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Research article

Analytical solutions of generalized differential equations using quadratic-phase Fourier transform

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Abstract: The aim of this study is to obtain the analytical solutions of some prominent differential equations including the generalized Laplace, heat and wave equations by using the quadratic-phase Fourier transform. To facilitate the narrative, we formulate the preliminary results vis-a-vis the differentiation properties of the quadratic-phase Fourier transform. The obtained results are reinforced with illustrative examples.

Keywords: quadratic-phase Fourier transform; differentiation theorem; Laplace equation; wave equation; heat equation
Mathematics Subject Classification: 35K05, 35L05, 42C20, 83C15

1. Introduction

While working on the problem of heat conduction, Joseph Fourier, made a detailed study of trigonometric series which appeared in his celebrated memoir "*Théorie Analytique de la Chaleur*". Thereafter, Fourier's work was well received from the research community and gained an admirable spot in diverse fields of physical, mathematical and engineering sciences [1]. The field of differential and integral equations, making effective use of Fourier transforms, has gained considerable attention since its inception because of their widespread applications in numerous disciplines of physical and engineering sciences [2]. Mathematically, the Fourier transform of any square integrable function f is defined as:

$$\mathscr{F}[f](\omega)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\,e^{-i\omega x}\,dx. \tag{1.1}$$

A much extreme generalization of the classical Fourier transform (1.1) was recently obtained by Saitoh [3] in the form of quadratic-phase Fourier transform (QPFT), while working on the solution of heat equation via the theory of reproducing kernels. For a given square integrable function f, the quadratic-phase Fourier transform is defined by

$$Q_\Omega[f](\omega)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\,\exp\left\{i(Ax^2+Bx\omega+C\omega^2+Dx+E\omega)\right\}dx, \tag{1.2}$$

where $A,B,C,D,E\in\mathbb{R}$, $B\neq 0$. The QPFT circumscribes a class of integral transforms ranging from the classical Fourier to the much recent special affine Fourier transforms [4]. Owing to the fact that the QPFT is governed by a set of free parameters, it has proved to be a reliable tool for an efficient treatment of problems demanding several controllable parameters arising in diverse branches of science and engineering, including harmonic analysis, theory of reproducing kernels, sampling, image processing, and several other fields [5–7].

On the flip side, it is well known that integral transforms are one of the most reliable mathematical tools for solving diverse classes of differential equations. Some of the commonly used integral transforms employed for obtaining the analytic solutions of differential equations include the Fourier,fractional Fourier, linear canonical and Laplace transforms [8–12]. Several other vital techniques for determining the solutions of the linear and non-linear differential equations can be found in recent literature including numerical methods, variational iteration methods and so on [13–15]. However, to the best of our knowledge, the solution of a generalized differential equation using the QPFT has not yet been reported in the literature. Keeping in view the fact that the QPFT is a new addition to the class of integral transforms which enjoys several extra degrees of freedom, we are deeply motivated to solve different classes of generalized differential equations in the QPFT domain. Taking this opportunity, our aim is to formulate some fundamental differentiation properties of the QPFT and employ the same for obtaining the analytical solutions of several well-known differential equations such as the laplace, wave, and heat equations. The obtained solutions are non-trivial generalizations of the solutions achieved via the classical tools governed by the Fourier, fractional Fourier, and linear canonical transforms. Nevertheless, the prolificacy of the obtained results is demonstrated via graphical simulations in the respective cases. Besides, the tabulated data is also provided for each of the differential equations at hand which portrays the advantages of QPFT over the existing transforms in the sense that it offers more degrees of freedom as the parameters in the set $\Omega=(A,B,C,D,E)$ can be customized to meet specific needs.

The remaining part of the paper is structured as follows: Section 2 deals with the recapitulation of preliminaries on the quadratic-phase Fourier transform, which will be used in subsequent sections. Section 3 is exclusively concerned with the analytic solutions of some generalized differential equations using the QPFT. The discourse is wrapped with some concluding remarks.

2. Quadratic-phase Fourier transform

We shall start this section with a brief overview of the quadratic-phase Fourier transform which serves as a cornerstone for the development of the subsequent sections. With minor modifications to (1.2), we have the following definition of quadratic phase Fourier transform:

Definition 2.1. [6] For a given set of parameters $\Omega = (A, B, C, D, E)$, the quadratic-phase Fourier transform of $f \in L^2(\mathbb{R})$ is denoted as $\mathcal{Q}_\Omega[f]$ and is defined by

$$\mathcal{Q}_\Omega[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, \mathcal{K}_\Omega(\omega, x)\, dx, \tag{2.1}$$

where $\mathcal{K}_\Omega(\omega, x)$ denotes the quadratic-phase Fourier kernel and is given by

$$\mathcal{K}_\Omega(\omega, x) = \exp\Big\{ - i(Ax^2 + B\omega x + C\omega^2 + Dx + E\omega)\Big\},\ A, B, C, D, E \in \mathbb{R},\ B \neq 0. \tag{2.2}$$

The backward transform and the Parseval's relation corresponding to (2.1) are given below:

$$f(x) = \frac{|B|}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{Q}_\Omega[f](\omega)\, \overline{\mathcal{K}_\Omega(\omega, x)}\, d\omega, \tag{2.3}$$

$$\langle f, g\rangle = \big|B\big| \Big\langle \mathcal{Q}_\Omega[f],\, \mathcal{Q}_\Omega[g]\Big\rangle, \quad \forall\, f, g \in L^2(\mathbb{R}). \tag{2.4}$$

By appropriately choosing parameters in $\Omega = (A, B, C, D, E)$, the expression (2.1) reduces to some of the prominent integral transforms as indicated below:

• For $A = C = D = E = 0$ and $B = 1$, the expression (2.1) yields the conventional Fourier transform defined in (1.1).

• The fractional Fourier transform is obtained by setting $D = E = 0$, $A = -\cot\theta/2$, $B = \csc\theta$, $C = -\cot\theta/2$, where $\theta \neq n\pi$, $n \in \mathbb{Z}$ and then amplifying (2.1) by $\sqrt{1 - i\cot\theta}$ [2]

$$\mathscr{F}^{\alpha}[f](\omega) = \int_{\mathbb{R}} f(x)\, \mathcal{K}_\alpha(x, \omega)\, dx, \tag{2.5}$$

where the kernel $\mathcal{K}_\alpha(\omega, x)$ is given by

$$\mathcal{K}_\alpha(\omega, x) = \begin{cases} \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, \exp\left\{\frac{i}{2}\left(\omega^2 + x^2\right)\cot\alpha - i\omega x\, \csc\alpha\right\}, & \alpha \neq k\pi, \\ \delta(x - \omega), & \alpha = 2k\pi, \\ \delta(x + \omega), & \alpha = (2k+1)\pi,\ k \in \mathbb{Z}. \end{cases}$$

• For $D = E = 0$ and $B \neq 0$, we consider the transformations $A \to -A/2B$, $B \to 1/B$, $C \to -C/2B$. Upon multiplying (2.1) by $1/\sqrt{iB}$, we obtain the linear canonical transform [9]

$$\mathcal{L}_\Omega[f](\omega) = \frac{1}{\sqrt{2\pi i B}} \int_{\mathbb{R}} f(x)\, \exp\left\{\frac{i(Ax^2 - 2\omega x + C\omega^2)}{2B}\right\} dx. \tag{2.6}$$

Next, we shall recall the novel convolution introduced by Shah and Tantary [7] for the quadratic-phase Fourier transforms. Here, our main motive is to utilize this novel convolution for obtaining the analytical solutions of some well-known differential equations in the next subsection.

Definition 2.2. [7] If $f, g \in L^2(\mathbb{R})$, then the quadratic-phase convolution $\otimes_\Omega$ with respect to the parameter set $\Omega = (A, B, C, D, E)$ is given as:

$$(f \otimes_\Omega g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi)\, g(x - \xi)\, e^{2iA\xi(x-\xi)}\, d\xi. \tag{2.7}$$

Note that, the quadratic-phase convolution (2.7) obeys the following properties:
• Commutativity: $(f \otimes_\Omega g)(x) = (g \otimes_\Omega f)(x)$
• Associativity: $((f \otimes_\Omega g) \otimes_\Omega h)(x) = (f \otimes_\Omega (g \otimes_\Omega h))(x)$
• Translation: $(f \otimes_\Omega g)(x-k) = e^{-2iAkx}(e^{2iAk\xi} f(\xi - k) \otimes_\Omega g(\xi))(x)$
• Scaling: $(f \otimes_\Omega g)(\lambda x) = |\lambda|(f(\lambda\xi) \otimes_{\Omega'} g(\lambda\xi))(x)$, where $\Omega' = (\lambda^2 A, B, C, D, E)$, $\lambda \in \mathbb{R}^*$

Moreover, the convolution theorem corresponding to (2.7) reads:

$$\mathcal{Q}_\Omega\Big[(f \otimes_\Omega g)(x)\Big](\omega) = e^{i(C\omega^2 + E\omega)}\, \mathcal{Q}_\Omega[f](\omega)\, \mathcal{Q}_\Omega[g](\omega). \tag{2.8}$$

3. Applications of quadratic-phase Fourier transform to generalized partial differential equations

In this section, our main concern is to express the analytical solutions of a class of generalized partial differential equations, including the Laplace, wave, and heat equations, by making use of the quadratic-phase Fourier transform. To facilitate the intention, we shall first define the notion of generalized Schwartz space and then prove some needful differential properties in the quadratic-phase Fourier domain.

The generalized Schwartz class $\mathbb{S}_\Omega(\mathbb{R})$ of rapidly decaying functions associated with the quadratic-phase Fourier transforms is defined as

$$\mathbb{S}_\Omega(\mathbb{R}) = \left\{ f(x) \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} x^\beta \mathbb{D}_x^\gamma f(x) < \infty \right\},$$

with $C^\infty(\mathbb{R})$ denoting the space of smooth functions, $\beta, \gamma \in \mathbb{Z}^+ \cup \{0\}$ and $\mathbb{D}_x = d/dx - i(2Ax + D)$, denotes the modified differential operator.

We are now ready to study some differentiable properties of the quadratic-phase Fourier transform (2.1) and its kernel $\mathcal{K}_\Omega(\omega, x)$:

Proposition 3.1. *Let $f \in \mathbb{S}_\Omega(\mathbb{R})$ and $\mathcal{K}_\Omega(\omega, x)$ be the kernel of quadratic-phase Fourier transform* (2.1). *Then, we have*
(i) $\overline{\mathbb{D}}_x^n \mathcal{K}_\Omega(\omega, x) = (iB\omega)^n \mathcal{K}_\Omega(\omega, x), \quad n \in \mathbb{N},$
(ii) $\overline{\mathbb{D}}_\omega^n \mathcal{K}_\Omega(\omega, x) = (iBx)^n \mathcal{K}_\Omega(\omega, x), \quad n \in \mathbb{N},$
(iii) $\displaystyle\int_{\mathbb{R}} (\mathbb{D}_x \mathcal{K}(\omega, x)) f(x)\, dx = \int_{\mathbb{R}} \mathcal{K}_\Omega(\omega, x)\, \overline{\mathbb{D}}_x f(x)\, dx$
(iv) $\mathcal{Q}_\Omega[\mathbb{D}_x^n f(x)](\omega) = (iB\omega)^n \mathcal{Q}_\Omega[f](\omega), \quad n \in \mathbb{N},$
(v) $\overline{\mathbb{D}}_\omega^n \mathcal{Q}_\Omega[f](\omega) = (iB)^n \mathcal{Q}_\Omega[x^n f(x)](\omega), \quad n \in \mathbb{N},$

where $\mathbb{D}_x = d/dx - i(2Ax + D), \mathbb{D}_\omega = d/d\omega - i(2C\omega + E), \overline{\mathbb{D}}_x = -d/dx - i(2Ax + D)$ and $\overline{\mathbb{D}}_\omega = -d/d\omega - i(2C\omega + E)$.

Proof. (i) Let $K_\Omega(\omega, x)$ be the kernel of quadratic-phase Fourier transform (2.1), then

$$\begin{aligned}\overline{\mathbb{D}}_x \mathcal{K}_\Omega(\omega, x) &= -\left(\frac{d}{dx} + i(2Ax + D)\right) \frac{1}{\sqrt{2\pi}} \exp\Big\{ -i(Ax^2 + Bx\omega + C\omega^2 + Dx + E\omega)\Big\} \\ &= \frac{iB\omega}{\sqrt{2\pi}} \exp\Big\{ -i(Ax^2 + Bx\omega + C\omega^2 + Dx + E\omega)\Big\}\end{aligned}$$

$$= iB\omega\,\mathcal{K}_\Omega(\omega, x).$$

Continuing like-this n-times, we can obtain

$$\overline{\mathbb{D}}_x^n\,\mathcal{K}_\Omega(\omega, x) = (iB\omega)^n\,\mathcal{K}_\Omega(\omega, x).$$

(ii) The proof of (ii) is quite straightforward.
(iii) For any $f \in \mathbb{S}_\Omega(\mathbb{R})$, we observe that

$$\begin{aligned}
\int_{\mathbb{R}} (\mathbb{D}_x\,\mathcal{K}_\Omega(\omega, x))f(x)\,dx &= \int_{\mathbb{R}} \left(\frac{d}{dx} - i(2Ax + D)\right)\mathcal{K}_\Omega(\omega, x)f(x)\,dx \\
&= \int_{\mathbb{R}} \frac{d}{dx}\,\mathcal{K}_\Omega(\omega, x)f(x)\,dx - \int_{\mathbb{R}} i(2Ax + D)\,\mathcal{K}(\omega, x)f(x)\,dx \\
&= -\int_{\mathbb{R}} \mathcal{K}_\Omega(\omega, x)\,\frac{d}{dx}\,f(x)\,dx - \int_{\mathbb{R}} i(2Ax + D)\,\mathcal{K}(\omega, x)f(x)\,dx \\
&= -\int_{\mathbb{R}} \mathcal{K}_\Omega(\omega, x)\left(\frac{d}{dx} + i(2Ax + D)\right)f(x)\,dx \\
&= \int_{\mathbb{R}} \mathcal{K}_\Omega(\omega, x)\,\overline{\mathbb{D}}_x f(x)\,dx.
\end{aligned}$$

(iv) By virtue of quadratic-phase Fourier transform (2.1), we have

$$\begin{aligned}
\mathcal{Q}_\Omega\Big[\mathbb{D}_x^n f(x)\Big](\omega) &= \int_{\mathbb{R}} \mathcal{K}_\Omega(\omega, x)\,\mathbb{D}_x^n f(x)\,dx \\
&= \int_{\mathbb{R}} \overline{\mathbb{D}}_x^n\,\mathcal{K}_\Omega(\omega, x)f(x)\,dx \\
&= (iB\omega)^n \int_{\mathbb{R}} \mathcal{K}_\Omega(\omega, x)f(x)\,dx \\
&= (iB\omega)^n\,\mathcal{Q}_\Omega[f](\omega).
\end{aligned}$$

(v) Finally, we have

$$\begin{aligned}
\overline{\mathbb{D}}_\omega^n\,\mathcal{Q}_\Omega[f](\omega) &= \int_{\mathbb{R}} \overline{\mathbb{D}}_\omega^n\,\mathcal{K}_\Omega(\omega, x)\,f(x)\,dx \\
&= \int_{\mathbb{R}} (iBx)^n\,\mathcal{K}_\Omega(\omega, x)f(x)\,dx \\
&= (iB)^n \int_{\mathbb{R}} \mathcal{K}_\Omega(\omega, x)\,x^n f(x)\,dx \\
&= (iB)^n\,\mathcal{Q}_\Omega[x^n f(x)](\omega).
\end{aligned}$$

The proof of Proposition 3.1 is thus completed. □

We are now in a position to derive the solution of generalized Laplace, wave, and heat equations by employing the quadratic-phase Fourier transform method. We shall also provide an example for a lucid illustration of the method.

3.1. The generalized Laplace equation

Here, we shall obtain an analytic solution of the Laplace equation in the quadratic-phase Fourier domain subjected to appropriate boundary conditions. Consider,

$$\nabla^2 U := \mathbb{D}_x^2 U + U_{tt} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{3.1}$$

$$U(x,0) = f(x), \quad -\infty < x < \infty, \tag{3.2}$$

$$U(x,t) \to 0 \quad \text{as} \quad |x| \to \infty \;\text{ and }\; t \to \infty. \tag{3.3}$$

For fixed t, applying quadratic-phase Fourier transform (2.1) on the above system of equations yields:

$$\frac{d^2}{dt^2}\mathcal{Q}_\Omega[U](\omega,t) - B^2\omega^2\mathcal{Q}_\Omega[U](\omega,t) = 0, \quad t > 0, \tag{3.4}$$

$$\mathcal{Q}_\Omega[U](\omega,0) = \mathcal{Q}_\Omega[f](\omega) \tag{3.5}$$

$$\mathcal{Q}_\Omega[U](\omega,t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.6}$$

Then, the solution of the transformed system of Eqs (3.4)–(3.6) is given by

$$\mathcal{Q}_\Omega[U](\omega,t) = e^{i(C\omega^2+E\omega)}\,\mathcal{Q}_\Omega[f](\omega)\,\mathcal{Q}_\Omega[g](\omega), \tag{3.7}$$

where $\mathcal{Q}_\Omega[g](\omega) = e^{-B|\omega|t - i(C\omega^2+E\omega)}$.

Invoking the inversion formula of the quadratic-phase Fourier transform (2.3) for (3.7), we obtain

$$U(x,t) = \mathcal{Q}_\Omega^{-1}\left[e^{i(C\omega^2+E\omega)}\,\mathcal{Q}_\Omega[f](\omega)\,\mathcal{Q}_\Omega[g](\omega)\right](x). \tag{3.8}$$

For obtaining an explicit form of the expression $U(x,t)$ appearing in (3.8), we shall use the convolution theorem for the quadratic-phase Fourier transform (2.8) as

$$U(x,t) = (f \otimes_\Omega g)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(\xi)\,g(x-\xi)\,e^{2iA\xi(x-\xi)}\,d\xi, \tag{3.9}$$

where

$$\begin{aligned} g(x) &= \frac{|B|}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-B|\omega|t - i(C\omega^2+E\omega)}\,\exp\left\{i(Ax^2 + Bx\omega + C\omega^2 + Dx + E\omega)\right\}d\omega \\ &= \frac{|B|}{\sqrt{2\pi}}\cdot e^{i(Ax^2+Dx)}\int_{\mathbb{R}} e^{-B|\omega|t + iBx\omega}\,d\omega \\ &= \frac{\operatorname{sgn}(B)}{\pi}\cdot\frac{e^{i(Ax^2+Dx)}\,t}{x^2+t^2}. \end{aligned} \tag{3.10}$$

After plugging (3.10) in (3.9), we get

$$U(x,t) = \frac{t\,\operatorname{sgn}(B)}{\pi}\cdot e^{i(Ax^2+Dx)}\int_{\mathbb{R}} \frac{f(\xi)}{(x-\xi)^2+t^2}\,e^{2iA\xi(x-\xi)}\,d\xi, \quad t > 0. \tag{3.11}$$

Expression (3.11) is the required integral solution of the generalized Laplace Eq (3.1) under the given boundary conditions in the quadratic-phase Fourier domain.

• For the parametric set $\Omega = (-A/2B, 1/B, -C/2B, 0, 0)$, solution (3.11) reduces to the solution of the Laplace equation in linear canonical domain as

$$U(x,t) = \frac{t\,\mathrm{sgn}\left(\frac{1}{B}\right)}{\pi} \cdot e^{-iAx^2/2B} \int_{\mathbb{R}} \frac{f(\xi)}{(x-\xi)^2 + t^2}\, e^{-iA\xi(x-\xi)/B}\, d\xi.$$

• For the parametric set $\Omega = (-\cot\theta/2, \csc\theta, -\cot\theta/2, 0, 0)$, $\theta \neq n\pi$, one can obtain the solution of the Laplace equation via fractional Fourier transform as

$$U(x,t) = \frac{t\,\mathrm{sgn}(\csc\theta)}{\pi} \cdot e^{-ix^2\cot\theta/2} \int_{\mathbb{R}} \frac{f(\xi)}{(x-\xi)^2 + t^2}\, e^{-iA\xi(x-\xi)\cot\theta}\, d\xi.$$

• For the parametric set $\Omega = (0, 1, -1, 0, 0)$, solution (3.11) boils down to the solution of the ordinary Laplace equation

$$U(x,t) = \frac{t}{\pi} \int_{\mathbb{R}} \frac{f(\xi)}{(x-\xi)^2 + t^2}\, d\xi.$$

Example 3.2. Let us consider the Laplace Eq (3.1) subjected to $U(x,0) = \delta(x)$, then according to (3.11), the solution has the following form:

$$\begin{aligned} U(x,t) &= \frac{t\,\mathrm{sgn}(B)}{\pi} \cdot e^{i(Ax^2+Dx)} \int_{\mathbb{R}} \frac{\delta(\xi)}{(x-\xi)^2 + t^2}\, e^{2iA\xi(x-\xi)}\, dx \\ &= \frac{t\,\mathrm{sgn}(B)}{\pi} \cdot \frac{e^{i(Ax^2+Dx)}}{x^2 + t^2}. \end{aligned} \tag{3.12}$$

The solution of an Example 3.2 for the case $A = B = D = 1$ is of the form

$$U(x,t) = \frac{t}{\pi} \cdot \frac{e^{i(x^2+x)}}{x^2 + t^2}. \tag{3.13}$$

Moreover, the solution of Example 3.2 for the case $A = B = D = 1$ is graphically shown in Figure 1. For fixed values of t, the solution of the Example 3.2 for the case $A = B = D = 1$ corresponding to different values of x are tabulated in Table 1.

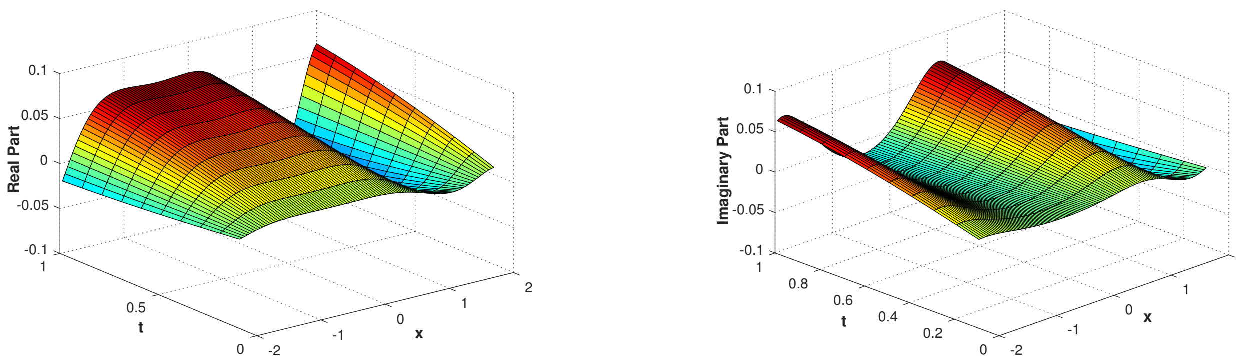


Figure 1. Real and imaginary parts of Example 3.2 for the case $A = B = D = 1$.

Table 1. Solution of Example 3.2 for different values of x and fixed t at $A = B = D = 1$.

	$t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2.0$
x	$u(x,t)$	$u(x,t)$	$u(x,t)$	$u(x,t)$
-1.6	$0.0325 + 0.0464i$	$0.0513 + 0.0732i$	$0.0569 + 0.0813i$	$0.0557 + 0.0795i$
-1.2	$0.0550 + 0.0135i$	$0.0869 + 0.0213i$	$0.0964 + 0.0236i$	$0.0943 + 0.0231i$
-0.8	$0.0559 - 0.0090i$	$0.0883 - 0.0142i$	$0.0980 - 0.0158i$	$0.0958 - 0.0155i$
-0.4	$0.550 - 0.0135i$	$0.0869 - 0.0213i$	$0.0964 - 0.0236i$	$0.0943 - 0.0231i$
0	$0.0566 + 0.0000i$	$0.0894 + 0.0000i$	$0.0993 + 0.0000i$	$0.0970 + 0.0000i$
0.4	$0.0480 + 0.0301i$	$0.0758 + 0.0475i$	$0.0841 + 0.0527i$	$0.0822 + 0.0515i$
0.8	$0.0074 + 0.0562i$	$0.0117 + 0.0886i$	$0.0129 + 0.0984i$	$0.0127 + 0.0962i$
1.2	$-0.0497 + 0.0272i$	$-0.0784 + 0.0430i$	$-0.0870 + 0.0477i$	$-0.0851 + 0.0467i$
1.6	$-0.0297 - 0.0482i$	$-0.0469 - 0.0761i$	$-0.0521 - 0.0845i$	$-0.0509 - 0.0826i$

The solution of an Example 3.2 for the case $A = D = 0$ and $B = 1$, yields the solution of the classical Laplace equation corresponding to the initial condition $U(x,0) = \delta(x)$ and is of the form

$$U(x,t) = \frac{t}{\pi} \cdot \frac{1}{x^2 + t^2}. \tag{3.14}$$

For a lucid illustration of the behavior of the solution of the traditional Laplace equation subjected to $U(x,0) = \delta(x)$, a graphical representation of (3.14) is presented in Figure 2.

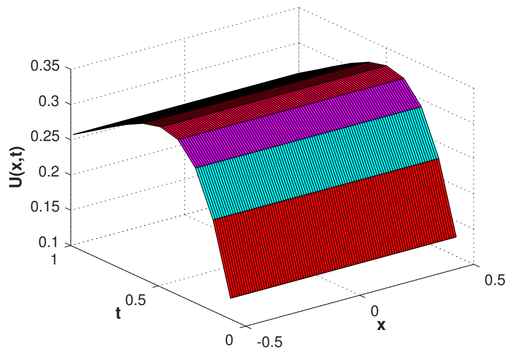


Figure 2. Solution of Example 3.2 for the case $A = D = 0, B = 1$ and $U(x,0) = \delta(x)$.

3.2. The generalized wave equation

The generalized wave equation defined on a Schwartz class $\mathbb{S}_\Omega(\mathbb{R})$ is given by

$$U_{tt} = k^2\, \overline{\mathbb{D}}_x^2\, U, \quad -\infty < k, x < \infty, \quad t > 0, \tag{3.15}$$

$$U(x,0) = f(x), \quad U_t(x,0) = g(x), \quad -\infty < x < \infty, \quad t > 0. \tag{3.16}$$

Suppose that $U(x,t)$ as well as its first partial derivatives vanish at infinity. Then, invoking the quadratic-phase Fourier transform yields

$$\frac{d^2}{dt^2}\, \mathcal{Q}_\Omega[U](\omega,t) + k^2B^2\omega^2\mathcal{Q}_\Omega[U](\omega,t) = 0, \tag{3.17}$$

$$Q_\Omega[U](\omega,0)=Q_\Omega[f](\omega),\qquad \frac{d}{dt}\,Q_\Omega[U]\,(\omega,t)\bigg|_{t=0}=Q_\Omega[g](\omega). \tag{3.18}$$

Consequently, we have

$$\begin{aligned} Q_\Omega[U](\omega,t)&=Q_\Omega[f](\omega)\,\cos(Bkt\omega)+Q_\Omega[g](\omega)\,\frac{\sin(Bkt\omega)}{Bk\omega}\\ &=\frac{Q_\Omega[f](\omega)}{2}\left\{e^{iBkt\omega}+e^{-iBkt\omega}\right\}+\frac{Q_\Omega[g](\omega)}{2iBk\omega}\left\{e^{iBkt\omega}-e^{-iBkt\omega}\right\}. \end{aligned} \tag{3.19}$$

Therefore, the desired solution is obtained by applying the backward quadratic-phase Fourier transformation to (3.19) as:

$$\begin{aligned} U(x,t)&=\frac{|B|}{\sqrt{2\pi}}\int_{\mathbb{R}}\left\{\frac{Q_\Omega[f](\omega)}{2}\left\{e^{iBkt\omega}+e^{-iBkt\omega}\right\}+\frac{Q_\Omega[g](\omega)}{2iBk\omega}\left\{e^{iBkt\omega}-e^{-iBkt\omega}\right\}\right\}\\ &\qquad\times\overline{\mathcal{K}_\Omega(\omega,x)}\,d\omega\\ &=\frac{|B|}{\sqrt{2\pi}}\int_{\mathbb{R}}\left\{\left(\frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}}f(\xi)\left\{e^{iBkt\omega}+e^{-iBkt\omega}\right\}\mathcal{K}_\Omega(\omega,\xi)\,d\xi\right)\right.\\ &\qquad\left.+\left(\frac{1}{2iBk\omega\sqrt{2\pi}}\int_{\mathbb{R}}g(\xi)\left\{e^{iBkt\omega}-e^{-iBkt\omega}\right\}\mathcal{K}_\Omega(\omega,\xi)\,d\xi\right)\right\}\overline{\mathcal{K}_\Omega(\omega,x)}\,d\omega\\ &=\frac{|B|}{2\pi}\int_{\mathbb{R}}\left\{\left(\frac{1}{2}\int_{\mathbb{R}}f(\xi)\left\{e^{iBkt\omega}+e^{-iBkt\omega}\right\}e^{-i(A\xi^2+B\omega\xi+D\xi)}\,e^{i(Ax^2+B\omega x+Dx)}\,d\xi\right)\right.\\ &\qquad\left.+\left(\frac{1}{2iBk\omega}\int_{\mathbb{R}}g(\xi)\left\{e^{iBkt\omega}-e^{-iBkt\omega}\right\}e^{-i(a\xi^2+B\omega\xi+D\xi)}\,e^{i(Ax^2+B\omega x+Dx)}\,d\xi\right)\right\}d\omega\\ &=\frac{|B|}{2\pi}\cdot e^{i(Ax^2+Dx)}\int_{\mathbb{R}}\left\{\left(\frac{1}{2}\int_{\mathbb{R}}f(\xi)\left\{e^{iBkt\omega}+e^{-iBkt\omega}\right\}e^{-i(A\xi^2+B\omega\xi+D\xi)}\,e^{iB\omega x}\,d\xi\right)\right.\\ &\qquad\left.+\left(\frac{1}{2iBk\omega}\int_{\mathbb{R}}g(\xi)\left\{e^{iBkt\omega}-e^{-iBkt\omega}\right\}e^{-i(A\xi^2+B\omega\xi+D\xi)}\,e^{iB\omega x}\,d\xi\right)\right\}d\omega. \end{aligned}$$

Setting $F(\xi)=f(\xi)\,e^{-i(A\xi^2+D\xi)}, G(\xi)=g(\xi)\,e^{-i(A\xi^2+D\xi)}$ and $v=B\omega$, we obtain

$$\begin{aligned} U(x,t)=\frac{\operatorname{sgn}(B)}{2\pi}\cdot e^{i(Ax^2+Dx)}&\int_{\mathbb{R}}\left\{\left(\frac{1}{2}\int_{\mathbb{R}}F(\xi)\,e^{-iv\xi}\,d\xi\left\{e^{iktv}+e^{-iktv}\right\}e^{ivx}\right)\right.\\ &\left.+\left(\frac{1}{2ikv}\int_{\mathbb{R}}G(\xi)\,e^{-iv\xi}\,d\xi\left\{e^{iktv}-e^{-iktv}\right\}e^{ivx}\right)\right\}dv. \end{aligned} \tag{3.20}$$

By invoking the definition of Fourier transform, (3.20) takes the form:

$$\begin{aligned} &U(x,t)\\ &=\frac{\operatorname{sgn}(B)}{\sqrt{2\pi}}\cdot e^{i(Ax^2+Dx)}\int_{\mathbb{R}}\left\{\frac{1}{2}\,\mathscr{F}[F](v)\left\{e^{i(x+kt)v}+e^{i(x-kt)v}\right\}\right.\\ &\qquad\left.+\frac{1}{2ikv}\,\mathscr{F}[G](v)\left\{e^{i(x+kt)v}-e^{i(x-kt)v}\right\}\right\}dv \end{aligned}$$

$$= \operatorname{sgn}(B)\, e^{i(Ax^2+Dx)} \left\{ \frac{1}{2}(F(x+kt)+F(x-kt)) + \frac{1}{2k} \int_{\mathbb{R}} \mathscr{F}[G](v) \left\{ \int_{x-kt}^{x+kt} e^{ivs}\, ds \right\} dv \right\}$$
$$= \operatorname{sgn}(B)\, e^{i(Ax^2+Dx)} \left\{ \frac{1}{2}(F(x+kt)+F(x-kt)) + \frac{1}{2k} \int_{x-kt}^{x+kt} \left\{ \int_{\mathbb{R}} \mathscr{F}[G](v)\, e^{ivs}\, dv \right\} ds \right\}$$
$$= \operatorname{sgn}(B)\, e^{i(Ax^2+Dx)} \left\{ \frac{1}{2}(F(x+kt)+F(x-kt)) + \frac{1}{2k} \int_{x-kt}^{x+kt} G(s)\, ds \right\}. \tag{3.21}$$

Substituting back the values of F and G in (3.21), we get

$$U(x,t) = \operatorname{sgn}(B)\, e^{i(Ax^2+Dx)} \left\{ \frac{1}{2} \Big(f(x+kt)\, e^{i(A(x+kt)^2+D(x+kt))} + f(x-kt) \right.$$
$$\left. \times e^{i(A(x-kt)^2+D(x-kt))} \Big) + \frac{1}{2k} \int_{x-kt}^{x+kt} g(s)\, e^{i(As^2+Ds)}\, ds \right\}. \tag{3.22}$$

Expression (3.22) is the desired solution corresponding to (3.15) in the quadratic-phase Fourier domain.

• For the parametric set $\Omega = (-A/2B, 1/B, -C/2B, 0, 0)$, Eq (3.22) reduces to the solution of the wave equation in the linear canonical domain

$$U(x,t) = \frac{\operatorname{sgn}\left(\frac{1}{B}\right)}{e^{iAx^2/2B}} \left\{ \frac{1}{2} \Big(f(x+kt)\, e^{-iA(x+kt)^2/2B} + f(x-kt)\, e^{-iA(x-kt)^2/2B} \Big) \right.$$
$$\left. + \frac{1}{2k} \int_{x-kt}^{x+kt} g(s)\, e^{-iAs^2/2B}\, ds \right\}.$$

• For the parametric set $\Omega = (-\cot\theta/2, \csc\theta, -\cot\theta/2, 0, 0)$, $\theta \neq n\pi$, one can obtain the solution of the generalized wave equation in fractional Fourier domain as

$$U(x,t) = \frac{\operatorname{sgn}(\csc\theta)}{e^{i\cot\theta\, x^2/2}} \left\{ \frac{1}{2} \Big(f(x+kt)\, e^{-i\cot\theta(x+kt)^2/2} + f(x-kt)\, e^{-i\cot\theta(x-kt)^2/2} \Big) \right.$$
$$\left. + \frac{1}{2k} \int_{x-kt}^{x+kt} g(s)\, e^{-i\cot\theta s^2/2}\, ds \right\}.$$

• For the parametric set $\Omega = (0, 1, -1, 0, 0)$, Eq (3.22) boils down to the classical solution of the wave equation

$$U(x,t) = \frac{1}{2}\big(f(x+kt)+f(x-kt)\big) + \frac{1}{2k} \int_{x-kt}^{x+kt} g(s)\, ds.$$

Example 3.3. Consider the initial value problem:

$$U_{tt} = \overline{\mathbb{D}}_x^2 U, \quad -\infty < x < \infty, \quad t > 0,$$
$$U(x,0) = e^{-x^2}, \quad U_t(x,0) = e^{-x^2}.$$

Then, according to (3.22), the solution of the above system is expressible as:

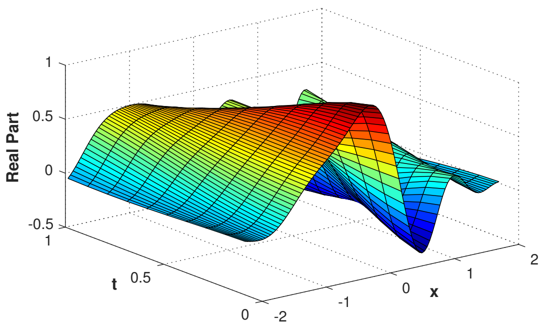
$$U(x,t) = \frac{\operatorname{sgn}(B)}{2} \cdot e^{i(Ax^2+Dx)} \Big\{ e^{-(x+t)^2}\, e^{i(A(x+t)^2+D(x+t))} + e^{-(x-t)^2} e^{i(A(x-t)^2+D(x-t))}$$

$$+\int_{x-t}^{x+t} e^{-s^2}\, e^{i(As^2+Ds)}\, ds\Big\}. \tag{3.23}$$

In particular, the solution (3.23) of an Example 3.3 for the case $A = B = D = 1$ takes the form:

$$\begin{aligned} U(x,t) &= \frac{e^{i(x^2+x)}}{2}\left\{e^{i(x+t)} + e^{i(x-t))} + \int_{x-t}^{x+t} e^{-s^2+i(s^2+s)}\, ds\right\} \\ &= \frac{e^{i(x^2+x)}}{2}\left\{e^{i(x+t)} + e^{i(x-t))} + \frac{i\sqrt{(1+i)\pi}}{2\sqrt{2}} \cdot e^{-(1+i)/8}\right. \\ &\times \left.\left(\operatorname{erfi}\left(\frac{\sqrt{1+i}}{2\sqrt{2}}\big(1+2(1+i)(x-t)\big)\right) - \operatorname{erfi}\left(\frac{\sqrt{1+i}}{2\sqrt{2}}\big(1+2(1+i)(x+t)\big)\right)\right)\right\}. \end{aligned} \tag{3.24}$$

The $3D$ plots of the solution of Example 3.3 are depicted in Figure 3, while, the estimates of $U(x,t)$ in (3.24) for different values of x and fixed t are presented in Table 2.



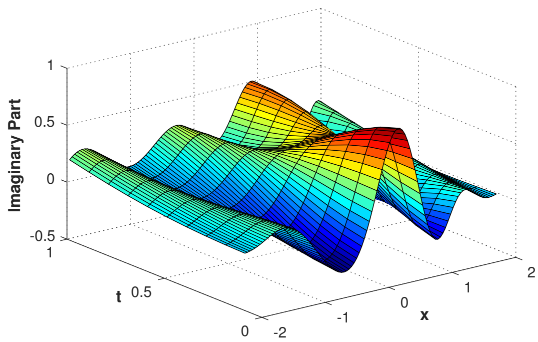


Figure 3. Real and imaginary parts corresponding to solution of Example 3.3 for $A = B = D = 1$.

Table 2. Numerical values of the solution of Example 3.3 for $A = B = D = 1$.

	$t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2.0$
x	$u(x,t)$	$u(x,t)$	$u(x,t)$	$u(x,t)$
−1.6	$0.0656 + 0.1300i$	$0.2625 + 0.2295i$	$0.3192 + 0.3784i$	$0.0216 + 0.4255i$
−1.2	$0.3101 + 0.0367i$	$0.4750 + 0.0394i$	$0.3693 + 0.2689i$	$-0.0287 + 0.2621i$
−0.8	$0.5159 - 0.1442i$	$0.4845 + 0.0572i$	$0.1553 + 0.2634i$	$-0.0934 + 0.0726i$
−0.4	$0.7013 - 0.1362i$	$0.3291 + 0.2522i$	$-0.0700 + 0.1444i$	$-0.0291 - 0.0271i$
0	$0.6622 + 0.1691i$	$0.1074 + 0.1673i$	$-0.0047 + 0.0058i$	$0.0050 + 0.0058i$
0.4	$0.2982 + 0.3944i$	$0.2810 + 0.0603i$	$-0.1301 + 0.0897i$	$0.0008 + 0.0396i$
0.8	$-0.1428 + 0.3531i$	$0.1569 + 0.4641i$	$0.1001 + 0.2897i$	$-0.0127 + 0.1177i$
1.2	$-0.2203 - 0.1721i$	$-0.4697 + 0.1232i$	$-0.3457 + 0.2984i$	$-0.2080 + 0.1620i$
1.6	$0.1446 + 0.0219i$	$0.1386 - 0.3198i$	$-0.2119 - 0.4474i$	$-0.3034 - 0.2992i$

Also for $A = D = 0$ and $B = 1$, solution (3.23) correspond to the classical case and is given by

$$U(x,t) = \frac{1}{2}\left(e^{-(x+t)^2} + e^{-(x-t)^2}\right) + \frac{1}{2}\int_{x-t}^{x+t} e^{-s^2}\, ds$$

$$= \frac{1}{2}\left(e^{-(x+t)^2} + e^{-(x-t)^2}\right) + \frac{\sqrt{\pi}}{4}\left(\text{erf}\,(x+t) - \text{erf}\,(x-t)\right). \tag{3.25}$$

For a lucid illustration of the behaviour of one dimensional wave equation, a graphical representation of the expression (3.25) is depicted in Figure 4.

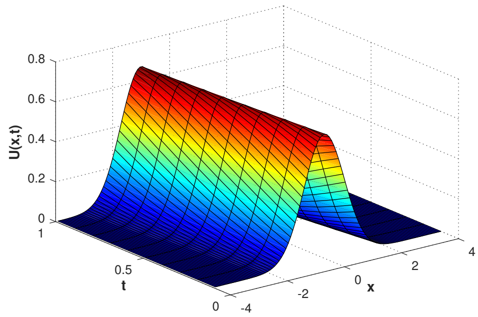


Figure 4. Solution of Example 3.3 for $A = D = 0$ and $B = 1$.

3.3. The generalized heat equation

Consider the heat equation associated with the quadratic-phase Fourier transform given by

$$U_t = k^2\,\overline{\mathbb{D}}_x^2\,U(x,t), \quad -\infty < k, x < \infty, \quad t > 0, \tag{3.26}$$

subjected to

$$U(x,0) = f(x). \tag{3.27}$$

Applying quadratic-phase Fourier transform (2.1) to the above system of equations, we obtain

$$\frac{d}{dt}\mathcal{Q}_\Omega[U](\omega,t) + k^2\,B^2\omega^2\,\mathcal{Q}_\Omega[U](\omega,t) = 0, \tag{3.28}$$

$$\mathcal{Q}_\Omega[U](\omega,0) = \mathcal{Q}_\Omega[f](\omega). \tag{3.29}$$

Subsequently, the complementary function of the Eq (3.28) is given by

$$\mathcal{Q}_\Omega[U](\omega,t) = e^{i(C\omega^2+E\omega)}\,\mathcal{Q}_\Omega[f](\omega)\,\mathcal{Q}_\Omega[g](\omega), \tag{3.30}$$

where $\mathcal{Q}_\Omega[g](\omega) = e^{-k^2B^2\omega^2t-i(C\omega^2+E\omega)}$.

Invoking the convolution theorem (2.8) for quadratic-phase Fourier transform, we obtain the solution of (3.30) as:

$$U(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(\xi)\,g(x-\xi)\,e^{2iA\xi(x-\xi)}\,d\xi, \tag{3.31}$$

where

$$g(x) = \frac{|B|}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-k^2B^2\omega^2t-i(C\omega^2+E\omega)}\,\overline{\mathcal{Q}_\Omega(\omega,x)}\,d\omega$$

$$= \frac{|B|}{\sqrt{2\pi}} \cdot e^{i(Ax^2+Dx)} \int_{\mathbb{R}} e^{-(k^2B^2t)\omega^2+(iBx)\omega}\, d\omega$$
$$= \frac{\operatorname{sgn}(B)}{k\sqrt{2t}} \cdot e^{i(Ax^2+Dx)-x^2/4k^2t}, \quad t > 0. \tag{3.32}$$

Implementing (3.32) in (3.31), we obtain

$$U(x,t) = \frac{\operatorname{sgn}(B)}{2k\sqrt{\pi t}} \int_{\mathbb{R}} f(\xi)\, \exp\left\{ i(x-\xi)(A(x-\xi)+2A\xi+D) - \frac{(x-\xi)^2}{4k^2t} \right\} d\xi. \tag{3.33}$$

Expression (3.33) is the desired solution of the heat equation in the quadratic-phase Fourier domain.
• For the case $\Omega = (-A/2B, 1/B, -C/2B, 0, 0)$, solution (3.33) boils down to the solution of the heat equation in linear canonical domain:

$$U(x,t) = \frac{\operatorname{sgn}\left(\frac{1}{B}\right)}{2k\sqrt{\pi t}} \int_{\mathbb{R}} f(\xi)\, \exp\left\{ \frac{-iA(x-\xi)(x-3\xi)}{2B} - \frac{(x-\xi)^2}{4k^2t} \right\} d\xi.$$

• For the case $\Omega = (-\cot\theta/2, \csc\theta, -\cot\theta/2, 0, 0)$, $\theta \neq n\pi$, one can obtain the solution in the context of fractional Fourier transform as

$$U(x,t) = \frac{\operatorname{sgn}(\csc\theta)}{2k\sqrt{\pi t}} \int_{\mathbb{R}} f(\xi)\, \exp\left\{ \frac{-i\cot\theta(x-\xi)(x-3\xi)}{2} - \frac{(x-\xi)^2}{4k^2t} \right\} d\xi.$$

• For the case $\Omega = (0, 1, -1, 0, 0)$, solution (3.33) reduces to the classical solution of the heat equation as

$$U(x,t) = \frac{1}{2k\sqrt{\pi t}} \int_{\mathbb{R}} f(\xi)\, e^{-(x-\xi)^2/4k^2t}\, d\xi.$$

Example 3.4. Consider the initial value problem:

$$U_t = \frac{1}{4}\overline{\mathbb{D}}_x^2\, U(x,t), \quad -\infty < x < \infty, \quad t > 0,$$
$$U(x,0) = \delta(x).$$

Then, according to (3.33), we can express the solution as:

$$U(x,t) = \frac{\operatorname{sgn}(B)}{\sqrt{\pi t}} \int_{\mathbb{R}} \delta(\xi)\, \exp\left\{ i(x-\xi)(A(x-\xi)+2A\xi+D) - \frac{(x-\xi)^2}{t} \right\} d\xi$$
$$= \frac{\operatorname{sgn}(B)}{\sqrt{\pi t}} \cdot \exp\left\{ i(Ax^2+Dx) - \frac{x^2}{t} \right\}. \tag{3.34}$$

In particular, for $A = B = D = 1$, solution (3.34) is expressible as under:

$$U(x,t) = \frac{1}{\sqrt{\pi t}} \cdot \exp\left\{ i(x^2+x) - \frac{x^2}{t} \right\}. \tag{3.35}$$

The graphical representation of the solution (3.35) of an Example 3.4 for the case $A = B = D = 1$ is depicted in Figure 5. To show the numerical values of the solution (3.35) for various values of x and fixed t are presented in Table 3.

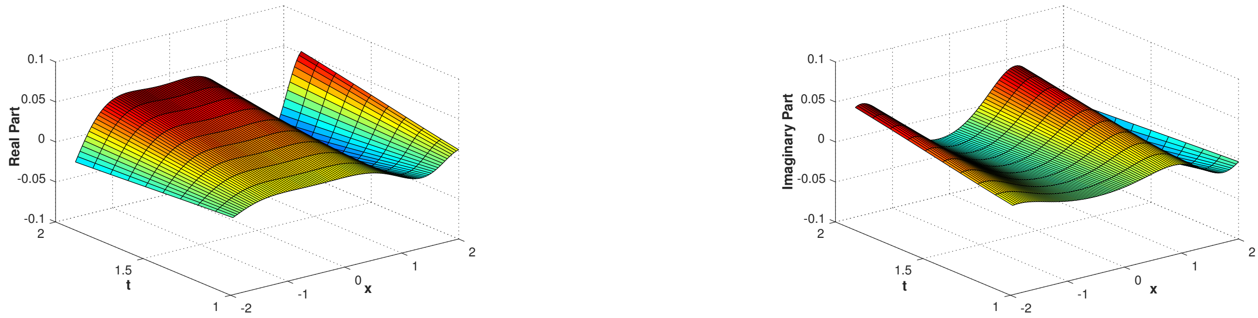


Figure 5. Real and imaginary parts of the solution of Example 3.4 for $A = B = D = 1$.

Table 3. The values of the solution of Example 3.4 for $A = B = D = 1$.

	$t = 0.5$	$t = 1.0$	$t = 1.5$	$t = 2.0$
x	$u(x,t)$	$u(x,t)$	$u(x,t)$	$u(x,t)$
-1.6	$0.0027 + 0.0039i$	$0.0250 + 0.0357i$	$0.0479 + 0.0685i$	$0.0636 + 0.0909i$
-1.2	$0.0046 + 0.0011i$	$0.0424 + 0.0104i$	$0.0812 + 0.0199i$	$0.1077 + 0.0264i$
-0.8	$0.0047 - 0.0008i$	$0.0431 - 0.0069i$	$0.0825 - 0.0133i$	$0.1095 - 0.0177i$
-0.4	$0.0046 - 0.0011i$	$0.0424 - 0.0104i$	$0.0812 - 0.0199i$	$0.1077 - 0.0264i$
0	$0.0048 + 0.0000i$	$0.0436 + 0.0000i$	$0.0836 + 0.0000i$	$0.1109 + 0.0000i$
0.4	$0.0040 + 0.0025i$	$0.0370 + 0.0232i$	$0.0708 + 0.0444i$	$0.0940 + 0.0589i$
0.8	$0.0006 + 0.0047i$	$0.0057 + 0.0432i$	$0.0109 + 0.0829i$	$0.0145 + 0.1100i$
1.2	$-0.0042 + 0.0023i$	$-0.0382 + 0.0210i$	$-0.0733 + 0.0402i$	$-0.0973 + 0.0533i$
1.6	$-0.0025 - 0.0041i$	$-0.0229 - 0.0371i$	$-0.0439 - 0.0712i$	$-0.0582 - 0.0944i$

Also for $A = D = 0, B = 1$, we have from (3.34)

$$U(x,t) = \frac{1}{\sqrt{\pi t}} \cdot e^{-x^2/t}, \tag{3.36}$$

which can be represented graphically in Figure 6 as:

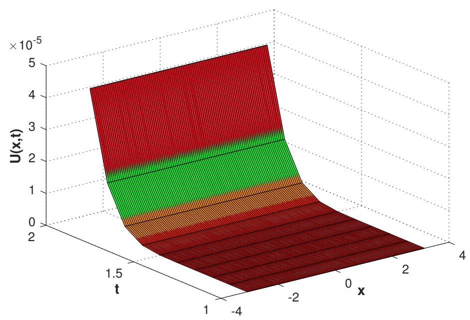


Figure 6. Representation of solution of the heat equation for $A = D = 0$ and $B = 1$.

4. Conclusions

In this article, we have investigated the analytic solutions of several prominent differential equations including the Laplace, wave, and the heat equations by employing the quadratic-phase Fourier transform. This strategy of obtaining the analytical solution of differential equations is novel to the literature and stands for its efficiency and simplicity. Moreover, this technique is advantageous over the existing ones in the sense that it is direct, compatible, and useful in investigating the problems in science and engineering demanding analytical solutions of non-linear systems. For lucid illustration of the novel technique, several examples and three-dimensional visualizations have been depicted in Figures 1–6. In addition, the solution of equations for various values is illustrated in Tables 1–3. These analytical solutions could have a wide range of applications in science and engineering. We observe that the obtained solutions are generalized versions of the solutions to the traditional Laplace, wave, and heat equations.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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