+2})]_\alpha, [\gimel(\Lambda_n)]_\alpha)}{2}\} \\ \leq \ & \max\{H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha), H([\gimel(\Lambda_{n+1})]_\alpha, \gimel([\Lambda_{n+2}]_\alpha)), \\ & \frac{H([\gimel(\Lambda_{n+2})]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha) + H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_n)]_\alpha)}{2}\} \\ \leq \ & \max\{H([F(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha), H([\gimel(\Lambda_{n+1})]_\alpha, \gimel([\Lambda_{n+2}]_\alpha))\} \end{aligned}$$

consequently, we get

$$\begin{aligned} & \tau(M(\Lambda_n, \Lambda_{n+1})) + F(H([[\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)) \\ \leq \ & F(\max\{H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha), H([\gimel(\Lambda_{n+1})]_\alpha, \gimel([\Lambda_{n+2}]_\alpha))\}). \end{aligned}$$

If

$$\max\{H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha), H([\gimel(\Lambda_{n+1})]_\alpha, \gimel([\Lambda_{n+2}]_\alpha))\} = H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha),$$

then we obtain

$$\begin{aligned} & \tau(H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)) + F(H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)) \\ \leq \ & F(H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)). \end{aligned}$$

This implies that

$$\begin{aligned} & F(H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)) \\ \leq \ & F(H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)) - \tau(H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)) \\ < \ & F(H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha)). \end{aligned}$$

Since F is strictly increasing. So, we have

$$H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha) < H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha),$$

a contradiction.

This means that

$$\max\{H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha),H([\daleth(\Lambda_{n+1})]_\alpha,\daleth([\Lambda_{n+2}]_\alpha))\}=H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha).$$

So, we have

$$\begin{aligned}
&\tau(M(\Lambda_n,\Lambda_{n+1}))+F(H([\daleth(\Lambda_{n+1})]_\alpha,[\daleth(\Lambda_{n+2})]_\alpha))\\
\leq\ &F(\max\{H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha),H([\daleth(\Lambda_{n+1})]_\alpha,\daleth([\Lambda_{n+2}]_\alpha)\})\\
\leq\ &F(H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)),
\end{aligned}$$

that is,

$$F(H([\daleth(\Lambda_{n+1})]_\alpha,[\daleth(\Lambda_{n+2})]_\alpha))\leq F(H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha))$$

for all $n\in\mathbb{N}$. Thus $\{H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)\}$ is a decreasing sequence. We now show that

$$\lim_{n\to\infty}H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)=0.$$

By the property of τ, there exists $c>0$ with $n_0\in\mathbb{N}$ such that $\tau(M(\Lambda_n,\Lambda_{n+1}))>c$ for all $n\geq n_0$. Now

$$\begin{aligned}
&F(H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha))\\
\leq\ &F(H([\daleth(\Lambda_{n-1})]_\alpha,[\daleth(\Lambda_n)]_\alpha))-\tau(M(\Lambda_n,\Lambda_{n-1}))\\
\leq\ &F(H([[\daleth(\Lambda_{n-2})]_\alpha,[\daleth(\Lambda_{n-1})]_\alpha))-\tau(M(\Lambda_{n-1},\Lambda_n))-\tau(M(\Lambda_{n-2},\Lambda_{n-1}))\\
\leq\ldots.\leq\ &F(H([\daleth(\Lambda_0)]_\alpha,[\daleth(\Lambda_1)]_\alpha))-[\tau(M(\Lambda_{n-1},\Lambda_n))+\tau(M(\Lambda_{n-2},\Lambda_{n-1}))+\\
&\ldots+\tau(M(\Lambda_0,\Lambda_1))]\\
\leq\ &F(H([\daleth(\Lambda_0)]_\alpha,[\daleth(\Lambda_1)]_\alpha))-nc,
\end{aligned}$$

this means that

$$\lim_{n\to\infty}F(H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha))=-\infty.$$

By (F_2), we get

$$\lim_{n\to 0}H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)=0.$$

Now by (F_3), there exists $h\in(0,1)$ such that

$$\lim_{n\to\infty}[H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)]^hF(H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)=0.$$

Consider

$$\begin{aligned}
&[H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)]^hF(H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)\\
-\ &[H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)]^hF(H([\daleth(\Lambda_0)]_\alpha,[\daleth(\Lambda_1)]_\alpha)\\
\leq\ &[H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)]^h[F(H([\daleth(\Lambda_0)]_\alpha,[\daleth(\Lambda_1)]_\alpha)-nc]\\
-\ &[H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Psi_{n+1})]_\alpha)]^hF(H([\daleth(\Lambda_0)]_\alpha,[\daleth(\Lambda_1)]_\alpha)\\
\leq\ &-nc[H([\daleth(\Lambda_n)]_\alpha,[\daleth(\Lambda_{n+1})]_\alpha)]^h\leq 0,
\end{aligned}$$

Now applying limit as $n \to \infty$ gives that

$$\lim_{n\to\infty} n[H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha)]^h = 0,$$

which means that

$$\lim_{n\to\infty} n^{\frac{1}{h}}[H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha)] = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$n^{\frac{1}{h}}[H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha)] \leq 1$$

for all $n \geq n_1$. So we get

$$[H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1})]_\alpha)] \leq \frac{1}{n^{\frac{1}{h}}}$$

for all $n \geq n_1$. For $m, n \in \mathbb{N}$ with $m > n \geq n_1$, we have

$$\begin{aligned} H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_m]_\alpha) &\leq H([\gimel(\Lambda_n)]_\alpha, [\gimel(\Lambda_{n+1}]_\alpha) + H([\gimel(\Lambda_{n+1})]_\alpha, [\gimel(\Lambda_{n+2})]_\alpha), \\ &\quad +.... + H([\gimel(\Lambda_{m-1})]_\alpha, [\gimel(\Lambda_m]_\alpha) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}. \end{aligned}$$

As the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{h}}}$ converges, so we have that $H([\gimel(A_n)]_\alpha, [\gimel(\Lambda_m)]_\alpha \to 0$ as $n, m \to \infty$. Hence $\{[\gimel(\Lambda_n)]_\alpha\}$ proves to be a Cauchy sequence in $[\gimel(\Psi)]_\alpha$. The completeness of $([\gimel(\Psi)]_\alpha, \varphi)$ in $I_{Fc}(\Psi)$ implies that $[\gimel(\Lambda_n)]_\alpha \to \Phi$ as $n \to \infty$ for some $\Phi \in I_{Fc}(\Psi)$. Also Θ in $CB(\Psi)$ can be found such that $[\gimel(\Theta)]_\alpha = \Phi$.

Let us suppose that $[\daleth(\Theta)]_\alpha = [\gimel(\Theta)]_\alpha$. Otherwise, as $([\gimel(\Lambda_{n+1})]_\alpha, ([\gimel(\Lambda_n)]_\alpha) \subseteq E(G)$, by property (P^*), there exists a subsequence $[\gimel(\Lambda_{n_k+1})]_\alpha$ of $[\gimel(\Lambda_{n+1})]_\alpha$ such that $([\gimel(\Theta)]_\alpha, \gimel(\Lambda_{n_k+1})]_\alpha) \subseteq E(G)$ for every $n \in \mathbb{N}$. As $(\Theta, [\gimel(\Theta)]_\alpha) \subseteq E(G)$ and $([\gimel(\Lambda_{n_k+1})]_\alpha, \Lambda_{n_k}) = ([\daleth(\Lambda_{n_k})]_\alpha, \Lambda_{n_k}) \subseteq E(G)$, we have $(\Theta, \Lambda_{n_k}) \subseteq E(G)$. Since the pair $(\daleth, \gimel)$ is generalized graphic fuzzy F-contractive mappings, so we get

$$\begin{aligned} &\tau(M(\Theta, \Lambda_{n_k})) + F(H([\daleth(\Theta)]_\alpha), [\gimel(\Lambda_{n_k+1})]_\alpha) \\ = \;&\tau(M(\Theta, \Lambda_{n_k})) + F(H([\daleth(\Theta)]_\alpha), [\daleth(\Lambda_{n_k})]_\alpha) \\ \leq \;&F(M(\Theta, \Lambda_{n_k})), \end{aligned}$$

where

$$\begin{aligned} &M(\Theta, \Lambda_{n_k})) \\ = \;&\max\{H([\gimel(\Theta)]_\alpha), [\gimel(\Lambda_{n_k})]_\alpha), H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha), \\ &H([\daleth(\Lambda_{n_k})]_\alpha, [\gimel(\Lambda_{n_k})]_\alpha, \frac{H([\daleth(\Theta)]_\alpha, [\gimel(\Lambda_{n_k})]_\alpha) + H([\gimel(\Theta)]_\alpha, [\daleth(\Lambda_{n_k}]_\alpha)}{2}\} \\ = \;&\max\{H(\Theta, [\gimel(\Lambda_{n_k})]_\alpha), H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha), \\ &H([\gimel(\Lambda_{n_k+1})]_\alpha, [\gimel(\Lambda_{n_k})]_\alpha), \frac{(H([\daleth(\Theta)]_\alpha, [\gimel(\Lambda_{n_k})]_\alpha) + H([\gimel(\Theta)]_\alpha, [\gimel(\Lambda_{n_k+1})]_\alpha)}{2}\}. \end{aligned}$$

Now we consider the following cases:

1) In case $M(\Theta, \Lambda_{n_k})) = H([\gimel(\Theta)]_\alpha), [\gimel(\Lambda_{n_k})]_\alpha)$, then we have

$$F(H([\daleth(\Theta)]_\alpha), [\gimel(\Lambda_{n_k+1})]_\alpha) = F(H([\daleth(\Theta)]_\alpha), [\gimel(\Lambda_{n_k})]_\alpha) - \tau(H([\gimel(\Theta)]_\alpha), [\gimel(\Lambda_{n_k})]_\alpha)),$$

applying an upper limit as $k \to \infty$ gives

$$F(H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)) < F(H([\gimel(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)),$$

a contradiction.

2) When $M(\Theta, \Lambda_{n_k})) = H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)$, then

$$F(H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)) \leq F(H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)) - \tau(H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)),$$

a contradiction.

3) If $M(\Theta, \Lambda_{n_k})) = H([\gimel(\Lambda_{n_k+1})]_\alpha, [\gimel(\Lambda_{n_k})]_\alpha)$, then we have that

$$F(H([\daleth(\Theta)]_\alpha), [\gimel(\Lambda_{n_k+1})]_\alpha) = F(H([\gimel(\Lambda_{n_k+1})]_\alpha, [\gimel(\Lambda_{n_k})]_\alpha)) - \tau(H([\gimel(\Lambda_{n_k+1})]_\alpha, [\gimel(\Lambda_{n_k})]_\alpha)),$$

applying an upper limit as $k \to \infty$ gives

$$F(H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)) < F(H([\gimel(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)),$$

a contradiction.

4) Lastly, if we take $M(\Theta, \Lambda_{n_k})) = \dfrac{H([\daleth(\Theta)]_\alpha, [\gimel(\Lambda_{n_k}]_\alpha) + H([\gimel(\Lambda_{n_k+1})]_\alpha, [\gimel(\Theta)]_\alpha))}{2}$,
so, we have

$$\begin{aligned} H([\daleth(\Theta)]_\alpha), [\gimel(\Lambda_{n_k})]_\alpha = F&\left(\frac{H([\daleth(\Theta)]_\alpha), [\gimel(\Lambda_{n_k})]_\alpha + H([\gimel(\Lambda_{n_k+1})]_\alpha, [\gimel(\Theta)]_\alpha)}{2}\right) \\ &- \tau\left(\frac{H([\daleth(\Theta)]_\alpha), [\gimel(\Lambda_{n_k})]_\alpha + H([\gimel(\Lambda_{n_k+1})]_\alpha, [\gimel(\Theta)]_\alpha)}{2}\right), \end{aligned}$$

applying an upper limit as $k \to \infty$ gives

$$\begin{aligned} F(H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)) \leq F&\left(\frac{H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)) + H([\gimel(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha}{2}\right) \\ &- \tau\left(\frac{H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha) + (H([\gimel(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)}{2}\right) \\ &< F(\frac{H([\daleth(\Theta)]_\alpha, [\gimel(\Theta)]_\alpha)}{2}), \end{aligned}$$

a contradiction.

All cases show that $[\daleth(\Theta)]_\alpha = [\gimel(\Theta)]_\alpha$, that is, $\Theta \in C_F(\daleth, \gimel)$.

To verify (2): Let $\Theta, \Phi \in C_F(\daleth, \gimel)$. Assume on contrary that the Pompeiu-Hausdorff weight assign to the $[\daleth(\Theta)]_\alpha$ and $[\daleth(\Phi)]_\alpha$ is not zero. Since the pair $(\daleth, \gimel)$ is generalized graphic fuzzy F-contractive mappings, we have

$$\tau(M(\Theta, \Phi)) + F(H([\daleth(\Theta)]_\alpha), [\daleth(\Phi)]_\alpha)) \leq F((M(\Theta, \Phi)),$$

where

$$
\begin{aligned}
& M(\Theta,\Phi) \\
= \ & \max\{H([\gimel(\Theta)]_\alpha,[\gimel(\Phi)]_\alpha), H([\daleth(\Theta)]_\alpha,[\gimel(\Theta)]_\alpha), H([\daleth(\Phi)]_\alpha,[\gimel(\Phi)]_\alpha), \\
& \frac{H([\daleth(\Theta)]_\alpha,[\gimel(\Phi)]_\alpha) + H([\daleth(\Phi)]_\alpha,[\gimel(\Theta)]_\alpha}{2})\} \\
= \ & \max\{H([\daleth(\Theta)]_\alpha,[\daleth(\Phi)]_\alpha), H([\daleth(\Theta)]_\alpha), [\daleth(\Theta)]_\alpha), H([\daleth(\Phi)]_\alpha,[\gimel(\Phi)]_\alpha), \\
& \frac{H([\daleth(\Theta)]_\alpha,[\daleth(\Phi)]_\alpha) + H([\daleth(\Phi)]_\alpha,[\daleth(\Theta)]_\alpha}{2})\} \\
= \ & H([\daleth(\Phi)]_\alpha,[\daleth(\Theta)]_\alpha).
\end{aligned}
$$

Hence

$$
\begin{aligned}
F(H([\daleth(\Theta)]_\alpha,[\daleth(\Phi)]_\alpha) &\leq F(H([\daleth(\Theta)]_\alpha,[\daleth(\Phi)]_\alpha)) - \tau(H([\daleth(\Theta)]_\alpha,[\daleth\Phi)]_\alpha)) \\
&< F(H([\daleth(\Theta)]_\alpha,[\daleth(\Phi)]_\alpha))
\end{aligned}
$$

a contradiction. Consequently (2) is verified.

To verify (3): First we are to prove that $Fuz(\gimel)\bigcap Fuz(\daleth)$ is nonempty. If $\beth = [S(\Theta)]_\alpha = [T(\Theta)]_\alpha$, then we get $[\gimel(\beth)]_\alpha = [\gimel([\daleth(\Theta)]_\alpha)]_\alpha = [\daleth[\gimel(\Theta)]_\alpha]_\alpha = [S(\beth)]_\alpha$ which implies that $\beth \in C_F(\daleth,\gimel)$. Hence the Pompeiu-Hausdorff weight assign to $[\daleth(\Theta)]_\alpha$ and $[S(\beth)]_\alpha$ is zero by (2). Thus $\beth = [\daleth(\beth)]_\alpha = [\gimel(\beth)]_\alpha$ that is $\beth \in Fuz(\gimel)\bigcap Fuz(\daleth)$. As $C_F(\daleth,\gimel)$ is a singleton set, so that $Fuz(\gimel)\bigcap Fuz(\daleth)$ is also singleton.

Finally, we verify (4): Let us assume that the set $Fuz(\gimel)\bigcap Fuz(\daleth)$ is complete. Now need to show that $Fuz(\gimel)\bigcap Fuz(\daleth)$ is singleton. On contrary, assume that there exists $\Theta,\Phi \in CB(\Psi)$ such that $\Theta,\Phi \in Fuz(T)\bigcap Fuz(\daleth)$ and $\Theta \neq \Phi$. By completeness of $Fuz(\daleth)\bigcap Fuz(\gimel)$, there exists an edge between Θ and Φ. As the pair $(\daleth,\gimel)$ is generalized graphic fuzzy F-contractive mappings, so we get

$$
\begin{aligned}
\tau(M(\Theta,\Phi)) + F(H(\Theta,\Phi)) &= \tau(M(\Theta,\Phi)) + F(H([\daleth(\Theta)]_\alpha,[\daleth(\Phi)]_\alpha)) \\
&\leq F(M(\Theta,\Phi)),
\end{aligned}
$$

where

$$
\begin{aligned}
M(\Theta,\Phi) &= \max\{H([\gimel(\Theta)]_\alpha),([\gimel(\Phi)]_\alpha), H([\daleth(\Theta)]_\alpha,[\gimel(\Theta)]_\alpha), H([\daleth(\Phi)]_\alpha,[\gimel(\Phi)]_\alpha), \\
&\quad \frac{H([\daleth(\Theta)]_\alpha,[\gimel(\Phi)]_\alpha) + H([\daleth(\Phi)]_\alpha,[\gimel(\Theta)]_\alpha}{2}\} \\
&= \max\{H(\Theta,\Phi), H(\Theta,\Theta), H(\Phi,\Phi), \frac{H(\Theta,\Phi)+H(\Phi,\Theta)}{2}\} \\
&= H(\Phi,\Theta).
\end{aligned}
$$

Thus

$$
F(H(\Theta,\Phi)) \leq F(H(\Theta,\Phi)) - \tau(H(\Theta,\Phi)),
$$

a contradiction. Hence $\Theta = \Phi$. Conversely, if $Fuz(\daleth)\bigcap Fuz(\gimel)$ is a singleton, then since $E(G) \supseteq \Delta$, implies $Fuz(\daleth)\bigcap Fuz(\gimel)$ is a complete set. □

Example 3.2. *Let* $\Psi = \{2n : n \in \{1, 2, 3, ..., m\}\} = V(G)$*,* $m \geq 1$*,* $E(G) = \{(i, j) \in \Psi \times \Psi : i < j\}$ *and* $\varphi : V(G) \times V(G) \to \mathbb{R}^+$ *be defined by*

$$\varphi(\xi, \zeta) = \begin{cases} 0 & if\ \xi = \zeta \\ \frac{1}{2n} & if\ \xi \in \{2, 4\}\ with\ \xi \neq \zeta \\ \frac{2n}{2n+1} & otherwise. \end{cases}$$

Moreover, the Pompeiu-Hausdorff metric is stated by

$$H(\Lambda, \Upsilon) = \begin{cases} \frac{1}{2n} & if\ \Lambda, \Upsilon \subseteq \{2, 4\}\ with\ \Lambda \neq \Upsilon \\ \frac{2n}{2n+1} & if\ \Lambda\ or\ \Upsilon (or\ both) \subsetneq \{2, 4\}\ with\ \Lambda \neq \Upsilon \\ 0 & if\ \Lambda = \Upsilon. \end{cases}$$

The Pompeiu-Hausdorff weights $(for\ n = 4)$ *assigned to* $\Lambda, \Upsilon \in CB(\Psi)$ *are exhibited in Figure* 1.

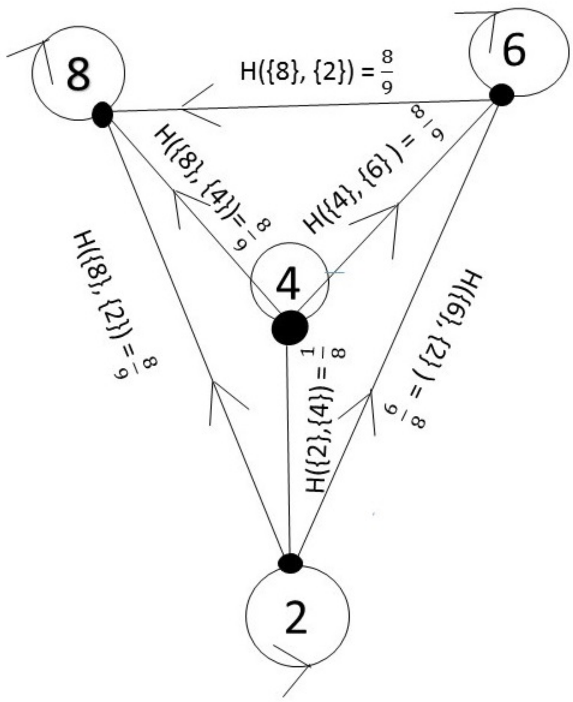

Figure 1. The Pompeiue-Hausdorff weights ($n = 4$).

Now we define $\daleth, \gimel : CB(\Psi) \to I_{Fc}(\Psi)$ *as follows.*
For $\Theta \subseteq \{2, 4\}$,

$$\daleth(\Theta)(\Phi) = \begin{cases} 1 & if\ \Phi = \{2\} \\ \frac{1}{2} & if\ \Phi \neq \{2\} \\ 0 & elswhere. \end{cases}$$

For $\Theta \subsetneq \{2,4\}$,

$$\daleth(\Theta)(\Phi) = \begin{cases} 1 & \textit{if } \Phi = \{2,4\} \\ \frac{1}{2} & \textit{if } \Phi \neq \{2,4\} \\ 0 & \textit{elswhere.} \end{cases}$$

Now

$$[\daleth(\Theta)]_1 = \{\Phi : \daleth(\Theta)(\Phi) = 1\} = \begin{cases} \{2\} & \textit{if } \Theta \subseteq \{2,4\} \\ \{2,4\} & \textit{if } \Theta \subsetneq \{2,4\}. \end{cases}$$

And for $\Theta = \{2\}$,

$$\gimel(\Theta)(\Phi) = \begin{cases} 1 & \textit{if } \Phi = \{2\} \\ \frac{1}{2} & \textit{if } \Phi \neq \{2\} \\ 0 & \textit{elswhere.} \end{cases}$$

For $\Theta \subseteq \{4,6\}$,

$$\gimel(\Theta)(\Phi) = \begin{cases} 1 & \textit{if } \Phi = \{2,4,6\} \\ \frac{1}{2} & \textit{if } \Phi \neq \{2,4,6\} \\ 0 & \textit{elswhere.} \end{cases}$$

and for $\Theta \subsetneq \{2,4,6\}$,

$$\gimel(\Theta)(\Phi) = \begin{cases} 1 & \textit{if } \Phi = \{2,4,6,...,2n\} \\ \frac{1}{2} & \textit{if } \Phi \neq \{2,4,6,...,2n\} \\ 0 & \textit{elswhere.} \end{cases}$$

Now

$$[\gimel(\Theta)]_1 = \{\Phi : \gimel(\Theta)(\Phi) = 1\} = \begin{cases} \{2\} & \textit{if } \Theta = \{2\} \\ \{2,4,6\} & \textit{if } \Theta \subseteq \{4,6\} \\ \{2,4,6...,2n\} & \textit{if } \Theta \subsetneq \{2,4,6\}. \end{cases}$$

Note that for all $\Phi \in CB(\Psi)$, $(\Phi, [\daleth(\Phi)]_\alpha) \subseteq E(G)$ *and* $(\Phi, [\gimel(\Phi)]_\alpha) \subseteq E(G)$.
Take

$$F(\beta) = \ln(\beta) + \beta$$

and

$$\tau(t) = \begin{cases} \frac{8}{11} & \textit{if } t \in [0,1] \\ \ln(1+2t) & \textit{if } t > 1. \end{cases}$$

Now for $\Theta_1, \Theta_2 \in CB(\Psi)$ *with* $[\daleth(\Theta_1)]_\alpha \neq [S(\Theta_2)]_\alpha$, *consider the following cases:*

(1) *For* $\alpha = 1$ *and* $\Theta_1 \subseteq \{2,4\}$ *and* $\Theta_2 = \{6\}$ *with* $(\Theta_1, \Theta_2) \in E(G)$, *we have*

$$\begin{aligned} & H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) e^{H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) - M(\Theta_1,\Theta_2)} \\ & \leq H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) e^{H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) - H([\daleth(\Theta_2)]_1, [\gimel(\Theta_2)]_1)} \\ & \leq H([\{2\}, \{2,4\}) e^{H([\{2\},\{2,4\}) - H([\{2\},\{2,4,6\})} < \frac{1}{2n} e^{\frac{1}{2n} - \frac{2n}{2n+1}} \\ & < \frac{2n}{2n+1} e^{-\frac{8}{11}} = e^{-\frac{8}{11}} \frac{2n}{2n+1} \end{aligned}$$

$$= e^{-\frac{8}{11}}H(\{2,4\},\{2,4,6\})$$
$$= e^{-\frac{8}{11}}H([\daleth(\Theta_2)]_1,[\gimel(\Theta_2)]_1)$$
$$\leq e^{-\tau(M(\Theta_1,\Theta_2))}M(\Theta_1,\Theta_2).$$

(2) *For* $\alpha = 1$ *and* $\Theta_1 \subsetneq \{2,4\}$ *and* $\Theta_2 \subseteq \{2,4,6\}$ *with* $(\Theta_1,\Theta_2) \in E(G)$ *then consider*

$$H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)e^{H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)-M(\Theta_1,\Theta_2)}$$
$$\leq H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)e^{H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)-H([\daleth(\Theta_2)]_1,[\gimel(\Theta_2)]_1)}$$
$$\leq H([\{2\},\{2,4\})e^{H([\{2\},\{2,4\})-H([\{2,4\},\{2,4,6...,2n\})}$$
$$< \frac{1}{2n}e^{\frac{1}{2n}-\frac{2n}{2n+1}} < \frac{2n}{2n+1}e^{-\frac{8}{11}} = e^{-\frac{8}{11}}\frac{2n}{2n+1}$$
$$= e^{-\frac{8}{11}}H(\{2,4\},\{2,4,6...2n\})$$
$$= e^{-\frac{8}{11}}H([\daleth(\Theta_2)]_1,[\gimel(\Theta_2)]_1)$$
$$\leq e^{-\tau(M(\Theta_1,\Theta_2))}M(\Theta_1,\Theta_2).$$

(3) *For* $\alpha = 1$ *and* $\Theta_1 = \{6\}$ *and* $\Theta_2 \subseteq \{2,4\}$ *with* $(\Theta_1,\Theta_2) \in E(G)$ *then consider*

$$H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)e^{H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)-M(\Theta_1,\Theta_2)}$$
$$\leq H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)e^{H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)-H([\daleth(\Theta_2)]_1,[\gimel(\Theta_2)]_1)}$$
$$\leq H([\{2,4\},\{2\})e^{H([\{2,4\},\{2\})-H([\{2,4\},\{2,4,6\})} < \frac{1}{2n}e^{\frac{1}{2n}-\frac{2n}{2n+1}}$$
$$< \frac{2n}{2n+1}e^{-\frac{8}{11}} = e^{-\frac{8}{11}}H(\{2,4\},\{2,4,6\})$$
$$= e^{-\frac{8}{11}}H([\daleth(\Theta_1)]_1,[\gimel(\Theta_1)]_1)$$
$$\leq e^{-\tau(M(\Theta_1,\Theta_2))}M(\Theta_1,\Theta_2).$$

(4) *For* $\alpha = 1$ *and* $\Theta_1 \subsetneq \{2,4,6\}$ *and* $U_2 \subseteq \{2,4\}$ *with* $(\Theta_1,\Theta_2) \in E(G)$ *then consider*

$$H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)e^{H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)-M(\Theta_1,\Theta_2)}$$
$$\leq H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)e^{H([\daleth(\Theta_1)]_1,[\daleth(\Theta_2)]_1)-H([\daleth(\Theta_2)]_1,[\gimel(\Theta_2)]_1)}$$
$$\leq H([\{2,4\},\{2\})e^{H([\{2,4\},\{2\})-H([\{2,4\},\{2,4,6...,2n\})}$$
$$< \frac{1}{2n}e^{\frac{1}{2n}-\frac{2n}{2n+1}} < \frac{2n}{2n+1}e^{-\frac{8}{11}}$$
$$= e^{-\frac{8}{11}}\frac{2n}{2n+1} = e^{-\frac{8}{11}}H(\{2,4\},\{2,4,6,...,2n\})$$
$$= e^{-\frac{8}{11}}H([\daleth(\Theta_1)]_1,[\gimel(\Theta_1)]_1)$$
$$\leq e^{-\tau(M(\Theta_1,\Theta_2))}M(\Theta_1,\Theta_2).$$

Hence, for all $\Theta_1,\Theta_2 \in CB(\Psi)$ *having edge between* Θ_1 *and* Θ_2*, (2.2) is satisfied. Thus all the conditions of Theorem 3.1 are satisfied. Furthermore* $\{2\}$ *is the common fuzzy fixed point of* $\daleth$ *and* $\gimel$*, and* $Fuz(\daleth)\bigcap Fuz(\gimel)$ *is complete.*

The following Example will show that it is not necessary the given graph $(V(G),E(G))$ will always be complete.

Example 3.3. *Let* $\Psi = \{2n : n \in \{1, 2, 3, ..., m\}\} = V(G)$, $m \geq 3$, $E(G) = \{(2, 2), (4, 4), (6, 6), ..., (2n, 2n), (2, 4), (2, 6), ..., (2, 2n)\}$ *and* $\varphi : V(G) \times V(G) \to \mathbb{R}^+$ *and Pompeiu-Hausdorff metric are same as explained in Example 3.2. The Pompieu Hausdorff weights for* $(n = 4)$ *assigned to* $\Lambda, \Upsilon \in CB(\Psi)$ *are exhibited in the Figure 2.*

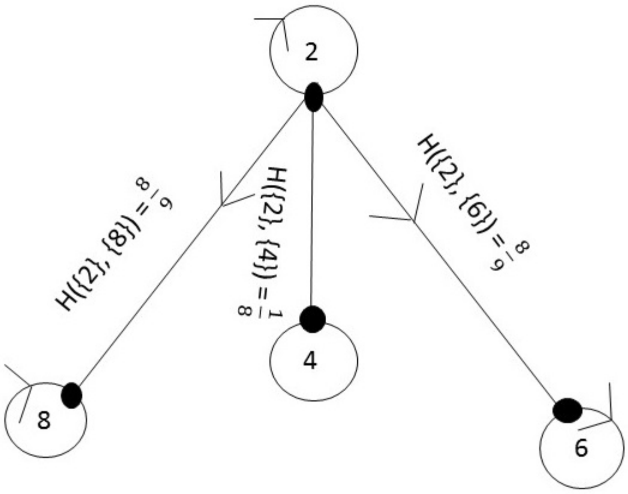

Figure 2. The Pompeiu-Haudorff weights $(n = 4)$ assigned to $\Lambda, \Upsilon \in CB(\Psi)$.

Now we define $\daleth, \gimel : CB(\Psi) \to I_{Fc}(\Psi)$ *as follows.*
For $\Theta = \{2\}$,

$$\daleth(\Theta)(\Phi) = \begin{cases} 1 & if\, \Phi = \{2\} \\ \frac{1}{2} & if\, \Phi \neq \{2\} \\ 0 & elswhere. \end{cases}$$

For $\Theta \neq \{2\}$,

$$\daleth(\Theta)(\Phi) = \begin{cases} 1 & if\, \Phi = \{2, 4\} \\ \frac{1}{2} & if\, \Phi \neq \{2, 4\} \\ 0 & elswhere. \end{cases}$$

Now

$$[\daleth(\Theta)]_1 = \{\Phi : \daleth(\Theta)(\Phi) = 1\} = \begin{cases} \{2\} & if\, \Theta = \{2\} \\ \{2, 4\} & if\, \Theta \neq \{2\}. \end{cases}$$

Also for $\Theta = \{2\}$,

$$\gimel(\Theta)(\Phi) = \begin{cases} 1 & if\, \Phi = \{2\} \\ \frac{1}{2} & if\, \Phi \neq \{2\} \\ 0 & elswhere. \end{cases}$$

and for $\Theta \neq \{2\}$,

$$\gimel(\Theta)(\Phi) = \begin{cases} 1 & \text{if } \Phi = \{2, 4, 6, ..., 2n\} \\ \frac{1}{2} & \text{if } \Phi \neq \{2, 4, 6, ..., 2n\} \\ 0 & \text{elswhere.} \end{cases}$$

Now

$$[\gimel(\Theta)]_1 = \{\Phi : \gimel(\Theta)(\Phi) = 1\} = \begin{cases} \{2\} & \text{if } \Theta = \{2\} \\ \{2, 4, 6, ..., 2n\} & \text{if } \Theta \neq \{2\}. \end{cases}$$

Note that for all $\Phi \in CB(\Psi)$, $(\Phi, [\daleth(\Phi)]_\alpha) \subseteq E(G)$ *and* $(\Phi, [\gimel(\Phi)]_\alpha) \subseteq E(G)$.
Take

$$F(\beta) = \ln(\beta) + \beta$$

and

$$\tau(t) = \begin{cases} \frac{2}{3} & \text{if } t \in [0, 1] \\ \ln(1 + t^2) & \text{if } t > 1. \end{cases}$$

Now, we consider the following cases:

(1) *For* $\alpha = 1$ *and* $\Theta_1 = \{2\}$ *and* $\Theta_2 \neq \{2\}$ *with* $(\Theta_1, \Theta_2) \in E(G)$ *then consider*

$$\begin{aligned} & H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) e^{H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) - M(\Theta_1, \Theta_2)} \\ & \leq H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) e^{H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) - H([\daleth(\Theta_2)]_1, [\gimel(\Theta_2)]_1)} \\ & < \frac{1}{2n} e^{\frac{1}{2n} - \frac{2n}{2n+1}} < \frac{2n}{2n+1} e^{-\frac{2}{3}} = e^{-\frac{2}{3}} \frac{2n}{2n+1} \\ & = e^{-\frac{2}{3}} H([\daleth(\Theta_2)]_1, [\gimel(\Theta_2)]_1) \\ & \leq e^{-\tau(M(\Theta_1, \Theta_2))} M(\Theta_1, \Theta_2). \end{aligned}$$

(2) *For* $\alpha = 1$ *and* $\Theta_1 \neq \{2\}$ *and* $\Theta_2 = \{2\}$ *with* $(\Theta_1, \Theta_2) \in E(G)$ *then consider*

$$\begin{aligned} & H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) e^{H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) - M(\Theta_1, \Theta_2)} \\ & \leq H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) e^{H([\daleth(\Theta_1)]_1, [\daleth(\Theta_2)]_1) - H([\daleth(\Theta_1)]_1, [\gimel(\Theta_1)]_1)} \\ & < \frac{1}{2n} e^{\frac{1}{2n} - \frac{2n}{2n+1}} < \frac{2n}{2n+1} e^{-\frac{2}{3}} \\ & = e^{-\frac{2}{3}} \frac{2n}{2n+1} = e^{-\frac{2}{3}} H([\daleth(\Theta_2)]_1, [\gimel(\Theta_2)]_1) \\ & \leq e^{-\tau(M(\Theta_1, \Theta_2))} M(\Theta_1, \Theta_2). \end{aligned}$$

Hence for all $\Theta_1, \Theta_2 \in CB(\Psi)$ *having an edge between* Θ_1 *and* Θ_2, *(2.2) is fulfilled. Hence all the conditions of Theorem 3.1 are fulfilled. Furthermore,* $\daleth$ *and* $\gimel$ *have a common fuzzy fixed point and* $Fuz(\daleth) \bigcap Fuz(\gimel)$ *is complete in* $CB(\Psi)$.

Theorem 3.4. *Let* (Ψ, φ) *be an* ε*-chainable complete metric space for* $\varepsilon > 0$ *and* $\daleth, \gimel : CB(\Psi) \to I_{Fc}(\Psi)$ *be fuzzy set valued-mappings. Assume that for all* $\Lambda, \Upsilon \in CB(\Psi)$ *and* $\alpha \in (0, 1]$ *such that* $0 < H([\daleth(\Lambda)]_\alpha, [\daleth(\Upsilon)]_\alpha) < \varepsilon$, *there exists a mapping* $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ *with*

$$\liminf_{\rho \to t^+} \tau(\rho) > 0$$

for all $t > 0$ *such that*

$$\tau(M(\Lambda, \Upsilon)) + F(H([\daleth(\Lambda)]_\alpha, [\daleth(\Upsilon)]_\alpha) \leq F(M(\Lambda, \Upsilon))$$

holds, where $F \in F$ and

$$M(\Lambda, \Upsilon) = \max\{H([\gimel(\Lambda)]_{\alpha}, [\gimel(\Upsilon)]_{\alpha}), H([\daleth(\Lambda)]_{\alpha}, [\gimel(\Lambda)]_{\alpha}), H([\daleth(\Upsilon)]_{\alpha}, [\gimel(\Upsilon)]_{\alpha}), \frac{H([\daleth(\Lambda)]_{\alpha}, [\gimel(\Upsilon)]_{\alpha}) + H([\daleth(\Upsilon)]_{\alpha}, [\gimel(\Lambda)]_{\alpha})}{2}\}.$$

Then $\daleth$ and $\gimel$ possess common fuzzy fixed point provided that $\daleth$ and $\gimel$ are weakly compatible.

Proof. We are given that $\daleth, \gimel : CB(\Psi) \to I_{Fc}(\Psi)$ be fuzzy set valued-mappings and $0 < H([\daleth(\Lambda)]_{\alpha}, [\daleth(\Upsilon)]_{\alpha}) < \varepsilon$ this implies that $H(\Lambda, \Upsilon) < \varepsilon$, now by Lemma 2.12 for each $\kappa \in \Lambda$, an element $\varrho \in \Upsilon$ can be chosen such that $\varphi(\kappa, \varrho) < \varepsilon$. Consider the graph G with $V(G) = \Psi$ and $E(G) = \{(\kappa, \varrho) \in \Psi \times \Psi : 0 < \varphi(\kappa, \varrho) < \varepsilon\}$.

Then ε-chainablity of (Ψ, φ) means that G is connected, the connectedness implies that for $\Lambda, \Upsilon \in CB(\Psi)$ we have $(\Lambda, \Upsilon) \subset E(G)$, therefore by hypothesis of the theorem we have

$$\tau(M(\Lambda, \Upsilon)) + F(H([\daleth(\Lambda)]_{\alpha}, [\daleth(\Upsilon)]_{\alpha})) \leq F(M(\Lambda, \Upsilon))$$

holds, where $F \in F$ and

$$M(\Lambda, \Upsilon) = \max\{H([\gimel(\Lambda)]_{\alpha}, [\gimel(\Upsilon)]_{\alpha}), H([\daleth(\Lambda)]_{\alpha}, [\gimel(\Lambda)]_{\alpha}), H([\daleth(\Upsilon)]_{\alpha}, [\gimel(\Upsilon)]_{\alpha}), \frac{H([\daleth(\Lambda)]_{\alpha}, [\gimel(\Upsilon)]_{\alpha}) + H([\daleth(\Upsilon)]_{\alpha}, [\gimel(\daleth)]_{\alpha})}{2}\}.$$

This shows that the pair $(\daleth, \gimel)$ is generalized graphic fuzzy F-contractive mappings. Also G has (P^*) property. Indeed if $\{\Psi_n\}$ is in $CB(\Psi)$ with $\Psi_n \to \Psi$ as $n \to \infty$ and $(\Psi_n, \Psi_{n+1}) \subset E(G)$ for $n \in \mathbb{N}$ means that there is subsequence $\{\Psi_{n_k}\}$ of $\{\Psi_n\}$ such that $(\Psi_{n_k}, \Psi) \subset E(G)$ for $n \in \mathbb{N}$. By employing Theorem 3.1(3), $\daleth$ and $\gimel$ possess a common fuzzy fixed point. □

Corollary 3.5. *Let (Ψ, φ) be a metric space equipped with a directed graph G with $V(G) = \Psi$ and $E(G) \supseteq \Delta$. Assume that $\daleth : CB(\Psi) \to I_{Fc}(\Psi)$ holds the following:*

1) *For every Φ in $CB(\Psi)$, $(\Phi, [\daleth(\Phi)]_{\alpha}) \subset E(G)$.*
2) *there exists a $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ with*

$$\liminf_{\rho \to t^+} \tau(\rho) > 0$$

for all $t \geq 0$ there is an edge between Λ, $\Upsilon \in CB(\Psi)$ with $[\daleth(\Lambda)]_{\alpha} \neq [\daleth(\Upsilon)]_{\alpha}$ such that
$\tau(M(\Lambda, \Upsilon)) + F(H([\daleth(\Lambda)]_{\alpha}, [\daleth(\Upsilon)]_{\alpha})) \leq F(M(\Lambda, \Upsilon))$
holds, where $F \in F$

$$M(\Lambda, \Upsilon) = \max\{H((\Lambda, \Upsilon), H(\Lambda, [\daleth(\Lambda)]_{\alpha}), H([\Upsilon, [\daleth(\Upsilon)]_{\alpha}), \frac{H(\Lambda, [\daleth(\Upsilon)]_{\alpha}) + H(\Upsilon, [\daleth(\Lambda)]_{\alpha})}{2}\}.$$

Then following statements satisfy.

(i) *$Fuz(\daleth)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $\Theta, \Phi \in Fuz(\daleth)$ is 0.*

(ii) *If the weakly connected graph G satisfies the property P^*, then $\daleth$ has fuzzy fixed point.*

(iii) *$Fuz(\daleth)$ is complete if and only if $Fuz(\daleth)$ is singleton.*

Proof. Take $\daleth = I$ (Identity map) in (2.2), then the Corollary 3.5 follows from Theorem 3.1. □

Remark 3.6. *Next, we deduce some consequences and comparative results of our main Theorem in the frame work of both single-valued and set-valued mappings. First we present set-valued analogues of Theorem 3.1.*

Corollary 3.7. *Let (Ψ, φ) be a metric space equipped with a directed graph G with $V(G) = \Psi$, $E(G) \supseteq \Delta$ and the relation R on $I_{Fc}(\Psi)$ is transitive. Suppose that $S, T : CB(\Psi) \to CB(\Psi)$ be set-valued mappings with $S(\Psi) \supseteq T(\Psi)$ and following statements are satisfied.*

1) *For any U in $CB(\Psi)$, $(S(U), U) \subseteq E(G)$ and $(U, T(U)) \subseteq E(G)$.*
2) *There is a function $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ with $\liminf_{\eta \to t^+} \tau(\eta) > 0$ for all $t \geq 0$ such that for $F \in F$ there is an edge between A and B with $S(A) \neq S(B)$ such that*

$$\tau(M(A, B)) + F(H(S(A), S(B))) \leq F(M(A, B))$$

holds, where

$$\begin{aligned} M(A, B) \;=\; & \max\{H(T(A), T(B)), H(S(A), T(A)), \\ & H(S(B), [T(B)), \frac{H(S(A), T(B)) + H(S(B), T(A))}{2}\}. \end{aligned}$$

Then the following statements hold

(i) *$CP(S, T) \neq \emptyset$ given that G is weakly connected which holds the property (P^*) and $T(\Psi)$ is a complete subspace of $CB(\Psi)$.*
(ii) *If $CP(S, T)$ is complete, then the Hausdorff weight assigned to $S(U)$ and $S(V)$ is 0 for some and for all $\Theta, \Phi \in CP(S, T)$.*
(iii) *If $CP(S, T)$ is complete and S, T are weakly compatible, then $Fix(S) \bigcap Fix(T)$ is singleton.*
(iv) *$Fix(S) \bigcap Fix(T)$ is complete if and only if $Fix(S) \bigcap Fix(T)$ is singleton.*

Proof. Consider the mappings $\omega, \iota : \Psi \to (0, 1]$ and fuzzy set-valued maps $\urcorner, \daleth : CB(\Psi) \to I_{Fc}(\Psi)$ defined by

$$\urcorner(\Lambda)(B) = \begin{cases} \omega x, & \text{if } B = \{S(A)\} \\ 0, & \text{if } B \neq \{S(A)\}. \end{cases}$$

and

$$\daleth(\Upsilon)(C) = \begin{cases} \iota y, & \text{if } C = \{T(B)\} \\ 0, & \text{if } C \neq \{T(B)\}. \end{cases}$$

Now for $\alpha(x) = \omega x \in (0, 1]$ for all $x \in \Psi$, we have

$$[\urcorner\Lambda]_{\alpha(x)} = \{B \in CB(\Psi) : \urcorner(\Lambda)(B) \geq \alpha(x)\} = \{S(A)\}$$

and for $\alpha(y) = \iota y \in (0, 1]$ for all $y \in \Psi$, we have

$$[\daleth B]_{\alpha(y)} = \{C \in CB(\Psi) : \daleth(y)(C) \geq \alpha(y)\} = \{T(B)\}.$$

Consequently by using Theorem 3.1, the result follows. □

Remark 3.8. *Let $\mathfrak{f}, \mathfrak{g} : X \to X$ be self mappings, by $F_{\mathfrak{f}}, F_{\mathfrak{g}}$ we mean the set of all fixed points of $\mathfrak{f}, \mathfrak{g}$, respectively, while by $cp(\mathfrak{f}, \mathfrak{g})$ we mean the set of coincidence points of $\mathfrak{f}$ and $\mathfrak{g}$. With this setting, we obtain the following result for single-valued maps.*

Corollary 3.9. *Let (Ψ, φ) be a metric space equipped with a directed graph G with $V(G) = \Psi$, $E(G) \supseteq \Delta$ and the relation R on Ψ is transitive. Let $\mathfrak{f}, \mathfrak{g} : \Psi \to \Psi$ be self maps satisfy the following.*

1) *For any μ in Ψ, $(\mathfrak{f}(\mu), \mu) \in E(G)$ and $(\mu, \mathfrak{g}(\mu) \in E(G)$.*
2) *There is a function $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ with $\liminf_{\eta \to t^+} \tau(\eta) > 0$ for all $t \geq 0$ such that for $F \in F$ there is an edge between a and b with $\mathfrak{f}(a) \neq \mathfrak{f}(b)$ such that*

$$\tau(M(a, b)) + F(\varphi(\mathfrak{f}(a), \mathfrak{f}(b))) \leq F(M(a, b))$$

holds, where

$$M(a, b) = \max\{\varphi(\mathfrak{g}(a), \mathfrak{g}(b)), \varphi(\mathfrak{f}(a), \mathfrak{g}(a)), \varphi(\mathfrak{f}(b), \mathfrak{g}(b)), \frac{\varphi(\mathfrak{f}(a), \mathfrak{g}(b)) + \varphi(\mathfrak{f}(b), \mathfrak{g}(a))}{2}\}.$$

Then the following statements satisfy with $\mathfrak{f}(\Psi) \supseteq \mathfrak{g}(\Psi)$.

i) *$cp(\mathfrak{f}, \mathfrak{g}) \neq \emptyset$ given that G is weakly connected which holds the property (P) and $\mathfrak{f}(\Psi)$ is a complete subspace of Ψ.*
ii) *If $cp(\mathfrak{f}, \mathfrak{g})$ is complete, then $\mathfrak{f}(\mu) = \mathfrak{f}(\nu)$ for all $\mu, \nu \in cp(\mathfrak{f}, \mathfrak{g})$.*
iii) *If $cp(\mathfrak{f}, \mathfrak{g})$ is complete and $\mathfrak{f}$, $\mathfrak{g}$ are weakly compatible, then $F_{\mathfrak{f}} \bigcap F_{\mathfrak{g}}$ is singleton.*
iv) *$F_{\mathfrak{f}} \bigcap F_{\mathfrak{g}}$ is complete if and only if $F_{\mathfrak{f}} \bigcap F_{\mathfrak{g}}$ is singleton.*

Proof. Consider the mappings $\omega, \iota : \Psi \to (0, 1]$ and a fuzzy set-valued maps $\urcorner, \beth : CB(\Psi) \to I_{Fc}\Psi$ defined by

$$\urcorner(\Lambda)(B) = \begin{cases} \omega x, & \text{if } B = \{\mathfrak{f}x\} \\ 0, & \text{if } B \neq \{\mathfrak{f}x\} \end{cases}$$

and

$$\beth(\Upsilon)(D) = \begin{cases} \iota y, & \text{if } D = \{\mathfrak{g}y\} \\ 0, & \text{if } D \neq \{\mathfrak{g}y\}. \end{cases}$$

Now for $\alpha(x) = \omega x \in (0, 1]$ for all $x \in \Psi$, we have

$$[\urcorner\Lambda]_{\alpha(x)} = \{B \in \Psi : \urcorner(\Lambda)(B) \geq \alpha(x)\} = \{\mathfrak{f}x\}$$

and for $\alpha(y) = \iota y \in (0, 1]$ for all $y \in \Psi$, we have

$$[\beth\Upsilon]_{\alpha(y)} = \{D \in \Psi : \beth(\Upsilon)(D) \geq \alpha(y)\} = \{\mathfrak{g}y\}.$$

Clearly $\{\mathfrak{f}x\}, \{\mathfrak{g}y\} \in CB(\Psi)$ for all $x, y \in \Psi$. Also, note that in this case $H([\urcorner(\Lambda)]_{\alpha(x)}, [\beth(\Upsilon)]_{\alpha(y)}) = \varphi(\mathfrak{f}x, \mathfrak{g}y)$ for all $x, y \in \Psi$. Consequently, by using Theorem 3.1, the result follows. □

4. Application to nonlinear integral equations

Consider the following nonlinear integral equation:

$$x(t) = f_1(t) - f_2(t) + \mu \int_a^t m(t,s)g(s,x(s))ds + \lambda \int_a^\infty k(t,s)h(s,x(s))ds \tag{4.1}$$

for all $t \in [0,\infty)$, where $f_1, f_2 \in L[a,\infty)$ are known such that $f_1(t) \geq f_2(t)$, and $m(t,s), k(t,s), g(s,s(x))$, $h(s,y(s))$ are real or complex valued function that are measurable both in t and s on $[0,\infty)$ and λ, μ are real or complex numbers. These functions satisfy the following.

(C_1) $\int_a^\infty \sup_{a\leq s} |m(t,s)|\, dt = M_1 < \infty$;
(C_2) $\int_a^\infty \sup_{a\leq s} |k(t,s)|\, dt = M_2 < \infty$;
(C_3) $g(s,x(s)) \in L[a,\infty)$ for all $x \in L[a,\infty)$ and there exists $K_1 > 0$ such that for all $s \in [a,\infty)$,

$$|g(s,x(s)) - g(s,y(s))| \leq K_1 |x(s) - y(s)|, \quad \text{for all } x,y \in L[a,\infty);$$

(C_4) $h(s,x(s)) \in L[a,\infty)$ for all $x \in L[a,\infty)$ and there exists $K_2 > 0$ such that for all $s \in [a,\infty)$,

$$|h(s,x(s)) - h(s,y(s))| \leq K_2 |x(s) - y(s)|, \quad \text{for all } x,y \in L[a,\infty).$$

The existence theorem regarding the solution of above nonlinear integral equation can be formulated as follows:

Theorem 4.1. *With the assumption $(C_1) - (C_4)$ if the following conditions are also satisfied.*

(a) $\lambda \int_a^\infty k(t,s)h(s, \mu \int_a^s m(s,\tau)g(\tau,x(\tau))d\tau + f_1(s) - f_2(s))ds = 0.$
(b) *For $x \in L[a,\infty)$,*

$$\begin{aligned}\mu \int_a^t m(t,s)g(s,x(s))ds &= x(t) - f_1(t) + f_2(t) - \lambda \int_a^\infty k(t,s)h(s,x(s))ds \\ &= \Gamma(t) \in L[a,\infty).\end{aligned}$$

(c) *For $\Gamma(t) \in L[a,\infty)$ there exists $\Theta(t) \in L[a,\infty)$ such that*

$$\begin{aligned}\mu \int_a^t m(t,s)g(s,x(s)-\Gamma(t))ds - f_2(t) &= f_1(t) + \lambda \int_a^\infty k(t,s)h(s,x(s)-\Gamma(t)-f_2(t))ds \\ &= \Theta(t),\end{aligned}$$

then the Eq (4.1) has a unique solution in $L[a,\infty)$ for the pair of real or complex numbers λ and μ with $|\lambda| K_2 M_2 < 1$ and $\frac{|\mu| K_1 M_1}{1-|\lambda| K_2 M_2} = \alpha < 1$.

Proof. For $x(t) \in L[a,\infty)$, we define

$$\begin{aligned}\mathfrak{f}x(t) &= \mu \int_a^t m(t,s)g(s,x(s))ds - f_2(t)), \\ \mathfrak{h}x(t) &= f_1(t) + \lambda \int_a^\infty k(t,s)h(s,x(s))ds \\ \mathfrak{g}x(t) &= (I - \mathfrak{h})x(t),\end{aligned}$$

where $f_1, f_2 \in L[0,\infty)$ are known and I is the identity operator on $L[0,\infty)$. Then $\mathfrak{f}, \mathfrak{g}$, and $\mathfrak{h}$ are self maps on $L[a,\infty)$. Indeed, we have

$$\begin{aligned} |\mathfrak{f}x(t)| &\le |\mu| \int_a^\infty |m(t,s)g(s,x(s))|\, ds + |f_2(t)| \\ &\le |\mu| \sup_{a\le s<\infty} |m(t,s)| \int_a^\infty |g(s,x(s))|\, ds + |f_2(t)| \end{aligned}$$

and by using (C_1) and (C_3), we obtain

$$\begin{aligned} \int_a^\infty |\mathfrak{f}x(t)|\, dt &\le |\mu| \int_a^\infty \sup_{a\le s<\infty} |m(t,s)|\, dt \int_a^\infty |g(s,x(s))|\, ds + \int_a^\infty |f_2(t)|\, dt \\ &< \infty \end{aligned}$$

and hence $\mathfrak{f}x \in L[a,\infty)$. For mapping $\mathfrak{h}$, we apply the conditions (C_2) and (C_4) to obtain

$$\begin{aligned} \int_a^\infty |\mathfrak{h}x(t)|\, dt &\le \int_a^\infty |f_1(t)|\, dt + |\lambda| \int_a^\infty \sup_{a\le s<\infty} |k(t,s)|\, dt \int_a^\infty |h(s,x(s))|\, ds \\ &< \infty. \end{aligned}$$

Hence $\mathfrak{h} \in L[a,\infty)$ and so, $\mathfrak{g}$ is also the self map on $L[a,\infty)$. Now, by using (C_2) and (C_3), we have for all $x, y \in L[a,\infty)$ that

$$\begin{aligned} \|\mathfrak{f}x - \mathfrak{f}y\| &= \int_a^\infty |\mathfrak{f}x(t) - \mathfrak{f}y(t)|\, dt \\ &= \int_a^\infty \left| \mu \int_a^t m(t,s)g(s,x(s))ds - \mu \int_a^t m(t,s)g(s,y(s))ds \right| dt \\ &= \int_a^\infty \left| \mu \int_a^t m(t,s)[g(s,x(s)) - g(s,y(s))]ds \right| dt \\ &\le \int_a^\infty |\mu| \sup_{a\le s<\infty} |m(t,s)| \left| \int_a^\infty [g(s,x(s)) - g(s,y(s))]ds \right| dt \\ &= |\mu|\, K_1 M_1 \int_a^\infty |x(t) - y(t)|\, ds \\ &\le |\mu|\, K_1 M_1 \|x - y\|. \end{aligned} \tag{4.2}$$

Similarly, by (C_2) and (C_4), we get

$$\|\mathfrak{h}x - \mathfrak{h}y\| \le |\lambda|\, K_2 M_2 \|x - y\|.$$

Hence we have

$$\begin{aligned} \|\mathfrak{g}x - \mathfrak{g}y\| &= \|(I - \mathfrak{h})x - (I - \mathfrak{h})y\| \\ &= \|(x - y) - (\mathfrak{h}x - \mathfrak{h}y)\| \\ &\ge \|x - y\| - \|\mathfrak{h}x - \mathfrak{h}y\| \\ &\ge \|x - y\| - |\lambda|\, K_2 M_2 \|x - y\| \end{aligned}$$

$$\geq \quad (1 - |\lambda|\, K_2 M_2)\, \|x - y\|\,,$$

which implies that

$$\|x - y\| \leq \frac{1}{1 - |\lambda|\, K_2 M_2}\, \|\mathfrak{g}x - \mathfrak{g}y\|\,. \tag{4.3}$$

From Eqs (4.2) and (4.3), we obtain

$$\begin{aligned}
\|\mathfrak{f}x - \mathfrak{f}y\| &\leq |\mu|\, K_1 M_1\, \|x - y\| \\
&\leq |\mu|\, K_1 M_1 \frac{1}{1 - |\lambda|\, K_2 M_2}\, \|\mathfrak{g}x - \mathfrak{g}y\| \\
&= \frac{|\mu|\, K_1 M_1}{1 - |\lambda|\, K_2 M_2}\, \|\mathfrak{g}x - \mathfrak{g}y\| \\
&= \alpha\, \|\mathfrak{g}x - \mathfrak{g}y\|\,.
\end{aligned} \tag{4.4}$$

Now we prove that $\mathfrak{f}(L[a,\infty)) \subset \mathfrak{g}(L[a,\infty))$ so let $x(t) \in L[a,\infty)$ be arbitrary. Then we have

$$\begin{aligned}
\mathfrak{g}(\mathfrak{f}x(t) + f_1(t)) &= (I - \mathfrak{h})(\mathfrak{f}x(t) + f_1(t)) \\
&= \mathfrak{f}x(t) + f_1(t) - f_1(t) - \lambda \int_a^\infty k(t,s)h(s, \mathfrak{f}x(s) + f_1(s))ds \\
&= \mathfrak{f}x(t) - \lambda \int_a^\infty k(t,s)h(s, \mu \int_a^s m(s,\tau)g(\tau, x(\tau))d\tau + f_1(s) - f_2(s))ds \\
&= \mathfrak{f}x(t),
\end{aligned}$$

by assumption (a) of the Theorem.
Now we prove that the pair $(\mathfrak{f}, \mathfrak{g})$ is weakly compitable. For this, we have

$$\begin{aligned}
\|\mathfrak{g}\mathfrak{f}x(t) - \mathfrak{f}\mathfrak{g}x(t)\| &= \|(I - \mathfrak{h})\mathfrak{f}x(t) - \mathfrak{f}(I - \mathfrak{h})x(t)\| \\
&= \|\mathfrak{f}x(t) - \mathfrak{h}\mathfrak{f}x(t) - \mathfrak{f}x(t) + \mathfrak{f}\mathfrak{h}x(t)\| \\
&= \|\mathfrak{f}\mathfrak{h}x(t) - \mathfrak{h}\mathfrak{f}x(t)\|\,.
\end{aligned} \tag{4.5}$$

Now for $\mathfrak{f}x(t) = \mathfrak{g}x(t)$, we have

$$\mu \int_a^t m(t,s)g(s, x(s))ds - f_2(t) = x(t) - f_1(t) - \lambda \int_a^\infty k(t,s)h(s, x(s))ds.$$

Therefore from (4.5), we get

$$\begin{aligned}
\|\mathfrak{g}\mathfrak{f}x(t) - \mathfrak{f}\mathfrak{g}x(t)\| &= \Big\|\mathfrak{f}\mathfrak{h}(f_1(t) - f_2(t) + \lambda \textstyle\int_a^\infty k(t,s)h(s,x(s))ds + \mu \int_a^t m(t,s)g(s,x(s))ds) \\
&\quad -\mathfrak{h}\mathfrak{f}(f_1(t) - f_2(t) + \lambda \textstyle\int_a^\infty k(t,s)h(s,x(s))ds + \mu \int_a^t m(t,s)g(s,x(s))ds)\Big\| \\
&= \Big\|\mathfrak{f}(f_1(t) + \lambda \textstyle\int_a^\infty k(t,s)h(s,x(s)) - \Gamma(s)ds) \\
&\quad -\mathfrak{h}(\mu \textstyle\int_a^t m(t,s)g(s,x(s) - \Gamma(s))ds - f_2(t))\Big\| \\
&= \Big\|-f_2(t) + \textstyle\int_a^t m(t,s)g(s, f_1(s) + \lambda \int_a^\infty k(s,\tau)h(\tau, x(\tau) - \Gamma(\tau))d\tau)ds \\
&\quad -f_1(t) - \lambda \textstyle\int_a^\infty k(t,s)h(s, \mu \int_a^s m(s,\tau)g(\tau, x(\tau) - f_2(\tau) - \Gamma(\tau))d\tau)ds\Big\| \\
&= 0.
\end{aligned}$$

This shows that maps $\mathfrak{f}$ and $\mathfrak{g}$ are weakly compitable. Thus all the conditions of Corollary 3.9 is satisfied. Consequently, there exists a unique $x^* \in L[a,\infty)$ such that $\mathfrak{f}x^* = \mathfrak{g}x^* = x^*$ that is the unique solution of Eq (4.1). □

5. Conclusions

The results of this paper broadened the scope of fuzzy fixed point theory and fixed point theory of multi valued mappings by incorporating the generalized fuzzy graphic F-contraction approaches. The ideas in this work, being discussed in the setting of metric spaces, are completely fundamental. Hence, they can be made better, when presented in the framework of generalized metric spaces such as b-metric spaces, G-metric spaces, F-metric spaces and some other pseudo-metric or quasi metric spaces. Also, the fuzzy set-valued map's component can be extended to L-fuzzy mappings, intuitionist fuzzy mappings, soft set-valued maps, and so on.

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Conflict of interest

The authors declare that they have no competing interests concerning the publication of this article.

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