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Research article

A two-grid mixed finite volume element method for nonlinear time fractional reaction-diffusion equations

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Abstract: In this paper, a two-grid mixed finite volume element (MFVE) algorithm is presented for the nonlinear time fractional reaction-diffusion equations, where the Caputo fractional derivative is approximated by the classical $L1$-formula. The coarse and fine grids (containing the primal and dual grids) are constructed for the space domain, then a nonlinear MFVE scheme on the coarse grid and a linearized MFVE scheme on the fine grid are given. By using the Browder fixed point theorem and the matrix theory, the existence and uniqueness for the nonlinear and linearized MFVE schemes are obtained, respectively. Furthermore, the stability results and optimal error estimates are derived in detailed. Finally, some numerical results are given to verify the feasibility and effectiveness of the proposed algorithm.

Keywords: two-grid mixed finite volume element algorithm; time fractional reaction-diffusion equations; $L1$-formula; Browder fixed point theorem; error estimate
Mathematics Subject Classification: 65M08, 65M12, 65M15

1. Introduction

In this paper, we consider the following nonlinear time fractional reaction-diffusion equations with the initial and Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial^\alpha u(\boldsymbol{x},t)}{\partial t^\alpha} - \mathrm{div}(\mathcal{A}(\boldsymbol{x})\nabla u(\boldsymbol{x},t)) + g(u(\boldsymbol{x},t)) = f(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Omega \times J, \\ u(\boldsymbol{x},t)|_{\partial\Omega} = 0, & (\boldsymbol{x},t) \in \partial\Omega \times \bar{J}, \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with the boundary $\partial\Omega$, $J = (0,T]$ with $0 < T < \infty$. Assume that the functions $u_0(\boldsymbol{x})$, $g(u(\boldsymbol{x},t))$ and $f(\boldsymbol{x},t)$ are smooth enough, and there exists a constant $L > 0$ such that $|g(u)| \le L|u|$. The diffusion coefficient matrix $\mathcal{A}(\boldsymbol{x}) = (a_{ij}(\boldsymbol{x}))_{2\times 2}$ is symmetric and uniformly positive definite, that is, there exist two constants $A_*, A^* > 0$ such that

$$A_* \boldsymbol{y}^T \boldsymbol{y} \leq \boldsymbol{y}^T \mathcal{A}(\boldsymbol{x}) \boldsymbol{y} \leq A^* \boldsymbol{y}^T \boldsymbol{y}, \ \forall \boldsymbol{y} \in \mathbb{R}^2, \ \forall \boldsymbol{x} \in \bar{\Omega}.$$

Moreover, we should assume that $\mathcal{A}^{-1}(\boldsymbol{x})$ satisfies the Lipschitz condition. In (1.1), the Caputo time fractional derivative $\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial t^\alpha}$ with order $\alpha \in (0, 1)$ is defined by

$$\frac{\partial^\alpha u(\boldsymbol{x}, t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(\boldsymbol{x}, s)}{\partial s} \frac{1}{(t-s)^\alpha} \mathrm{d}s, \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function.

Fractional differential equations (FDEs) can be applied to simulate various natural phenomena in chemistry, physics and biology and so on [1–3], which have attracted great interest of more and more scholars. However, it is very difficult to obtain the exact solutions for a large number of FDEs due to the nonlocality of fractional integrals and derivatives and other reasons, such as complex nonlinear terms, initial or boundary conditions. Therefore, a lot of numerical algorithms have been proposed and applied to solve FDEs [4–23], including finite element (FE) methods, finite difference (FD) methods, finite volume/element (FV/FVE) methods, discontinuous Galerkin (DG) methods, spectral methods and so on. In this paper, we establish a two-grid algorithm to solve the nonlinear time fractional reaction-diffusion Eq (1.1).

The two-grid method is proposed and developed by Xu [24, 25] to solve nonlinear elliptic partial differential equations based on FE methods. Because of the advantage of saving computing time, many scholars have extended and applied it to integer order partial differential equations. Dawson et al. [26] presented a two-grid mixed finite element (MFE) method for nonlinear parabolic equations which arises in flow through porous media, and gave the error analysis. Yan et al. [27] proposed a two-grid FVE method for the nonlinear Sobolev equations, and obtained optimal H^1-norm error estimate. Hou et al. [28] applied a two-grid expanded MFE method to solve semi-linear parabolic integro-differential equations, and gave the convergence analysis and some numerical results. Liu [29] presented a two-grid FVE method for semi-linear reaction-diffusion system of the solutes in the groundwater flow, and obtained the error estimates in L^2-norm and H^1-norm. In recent years, the two-grid method was also applied to solve fractional partial differential equations. Liu et al. [30] proposed a two-grid MFE algorithm for a nonlinear fourth-order reaction-diffusion model with the Caputo time fractional derivative, and obtained the unconditional stability and error estimates. Liu et al. [31] presented a two-grid FE algorithm for a time fractional Cable equation, in which the Riemann-Liouville fractional derivative was approximated by the second-order weighted and shifted Grünwald difference (WSGD) scheme. Li et al. [32] constructed a two-grid expanded MFE scheme to solve a semilinear time fractional reaction-diffusion equation, in which the Caputo fractional derivative was approximated by the $L1$-formula. Li et al. [33] proposed a two-grid FE method for a nonlinear time fractional diffusion equation, and gave some numerical results to confirm the theoretical results. Chen et al. [34] studied a two-grid modified method of characteristics scheme to solve nonlinear variable-order time fractional advection-diffusion equations, and obtained the optimal L^2-norm error estimates. Liu et al. [35] presented a two-grid FE fast algorithm to solve a nonlinear space-time fractional diffusion equation, and gave the stability and convergence analysis. From the current literatures, we find that there is no report about the two-grid fast algorithm based on the mixed finite volume element (MFVE) method [36–39] for solving the FDEs.

In this paper, we will construct a two-grid MFVE algorithm to solve the nonlinear time fractional reaction-diffusion equations. In temporal discretization, we select the classical $L1$-formula to approximate the Caputo time fractional derivative. In spatial discretization, we construct coarse and fine grids (containing primal and dual grids), and establish a two-grid MFVE scheme by introducing an auxiliary variable $\boldsymbol{\lambda}$ and using the transfer operator. The calculation process is divided into two steps: firstly, the coarse solution is computed iteratively by using the nonlinear MFVE scheme on the space coarse grid, then a linearized scheme is constructed by using the coarse solution, and finally solution on the space fine grid is obtained. In our theoretical analysis, we give the existence and uniqueness results of the fully discrete solutions for the two-grid MFVE scheme by applying the Browder fixed point theorem and the matrix theory, and obtain unconditional stability results and error estimates in $L^2(\Omega)$-norm for the variable u. Moreover, we derive the conditional stability results and error estimates in $(L^2(\Omega))^2$-norm and $\boldsymbol{H}(\text{div})$-norm for the variable $\boldsymbol{\lambda}$ by using a special analytical technique. Finally, we give some numerical results to verify the feasibility and effectiveness, and find that the proposed two-grid MFVE algorithm can greatly save the computing time.

The layout of this paper is as follows: By constructing coarse and fine grids (primal and dual) and introducing the transfer operator, a two-grid MFVE algorithm for the nonlinear time fractional reaction-diffusion equation is proposed in Section 2. Some properties of the transfer operator $\gamma_\hbar$ and the fractional Gronwall inequality are given, and the existence and uniqueness results are obtained in Section 3. In Sections 4 and 5, the stability and error estimates are derived in detailed. In Section 6, two numerical examples are given to verify the feasibility and effectiveness.

2. Two-grid MFVE scheme

We shall use the standard Sobolev spaces $W^{m,p}(\Omega)$ with the norm $\|\cdot\|_{m,p}$. For $p = 2$, we define $H^m(\Omega) = W^{m,2}(\Omega)$ with the norm $\|\cdot\|_m$, and $H^0(\Omega) = L^2(\Omega)$ with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. We also use $\boldsymbol{H}(\text{div},\Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2, \text{div}\boldsymbol{v} \in L^2(\Omega)\}$ with the norm $\|\cdot\|_{\boldsymbol{H}(\text{div})}$. Furthermore, throughout this paper, the mark C is a generic positive constant, which is independent of the mesh parameters.

In order to get the MFVE scheme, by introducing an auxiliary variable $\boldsymbol{\lambda}(\boldsymbol{x},t) = -\mathcal{A}(\boldsymbol{x})\nabla u(\boldsymbol{x},t)$, we can rewrite the primal problem (1.1) as

$$\begin{cases} (a)\ \ \dfrac{\partial^\alpha u(\boldsymbol{x},t)}{\partial t^\alpha} + \text{div}\boldsymbol{\lambda}(\boldsymbol{x},t) + g(u(\boldsymbol{x},t)) = f(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Omega \times J, \\ (b)\ \ \mathcal{A}^{-1}\boldsymbol{\lambda}(\boldsymbol{x},t) + \nabla u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \Omega \times J, \\ (c)\ \ u(\boldsymbol{x},t)|_{\partial\Omega} = 0, & (\boldsymbol{x},t) \in \partial\Omega \times \bar{J}, \\ (d)\ \ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \bar{\Omega}. \end{cases} \tag{2.1}$$

Then, we can obtain the weak formulation of (2.1): Find $(\boldsymbol{\lambda}, u) \in \boldsymbol{V} \times W$ such that

$$\begin{cases} (a)\ \ (\dfrac{\partial^\alpha u}{\partial t^\alpha}, w) + (\text{div}\boldsymbol{\lambda}, w) + (g(u), w) = (f, w), & \forall w \in W, \\ (b)\ \ (\mathcal{A}^{-1}\boldsymbol{\lambda}, \boldsymbol{v}) - (\text{div}\boldsymbol{v}, u) = 0, & \forall \boldsymbol{v} \in \boldsymbol{V}, \\ (c)\ \ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \bar{\Omega}, \end{cases} \tag{2.2}$$

where $\boldsymbol{V} = \boldsymbol{H}(\text{div},\Omega)$ and $W = L^2(\Omega)$.

Now, we use $\mathscr{K}_h$ to denote a quasiuniform triangulation partition of the domain Ω, that is $\mathscr{K}_h = \cup K_B$, where K_B stands for the triangle with the barycenter B, referring to Figure 1. Let $h = \max\{h_{K_B}\}$, where h_{K_B} is the diameter of the triangle K_B. Moreover, we should define the nodes of a triangular element to be its midpoints of three sides, and mark $P_1, P_2, ..., P_{M_S}$ as the inner nodes and $P_{M_S+1}, P_{M_S+2}, ..., P_M$ as the boundary nodes.

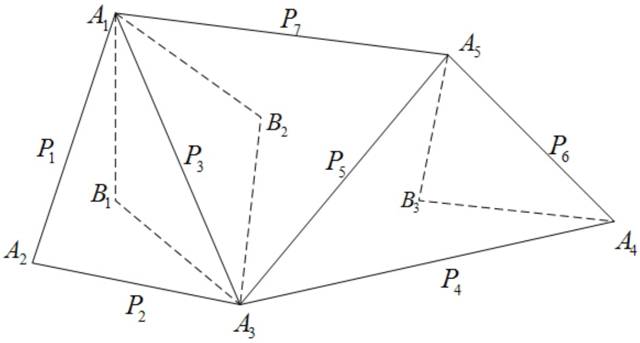

Figure 1. Primal and dual partitions.

We select the lowest order Raviart-Thomas space $\boldsymbol{V}_h$ and piecewise constant function space W_h as the trial function spaces for $\boldsymbol{\lambda}$ and u, respectively, where

$$\boldsymbol{V}_h = \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div}, \Omega) : \boldsymbol{v}_h|_K = (a + bx, c + bx), \forall K \in \mathscr{K}_h\},$$
$$W_h = \{w_h \in W : w_h|_K \in \mathcal{P}_0(K), \forall K \in \mathscr{K}_h\}.$$

Based on the primal partition $\mathscr{K}_h$, we construct the dual partition $\mathscr{K}_h^*$. Referring to Figure 1, the interior node P_3 belongs to the common side of two adjacent triangles $K_{B_1} = \Delta A_1A_2A_3$ and $K_{B_2} = \Delta A_1A_3A_5$, then we define the quadrilateral $A_1B_1A_3B_2$ to be the dual element for P_3. In general, for an interior node P, the dual element K_P^* is the union of two triangles K_L (with $\Delta A_1B_1A_3$) and K_R (with $\Delta A_1A_3B_2$). For a boundary node such as P_6, the associated dual element is a triangle K_I (with $\Delta A_5B_3A_4$).

Integrating (2.1) on all the primal and dual partitions, respectively, we obtain

$$\begin{cases} (a) \ \ \int_{K_B}(\dfrac{\partial^\alpha u(\boldsymbol{x}, t)}{\partial t^\alpha} + \mathrm{div}\boldsymbol{\lambda}(\boldsymbol{x}, t) + g(u(\boldsymbol{x}, t)))\mathrm{d}x = \int_{K_B} f(\boldsymbol{x}, t)\mathrm{d}x, \\ (b) \ \ \int_{K_P^*}(\mathcal{A}^{-1}\boldsymbol{\lambda}(\boldsymbol{x}, t) + \nabla u(\boldsymbol{x}, t))\mathrm{d}x = 0. \end{cases} \tag{2.3}$$

We define the transfer operator $\gamma_h : \boldsymbol{V}_h \to (L^2(\Omega))^2$ as follows

$$\gamma_h\boldsymbol{v}_h = \sum_{j=1}^{M_S} \boldsymbol{v}_h|_{K_L}(P_j)\chi^*_{K_j^*\cap K_L} + \boldsymbol{v}_h|_{K_R}(P_j)\chi^*_{K_j^*\cap K_R} + \sum_{j=M_S+1}^{M} \boldsymbol{v}_h|_{K_I}(P_j)\chi^*_{K_j^*}, \text{ for } \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

where χ_K^* is characteristic function of a set K. We use $\bar{\boldsymbol{Y}}_h = \gamma_h\boldsymbol{V}_h$ as the test function space, and rewrite (2.3) as

$$\begin{cases} (a) \ \ (\dfrac{\partial^\alpha u}{\partial t^\alpha}, w_h) + (\mathrm{div}\boldsymbol{\lambda}, w_h) + (g(u), w_h) = (f, w_h), \quad \forall w_h \in W_h, \\ (b) \ \ (\mathcal{A}^{-1}\boldsymbol{\lambda} + \nabla u, \gamma_h\boldsymbol{v}_h) = 0, \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{2.4}$$

Similar to [37], making use of the operator γ_h and the Green theorem, we have $(\nabla w_h, \gamma_h \boldsymbol{v}_h) = -(\mathrm{div}\boldsymbol{v}_h, w_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \forall w_h \in W_h$. Then, we obtain the nonlinear semi-discrete MFVE scheme: For the selected appropriate $(\boldsymbol{\lambda}_h(0), u_h(0))$, find $(\boldsymbol{\lambda}_h(t), u_h(t)) \in \boldsymbol{V}_h \times W_h$ such that

$$\begin{cases} (a)\ (\dfrac{\partial^\alpha u_h}{\partial t^\alpha}, w_h) + (\mathrm{div}\boldsymbol{\lambda}_h, w_h) + (g(u_h), w_h) = (f, w_h), & \forall w_h \in W_h, \\ (b)\ (\mathcal{A}^{-1}\boldsymbol{\lambda}_h, \gamma_h \boldsymbol{v}_h) - (\mathrm{div}\boldsymbol{v}_h, u_h) = 0, & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{2.5}$$

In order to approximate the Caputo time fractional derivative and give the fully discrete scheme, we should give the grid points $t_n = n\tau$ $(n = 0, 1, \cdots, N)$ in time interval $[0, T]$, where N is a positive integer and $\tau = T/N$. We denote $\varphi^n = \varphi(\cdot, t_n)$ for a function φ. Following [4, 5], we will approximate the fractional derivative $\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial t^\alpha}$ at $t = t_n$ by using the $L1$-formula as follows

$$\begin{aligned} \frac{\partial^\alpha u(\boldsymbol{x}, t_n)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial u(\boldsymbol{x}, s)}{\partial s} \frac{1}{(t_n - s)^\alpha} \mathrm{d}s \\ &= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \frac{u(\boldsymbol{x}, t^{n-k}) - u(\boldsymbol{x}, t^{n-k-1})}{\tau} + R_t^n(\boldsymbol{x}) \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n} b_k^n u^k + R_t^n(\boldsymbol{x}), \end{aligned} \tag{2.6}$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$, $b_0^n = (n-1)^{1-\alpha} - n^{1-\alpha}$, $b_n^n = 1$, $b_k^n = b_{n-k} - b_{n-k-1}$ $(0 < k < n)$. Setting $D_\tau^\alpha u^n = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum\limits_{k=0}^{n} b_k^n u^k$, we have $\frac{\partial^\alpha u(\boldsymbol{x},t_n)}{\partial t^\alpha} = D_\tau^\alpha u^n + R_t^n(\boldsymbol{x})$. Following [4, 5], we can get that if $u \in C^2(\bar{J}, L^2(\Omega))$, then there exist a constant $C > 0$ independent of τ such that $\|R_t^n(\boldsymbol{x})\| \leq C\tau^{2-\alpha}$.

Let $\boldsymbol{\lambda}_h^n$ and u_h^n be the numerical solutions of $\boldsymbol{\lambda}$ and u at $t = t_n$, respectively. Then, we can obtain the nonlinear fully discrete MFVE scheme for the problem (1.1): For the properly selected $(\boldsymbol{\lambda}_h^0, u_h^0)$, find $(\boldsymbol{\lambda}_h^n, u_h^n) \in \boldsymbol{V}_h \times W_h$, $n = 1, 2, \cdots, N$, such that

$$\begin{cases} (a)\ (D_\tau^\alpha u_h^n, w_h) + (\mathrm{div}\boldsymbol{\lambda}_h^n, w_h) + (g(u_h^n), w_h) = (f^n, w_h), & \forall w_h \in W_h, \\ (b)\ (\mathcal{A}^{-1}\boldsymbol{\lambda}_h^n, \gamma_h \boldsymbol{v}_h) - (\mathrm{div}\boldsymbol{v}_h, u_h^n) = 0, & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{cases} \tag{2.7}$$

For improving the nonlinear fully discrete MFVE scheme (2.7), we consider the following two-grid MFVE system based on the coarse grid $\mathscr{K}_H$ and the fine grid $\mathscr{K}_h$ with the corresponding dual grids $\mathscr{K}_H^*$ and $\mathscr{K}_h^*$, where $h \ll H$.

STEP I. On the coarse primal and dual grids ($\mathscr{K}_H$ and $\mathscr{K}_H^*$), solve the following nonlinear system for $(\boldsymbol{\lambda}_H^n, u_H^n) \in \boldsymbol{V}_H \times W_H$, $n = 1, 2, \cdots, N$, such that

$$\begin{cases} (a)\ (D_\tau^\alpha u_H^n, w_H) + (\mathrm{div}\boldsymbol{\lambda}_H^n, w_H) + (g(u_H^n), w_H) = (f^n, w_H), & \forall w_H \in W_H, \\ (b)\ (\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n, \gamma_H \boldsymbol{v}_H) - (\mathrm{div}\boldsymbol{v}_H, u_H^n) = 0, & \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \end{cases} \tag{2.8}$$

where $(\boldsymbol{\lambda}_H^0, u_H^0) \in \boldsymbol{V}_H \times W_H$ is defined in Section 5.

STEP II. On the fine primal and dual grids ($\mathscr{K}_h$ and $\mathscr{K}_h^*$), solve the following linearized system for $(\hat{\boldsymbol{\lambda}}_h^n, \hat{u}_h^n) \in \boldsymbol{V}_h \times W_h$, $n = 1, 2, \cdots, N$, such that

$$\begin{cases} (a)\ (D_\tau^\alpha \hat{u}_h^n, w_h) + (\mathrm{div}\hat{\boldsymbol{\lambda}}_h^n, w_h) + (g(u_H^n) + g'(u_H^n)(\hat{u}_h^n - u_H^n), w_h) = (f^n, w_h), & \forall w_h \in W_h, \\ (b)\ (\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^n, \gamma_h \boldsymbol{v}_h) - (\mathrm{div}\boldsymbol{v}_h, \hat{u}_h^n) = 0, & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{cases} \tag{2.9}$$

where $(\hat{\boldsymbol{\lambda}}_h^0, \hat{u}_h^0) \in \boldsymbol{V}_h \times W_h$ is defined in Section 5.

Remark 2.1. *In the actual numerical calculation of the two-grid systems* (2.8) *and* (2.9)*, we can find a solution* $(\boldsymbol{\lambda}_H^n, u_H^n) \in \boldsymbol{V}_H \times W_H$ *on the coarse primal and dual grids* ($\mathscr{K}_H$ *and* $\mathscr{K}_H^*$) *by calculating the nonlinear implicit system* (2.8)*, then obtain the final solution* $(\hat{\boldsymbol{\lambda}}_h^n, \hat{u}_h^n) \in \boldsymbol{V}_h \times W_h$ *on the fine primal and dual grids* ($\mathscr{K}_h$ *and* $\mathscr{K}_h^*$) *by calculating the linearized system* (2.9)*. This calculation method will be more efficient than the standard nonlinear implicit system* (2.7)*, and we will see this advantage from the numerical results.*

3. Existence and uniqueness

In the proof of existence and uniqueness and subsequent theoretical analysis, we need to use some important properties of transfer operator $\gamma_\hbar$ ($\hbar = H$ or h), which are as follows:

Lemma 3.1. *[37] The transfer operator* $\gamma_\hbar$ *is bounded*

$$\|\gamma_\hbar \boldsymbol{v}_\hbar\| \leq \|\boldsymbol{v}_\hbar\|, \ \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar.$$

Lemma 3.2. *[38] The following symmetry relations holds*

$$(\bar{\mathcal{A}}^{-1}\boldsymbol{z}_\hbar, \gamma_\hbar \boldsymbol{v}_\hbar) = (\bar{\mathcal{A}}^{-1}\boldsymbol{v}_\hbar, \gamma_\hbar \boldsymbol{z}_\hbar), \ \forall \boldsymbol{z}_\hbar, \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar,$$

where $\bar{\mathcal{A}}^{-1}(\boldsymbol{x}) = \mathcal{A}^{-1}(B)$, $\forall \boldsymbol{x} \in K_B$,

Lemma 3.3. *[38] There exists three constants* $\mu_1, \mu_2, \mu_3 > 0$ *independent of* $\hbar$ *such that*

$$\begin{aligned}
&(\mathcal{A}^{-1}\boldsymbol{v}_\hbar, \gamma_\hbar \boldsymbol{v}_\hbar) \geq \mu_1 \|\boldsymbol{v}_\hbar\|^2, \ \forall \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar,\\
&(\bar{\mathcal{A}}^{-1}\boldsymbol{v}_\hbar, \gamma_\hbar \boldsymbol{v}_\hbar) \geq \mu_2 \|\boldsymbol{v}_\hbar\|^2, \ \forall \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar,\\
&|(\mathcal{A}^{-1}\boldsymbol{z}_\hbar, \gamma_\hbar \boldsymbol{v}_\hbar) - (\bar{\mathcal{A}}^{-1}\boldsymbol{z}_\hbar, \gamma_\hbar \boldsymbol{v}_\hbar)| \leq \mu_3 \hbar \|\boldsymbol{z}_\hbar\| \|\boldsymbol{v}_\hbar\|, \ \forall \boldsymbol{z}_\hbar, \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar.
\end{aligned}$$

Lemma 3.4. *[38] There exists two constants* $\mu_4, \mu_5 > 0$ *independent of* $\hbar$ *such that*

$$\begin{aligned}
&\|(I - \gamma_\hbar)\boldsymbol{v}_\hbar\| \leq \mu_4 \hbar \|\boldsymbol{v}_\hbar\|_{1,\hbar}, \ \forall \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar,\\
&|(\mathcal{A}^{-1}\boldsymbol{z}_\hbar, (I - \gamma_\hbar)\boldsymbol{v}_\hbar)| \leq \mu_5 \hbar \|\boldsymbol{z}_\hbar\|_{1,\hbar} \|\boldsymbol{v}_\hbar\|, \ \forall \boldsymbol{z}_\hbar, \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar,\\
&|(\mathcal{A}^{-1}\boldsymbol{z}, (I - \gamma_\hbar)\boldsymbol{v}_\hbar)| \leq \mu_5 \hbar \|\boldsymbol{z}\|_1 \|\boldsymbol{v}_\hbar\|, \ \forall \boldsymbol{z} \in (H^1(\Omega))^2, \forall \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar,
\end{aligned}$$

where $\|\boldsymbol{z}_\hbar\|_{1,\hbar}^2 = \|\boldsymbol{z}_\hbar\|^2 + |\boldsymbol{z}|_{1,\hbar}^2$ *and* $|\boldsymbol{z}|_{1,\hbar}^2 = \sum_{K \in \mathscr{K}_\hbar} (\|\nabla z_\hbar^1\|_{0,K}^2 + \|\nabla z_\hbar^2\|_{0,K}^2)$, $\forall \boldsymbol{z}_\hbar = (z_\hbar^1, z_\hbar^2) \in \boldsymbol{V}_\hbar$.

Lemma 3.5. *Let* $\{\boldsymbol{\lambda}^n\}_{n=0}^\infty$ *be a function sequence on* $\boldsymbol{V}_\hbar$*, then we have*

$$\begin{aligned}
\sum_{k=0}^{n} b_k^n (\mathcal{A}^{-1}\boldsymbol{\lambda}^k, \gamma_\hbar \boldsymbol{\lambda}^n) =& \frac{1}{2}[(\mathcal{A}^{-1}\boldsymbol{\lambda}^n, \gamma_\hbar \boldsymbol{\lambda}^n) + \sum_{k=0}^{n-1} b_k^n (\mathcal{A}^{-1}\boldsymbol{\lambda}^k, \gamma_\hbar \boldsymbol{\lambda}^k) - \sum_{k=0}^{n-1} b_k^n (\mathcal{A}^{-1}(\boldsymbol{\lambda}^n - \boldsymbol{\lambda}^k), \gamma_\hbar (\boldsymbol{\lambda}^n - \boldsymbol{\lambda}^k))\\
&+ \sum_{k=0}^{n-1} b_k^n ((\mathcal{A}^{-1}\boldsymbol{\lambda}^n, \gamma_\hbar \boldsymbol{\lambda}^k) - (\mathcal{A}^{-1}\boldsymbol{\lambda}^k, \gamma_\hbar \boldsymbol{\lambda}^n))].
\end{aligned}$$

Lemma 3.6. *[40] Let* $\varphi^k \geq 0$, $k = 0, 1, \ldots, N$, $\zeta > 0$ *and* $C_0 \geq 1$ *be two constants, which satisfy*

$$\varphi^n \leq -C_0 \sum_{k=0}^{n-1} \tilde{b}_k^n \varphi^k + \zeta.$$

Then, the following relation holds

$$\varphi^n \le C_0^n(\varphi^0 + b_{n-1}^{-1}\zeta),\ n = 1, 2, \cdots, N.$$

Furthermore, the above result can be further written as

$$\varphi^n \le C_0^n(\varphi^0 + \frac{t_n^\alpha}{1-\alpha}\tau^{-\alpha}\zeta).$$

Lemma 3.7. *[41] Let φ^n be a function on Ω, then*

$$(D_\tau^\alpha \varphi^n, \varphi^n) \ge \frac{1}{2} D_\tau^\alpha \|\varphi^n\|^2.$$

Lemma 3.8. *[41] Let $\varphi^n, \varsigma^n \ge 0$, $n = 0, 1, \cdots$, satisfy*

$$D_\tau^\alpha \varphi^n \le \lambda_1 \varphi^n + \lambda_2 \varphi^{n-1} + \varsigma^n,$$

where $\lambda_1, \lambda_2 \ge 0$ are two constants independent of τ. There exists a constant $\tau^* > 0$ such that, if $\tau \le \tau^*$, then

$$\varphi^n \le 2(\varphi^0 + \frac{t_n^\alpha}{\Gamma(1+\alpha)} \max_{0\le j\le n} \varsigma^j) E_\alpha(2\lambda t_n^\alpha),\ 1 \le n \le N,$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function and $\lambda = \lambda_1 + \frac{\lambda_2}{(2-2^{1-\alpha})}$.

Lemma 3.9. *[42] [Browder fixed point theorem] Let S be a finite dimensional space with the inner product $(\cdot,\cdot)_S$ and the norm $\|\cdot\|_S$, and the map $G : S \to S$ be continuous. Suppose the there exists $\mu > 0$ such that $(G(\xi), \xi)_S \ge 0$ for $\forall \xi \in S$ with $\|\xi\|_S = \mu$. Then, there exists $\xi^* \in S$ such that $G(\xi^*) = 0$ and $\|\xi^*\|_S \le \mu$.*

We first give the existence and uniqueness results for the nonlinear MFVE scheme (2.8) by using Lemma 3.9.

Theorem 3.1. *Assume that $(\boldsymbol{\lambda}_H^i, u_H^i)$ $(i = 0, 1, \cdots, n-1)$ are given. There exists a constant $\tau_0 > 0$ such that, if $\tau < \tau_0$, then there exists a unique solution $(\boldsymbol{\lambda}_H^n, u_H^n) \in \boldsymbol{V}_H \times W_H$ for the nonlinear MFVE scheme* (2.8) *on the coarse primal and dual grids.*

Proof. Let $G : \boldsymbol{V}_H \times W_H \to \boldsymbol{V}_H \times W_H$ be the map. For $\bar{\boldsymbol{\lambda}}_H, \bar{u}_H \in \boldsymbol{V}_H \times W_H$, we define $G(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H)$ as follows:

$$\begin{aligned}(G(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H), (\boldsymbol{v}_H, w_H))_{\boldsymbol{V}_H\times W_H} =& \frac{1}{\Gamma(2-\alpha)}(\bar{u}_H, w_H) + \tau^\alpha(g(\bar{u}_H), w_H) - \tau^\alpha(f^n, w_H)\\ &+ \tau^\alpha[(\mathrm{div}\bar{\boldsymbol{\lambda}}_H, w_H) + (\mathcal{A}^{-1}\bar{\boldsymbol{\lambda}}_H, \gamma_H \boldsymbol{v}_H) - (\mathrm{div}\boldsymbol{v}_H, \bar{u}_H)]\\ &+ \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k^n(u_H^k, w_H),\ \forall(\boldsymbol{v}_H^n, w_H^n) \in \boldsymbol{V}_H \times W_H.\end{aligned} \tag{3.1}$$

The map G is obviously continuous. Furthermore, setting $\boldsymbol{v}_H = \bar{\boldsymbol{\lambda}}_H, w_H = \bar{u}_H$ in (3.1), and applying Lemma 3.3, we have

$$\begin{aligned}(G(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H), (\bar{\boldsymbol{\lambda}}_H, \bar{u}_H))_{\boldsymbol{V}_H\times W_H} \geq & \frac{1}{\Gamma(2-\alpha)}\|\bar{u}_H\|^2 + \mu_1\tau^\alpha\|\bar{\boldsymbol{\lambda}}_H\|^2 - L\tau^\alpha\|\bar{u}_H\|^2 - \frac{\tau^\alpha}{2}\|f^n\|^2 \\ & - \frac{\tau^\alpha}{2}\|\bar{u}_H\|^2 + \frac{1}{\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k^n(\frac{\|u_H^k\|^2}{2} + \frac{\|\bar{u}_H\|^2}{2}).\end{aligned} \tag{3.2}$$

Noting that $\sum\limits_{k=0}^{n-1} b_k^n = -1$, we have

$$\begin{aligned}(G(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H), (\bar{\boldsymbol{\lambda}}_H, \bar{u}_H))_{\boldsymbol{V}_H\times W_H} \geq & (\frac{1}{2\Gamma(2-\alpha)} - L\tau^\alpha - \frac{\tau^\alpha}{2})\|\bar{u}_H\|^2 + \mu_1\tau^\alpha\|\bar{\boldsymbol{\lambda}}_H\|^2 \\ & - \frac{\tau^\alpha}{2}\|f^n\|^2 + \frac{1}{2\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k^n\|u_H^k\|^2.\end{aligned} \tag{3.3}$$

Thus, there exists a constant $\tau_{0,1} > 0$ such that, if $\tau < \tau_{0,1}$, then

$$(G(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H), (\bar{\boldsymbol{\lambda}}_H, \bar{u}_H))_{\boldsymbol{V}_H\times W_H} \geq \frac{1}{4\Gamma(2-\alpha)}\|\bar{u}_H\|^2 + \mu_1\tau^\alpha\|\bar{\boldsymbol{\lambda}}_H\|^2 - \frac{\tau^\alpha}{2}\|f^n\|^2 + \frac{1}{2\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k^n\|u_H^k\|^2. \tag{3.4}$$

Because of the norm equivalence in finite dimensional normed linear space, there exists a constant $C_0 > 0$ such that $\|\bar{\boldsymbol{\lambda}}_H\| \geq C_0\|\bar{\boldsymbol{\lambda}}_H\|_{\boldsymbol{H}(\mathrm{div})}$. Thus, we have

$$(G(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H), (\bar{\boldsymbol{\lambda}}_H, \bar{u}_H))_{\boldsymbol{V}_H\times W_H} \geq C_1\|(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H)\|^2_{\boldsymbol{V}_H\times W_H} - \frac{\tau^\alpha}{2}\|f^n\|^2 + \frac{1}{2\Gamma(2-\alpha)}\sum_{k=0}^{n-1} b_k^n\|u_H^k\|^2, \tag{3.5}$$

where $\|(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H)\|^2_{\boldsymbol{V}_H\times W_H} = \|\bar{\boldsymbol{\lambda}}_H\|^2_{\boldsymbol{H}(\mathrm{div})} + \|\bar{u}\|^2$ and $C_1 = \min\{\frac{1}{4\Gamma(2-\alpha)}, \mu_1 C_0^2\tau^\alpha\}$. Let $\mu = \frac{1}{C_1}(1 + \frac{\tau^\alpha}{2}\|f^n\|^2 - \frac{1}{2\Gamma(2-\alpha)}\sum\limits_{k=0}^{n-1} b_k^n\|u_H^k\|^2)$. Based on above analysis, we know that if $\|(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H)\|^2_{\boldsymbol{V}_H\times W_H} = \mu$, then $(G(\bar{\boldsymbol{\lambda}}_H, \bar{u}_H), (\bar{\boldsymbol{\lambda}}_H, \bar{u}_H))_{\boldsymbol{V}_H\times W_H} \geq 0$. Applying Lemma 3.9, we obtain that there exists $(\bar{\boldsymbol{\lambda}}_H^*, \bar{u}_H^*) \in \boldsymbol{V}_H \times W_H$ such that $G(\bar{\boldsymbol{\lambda}}_H^*, \bar{u}_H^*) = \mathbf{0}$. Then $(\boldsymbol{\lambda}_H^n, u_H^n) = (\bar{\boldsymbol{\lambda}}_H^*, \bar{u}_H^*)$ satisfies (2.8).

Next, we prove the uniqueness of the solution. Let $(\boldsymbol{\Lambda}_H^n, U_H^n) \in \boldsymbol{V}_H \times W_H$ be another solution of (2.8), and $(\boldsymbol{\Lambda}_H^0, U_H^0) = (\boldsymbol{\lambda}_H^0, u_H^0)$. Making use of (2.8), we obtain

$$\begin{cases} (a) \ \frac{1}{\Gamma(2-\alpha)}(p_H^n, w_h) + \tau^\alpha(\mathrm{div}\boldsymbol{q}_H^n, w_h) + \tau^\alpha(g(u_H^n) - g(U_H^n), w_h) = 0, & \forall w_h \in W_h, \\ (b) \ (\mathcal{A}^{-1}\boldsymbol{q}_H^n, \gamma_h\boldsymbol{v}_h) - (\mathrm{div}\boldsymbol{v}_h, p_H^n) = 0, & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{cases} \tag{3.6}$$

where $p_H^n = u_H^n - U_H^n, \boldsymbol{q}_H^n = \boldsymbol{\lambda}_H^n - \boldsymbol{\Lambda}_H^n$. Choose $w_h = p_H^n, \boldsymbol{v}_h = \boldsymbol{q}_H^n$ in (3.6) to obtain

$$\frac{1}{\Gamma(2-\alpha)}\|p_H^n\|^2 + \tau^\alpha(\mathcal{A}^{-1}\boldsymbol{q}_H^n, \gamma_h\boldsymbol{q}_H^n) + \tau^\alpha(g(u_H^n) - g(U_H^n), p_H^n) = 0. \tag{3.7}$$

Applying Lemma 3.3, we have

$$\frac{1}{\Gamma(2-\alpha)}\|p_H^n\|^2 + \mu_1\tau^\alpha\|\boldsymbol{q}_H^n\|^2 \leq \|g\|_{1,\infty}\tau^\alpha\|p_H^n\|^2. \tag{3.8}$$

There exits a constant $\tau_{0,2} > 0$ such that, if $\tau \le \tau_{0,2}$, then $\|g\|_{1,\infty}\tau^\alpha \le \frac{1}{2\Gamma(2-\alpha)}$, and

$$\frac{1}{2\Gamma(2-\alpha)}\|p_H^n\|^2 + \mu_1\tau^\alpha\|\boldsymbol{q}_H^n\|^2 \le 0. \tag{3.9}$$

It follows that $\|p_H^n\| = 0$ and $\|\boldsymbol{q}_H^n\| = \mathbf{0}$. Setting $\tau_0 = \min\{\tau_{0,1}, \tau_{0,2}\}$, we have completed the proof of the theorem. □

Now, we give the existence and uniqueness results for the linearized scheme (2.9).

Theorem 3.2. *Assume that* $(\hat{\boldsymbol{\lambda}}_h^i, \hat{u}_h^i)$ $(i = 0, 1, \cdots, n-1)$ *are given. There exists a constant* τ_1 $(0 < \tau_1 \le \tau_0)$ *such that, if* $\tau < \tau_1$, *then there exists a unique solution* $(\hat{\boldsymbol{\lambda}}_h^n, \hat{u}_h^n) \in \boldsymbol{V}_h \times W_h$ *for linearized scheme* (2.9) *on the fine primal and dual grids.*

Proof. Let $\{\boldsymbol{\phi}_i\}_{i=1}^{M_1}$ and $\{\varphi_j\}_{j=1}^{M_2}$ be the basis functions of $\boldsymbol{V}_h$ and W_h, respectively. Then $(\hat{\boldsymbol{\lambda}}_h^n, \hat{u}_h^n)$ can be expressed as

$$\hat{\boldsymbol{\lambda}}_h^n = \sum_{i=1}^{M_1} r_i^n \boldsymbol{\phi}_i, \ \hat{u}_h^n = \sum_{j=1}^{M_2} u_j^n \varphi_j. \tag{3.10}$$

Substituting (3.10) into (2.9), and taking $\boldsymbol{v}_h = \boldsymbol{\phi}_i$ $(i = 1, 2, \cdots, M_1)$ and $w_h = \varphi_j$ $(j = 1, 2, \cdots, M_2)$, we have

$$\begin{bmatrix} \frac{1}{\Gamma(2-\alpha)}B_1 + \tau^\alpha B_3 & \tau^\alpha C \\ -C^T & B_2 \end{bmatrix} \begin{bmatrix} \hat{U}^n \\ \hat{\Lambda}^n \end{bmatrix} = \begin{bmatrix} \tau^\alpha F^n - \tau^\alpha P^n - \frac{1}{\Gamma(2-\alpha)} \sum\limits_{k=0}^{n-1} b_k^n B_1 \hat{U}^k \\ 0 \end{bmatrix}, \tag{3.11}$$

where

$$\begin{aligned} &\hat{\Lambda}^n = (r_1^n, r_2^n, \cdots, r_{M_1}^n)^T, && \hat{U}^n = (u_1^n, u_2^n, \cdots, u_{M_2}^n)^T, \\ &B_1 = ((\varphi_i, \varphi_j))_{i,j=1,2,\cdots,M_2}, && B_2 = ((\mathcal{A}^{-1}\boldsymbol{\phi}_i, \gamma_h\boldsymbol{\phi}_j))_{i,j=1,2,\cdots,M_1}, \\ &B_3 = ((g'(u_H^n)\varphi_i, \varphi_j))_{i,j=1,2,\cdots,M_2}, && C = ((\mathrm{div}\boldsymbol{\phi}_i, \varphi_j))_{i=1,2,\cdots,M_1; j=1,2,\cdots,M_2}, \\ &P^n = ((g(u_H^n) - g'(u_H^n)u_H^n, \varphi_j))_{j=1,2,\cdots,M_2}^T, && F^n = ((f^n, \varphi_j))_{j=1,2,\cdots,M_2}^T. \end{aligned}$$

Noting that B_1 and B_2 are invertible, and applying the multiplication of partitioned matrices, we can get

$$\begin{bmatrix} E & -\tau^\alpha C B_2^{-1} \\ 0 & E \end{bmatrix} \begin{bmatrix} \frac{1}{\Gamma(2-\alpha)}B_1 + \tau^\alpha B_3 & \tau^\alpha C \\ -C^T & B_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\Gamma(2-\alpha)}B_1 + \tau^\alpha B_3 + \tau^\alpha C B_2^{-1} C^T & 0 \\ -C^T & B_2 \end{bmatrix}. \tag{3.12}$$

Let $\psi(\tau) = \det(\frac{1}{\Gamma(2-\alpha)}B_1 + \tau^\alpha B_3 + \tau^\alpha C B_2^{-1} C^T)$, then $\psi(\tau)$ is a continuous function. Noting that $\psi(0) = \det(\frac{1}{\Gamma(2-\alpha)}B_1) > 0$. According to the property of continuous function, there exists a constant τ_1 $(0 < \tau_1 \le \tau_0)$ such that, if $\tau < \tau_1$, then $\psi(\tau) > \frac{1}{2}\det(\frac{1}{\Gamma(2-\alpha)}B_1) > 0$. So the coefficient matrix of (3.11) is invertible, then there exists a unique solution for the linearized scheme (2.9). □

4. Stability analysis

We will give the stability results for the nonlinear MFVE scheme (2.8) and linearized MFVE scheme (2.9) on the coarse and fine grids, respectively.

Theorem 4.1. *Let* $(\boldsymbol{\lambda}_H^n, u_H^n)_{n=0}^N \in \boldsymbol{V}_H \times W_H$ *be the solution of system* (2.8)*, then there exist a constant* C *independent of* H *and* τ *such that*

$$\|u_H^n\| \le C(\|u_H^0\| + \sup_{t\in[0,T]} \|f(t)\|). \tag{4.1}$$

Moreover, for a constant $c_0 > 0$*, there exist a constant* $\tau_2 > 0$ *independent of* H *and* τ *such that, if* $H \le c_0\tau \le c_0 \min\{\tau_2, \tau_0\}$ *and* $H \le \hbar_0$*, then*

$$\|\boldsymbol{\lambda}_H^n\| \le Ce^{\frac{c_0 T\mu_3}{\mu_1}}(\|u_H^0\| + \|\boldsymbol{\lambda}_H^0\| + \sup_{t\in[0,T]} \|f(t)\|), \tag{4.2}$$

where τ_0 *is defined in Theorem 3.1,* $\hbar_0 = \frac{\mu_1}{2\mu_3}$*,* $C > 0$ *is a constant independent of* H*,* τ *and* c_0*.*

Proof. Choosing $w_H = u_H^n$ and $\boldsymbol{v}_H = \boldsymbol{\lambda}_H^n$ in (2.8), we can get

$$(D_\tau^\alpha u_H^n, u_H^n) + (\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n, \gamma_H\boldsymbol{\lambda}_H^n) + (g(u_H^n), u_H^n) = (f^n, u_H^n). \tag{4.3}$$

Apply the Lemma 3.3 and Lemma 3.7 in (4.3) to obtain

$$\frac{1}{2}D_\tau^\alpha\|u_H^n\|^2 + \mu_1\|\boldsymbol{\lambda}_H^n\|^2 \le \frac{1}{2}\|f^n\|^2 + (\frac{1}{2} + L)\|u_H^n\|^2. \tag{4.4}$$

Apply Lemma 3.8 in (4.4) to obtain

$$\|u_H^n\| \le C(\|u_H^0\| + \sup_{t\in[0,T]} \|f(t)\|). \tag{4.5}$$

Now, making use of (2.8)(b), we have

$$(\mathcal{A}^{-1}D_\tau^\alpha\boldsymbol{\lambda}_H^n, \gamma_H\boldsymbol{v}_H) - (\mathrm{div}\boldsymbol{v}_H, D_\tau^\alpha u_H^n) = 0, \forall \boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{4.6}$$

Choosing $w_H = D_\tau^\alpha u_H^n$ in (2.8)(a) and $\boldsymbol{v}_H = \boldsymbol{\lambda}_H^n$ in (4.6), we have

$$\|D_\tau^\alpha u_H^n\|^2 + (\mathcal{A}^{-1}D_\tau^\alpha\boldsymbol{\lambda}_H^n, \gamma_H\boldsymbol{\lambda}_H^n) + (g(u_H^n), D_\tau^\alpha u_H^n) = (f^n, D_\tau^\alpha u_H^n). \tag{4.7}$$

Applying Lemma 3.5 in (4.7), and noting that $b_k^n < 0$ $(0 \le k < n)$, we have

$$\begin{aligned}
&\|D_\tau^\alpha u_H^n\|^2 + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n, \gamma_H\boldsymbol{\lambda}_H^n) \\
&\quad \le -\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}[\sum_{k=0}^{n-1} b_k^n(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^k, \gamma_H\boldsymbol{\lambda}_H^k) + \sum_{k=0}^{n-1} b_k^n((\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n, \gamma_H\boldsymbol{\lambda}_H^k) - (\mathcal{A}^{-1}\boldsymbol{\lambda}_H^k, \gamma_H\boldsymbol{\lambda}_H^n))] \\
&\qquad + \frac{1}{2}\|D_\tau^\alpha u_H^n\|^2 + C[\|f^n\|^2 + L^2\|u_H^n\|^2].
\end{aligned} \tag{4.8}$$

Apply Lemma 3.2 and Lemma 3.3 in (4.8) to obtain

$$\begin{aligned}
|(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n, \gamma_H\boldsymbol{\lambda}_H^k) - (\mathcal{A}^{-1}\boldsymbol{\lambda}_H^k, \gamma_H\boldsymbol{\lambda}_H^n)| &\le 2\mu_3 H\|\boldsymbol{\lambda}_H^k\|\|\boldsymbol{\lambda}_H^n\| \\
&\le \frac{\mu_3}{\mu_1}H[(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^k, \gamma_H\boldsymbol{\lambda}_H^k) + (\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n, \gamma_H\boldsymbol{\lambda}_H^n)].
\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8), we have

$$\begin{aligned}(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n,\gamma_H\boldsymbol{\lambda}_H^n)\leq&-\sum_{k=0}^{n-1}b_k^n(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^k,\gamma_H\boldsymbol{\lambda}_H^k)-\frac{\mu_3}{\mu_1}H\sum_{k=0}^{n-1}b_k^n(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^k,\gamma_H\boldsymbol{\lambda}_H^k)\\&+\frac{\mu_3}{\mu_1}H(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n,\gamma_H\boldsymbol{\lambda}_H^n)+C\Gamma(2-\alpha)\tau^\alpha[\|f^n\|^2+L^2\|u_H^n\|^2].\end{aligned}\tag{4.10}$$

Setting $\hbar_0=\frac{\mu_1}{2\mu_3}$, when $H\leq\hbar_0$, we have $1-\frac{\mu_3}{\mu_1}H\geq\frac{1}{2}$ and

$$(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n,\gamma_H\boldsymbol{\lambda}_H^n)\leq-\frac{1+\frac{\mu_3}{\mu_1}H}{1-\frac{\mu_3}{\mu_1}H}\sum_{k=0}^{n-1}b_k^n(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^k,\gamma_H\boldsymbol{\lambda}_H^k)+C\Gamma(2-\alpha)\tau^\alpha(\|u_H^0\|^2+\sup_{t\in[0,T]}\|f(t)\|^2).\tag{4.11}$$

Applying Lemma 3.6 to have

$$(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n,\gamma_H\boldsymbol{\lambda}_H^n)\leq C(\frac{1+\frac{\mu_3}{\mu_1}H}{1-\frac{\mu_3}{\mu_1}H})^n((\mathcal{A}^{-1}\boldsymbol{\lambda}_H^0,\gamma_H\boldsymbol{\lambda}_H^0)+\|u_H^0\|^2+\sup_{t\in[0,T]}\|f(t)\|^2).\tag{4.12}$$

Let $c_0>0$ be a constant. Selecting H and τ to satisfy $H\leq c_0\tau$ in (4.12), we have

$$(\mathcal{A}^{-1}\boldsymbol{\lambda}_H^n,\gamma_H\boldsymbol{\lambda}_H^n)\leq C(\frac{1+\frac{c_0\mu_3}{\mu_1}\tau}{1-\frac{c_0\mu_3}{\mu_1}\tau})^{\frac{T}{\tau}}((\mathcal{A}^{-1}\boldsymbol{\lambda}_H^0,\gamma_H\boldsymbol{\lambda}_H^0)+\|u_H^0\|^2+\sup_{t\in[0,T]}\|f(t)\|^2).\tag{4.13}$$

Noting that

$$\lim_{\tau\to0}(\frac{1+\frac{c_0\mu_3}{\mu_1}\tau}{1-\frac{c_0\mu_3}{\mu_1}\tau})^{\frac{T}{\tau}}=e^{\frac{2c_0T\mu_3}{\mu_1}},\tag{4.14}$$

then, there exists a constant $\tau_2>0$ such that, if $\tau<\min\{\tau_2,\tau_0\}$, where τ_0 is defined in Theorem 3.1, then

$$\|\boldsymbol{\lambda}_H^n\|\leq Ce^{\frac{c_0T\mu_3}{\mu_1}}(\|\boldsymbol{\lambda}_H^0\|+\|u_H^0\|+\sup_{t\in[0,T]}\|f(t)\|).\tag{4.15}$$

Thus, we complete the proof of this theorem. □

Theorem 4.2. *Let* $(\hat{\boldsymbol{\lambda}}_h^n,\hat{u}_h^n)_{n=0}^N\in\boldsymbol{V}_h\times W_h$ *be the solution of system* (2.9)*, then there exist a constant* $C>0$ *independent of h and* τ *such that*

$$\|\hat{u}_h^n\|\leq C(\|u_H^0\|+\|\hat{u}_h^0\|+\sup_{t\in[0,T]}\|f(t)\|).\tag{4.16}$$

Moreover, there exists a constant $C>0$ *independent of h,* τ *and* c_0 *such that, if* $h\leq c_0\tau\leq c_0\min\{\tau_2,\tau_1\}$ *and* $h<\hbar_0$*, then*

$$\|\hat{\boldsymbol{\lambda}}_h^n\|\leq Ce^{\frac{c_0T\mu_3}{\mu_1}}(\|u_H^0\|+\|\hat{u}_h^0\|+\|\hat{\boldsymbol{\lambda}}_h^0\|+\sup_{t\in[0,T]}\|f(t)\|),\tag{4.17}$$

where τ_1 *is defined in Theorem 3.2,* $c_0,\hbar_0$ *and* τ_2 *are defined in Theorem 4.1.*

Proof. Choosing $w_h = \hat{u}_h^n$ and $\boldsymbol{v}_h = \hat{\boldsymbol{\lambda}}_h^n$ in (2.9), we have

$$(D_\tau^\alpha \hat{u}_h^n, \hat{u}_h^n) + (\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^n, \gamma_h \hat{\boldsymbol{\lambda}}_h^n) + (g(u_H^n) + g'(u_H^n)(\hat{u}_h^n - u_H^n), \hat{u}_h^n) = (f^n, \hat{u}_h^n). \tag{4.18}$$

Apply Lemma 3.3 and Lemma 3.7 in (4.18) to obtain

$$\frac{1}{2}D_\tau^\alpha \|\hat{u}_h^n\|^2 + \mu_1 \|\hat{\boldsymbol{\lambda}}_h^n\|^2 \leq \frac{1}{2}\|f^n\|^2 + (\frac{L}{2} + \frac{1}{2}\|g\|_{1,\infty})\|u_H^n\|^2 + (\frac{L+1}{2} + \frac{3}{2}\|g\|_{1,\infty})\|\hat{u}_h^n\|^2. \tag{4.19}$$

Apply Lemma 3.8 and Theorem 4.1 to obtain

$$\|\hat{u}_h^n\| \leq C(\|\hat{u}_h^0\| + \|u_H^0\| + \sup_{t\in[0,T]} \|f(t)\|). \tag{4.20}$$

Now, making use of (2.9)(b), we have

$$(\mathcal{A}^{-1}D_\tau^\alpha \hat{\boldsymbol{\lambda}}_h^n, \gamma_h \boldsymbol{v}_h) - (\mathrm{div}\boldsymbol{v}_h, D_\tau^\alpha \hat{u}_h^n) = 0, \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.21}$$

Choosing $w_H = D_\tau^\alpha \hat{u}_h^n$ in (2.9)(a) and $\boldsymbol{v}_H = \hat{\boldsymbol{\lambda}}_h^n$ in (4.21), we have

$$\|D_\tau^\alpha \hat{u}_h^n\|^2 + (\mathcal{A}^{-1}D_\tau^\alpha \hat{\boldsymbol{\lambda}}_h^n, \gamma_h \hat{\boldsymbol{\lambda}}_h^n) + (g(u_H^n) + g'(u_H^n)(\hat{u}_h^n - u_H^n), D_\tau^\alpha \hat{u}_h^n) = (f^n, D_\tau^\alpha \hat{u}_h^n). \tag{4.22}$$

Apply Lemma 3.5 to get

$$\begin{aligned}
&\|D_\tau^\alpha \hat{u}_h^n\|^2 + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}[(\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^n, \gamma_h \hat{\boldsymbol{\lambda}}_h^n) + \sum_{k=0}^{n-1} b_k^n (\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^k, \gamma_h \hat{\boldsymbol{\lambda}}_h^k) \\
&+ \sum_{k=0}^{n-1} b_k^n ((\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^n, \gamma_h \hat{\boldsymbol{\lambda}}_h^k) - (\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^k, \gamma_h \hat{\boldsymbol{\lambda}}_h^n)) - \sum_{k=0}^{n-1} b_k^n (\mathcal{A}^{-1}(\hat{\boldsymbol{\lambda}}_h^k - \hat{\boldsymbol{\lambda}}_h^n), \gamma_h(\hat{\boldsymbol{\lambda}}_h^k - \hat{\boldsymbol{\lambda}}_h^n))] \\
&\leq (f^n, D_\tau^\alpha \hat{u}_H^n) - (g(u_H^n) + g'(u_H^n)(\hat{u}_h^n - u_H^n), D_\tau^\alpha \hat{u}_h^n).
\end{aligned} \tag{4.23}$$

Noting that $b_k^n < 0$ $(0 \leq k < n)$, we have

$$\begin{aligned}
&\|D_\tau^\alpha \hat{u}_h^n\|^2 + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}(\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^n, \gamma_h \hat{\boldsymbol{\lambda}}_h^n) \\
&\quad \leq -\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k^n (\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^k, \gamma_h \hat{\boldsymbol{\lambda}}_h^k) - \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k^n ((\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^n, \gamma_h \hat{\boldsymbol{\lambda}}_h^k) - (\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^k, \gamma_h \hat{\boldsymbol{\lambda}}_h^n)) \\
&\quad + 2\|f^n\|^2 + (2L^2 + 2\|g\|_{1,\infty}^2)\|u_H^n\|^2 + 2\|g\|_{1,\infty}^2 \|\hat{u}_h^n\|^2 + \frac{1}{2}\|D_\tau^\alpha \hat{u}_h^n\|^2.
\end{aligned} \tag{4.24}$$

Making use of (4.1) and (4.20), we can apply the technique of (4.9) to obtain

$$\begin{aligned}
(1 - \frac{\mu_3}{\mu_1}h)(\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^n, \gamma_h \hat{\boldsymbol{\lambda}}_h^n) \leq &- (1 + \frac{\mu_3}{\mu_1}h) \sum_{k=0}^{n-1} b_k^n (\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^k, \gamma_h \hat{\boldsymbol{\lambda}}_h^k) \\
&+ C\Gamma(2-\alpha)\tau^\alpha [\|u_H^0\|^2 + \|\hat{u}_h^0\|^2 + \sup_{t\in[0,T]} \|f(t)\|^2].
\end{aligned} \tag{4.25}$$

Selecting h to satisfy $h \leq \hbar_0$, where $\hbar_0 = \frac{\mu_1}{2\mu_3}$, we have $1 - \frac{\mu_3}{\mu_1} h \geq \frac{1}{2}$ and

$$(\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^n, \gamma_h \hat{\boldsymbol{\lambda}}_h^n) \leq -\frac{(1+\frac{\mu_3}{\mu_1}h)}{(1-\frac{\mu_3}{\mu_1}h)} \sum_{k=0}^{n-1} b_k^n (\mathcal{A}^{-1}\hat{\boldsymbol{\lambda}}_h^k, \gamma_h \hat{\boldsymbol{\lambda}}_h^k) + C\Gamma(2-\alpha)\tau^\alpha [\|u_H^0\|^2 + \|\hat{u}_h^0\|^2 + \sup_{t\in[0,T]} \|f(t)\|^2]. \quad (4.26)$$

Applying the technique of (4.11)–(4.15), for the positive constant c_0 and τ_2 which defined in Theorem 4.1, we can obtain that if $\tau < \min\{\tau_2, \tau_1\}$ and $h \leq c_0\tau$, then

$$\|\hat{\boldsymbol{\lambda}}_h^n\| \leq C e^{\frac{c_0 T \mu_3}{\mu_1}} (\|\hat{\boldsymbol{\lambda}}_h^0\| + \|u_H^0\| + \|\hat{u}_h^0\| + \sup_{t\in[0,T]} \|f(t)\|). \quad (4.27)$$

We complete the proof of the stability. □

Remark 4.1. *(I) In Theorems 4.1 and 4.2, we also need to select τ to satisfy $\tau < \tau_0$ and $\tau < \tau_1$, respectively, because of the existence and uniqueness of the MFVE solutions in Theorems 3.1 and 3.2.*

(II) From Theorems 4.1 and 4.2, we can see that $\|u_H^n\|$ and $\|\hat{u}_h^n\|$ are unconditionally stable, and $\|\boldsymbol{\lambda}_H^n\|$ and $\|\hat{\boldsymbol{\lambda}}_h^n\|$ are conditionally stable, because the bilinear $(\mathcal{A}^{-1}\boldsymbol{z}_\hbar, \gamma_q^* \boldsymbol{w}_\hbar)$ $(\forall \boldsymbol{z}_\hbar, \boldsymbol{w}_\hbar \in \boldsymbol{V}_\hbar, \hbar = H$ or $h)$ does not necessarily satisfy symmetry. When the coefficient $\mathcal{A}(\boldsymbol{x})$ is a symmetry and positive definite constant matrix, making use of Lemma 3.2, we can see that $(\mathcal{A}^{-1}\boldsymbol{z}_\hbar, \gamma_\hbar^* \boldsymbol{w}_\hbar)$ is symmetry, Under this condition, we can obtain that $\|\boldsymbol{\lambda}_H^n\|$ and $\|\hat{\boldsymbol{\lambda}}_h^n\|$ are also unconditionally stable.

5. Error estimates

In order to get the error estimates for two-grid MFVE systems (2.8) and (2.9), we should introduce the standard L^2-projection [44] $P_\hbar : W \to W_\hbar$, which satisfies

$$(P_\hbar \chi - \chi, w_\hbar) = 0, \ \forall w_\hbar \in W_\hbar, \ \text{for any } \chi \in W, \quad (5.1)$$

$$\|\chi - P_\hbar \chi\|_{-s,q} \leq C\hbar^{1+s} \|\chi\|_{1,q}, \ s = 0, 1, \ 2 \leq q \leq \infty, \ \chi \in W^{1,q}(\Omega), \quad (5.2)$$

where $\hbar = H$ or h.

We introduce a generalized MFVE projection $(\tilde{\boldsymbol{\lambda}}_\hbar, \tilde{u}_\hbar) : \bar{J} \to \boldsymbol{V}_\hbar \times W_\hbar$ such that, for $\hbar = H$ or h,

$$\begin{cases} (a) \ (\mathrm{div}(\tilde{\boldsymbol{\lambda}}_\hbar - \boldsymbol{\lambda}), w_\hbar) = 0, & \forall w_\hbar \in W_\hbar, \\ (b) \ (\mathcal{A}^{-1}(\tilde{\boldsymbol{\lambda}}_\hbar - \boldsymbol{\lambda}), \gamma_\hbar \boldsymbol{v}_\hbar) - (\mathrm{div}\boldsymbol{v}_\hbar, \tilde{u}_\hbar - u) = (\mathcal{A}^{-1}\boldsymbol{\lambda}, (I - \gamma_\hbar)\boldsymbol{v}_\hbar), & \forall \boldsymbol{v}_\hbar \in \boldsymbol{V}_\hbar. \end{cases} \quad (5.3)$$

Then the above projection satisfies the following estimates.

Lemma 5.1. *[43] There exists a constant $C > 0$ independent of $\hbar$ and t such that, for $j = 0, 1$ and $\hbar = H$ or h*

$$\|\frac{\partial^j \boldsymbol{\lambda}}{\partial t^j} - \frac{\partial^j \tilde{\boldsymbol{\lambda}}_\hbar}{\partial t^j}\| \leq C\hbar \|\frac{\partial^j \boldsymbol{\lambda}}{\partial t^j}\|_1, \ \frac{\partial^j \boldsymbol{\lambda}}{\partial t^j} \in (H^1(\Omega))^2,$$

$$\|\mathrm{div}\frac{\partial^j \boldsymbol{\lambda}}{\partial t^j} - \mathrm{div}\frac{\partial^j \tilde{\boldsymbol{\lambda}}_\hbar}{\partial t^j}\| \leq C\hbar \|\mathrm{div}\frac{\partial^j \boldsymbol{\lambda}}{\partial t^j}\|_1, \ \frac{\partial^j \boldsymbol{\lambda}}{\partial t^j} \in \boldsymbol{H}^1(\mathrm{div}, \Omega),$$

$$\|\frac{\partial^j u}{\partial t^j} - \frac{\partial^j \tilde{u}_\hbar}{\partial t^j}\| \leq C\hbar (\|\frac{\partial^j \boldsymbol{\lambda}}{\partial t^j}\|_1 + \|\frac{\partial^j u}{\partial t^j}\|_1), \ \frac{\partial^j \boldsymbol{\lambda}}{\partial t^j} \in (H^1(\Omega))^2, \frac{\partial^j u}{\partial t^j} \in H^1(\Omega),$$

where $\boldsymbol{H}^1(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in (L^2(\Omega))^2 : \mathrm{div}\boldsymbol{v} \in H^1(\Omega)\}$.

Lemma 5.2. *[38] For $2 < q \leq \infty$ and $\hbar = H$ or h, the following L^q-estimate holds*

$$\|u - \tilde{u}_\hbar\|_{0,q} \leq C\hbar(\|u\|_{1,q} + \|\boldsymbol{\lambda}\|_1 + \|\mathrm{div}\boldsymbol{\lambda}\|_1),\ u \in W^{1,q}(\Omega), \boldsymbol{\lambda} \in (H^1(\Omega))^2 \cap \boldsymbol{H}^1(\mathrm{div}, \Omega).$$

Moreover, for $j = 0, 1$, the following superconvergence result holds

$$\|\frac{\partial^j P_\hbar u}{\partial t^j} - \frac{\partial^j \tilde{u}_\hbar}{\partial t^j}\| \leq C\hbar^2(\|\frac{\partial^j \boldsymbol{\lambda}}{\partial t^j}\|_1 + \|\frac{\partial^j \mathrm{div}\boldsymbol{\lambda}}{\partial t^j}\|_1),\ \frac{\partial^j u}{\partial t^j} \in H^1(\Omega), \frac{\partial^j \boldsymbol{\lambda}}{\partial t^j} \in (H^1(\Omega))^2 \cap \boldsymbol{H}^1(\mathrm{div}, \Omega).$$

Now, let $\beta^n = u^n - \tilde{u}_H^n$, $\sigma^n = \tilde{u}_H^n - u_H^n$, $\boldsymbol{\zeta}^n = \boldsymbol{\lambda}^n - \tilde{\boldsymbol{\lambda}}_H^n$, $\boldsymbol{\delta}^n = \tilde{\boldsymbol{\lambda}}_H^n - \boldsymbol{\lambda}_H^n$, where $(\tilde{\boldsymbol{\lambda}}_H^n, \tilde{u}_H^n) \in \boldsymbol{V}_H \times W_H$ is the generalized MFVE projection of $(\boldsymbol{\lambda}, u)$, then we can obtain the error equations as follows

$$\begin{cases} (a)\ (D_\tau^\alpha \sigma^n, w_H) + (\mathrm{div}\boldsymbol{\delta}^n, w_H) + (g(u^n) - g(u_H^n), w_H) = -(D_\tau^\alpha \beta^n, w_H) - (R_t^n(\boldsymbol{x}), w_H), & \forall w_H \in W_H, \\ (b)\ (\mathcal{A}^{-1}\boldsymbol{\delta}^n, \gamma_H \boldsymbol{v}_H) - (\mathrm{div}\boldsymbol{v}_H, \sigma^n) = 0, & \forall \boldsymbol{v}_H \in \boldsymbol{V}_H. \end{cases} \tag{5.4}$$

Theorem 5.1. *Let $(\boldsymbol{\lambda}_H^n, u_H^n) \in \boldsymbol{V}_H \times W_H$ and $(\boldsymbol{\lambda}^n, u^n) \in \boldsymbol{V} \times W$ be the solutions of systems* (2.8) *and* (2.2)*, respectively. Assume that $u, \mathrm{div}\boldsymbol{\lambda} \in C^2(\bar{J}, H^1(\Omega))$, $\boldsymbol{\lambda} \in C^2(\bar{J}, (H^1(\Omega))^2)$, and $(\boldsymbol{\lambda}_H^0, u_H^0) = (\tilde{\boldsymbol{\lambda}}_H^0, \tilde{u}_H^0)$, then there exists a constant $C > 0$ independent of H and τ such that*

$$\max_{1\leq n\leq N} \|u^n - u_H^n\| \leq C(\tau^{2-\alpha} + H), \tag{5.5}$$

$$\max_{1\leq n\leq N} \|\tilde{u}_H^n - u_H^n\| \leq C(\tau^{2-\alpha} + H^2). \tag{5.6}$$

Moreover, there exist a constant $C > 0$ independent of H, τ and c_0 such that, if $H \leq c_0\tau \leq c_0 \min\{\tau_2, \tau_0\}$ and $H < \hbar_0$, then

$$\max_{1\leq n\leq N} \|\boldsymbol{\lambda}^n - \boldsymbol{\lambda}_H^n\| \leq C(H + e^{\frac{c_0 T\mu_3}{\mu_1}}(\tau^{2-\alpha} + H^2)), \tag{5.7}$$

$$\max_{1\leq n\leq N} \|\tilde{\boldsymbol{\lambda}}_H^n - \boldsymbol{\lambda}_H^n\| \leq Ce^{\frac{c_0 T\mu_3}{\mu_1}}(\tau^{2-\alpha} + H^2), \tag{5.8}$$

$$\max_{1\leq n\leq N} \|(\boldsymbol{\lambda}^n - \boldsymbol{\lambda}_H^n)\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \leq C(H + e^{\frac{c_0 T\mu_3}{\mu_1}}(1 + \tau^{-\frac{\alpha}{2}})(\tau^{2-\alpha} + H^2)), \tag{5.9}$$

where τ_0 is defined in Theorem 3.1, $c_0, \hbar_0$ and τ_2 are defined in Theorem 4.1.

Proof. Choosing $\boldsymbol{v}_H = \boldsymbol{\delta}^n$ and $w_H = \sigma^n$ in (5.4), we have

$$(D_\tau^\alpha \sigma^n, \sigma^n) + (\mathcal{A}^{-1}\boldsymbol{\delta}^n, \gamma_H \boldsymbol{\delta}^n) = -(g(u^n) - g(u_H^n), \sigma^n) - (D_\tau^\alpha \beta^n, \sigma^n) - (R_t^n(\boldsymbol{x}), \sigma^n). \tag{5.10}$$

Making use of the Lagrange mean value theorem, L^2-projection P_H and Lemma 5.2, we have

$$\begin{aligned} -(g(u^n) - g(u_H^n), \sigma^n) &= -(g'(u_*^n)(u^n - u_H^n), \sigma^n) \\ &= -(g'(u_*^n)(u^n - P_H u^n + P_H u^n - \tilde{u}_H^n + \sigma^n), \sigma^n) \\ &= -((g'(u_*^n) - P_H g'(u_*^n))(u^n - P_H u^n) + g'(u_*^n)(P_H u^n - \tilde{u}_H^n + \sigma^n), \sigma^n) \\ &\leq CH^4(\|g\|_{2,\infty}^2 \|u\|_{L^\infty(H^1(\Omega))}^2 + \|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2) \\ &\quad + (1 + \|g\|_{1,\infty})\|\sigma^n\|^2, \end{aligned} \tag{5.11}$$

where u_*^n is located between u^n and u_H^n. And we can also obtain

$$\begin{aligned}-(D_\tau^\alpha\beta^n,\sigma^n) &= -(D_\tau^\alpha(u^n - P_H u^n + P_H u^n - \tilde{u}_H^n),\sigma^n)\\ &= -(D_\tau^\alpha(P_H u^n - \tilde{u}_H^n),\sigma^n)\\ &\le Ct_n^{1-\alpha}H^2(\|\boldsymbol{\lambda}_t\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}_t\|_{L^\infty(H^1(\Omega))})\|\sigma^n\|.\end{aligned} \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.10), making use of Lemma 3.7, we obtain

$$\begin{aligned}D_\tau^\alpha\|\sigma^n\|^2 \le& CH^4(\|g\|_{2,\infty}^2\|u\|_{L^\infty(H^1(\Omega))}^2 + \|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2)\\ &+ Ct_n^{2-2\alpha}H^4(\|\boldsymbol{\lambda}_t\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}_t\|_{L^\infty(H^1(\Omega))})^2 + 2(2+\|g\|_{1,\infty})\|\sigma^n\|^2 + \|R_t^n(\boldsymbol{x})\|^2.\end{aligned} \tag{5.13}$$

Noting that $\sigma^0 = 0$, applying Lemma 3.8, we obtain

$$\|\sigma^n\| \le C(\tau^{2-\alpha} + H^2). \tag{5.14}$$

Making use of (5.4)(b), we have

$$(\mathcal{A}^{-1}D_\tau^\alpha\boldsymbol{\delta}^n,\gamma_H\boldsymbol{v}_H) - (\mathrm{div}\boldsymbol{v}_H, D_\tau^\alpha\sigma^n) = 0, \forall\boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.15}$$

Choosing $w_H = D_\tau^\alpha\sigma^n$ in (5.4)(a) and $\boldsymbol{v}_H = \boldsymbol{\delta}^n$ in (5.15), we have

$$\|D_\tau^\alpha\sigma^n\|^2 + (\mathcal{A}^{-1}D_\tau^\alpha\boldsymbol{\delta}^n,\gamma_H\boldsymbol{\delta}^n) = -(g(u^n) - g(u_H^n), D_\tau^\alpha\sigma^n) - (D_\tau^\alpha\beta^n, D_\tau^\alpha\sigma^n) - (R_t^n(\boldsymbol{x}), D_\tau^\alpha\sigma^n). \tag{5.16}$$

For the term $-(g(u^n) - g(u_H^n), D_\tau^\alpha\sigma^n)$, similar to (5.11), we have

$$\begin{aligned}-(g(u^n) - g(u_H^n), D_\tau^\alpha\sigma^n) \le& CH^4(\|g\|_{2,\infty}^2\|u\|_{L^\infty(H^1(\Omega))}^2 + \|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2)\\ &+ C\|g\|_{1,\infty}^2\|\sigma^n\|^2 + \frac{1}{6}\|D_\tau^\alpha\sigma^n\|^2.\end{aligned} \tag{5.17}$$

For the term $-(D_\tau^\alpha\beta^n, D_\tau^\alpha\sigma^n)$, similar to (5.12), we have

$$-(D_\tau^\alpha\beta^n, D_\tau^\alpha\sigma^n) \le Ct_n^{2-2\alpha}H^4(\|\boldsymbol{\lambda}_t\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}_t\|_{L^\infty(H^1(\Omega))})^2 + \frac{1}{6}\|D_\tau^\alpha\sigma^n\|^2. \tag{5.18}$$

Substituting (5.17) and (5.18) into (5.16), applying Lemma 3.5, we obtain

$$\begin{aligned}&\frac{1}{2}\|D_\tau^\alpha\sigma^n\|^2 + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}[(\mathcal{A}^{-1}\boldsymbol{\delta}^n,\gamma_H\boldsymbol{\delta}^n) + \sum_{k=0}^{n-1} b_k^n(\mathcal{A}^{-1}\boldsymbol{\delta}^k,\gamma_H\boldsymbol{\delta}^k)\\ &- \sum_{k=0}^{n-1} b_k^n((\mathcal{A}^{-1}(\boldsymbol{\delta}^n - \boldsymbol{\delta}^k),\gamma_H(\boldsymbol{\delta}^n - \boldsymbol{\delta}^k)) + \sum_{k=0}^{n-1} b_k^n((\mathcal{A}^{-1}\boldsymbol{\delta}^n,\gamma_H\boldsymbol{\delta}^k) - (\mathcal{A}^{-1}\boldsymbol{\delta}^k,\gamma_H\boldsymbol{\delta}^n))]\\ &\quad\le CH^4(\|g\|_{2,\infty}^2\|u\|_{L^\infty(H^1(\Omega))}^2 + \|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2)\\ &\quad\quad + Ct_n^{2-2\alpha}H^4(\|\boldsymbol{\lambda}_t\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}_t\|_{L^\infty(H^1(\Omega))})^2 + C\|g\|_{1,\infty}^2\|\sigma^n\|^2 + \frac{3}{2}\|R_t^n(\boldsymbol{x})\|^2.\end{aligned} \tag{5.19}$$

Noting that $b_k^n < 0$ $(0 \le k < n)$, making use of (4.9) and (5.14), we have

$$(1 - \frac{\mu_3}{\mu_1}H)(\mathcal{A}^{-1}\boldsymbol{\delta}^n,\gamma_H\boldsymbol{\delta}^n) \le -(1 + \frac{\mu_3}{\mu_1}H)\sum_{k=0}^{n-1} b_k^n(\mathcal{A}^{-1}\boldsymbol{\delta}^k,\gamma_H\boldsymbol{\delta}^k) + C\tau^\alpha(\tau^{2(2-\alpha)} + H^4). \tag{5.20}$$

Selecting H to satisfy $H \leq \hbar_0$, where $\hbar_0 = \frac{\mu_1}{2\mu_3}$, we have $1 - \frac{\mu_3}{\mu_1}H \geq \frac{1}{2}$ and

$$(\mathcal{A}^{-1}\boldsymbol{\delta}^n, \gamma_H \boldsymbol{\delta}^n) \leq -\frac{1 + \frac{\mu_3}{\mu_1}H}{1 - \frac{\mu_3}{\mu_1}H} \sum_{k=0}^{n-1} b_k^n (\mathcal{A}^{-1}\boldsymbol{\delta}^k, \gamma_H \boldsymbol{\delta}^k) + C\tau^\alpha(\tau^{2(2-\alpha)} + H^4). \tag{5.21}$$

Applying the technique of (4.11)–(4.15), for the positive constant c_0 and τ_2 which defined in Theorem 4.1, noting that $\boldsymbol{\delta}^0 = \mathbf{0}$, we can obtain that if $\tau < \min\{\tau_2, \tau_0\}$ and $H \leq c_0\tau$, then

$$\|\boldsymbol{\delta}^n\| \leq C e^{\frac{c_0 T \mu_3}{\mu_1}} (\tau^{2-\alpha} + H^2). \tag{5.22}$$

Finally, we estimate $\|\boldsymbol{\lambda}^n - \boldsymbol{\lambda}_H^n\|_{\boldsymbol{H}(\mathrm{div},\Omega)}$. Choosing $w_H = \mathrm{div}\boldsymbol{\delta}^n$ in (5.4)(a) and $v_H = \boldsymbol{\delta}^n$ in (5.15), we have

$$(\mathcal{A}^{-1}D_\tau^\alpha \boldsymbol{\delta}^n, \gamma_H \boldsymbol{\delta}^n) + \|\mathrm{div}\boldsymbol{\delta}^n\|^2 + (g(u^n) - g(u_H^n), \mathrm{div}\boldsymbol{\delta}^n) = -(D_\tau^\alpha \beta^n, \mathrm{div}\boldsymbol{\delta}^n) - (R_t^n(\boldsymbol{x}), \mathrm{div}\boldsymbol{\delta}^n). \tag{5.23}$$

Noting that

$$\begin{aligned} -(\mathcal{A}^{-1}D_\tau^\alpha \boldsymbol{\delta}^n, \gamma_H \boldsymbol{\delta}^n) &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[\sum_{k=0}^{n-1}(-b_k^n)(\mathcal{A}^{-1}\boldsymbol{\delta}^k, \gamma_H \boldsymbol{\delta}^n) - (\mathcal{A}^{-1}\boldsymbol{\delta}^n, \gamma_H \boldsymbol{\delta}^n)] \\ &\leq C\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1}(-b_k^n)\|\boldsymbol{\delta}^k\|\|\boldsymbol{\delta}^n\| \\ &\leq C e^{\frac{2c_0 T\mu_3}{\mu_1}} \frac{1}{\Gamma(2-\alpha)} \tau^{-\alpha}(\tau^{2-\alpha} + H^2)^2, \end{aligned} \tag{5.24}$$

similar to the proof of (5.17) and (5.18), we obtain

$$\begin{aligned} \frac{1}{2}\|\mathrm{div}\boldsymbol{\delta}^n\|^2 \leq & C e^{\frac{2c_0 T\mu_3}{\mu_1}} \tau^{-\alpha}(\tau^{2-\alpha} + H^2)^2 + C\|g\|_{1,\infty}^2 \|\sigma^n\|^2 + \frac{3}{2}\|R_t^n(\boldsymbol{x})\|^2 \\ & + CH^4(\|g\|_{2,\infty}^2 \|u\|_{L^\infty(H^1(\Omega))}^2 + \|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2) \\ & + Ct_n^{2-2\alpha}H^4(\|\boldsymbol{\lambda}_t\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}_t\|_{L^\infty(H^1(\Omega))})^2. \end{aligned} \tag{5.25}$$

Making use of (5.14), we have

$$\|\mathrm{div}\boldsymbol{\delta}^n\| \leq C(\tau^{2-\alpha} + H^2 + e^{\frac{c_0 T\mu_3}{\mu_1}} \tau^{-\frac{\alpha}{2}}(\tau^{2-\alpha} + H^2)). \tag{5.26}$$

Then, apply Lemmas 5.1 and 5.2 to complete the proof. □

Remark 5.1. *For* $2 < q \leq \infty$*, making use of the inverse estimate and Lemma 5.2, we obtain*

$$\begin{aligned} \|u^n - u_H^n\|_{0,q} \leq & \|u^n - \tilde{u}_H^n\|_{0,q} + \|\tilde{u}_H^n - u_H^n\|_{0,q} \\ \leq & \|u^n - \tilde{u}_H^n\|_{0,q} + H^{\frac{2}{q}-1}\|\tilde{u}_H^n - u_H^n\| \\ \leq & C(H + H^{\frac{2}{q}-1}(\tau^{2-\alpha} + H^2)). \end{aligned} \tag{5.27}$$

Moreover, when $q = 4$*, we have*

$$\|u^n - u_H^n\|_{0,4} \leq C(H + H^{-\frac{1}{2}}\tau^{2-\alpha})),$$

which will be applied to the following estimates for the linearized MFVE scheme (2.9).

Next, we give the error estimates for the linearized MFVE scheme (2.9). Let $\vartheta^n = u^n - \tilde{u}_h^n$, $\xi^n = \tilde{u}_h^n - \hat{u}_h^n$, $\boldsymbol{\rho}^n = \boldsymbol{\lambda}^n - \tilde{\boldsymbol{\lambda}}_h^n$, $\boldsymbol{\theta}^n = \tilde{\boldsymbol{\lambda}}_h^n - \hat{\boldsymbol{\lambda}}_h^n$, where $(\tilde{\boldsymbol{\lambda}}_h^n, \tilde{u}_h^n) \in \boldsymbol{V}_h \times W_h$ is the generalized MFVE projection of $(\boldsymbol{\lambda}, u)$, then we can obtain the error equations as follows

$$\begin{cases} (D_\tau^\alpha \xi^n, w_h) + (\mathrm{div}\boldsymbol{\theta}^n, w_h) = -(G, w_h) - (D_\tau^\alpha \vartheta^n, w_h) - (R_t^n(\boldsymbol{x}), w_h), & \forall w_h \in W_h, \\ (\mathcal{A}^{-1}\boldsymbol{\theta}^n, \gamma_h \boldsymbol{v}_h) - (\mathrm{div}\boldsymbol{v}_h, \xi^n) = 0, & \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{cases} \tag{5.28}$$

where $G = g(u^n) - g(u_H^n) - g'(u_H^n)(\hat{u}_h^n - u_H^n)$.

Theorem 5.2. *Let $(\hat{\boldsymbol{\lambda}}_h^n, \hat{u}_h^n) \in \boldsymbol{V}_h \times W_h$ and $(\boldsymbol{\lambda}^n, u^n) \in \boldsymbol{V} \times W$ be the solutions of systems* (2.9) *and* (2.2)*, respectively. Assume that $u \in C^2(\bar{J}, W^{1,4}(\Omega))$, $\boldsymbol{\lambda} \in C^2(\bar{J}, (H^1(\Omega))^2)$, $\mathrm{div}\boldsymbol{\lambda} \in C^2(\bar{J}, H^1(\Omega))$, and $(\hat{\boldsymbol{\lambda}}_h^0, \hat{u}_h^0) = (\tilde{\boldsymbol{\lambda}}_h^0, \tilde{u}_h^0)$, then there exists a constant $C > 0$ independent of h and τ such that*

$$\max_{1\le n\le N} \|u^n - \hat{u}_h^n\| \le C(\tau^{2-\alpha} + h + H^2 + H^{-1}\tau^{4-2\alpha}). \tag{5.29}$$

Moreover, there exist a constant $C > 0$ independent of h, τ and c_0 such that, if $h \le c_0\tau \le c_0 \min\{\tau_2, \tau_1\}$ and $h < \hbar_0$, then

$$\max_{1\le n\le N} \|\boldsymbol{\lambda}^n - \hat{\boldsymbol{\lambda}}_h^n\| \le C(h + e^{\frac{c_0 T \mu_3}{\mu_1}}(\tau^{2-\alpha} + h^2 + H^2 + H^{-1}\tau^{4-2\alpha})), \tag{5.30}$$

$$\max_{1\le n\le N} \|(\boldsymbol{\lambda}^n - \hat{\boldsymbol{\lambda}}_h^n)\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \le C(h + e^{\frac{c_0 T \mu_3}{\mu_1}}(1 + \tau^{-\frac{\alpha}{2}})(\tau^{2-\alpha} + h^2 + H^2 + H^{-1}\tau^{4-2\alpha})), \tag{5.31}$$

where τ_1 is defined in Theorem 3.2, $c_0, \hbar_0$ and τ_2 are defined in Theorem 4.1.

Proof. Choosing $\boldsymbol{v}_h = \boldsymbol{\theta}^n$ and $w_h = \xi^n$ in (5.28), we have

$$(D_\tau^\alpha \xi^n, \xi^n) + (\mathcal{A}^{-1}\boldsymbol{\theta}^n, \gamma_h \boldsymbol{\theta}^n) = -(G, \xi^n) - (D_\tau^\alpha \vartheta^n, \xi^n) - (R_t^n(\boldsymbol{x}), \xi^n). \tag{5.32}$$

Making use of the Taylor expansion for $g(u^n)$ on $u = u_H^n$, we have

$$g(u^n) = g(u_H^n) + g'(u_H^n)(u^n - u_H^n) + \frac{1}{2}g''(u_\diamond^n)(u^n - u_H^n)^2, \tag{5.33}$$

where $u_\diamond^n$ is located between u^n and u_H^n. Noting that

$$\begin{aligned} -(G, \xi^n) &= -(g'(u_H^n)(u^n - \hat{u}_h^n) + \frac{1}{2}g''(u_\diamond^n)(u^n - u_H^n)^2, \xi^n) \\ &\le Ch^4(\|g\|_{2,\infty}^2 \|u\|_{L^\infty(H^1(\Omega))}^2 + \|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2) \\ &\quad + C\|g\|_{2,\infty}^2 \|u^n - u_H^n\|_{0,4}^4 + (1 + \|g\|_{1,\infty})\|\xi^n\|^2, \end{aligned} \tag{5.34}$$

similar to (5.12) for $(D_\tau^\alpha \vartheta^n, \xi^n)$, applying Lemma 3.7, we obtain

$$\begin{aligned} D_\tau^\alpha \|\xi^n\|^2 \le &Ch^4(\|g\|_{2,\infty}^2 \|u\|_{L^\infty(H^1(\Omega))}^2 + \|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2) \\ &+ Ct_n^{2-2\alpha}h^4(\|\boldsymbol{\lambda}_t\|_{L^\infty((H^1(\Omega))^2)} + \|\mathrm{div}\boldsymbol{\lambda}_t\|_{L^\infty(H^1(\Omega))})^2 \\ &+ C\|g\|_{2,\infty}^2 \|u^n - u_H^n\|_{0,4}^4 + 2(2 + \|g\|_{1,\infty})\|\xi^n\|^2 + \|R_t^n(\boldsymbol{x})\|^2. \end{aligned} \tag{5.35}$$

Applying Remark 5.1, Lemma 5.1 and Lemma 3.8, noting that $\xi^0 = 0$, we obtain

$$\|\xi^n\| \le C(\tau^{2-\alpha} + h^2 + H^2 + H^{-1}\tau^{4-2\alpha}). \tag{5.36}$$

Now, making use of (5.28)(b), we have

$$(\mathcal{A}^{-1}D_\tau^\alpha\boldsymbol{\theta}^n,\gamma_h\boldsymbol{v}_h)-(\mathrm{div}\boldsymbol{v}_h,D_\tau^\alpha\xi^n)=0,\forall\boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{5.37}$$

Choosing $w_h=D_\tau^\alpha\xi^n$ in (5.28)(a) and $\boldsymbol{v}_h=\boldsymbol{\theta}^n$ in (5.37), we have

$$\|D_\tau^\alpha\xi^n\|^2+(\mathcal{A}^{-1}D_\tau^\alpha\boldsymbol{\theta}^n,\gamma_h\boldsymbol{\theta}^n)=-(G,D_\tau^\alpha\xi^n)-(D_\tau^\alpha\vartheta^n,D_\tau^\alpha\xi^n)-(R_t^n(\boldsymbol{x}),D_\tau^\alpha\xi^n). \tag{5.38}$$

For the term $-(G,D_\tau^\alpha\xi^n)$, similar to (5.34), we have

$$\begin{aligned}-(G,D_\tau^\alpha\xi^n)&=-(g'(u_H^n)(u^n-\hat{u}_h^n)+\frac{1}{2}g''(u_\diamond^n)(u^n-u_H^n)^2,D_\tau^\alpha\xi^n)\\&\le Ch^4(\|g\|_{2,\infty}^2\|u\|_{L^\infty(H^1(\Omega))}^2+\|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)}+\|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2)\\&\quad+C\|g\|_{2,\infty}^2\|u^n-u_H^n\|_{0,4}^4+C\|g\|_{1,\infty}^2\|\xi^n\|^2+\frac{1}{6}\|D_\tau^\alpha\xi^n\|^2.\end{aligned} \tag{5.39}$$

Similar to (5.12) for $(D_\tau^\alpha\vartheta^n,D_\tau^\alpha\xi^n)$, applying Lemma 3.5 in (5.38), we obtain

$$\begin{aligned}&\|D_\tau^\alpha\xi^n\|^2+\frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}[(\mathcal{A}^{-1}\boldsymbol{\theta}^n,\gamma_h\boldsymbol{\theta}^n)+\sum_{k=0}^{n-1}b_k^n(\mathcal{A}^{-1}\boldsymbol{\theta}^k,\gamma_h\boldsymbol{\theta}^k)\\&-\sum_{k=0}^{n-1}b_k^n(\mathcal{A}^{-1}(\boldsymbol{\theta}^n-\boldsymbol{\theta}^k),\gamma_h(\boldsymbol{\theta}^n-\boldsymbol{\theta}^k))+\sum_{k=0}^{n-1}b_k^n((\mathcal{A}^{-1}\boldsymbol{\theta}^n,\gamma_h\boldsymbol{\theta}^k)-(\mathcal{A}^{-1}\boldsymbol{\theta}^k,\gamma_h\boldsymbol{\theta}^n))]\\&\quad\le Ch^4(\|g\|_{2,\infty}^2\|u\|_{L^\infty(H^1(\Omega))}^2+\|g\|_{1,\infty}^2(\|\boldsymbol{\lambda}\|_{L^\infty((H^1(\Omega))^2)}+\|\mathrm{div}\boldsymbol{\lambda}\|_{L^\infty(H^1(\Omega))})^2)\\&\quad+C\|g\|_{2,\infty}^2\|u^n-u_H^n\|_{0,4}^4+C\|g\|_{1,\infty}^2\|\xi^n\|^2+C\|R_t^n(\boldsymbol{x})\|^2\\&\quad+Ct_n^{2-2\alpha}h^4(\|\boldsymbol{\lambda}_t\|_{L^\infty((H^1(\Omega))^2)}+\|\mathrm{div}\boldsymbol{\lambda}_t\|_{L^\infty(H^1(\Omega))})^2+\frac{1}{2}\|D_\tau^\alpha\xi^n\|^2.\end{aligned} \tag{5.40}$$

Noting that $b_k^n<0$ $(0\le k<n)$, and making use of (5.36), we get

$$(1-\frac{\mu_3}{\mu_1}h)(\mathcal{A}^{-1}\boldsymbol{\theta}^n,\gamma_h\boldsymbol{\theta}^n)\le-(1+\frac{\mu_3}{\mu_1}h)\sum_{k=0}^{n-1}b_k^n(\mathcal{A}^{-1}\boldsymbol{\theta}^k,\gamma_h\boldsymbol{\theta}^k)+C\tau^\alpha(\tau^{4-2\alpha}+h^4+(H+H^{-\frac{1}{2}}\tau^{2-\alpha})^4). \tag{5.41}$$

Selecting h to satisfy $h\le\hbar_0$, where $\hbar_0=\frac{\mu_1}{2\mu_3}$, we have $1-\frac{\mu_3}{\mu_1}h\ge\frac{1}{2}$ and

$$(\mathcal{A}^{-1}\boldsymbol{\theta}^n,\gamma_h\boldsymbol{\theta}^n)\le-\frac{1+\frac{\mu_3}{\mu_1}h}{1-\frac{\mu_3}{\mu_1}h}\sum_{k=0}^{n-1}b_k^n(\mathcal{A}^{-1}\boldsymbol{\theta}^k,\gamma_h\boldsymbol{\theta}^k)+C\tau^\alpha(\tau^{4-2\alpha}+h^4+(H+H^{-\frac{1}{2}}\tau^{2-\alpha})^4). \tag{5.42}$$

Applying the technique of (4.11)–(4.15), for the positive constant c_0 and τ_2 which defined in Theorem 4.1, we can obtain that if $\tau<\min\{\tau_2,\tau_1\}$ and $h\le c_0\tau$, then

$$\|\boldsymbol{\theta}^n\|\le Ce^{\frac{c_0T\mu_3}{\mu_1}}(\tau^{2-\alpha}+h^2+H^2+H^{-1}\tau^{4-2\alpha}). \tag{5.43}$$

Apply Lemma 5.1 to complete the proof of (5.29) and (5.30).

Then, we estimate $\|\mathrm{div}(\boldsymbol{\lambda}^n-\boldsymbol{\lambda}_h^n)\|$. Choosing $w_h=\mathrm{div}\boldsymbol{\theta}^n$ in (5.28) and $\boldsymbol{v}_h=\boldsymbol{\theta}^n$ in (5.37), we have

$$(\mathcal{A}^{-1}D_\tau^\alpha\boldsymbol{\theta}^n,\gamma_h\boldsymbol{\theta}^n)+\|\mathrm{div}\boldsymbol{\theta}^n\|^2=-(G,\mathrm{div}\boldsymbol{\theta}^n)-(D_\tau^\alpha\vartheta^n,\mathrm{div}\boldsymbol{\theta}^n)-(R_t^n(\boldsymbol{x}),\mathrm{div}\boldsymbol{\theta}^n). \tag{5.44}$$

Apply Lemma 3.7 to obtain

$$\frac{1}{2}\|\mathrm{div}\boldsymbol{\theta}^n\|^2 \leq -(\mathcal{A}^{-1}D_\tau^\alpha\boldsymbol{\theta}^n, \gamma_h\boldsymbol{\theta}^n) - (G, \mathrm{div}\boldsymbol{\theta}^n) - (D_\tau^\alpha\vartheta^n, \mathrm{div}\boldsymbol{\theta}^n) - (R_t^n(\boldsymbol{x}), \mathrm{div}\boldsymbol{\theta}^n). \tag{5.45}$$

Applying the technique of (5.24) and (5.25), we have

$$\|\mathrm{div}\boldsymbol{\theta}^n\| \leq C(\tau^{2-\alpha} + h^2 + H^2 + H^{-1}\tau^{4-2\alpha} + e^{\frac{c_0T\mu_3}{\mu_1}}\tau^{-\frac{\alpha}{2}}(\tau^{2-\alpha} + h^2 + H^2 + H^{-1}\tau^{4-2\alpha})). \tag{5.46}$$

Finally, apply Lemma 5.1 and Remark 5.1 to complete the proof of (5.31). □

Remark 5.2. *(I) In Theorem 5.1, we should assume that* $u, \mathrm{div}\boldsymbol{\lambda} \in C^2(\bar{J}, H^1(\Omega))$, $\boldsymbol{\lambda} \in C^2(\bar{J}, (H^1(\Omega))^2)$. *We also need add the regularity* $u \in C^2(\bar{J}, W^{1,4}(\Omega))$ *in Theorem 5.2. Moreover, it should be pointed out that the solutions of FDEs usually show the initial weak singularity, some numerical methods [45–50] were proposed to deal with this problem.*

(II) Similar to Remark 4.1, when the coefficient $\mathcal{A}(\boldsymbol{x})$ *is a symmetry and positive definite constant matrix, we can remove the conditions* $H \leq c_0\tau$ *and* $h \leq c_0\tau$ *in the analysis and results of Theorems 5.1 and 5.2, respectively.*

6. Numerical examples

In this section, we will give two examples with some numerical results to test the convergence rates and the influence of the fractional parameters. In (1.1), we choose $\Omega = (0, 1)^2$, $J = (0, T]$, $\mathcal{A}(\boldsymbol{x})$ as the identity matrix, and the exact solution (similar as in [12, 31])

$$u(\boldsymbol{x}, t) = t^{\varpi}\sin(2\pi x_1)\sin(2\pi x_2), \boldsymbol{x} = (x_1, x_2) \in \bar{\Omega}, t \in \bar{J},$$

where ϖ is a parameter. Then, we can get the auxiliary variable

$$\boldsymbol{\lambda}(\boldsymbol{x}, t) = (-t^{\varpi}\cos(2\pi x_1)\sin(2\pi x_2), -t^{\varpi}\sin(2\pi x_1)\cos(2\pi x_2)),$$

and the source function

$$f(\boldsymbol{x}, t) = (\frac{\Gamma(\varpi + 1)}{\Gamma(\varpi + 1 - \alpha)}t^{\varpi-\alpha} + 8\pi^2t^{\varpi})\sin(2\pi x_1)\sin(2\pi x_2) + g(t^{\varpi}\sin(2\pi x_1)\sin(2\pi x_2)).$$

Example 6.1. By choosing $T = 1$, $g(u) = \sin(u)$, and $\varpi = 2$, we carry out numerical simulation for some different fractional parameters $\alpha = 0.2, 0.4, 0.6.0.8$ and grid sizes. In Tables 1 and 2, we take $\tau = 1/5, 1/8, 1/10$, $h \approx \sqrt{2}\tau^{2-\alpha}, H^2 \approx 2\tau^{2-\alpha}$ (in two-grid MFVE algorithm), and $h \approx \sqrt{2}\tau^{2-\alpha}$ (in MFVE algorithm (2.7)), and obtain that the convergence rates in time direction are close to $2 - \alpha$ for u in $L^2(\Omega)$-norm and $\boldsymbol{\lambda}$ in $(L^2(\Omega))^2$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$-norms, which is consistent with the theoretical results in Theorems 5.1 and 5.2. For testing convergence rates in space direction, we fix the time step length $\tau = 1/100$, select the coarse and fine grid sizes to satisfy $h = H^2/\sqrt{2} = \sqrt{2}/4, \sqrt{2}/16, \sqrt{2}/25, \sqrt{2}/36$, and give numerical results and computing time for the two-grid MFVE algorithm in Table 3. At the same time, in Table 4, we give some numerical results for the MFVE algorithm (2.7) with grid sizes $h = \sqrt{2}/4, \sqrt{2}/16, \sqrt{2}/25, \sqrt{2}/36$. We can see that the convergence rates are close to 1. Moreover, we choose the coarse and fine grid sizes to satisfy $h = H^2/\sqrt{2} = \sqrt{2}/4, \sqrt{2}/16, \sqrt{2}/25, \sqrt{2}/36$ and

$h = \sqrt{2}\tau$, give numerical results for the two-grid MFVE algorithm in Table 5, and the corresponding numerical results for the MFVE algorithm (2.7) in Table 6. Then we obtain the same conclusions as that discussed in Tables 3 and 4. Furthermore, for the time parameter $t = 1$, we show the graphs of the exact solutions for u and $\boldsymbol{\lambda}$ with $h = \sqrt{2}/32$ in Figures 2 and 4, respectively, also show the graphs of the numerical solutions based on the two-grid MFVE algorithm with $h = \sqrt{2}\tau = H^2/\sqrt{2} = \sqrt{2}/25$ in Figures 3 and 5. We find that the numerical solutions and the exact solutions have the same numerical behaviors.

Table 1. Numerical results of two-grid MFVE method with $\sqrt{2}h \approx H^2 \approx 2\tau^{2-\alpha}$ in Example 6.1.

α	τ	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}(\mathrm{div})$	Rates	CPU(s)
0.2	1/5	5.8737E-02		4.4976E-01		4.6277E+00		0.65
	1/8	2.6515E-02	1.6922	1.9978E-01	1.7266	2.0305E+00	1.7527	37.96
	1/10	1.8318E-02	1.6574	1.3464E-01	1.7684	1.3398E+00	1.8632	563.84
0.4	1/5	8.0866E-02		6.1961E-01		6.3754E+00		0.26
	1/8	3.8300E-02	1.5901	2.9246E-01	1.5973	3.0012E+00	1.6030	5.71
	1/10	2.7028E-02	1.5621	2.0417E-01	1.6106	2.0779E+00	1.6477	44.57
0.6	1/5	1.0468E-01		8.0240E-01		8.2477E+00		0.12
	1/8	5.8699E-02	1.2309	4.4978E-01	1.2316	4.6292E+00	1.2288	0.98
	1/10	4.2683E-02	1.4279	3.2635E-01	1.4377	3.3528E+00	1.4456	4.55
0.8	1/5	1.4887E-01		1.1400E+00		1.1662E+01		0.04
	1/8	8.7251E-02	1.1368	6.6995E-01	1.1310	6.8995E+00	1.1169	0.22
	1/10	6.5859E-02	1.2605	5.0497E-01	1.2669	5.1991E+00	1.2680	0.82

Table 2. Numerical results of MFVE method with $h \approx \sqrt{2}\tau^{2-\alpha}$ in Example 6.1.

α	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}(\mathrm{div})$	Rates	CPU(s)
0.2	1/5	5.8136E-02		4.4711E-01		4.6025E+00		2.28
	1/8	2.4930E-02	1.8015	1.9182E-01	1.8005	1.9770E+00	1.7979	199.75
	1/10	1.6621E-02	1.8167	1.2790E-01	1.8165	1.3184E+00	1.8158	2421.55
0.4	1/5	8.0427E-02		6.1829E-01		6.3565E+00		0.48
	1/8	3.7386E-02	1.6299	2.8763E-01	1.6282	2.9636E+00	1.6236	24.77
	1/10	2.6175E-02	1.5977	2.0140E-01	1.5971	2.0757E+00	1.5958	201.41
0.6	1/5	1.0441E-01		8.0223E-01		8.2333E+00		0.19
	1/8	5.8123E-02	1.2462	4.4704E-01	1.2441	4.6026E+00	1.2374	3.15
	1/10	4.1866E-02	1.4703	3.2209E-01	1.4692	3.3183E+00	1.4662	16.84
0.8	1/5	1.4856E-01		1.1405E+00		1.1652E+01		0.05
	1/8	8.7070E-02	1.1367	6.6933E-01	1.1339	6.8801E+00	1.1210	0.38
	1/10	6.5363E-02	1.2851	5.0269E-01	1.2831	5.1742E+00	1.2770	2.43

Table 3. Numerical results of two-grid MFVE method with $\tau = 1/100$ in Example 6.1.

α	H	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}$(div)	Rates	CPU(s)
0.2	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5504E-01		1.9551E+00		1.9629E+01		0.12
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5918E-02	0.9760	5.0488E-01	0.9766	5.1957E+00	0.9588	10.79
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2717E-02	0.9721	3.2625E-01	0.9784	3.3509E+00	0.9828	67.58
	$\sqrt{2}/6$	$\sqrt{2}/36$	3.0397E-02	0.9331	2.3037E-01	0.9544	2.3518E+00	0.9710	329.64
0.4	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5492E-01		1.9546E+00		1.9629E+01		0.12
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5894E-02	0.9759	5.0481E-01	0.9765	5.1956E+00	0.9588	11.23
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2689E-02	0.9727	3.2618E-01	0.9786	3.3507E+00	0.9829	66.57
	$\sqrt{2}/6$	$\sqrt{2}/36$	3.0358E-02	0.9349	2.3026E-01	0.9550	2.3515E+00	0.9712	331.08
0.6	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5479E-01		1.9541E+00		1.9629E+01		0.12
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5864E-02	0.9759	5.0473E-01	0.9764	5.1955E+00	0.9588	10.95
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2658E-02	0.9733	3.2610E-01	0.9788	3.3505E+00	0.9830	66.47
	$\sqrt{2}/6$	$\sqrt{2}/36$	3.0313E-02	0.9369	2.3014E-01	0.9558	2.3512E+00	0.9713	329.51
0.8	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5465E-01		1.9535E+00		1.9630E+01		0.11
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5830E-02	0.9758	5.0466E-01	0.9763	5.1956E+00	0.9588	10.66
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2624E-02	0.9740	3.2602E-01	0.9790	3.3506E+00	0.9829	66.49
	$\sqrt{2}/6$	$\sqrt{2}/36$	3.0264E-02	0.9391	2.3004E-01	0.9564	2.3512E+00	0.9713	330.54

Table 4. Numerical results of MFVE method with $\tau = 1/100$ in Example 6.1.

α	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}$(div)	Rates	CPU(s)
0.2	$\sqrt{2}/4$	2.5479E-01		1.9571E+00		1.9624E+01		0.23
	$\sqrt{2}/16$	6.5395E-02	0.9810	5.0286E-01	0.9802	5.1740E+00	0.9617	27.36
	$\sqrt{2}/25$	4.1874E-02	0.9989	3.2213E-01	0.9979	3.3182E+00	0.9953	190.66
	$\sqrt{2}/36$	2.9084E-02	0.9995	2.2378E-01	0.9991	2.3060E+00	0.9980	1181.80
0.4	$\sqrt{2}/4$	2.5470E-01		1.9567E+00		1.9625E+01		0.18
	$\sqrt{2}/16$	6.5393E-02	0.9808	5.0285E-01	0.9801	5.1740E+00	0.9617	27.61
	$\sqrt{2}/25$	4.1874E-02	0.9988	3.2213E-01	0.9979	3.3182E+00	0.9953	190.46
	$\sqrt{2}/36$	2.9084E-02	0.9995	2.2378E-01	0.9991	2.3060E+00	0.9980	1180.70
0.6	$\sqrt{2}/4$	2.5460E-01		1.9562E+00		1.9625E+01		0.23
	$\sqrt{2}/16$	6.5390E-02	0.9805	5.0284E-01	0.9799	5.1740E+00	0.9617	27.26
	$\sqrt{2}/25$	4.1873E-02	0.9988	3.2212E-01	0.9979	3.3182E+00	0.9953	191.45
	$\sqrt{2}/36$	2.9084E-02	0.9995	2.2377E-01	0.9991	2.3060E+00	0.9980	1187.40
0.8	$\sqrt{2}/4$	2.5449E-01		1.9556E+00		1.9625E+01		0.21
	$\sqrt{2}/16$	6.5386E-02	0.9803	5.0282E-01	0.9798	5.1740E+00	0.9617	26.96
	$\sqrt{2}/25$	4.1871E-02	0.9987	3.2212E-01	0.9978	3.3182E+00	0.9953	190.56
	$\sqrt{2}/36$	2.9083E-02	0.9995	2.2377E-01	0.9991	2.3060E+00	0.9980	1193.80

Table 5. Numerical results of two-grid MFVE method with $h = \sqrt{2}\tau$ in Example 6.1.

α	H	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}(\text{div})$	Rates	CPU(s)
0.2	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5504E-01		1.9551E+00		1.9629E+01		0.020
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5919E-02	0.9760	5.0488E-01	0.9766	5.1958E+00	0.9588	1.42
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2717E-02	0.9721	3.2626E-01	0.9784	3.3509E+00	0.9828	11.87
	$\sqrt{2}/6$	$\sqrt{2}/36$	3.0397E-02	0.9331	2.3037E-01	0.9544	2.3518E+00	0.9710	87.79
0.4	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5492E-01		1.9545E+00		1.9630E+01		0.021
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5896E-02	0.9759	5.0484E-01	0.9765	5.1959E+00	0.9588	1.34
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2691E-02	0.9726	3.2620E-01	0.9786	3.3509E+00	0.9829	11.92
	$\sqrt{2}/6$	$\sqrt{2}/36$	3.0360E-02	0.9349	2.3027E-01	0.9550	2.3516E+00	0.9712	87.84
0.6	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5479E-01		1.9538E+00		1.9631E+01		0.020
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5870E-02	0.9758	5.0480E-01	0.9763	5.1963E+00	0.9588	1.33
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2664E-02	0.9732	3.2616E-01	0.9787	3.3511E+00	0.9829	11.96
	$\sqrt{2}/6$	$\sqrt{2}/36$	3.0318E-02	0.9368	2.3019E-01	0.9557	2.3516E+00	0.9713	88.40
0.8	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5464E-01		1.9530E+00		1.9632E+01		0.019
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5845E-02	0.9757	5.0482E-01	0.9759	5.1975E+00	0.9587	1.34
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2639E-02	0.9737	3.2619E-01	0.9786	3.3520E+00	0.9828	11.95
	$\sqrt{2}/6$	$\sqrt{2}/36$	3.0278E-02	0.9388	2.3017E-01	0.9561	2.3524E+00	0.9712	88.11

Table 6. Numerical results of MFVE method with $h = \sqrt{2}\tau$ in Example 6.1.

α	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}(\text{div})$	Rates	CPU(s)
0.2	$\sqrt{2}/4$	2.5478E-01		1.9570E+00		1.9624E+01		0.016
	$\sqrt{2}/16$	6.5394E-02	0.9810	5.0286E-01	0.9802	5.1740E+00	0.9617	4.12
	$\sqrt{2}/25$	4.1874E-02	0.9988	3.2213E-01	0.9979	3.3182E+00	0.9953	44.61
	$\sqrt{2}/36$	2.9084E-02	0.9995	2.2378E-01	0.9991	2.3060E+00	0.9980	407.31
0.4	$\sqrt{2}/4$	2.5468E-01		1.9565E+00		1.9625E+01		0.013
	$\sqrt{2}/16$	6.5390E-02	0.9808	5.0284E-01	0.9800	5.1740E+00	0.9617	4.13
	$\sqrt{2}/25$	4.1873E-02	0.9988	3.2212E-01	0.9979	3.3182E+00	0.9953	45.10
	$\sqrt{2}/36$	2.9084E-02	0.9995	2.2377E-01	0.9991	2.3060E+00	0.9980	425.11
0.6	$\sqrt{2}/4$	2.5456E-01		1.9557E+00		1.9625E+01		0.012
	$\sqrt{2}/16$	6.5384E-02	0.9805	5.0280E-01	0.9798	5.1740E+00	0.9617	4.10
	$\sqrt{2}/25$	4.1871E-02	0.9987	3.2211E-01	0.9978	3.3182E+00	0.9954	45.55
	$\sqrt{2}/36$	2.9083E-02	0.9994	2.2377E-01	0.9990	2.3060E+00	0.9980	424.78
0.8	$\sqrt{2}/4$	2.5442E-01		1.9548E+00		1.9625E+01		0.013
	$\sqrt{2}/16$	6.5373E-02	0.9802	5.0274E-01	0.9796	5.1741E+00	0.9617	4.23
	$\sqrt{2}/25$	4.1867E-02	0.9985	3.2209E-01	0.9977	3.3183E+00	0.9954	45.52
	$\sqrt{2}/36$	2.9081E-02	0.9993	2.2376E-01	0.9989	2.3061E+00	0.9980	426.82

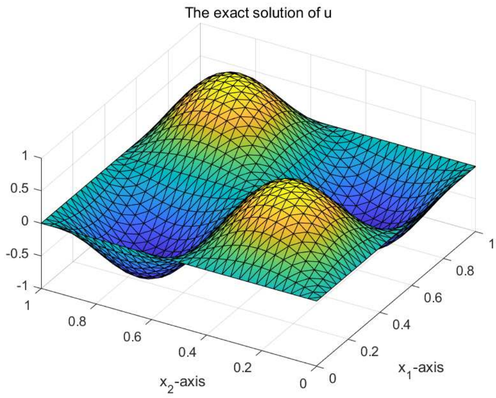

Figure 2. The exact solution of u at $t = 1$ with $h = \sqrt{2}/32$.

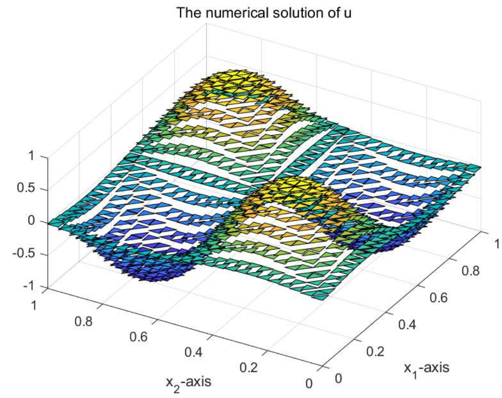

Figure 3. The numerical solution of u at $t = 1$ with $h = \sqrt{2}\tau = H^2/\sqrt{2} = \sqrt{2}/25$.

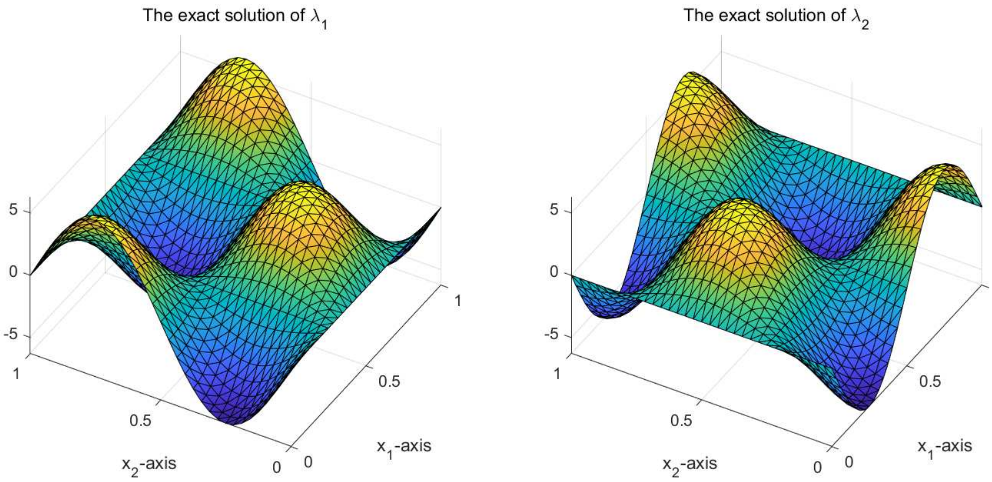

Figure 4. The exact solution of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ at $t = 1$ with $h = \sqrt{2}/32$.

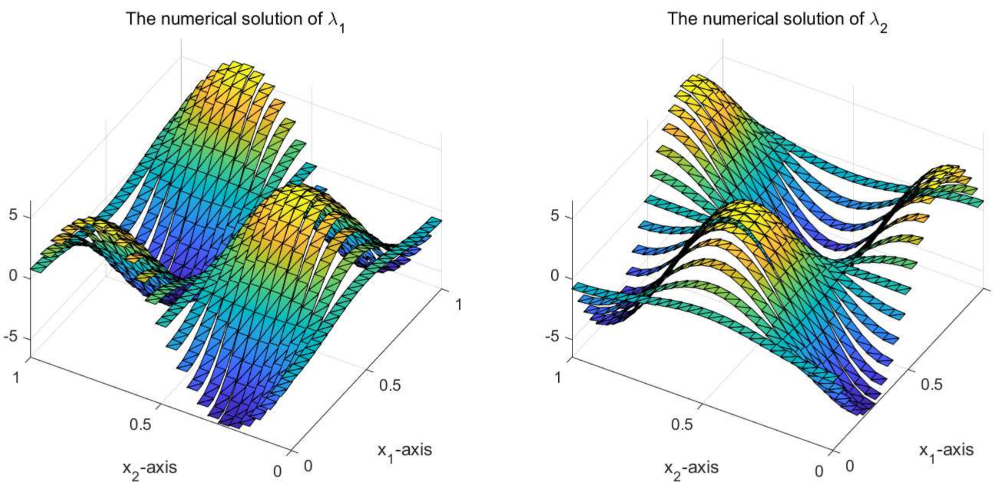

Figure 5. The numerical solution of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ at $t = 1$ with $h = \sqrt{2}\tau = H^2/\sqrt{2} = \sqrt{2}/25$.

Example 6.2. In this example, we take $T = 1$, $g(u) = u^3 - u$, and $\varpi = 2 + \alpha$, then obtain the exact solution $u(\boldsymbol{x}, t) = t^{2+\alpha}\sin(2\pi x_1)\sin(2\pi x_2), \boldsymbol{x} = (x_1, x_2) \in [0,1]^2, t \in [0,1]$, the auxiliary variable $\boldsymbol{\lambda}(\boldsymbol{x}, t) = -\nabla u(\boldsymbol{x}, t)$. For some different fractional parameters $\alpha = 0.2, 0.4, 0.6.0.8$ and grid sizes, we conduct numerical experiments as in Example 6.1. For the two-grid MFVE algorithm and MFVE algorithm (2.7), we can see that the convergence rates in time direction are close to $2 - \alpha$ (in Tables 7 and 8), and the convergence rates in space direction are close to 1 (in Tables 9 and 10). Moreover, in Tables 11 and 12, we choose $h = \sqrt{2}\tau = H^2/\sqrt{2}$ (in two-grid MFVE algorithm) and $h = \sqrt{2}\tau$ (in MFVE algorithm), then obtain the same convergence rates as in Tables 9 and 10.

Table 7. Numerical results of two-grid MFVE method with $\sqrt{2}h \approx H^2 \approx 2\tau^{2-\alpha}$ in Example 6.2.

α	τ	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}$(div)	Rates	CPU(s)
0.2	1/5	5.8163E-02		4.4738E-01		4.6060E+00		0.76
	1/8	2.5745E-02	1.7341	1.9636E-01	1.7520	2.0493E+00	1.7231	38.24
	1/10	1.7569E-02	1.7122	1.3248E-01	1.7635	1.3984E+00	1.7128	481.85
0.4	1/5	8.0430E-02		6.1858E-01		6.3587E+00		0.203
	1/8	3.7676E-02	1.6135	2.8895E-01	1.6195	2.9888E+00	1.6063	5.61
	1/10	2.6375E-02	1.5981	2.0111E-01	1.6242	2.0981E+00	1.5856	44.01
0.6	1/5	1.0442E-01		8.0368E-01		8.2469E+00		0.09
	1/8	5.8141E-02	1.2458	4.4727E-01	1.2469	4.6059E+00	1.2393	0.98
	1/10	4.2132E-02	1.4434	3.2328E-01	1.4549	3.3406E+00	1.4394	4.50
0.8	1/5	1.4832E-01		1.1422E+00		1.1672E+01		0.09
	1/8	8.7128E-02	1.1318	6.7024E-01	1.1343	6.8830E+00	1.1237	0.22
	1/10	6.5371E-02	1.2875	5.0289E-01	1.2873	5.1769E+00	1.2766	0.82

Table 8. Numerical results of MFVE method with $h \approx \sqrt{2}\tau^{2-\alpha}$ in Example 6.2.

α	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}$(div)	Rates	CPU(s)
0.2	1/5	5.8141E-02		4.4713E-01		4.6025E+00		2.04
	1/8	2.4930E-02	1.8017	1.9182E-01	1.8006	1.9770E+00	1.7979	205.44
	1/10	1.6621E-02	1.8168	1.2790E-01	1.8165	1.3184E+00	1.8158	2463.05
0.4	1/5	8.0441E-02		6.1834E-01		6.3565E+00		0.38
	1/8	3.7388E-02	1.6302	2.8764E-01	1.6284	2.9636E+00	1.6236	25.03
	1/10	2.6175E-02	1.5977	2.0140E-01	1.5972	2.0757E+00	1.5958	185.43
0.6	1/5	1.0443E-01		8.0230E-01		8.2334E+00		0.17
	1/8	5.8126E-02	1.2466	4.4705E-01	1.2443	4.6026E+00	1.2374	3.11
	1/10	4.1867E-02	1.4704	3.2209E-01	1.4692	3.3183E+00	1.4662	16.77
0.8	1/5	1.4859E-01		1.1405E+00		1.1653E+01		0.18
	1/8	8.7070E-02	1.1372	6.6931E-01	1.1340	6.8804E+00	1.1210	0.44
	1/10	6.5361E-02	1.2852	5.0267E-01	1.2831	5.1744E+00	1.2770	2.50

Table 9. Numerical results of two-grid MFVE method with $\tau = 1/100$ in Example 6.2.

α	H	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}$(div)	Rates	CPU(s)
0.2	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5509E-01		1.9610E+00		1.9625E+01		0.13
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5423E-02	0.9816	5.0316E-01	0.9813	5.1769E+00	0.9613	11.55
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2207E-02	0.9821	3.2345E-01	0.9901	3.3406E+00	0.9816	67.63
	$\sqrt{2}/6$	$\sqrt{2}/36$	2.9633E-02	0.9700	2.2714E-01	0.9694	2.3749E+00	0.9357	337.08
0.4	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5500E-01		1.9605E+00		1.9625E+01		0.12
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5418E-02	0.9814	5.0315E-01	0.9811	5.1769E+00	0.9613	11.09
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2163E-02	0.9843	3.2332E-01	0.9909	3.3401E+00	0.9819	68.19
	$\sqrt{2}/6$	$\sqrt{2}/36$	2.9567E-02	0.9732	2.2693E-01	0.9709	2.3739E+00	0.9365	335.08
0.6	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5488E-01		1.9599E+00		1.9625E+01		0.12
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5411E-02	0.9811	5.0314E-01	0.9809	5.1770E+00	0.9613	11.02
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2099E-02	0.9874	3.2314E-01	0.9921	3.3394E+00	0.9824	67.74
	$\sqrt{2}/6$	$\sqrt{2}/36$	2.9471E-02	0.9780	2.2661E-01	0.9732	2.3723E+00	0.9378	337.56
0.8	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5473E-01		1.9592E+00		1.9625E+01		0.12
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5404E-02	0.9808	5.0313E-01	0.9806	5.1770E+00	0.9613	11.56
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2019E-02	0.9914	3.2291E-01	0.9937	3.3385E+00	0.9830	67.61
	$\sqrt{2}/6$	$\sqrt{2}/36$	2.9351E-02	0.9840	2.2621E-01	0.9761	2.3701E+00	0.9395	336.04

Table 10. Numerical results of MFVE method with $\tau = 1/100$ in Example 6.2.

α	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}$(div)	Rates	CPU(s)
0.2	$\sqrt{2}/4$	2.5527E-01		1.9592E+00		1.9625E+01		0.18
	$\sqrt{2}/16$	6.5402E-02	0.9823	5.0288E-01	0.9810	5.1740E+00	0.9617	27.90
	$\sqrt{2}/25$	4.1876E-02	0.9990	3.2214E-01	0.9980	3.3182E+00	0.9953	191.53
	$\sqrt{2}/36$	2.9085E-02	0.9996	2.2378E-01	0.9991	2.3060E+00	0.9980	1166.74
0.4	$\sqrt{2}/4$	2.5516E-01		1.9587E+00		1.9625E+01		0.16
	$\sqrt{2}/16$	6.5399E-02	0.9820	5.0287E-01	0.9808	5.1740E+00	0.9617	28.73
	$\sqrt{2}/25$	4.1875E-02	0.9989	3.2213E-01	0.9979	3.3182E+00	0.9954	198.13
	$\sqrt{2}/36$	2.9085E-02	0.9996	2.2378E-01	0.9991	2.3060E+00	0.9980	1216.14
0.6	$\sqrt{2}/4$	2.5502E-01		1.9581E+00		1.9625E+01		0.19
	$\sqrt{2}/16$	6.5395E-02	0.9817	5.0286E-01	0.9806	5.1740E+00	0.9617	28.53
	$\sqrt{2}/25$	4.1874E-02	0.9989	3.2213E-01	0.9979	3.3182E+00	0.9954	198.66
	$\sqrt{2}/36$	2.9084E-02	0.9995	2.2378E-01	0.9991	2.3060E+00	0.9980	1220.80
0.8	$\sqrt{2}/4$	2.5484E-01		1.9572E+00		1.9625E+01		0.19
	$\sqrt{2}/16$	6.5390E-02	0.9812	5.0283E-01	0.9803	5.1740E+00	0.9617	28.19
	$\sqrt{2}/25$	4.1872E-02	0.9988	3.2212E-01	0.9979	3.3182E+00	0.9954	198.22
	$\sqrt{2}/36$	2.9083E-02	0.9995	2.2377E-01	0.9991	2.3061E+00	0.9980	1221.50

Table 11. Numerical results of two-grid MFVE method with $h = \sqrt{2}\tau$ in Example 6.2.

α	H	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}$(div)	Rates	CPU(s)
0.2	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5509E-01		1.9609E+00		1.9625E+01		0.011
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5423E-02	0.9816	5.0316E-01	0.9812	5.1769E+00	0.9613	1.32
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2208E-02	0.9820	3.2345E-01	0.9900	3.3406E+00	0.9816	12.01
	$\sqrt{2}/6$	$\sqrt{2}/36$	2.9634E-02	0.9700	2.2715E-01	0.9693	2.3749E+00	0.9357	90.82
0.4	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5498E-01		1.9603E+00		1.9625E+01		0.011
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5416E-02	0.9813	5.0314E-01	0.9810	5.1769E+00	0.9613	1.32
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2166E-02	0.9840	3.2334E-01	0.9908	3.3402E+00	0.9818	12.06
	$\sqrt{2}/6$	$\sqrt{2}/36$	2.9570E-02	0.9732	2.2694E-01	0.9708	2.3740E+00	0.9365	92.40
0.6	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5483E-01		1.9594E+00		1.9625E+01		0.012
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5405E-02	0.9810	5.0309E-01	0.9808	5.1769E+00	0.9613	1.33
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2109E-02	0.9867	3.2318E-01	0.9917	3.3397E+00	0.9821	11.98
	$\sqrt{2}/6$	$\sqrt{2}/36$	2.9481E-02	0.9777	2.2666E-01	0.9729	2.3726E+00	0.9376	91.20
0.8	$\sqrt{2}/2$	$\sqrt{2}/4$	2.5462E-01		1.9579E+00		1.9625E+01		0.012
	$\sqrt{2}/4$	$\sqrt{2}/16$	6.5385E-02	0.9807	5.0299E-01	0.9804	5.1768E+00	0.9613	1.32
	$\sqrt{2}/5$	$\sqrt{2}/25$	4.2039E-02	0.9897	3.2300E-01	0.9924	3.3393E+00	0.9824	12.03
	$\sqrt{2}/6$	$\sqrt{2}/36$	2.9374E-02	0.9831	2.2634E-01	0.9753	2.3711E+00	0.9390	91.93

Table 12. Numerical results of MFVE method with $h = \sqrt{2}\tau$ in Example 6.2.

α	h	u–L^2	Rates	$\boldsymbol{\lambda}$–$(L^2)^2$	Rates	$\boldsymbol{\lambda}$–$\boldsymbol{H}$(div)	Rates	CPU(s)
0.2	$\sqrt{2}/4$	2.5527E-01		1.9592E+00		1.9625E+01		0.011
	$\sqrt{2}/16$	6.5401E-02	0.9823	5.0288E-01	0.9810	5.1740E+00	0.9617	4.00
	$\sqrt{2}/25$	4.1876E-02	0.9990	3.2214E-01	0.9980	3.3182E+00	0.9954	43.04
	$\sqrt{2}/36$	2.9085E-02	0.9996	2.2378E-01	0.9991	2.3060E+00	0.9980	393.60
0.4	$\sqrt{2}/4$	2.5515E-01		1.9586E+00		1.9625E+01		0.015
	$\sqrt{2}/16$	6.5397E-02	0.9820	5.0286E-01	0.9808	5.1740E+00	0.9617	4.02
	$\sqrt{2}/25$	4.1875E-02	0.9989	3.2213E-01	0.9979	3.3182E+00	0.9954	42.51
	$\sqrt{2}/36$	2.9084E-02	0.9995	2.2378E-01	0.9991	2.3060E+00	0.9980	389.52
0.6	$\sqrt{2}/4$	2.5499E-01		1.9577E+00		1.9625E+01		0.012
	$\sqrt{2}/16$	6.5389E-02	0.9817	5.0282E-01	0.9805	5.1740E+00	0.9617	3.91
	$\sqrt{2}/25$	4.1872E-02	0.9988	3.2212E-01	0.9978	3.3182E+00	0.9954	42.56
	$\sqrt{2}/36$	2.9083E-02	0.9995	2.2377E-01	0.9990	2.3060E+00	0.9980	386.93
0.8	$\sqrt{2}/4$	2.5477E-01		1.9563E+00		1.9626E+01		0.015
	$\sqrt{2}/16$	6.5373E-02	0.9812	5.0274E-01	0.9801	5.1742E+00	0.9617	3.91
	$\sqrt{2}/25$	4.1866E-02	0.9985	3.2209E-01	0.9977	3.3183E+00	0.9954	42.67
	$\sqrt{2}/36$	2.9081E-02	0.9993	2.2376E-01	0.9989	2.3061E+00	0.9980	412.74

Base on the above the numerical results in Tables 1–6 and Figures 2–5 for Example 6.1 and Tables 7–12 for Example 6.2, we can know that the convergence rates are consistent with the theoretical results in Theorems 5.1 and 5.2. We also find that the two-grid MFVE algorithm can save the computing time compared with the MFVE algorithm while maintaining the same convergence rates. Finally, numerical results and the figures show that the proposed two-grid MFVE algorithm for the nonlinear time fractional reaction-diffusion equations is feasible and effective.

7. Conclusions

In this paper, we construct the two-grid MFVE fast algorithm to solve the nonlinear time fractional reaction-diffusion equations with the Caputo time fractional derivative. We obtain the stability results and the optimal error estimates for u (in $L^2(\Omega)$-norm) and $\boldsymbol{\lambda}$ (in $(L^2(\Omega))^2$-norm), and the sub-optimal error estimates for $\boldsymbol{\lambda}$ (in $H(\text{div}, \Omega)$-norm). Furthermore, we also give two numerical examples to verify that the proposed algorithm can greatly save the computing time. In future works, for the Caputo fractional derivative (1.2) with $\alpha \in (0, 1)$, we will try to use other approximation methods (such as $L1$-2, $L2$-1_σ, $L1$-2-3 formulas [17–20]) and the two-grid MFVE method to solve more fractional partial differential equations in scientific and engineering fields.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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