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Research article

Some approximation results for the new modification of Bernstein-Beta operators

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Abstract: This paper deals with the newly modification of Beta-type Bernstein operators, preserving constant and Korovkin's other test functions $e_i = t^i$, $i = 1, 2$ in limit case. Then the uniform convergence of the constructed operators is given. The rate of convergence is obtained in terms of modulus of continuity, Peetre-$\mathcal{K}$ functionals and Lipschitz class functions. After that, the Voronovskaya-type asymptotic result for these operators is established. At last, the graphical results of the newly defined operators are discussed.

Keywords: Korovkin type approximation theorem; modulus of continuity; functions of Lipschitz class; Peetre's $\mathcal{K}$-functionals
Mathematics Subject Classification: 41A25, 41A36, 47A58

1. Introduction

Approximation theory, an old area of mathematical research, has an extensive potential for applications to a wide variety of areas. Bernstein operators are one of the most widely-investigated linear and positive operators in the theory of approximation. Bernstein operators are defined by [2] as

$$B_m(k;x) = \sum_{p=0}^{m} \binom{m}{p} x^m (1-x)^{m-p} k\left(\frac{p}{m}\right), \quad m \geq 1 \tag{1.1}$$

for function $k \in C[0,1]$ and $x \in [0,1]$. After that, numereous variations of linear positive operators are studied by researchers such as [3–8, 11–17].

In 2020, Usta [18] introduced a new modification of Bernstein operators. For $k \in C(0,1)$, $m \in \mathbb{N}$ and $x \in (0,1)$

$$B_m^*(k;x) = \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2 x^{p-1}(1-x)^{m-p-1}k\left(\frac{p}{m}\right). \quad (1.2)$$

In this modification, the constructed operators preserve constant test function and Korovkin's other test functions t^i, $i = 1,2$ in limit case. Usta [18] gave $B_m^*(e_0;x) = 1$, $B_m^*(e_1;x) = \frac{m-2}{m}x + \frac{1}{m}$, $B_m^*(e_2;x) = \frac{m^2-7m+6}{m^2}x^2 + \frac{5m-6}{m^2}x + \frac{1}{m^2}$ and the approximation results of the $B_m^*(k;x)$ operators. Motivated by this work, we develop a Beta-type modification of Bernstein operators. The newly constructed Bernstein-Beta operators are presented for $k \in C[0,1]$ as follows:

$$\widetilde{B_m}(k;x) = \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2 x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)}\int_0^1 u^p(1-u)^{m-p}k(u)\,du, \quad (1.3)$$

where $x \in (0,1)$, $m \in \mathbb{N}$ and $\beta(p+1,m-p+1)$ denotes the Beta function. For $r,s > 0$ Beta function is defined by

$$\beta(r,s) = \int_0^1 u^{r-1}(1-u)^{s-1}du. \quad (1.4)$$

We can easily see that for $a,b \in \mathbb{R}$ and $k,f \in C[0,1]$

$$\begin{aligned}
\widetilde{B_m}(ak+bf;x) &= \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2 x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)} \\
&\quad \times\int_0^1 u^p(1-u)^{m-p}\left(ak(u)+bf(u)\right)du \\
&= \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2 x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)} \\
&\quad \times\int_0^1 u^p(1-u)^{m-p}\left(ak(u)\right)du \\
&\quad +\frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2 x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)} \\
&\quad \times\int_0^1 u^p(1-u)^{m-p}\left(bf(u)\right)du \\
&= a\widetilde{B_m}(k;x) + b\widetilde{B_m}(f;x).
\end{aligned}$$

Also, for $k \geq 0$, $\widetilde{B_m}(k;x) \geq 0$. Thus, $\widetilde{B_m}(k;x)$ operators are linear and positive.

2. Approximation properties of $\widetilde{B_m}$

Lemma 2.1. *For each $x \in (0,1)$, we obtain*

$$\begin{aligned}
\widetilde{B_m}(e_0;x) &= 1, \\
\widetilde{B_m}(e_1;x) &= \frac{m-2}{m+2}x + \frac{2}{m+2},
\end{aligned}$$

$$\widetilde{B_m}(e_2;x) = \frac{(m-6)(m-1)}{(m+3)(m+2)}x^2 + \frac{8m-12}{(m+3)(m+2)}x + \frac{6}{(m+3)(m+2)},$$

$$\widetilde{B_m}(e_3;x) = \frac{(m-1)(m-2)(m-12)}{(m+4)(m+3)(m+2)}x^3 + \frac{18(m-4)(m-1)}{(m+4)(m+3)(m+2)}x^2 + \frac{18(3m-4)}{(m+4)(m+3)(m+2)}x + \frac{24}{(m+4)(m+3)(m+2)},$$

$$\begin{aligned}\widetilde{B_m}(e_4;x) = {} & \frac{(m-1)(m-2)(m-3)(m-20)}{(m+5)(m+4)(m+3)(m+2)}\frac{x^5}{x-1} - \frac{(m-1)(m-2)(m^2-55m+300)}{(m+5)(m+4)(m+3)(m+2)}\frac{x^4}{x-1} \\ & + \frac{8(m-1)(4m^2-65m+150)}{(m+5)(m+4)(m+3)(m+2)}\frac{x^3}{x-1} - \frac{24(m-5)(9m-10)}{(m+5)(m+4)(m+3)(m+2)}\frac{x^2}{x-1} \\ & + \frac{579-363m}{(m+5)(m+4)(m+3)(m+2)}\frac{x}{x-1} - \frac{120}{(m+5)(m+4)(m+3)(m+2)}\frac{1}{x-1}\end{aligned}$$

where $e_i = t^i$ for $i = 0,1,2,3,4$.

Proof. By using the definition of Beta function (1.4), it is clear that

$$\widetilde{B_m}(e_0;x) = B_m^*(e_0;x) = 1.$$

For $i = 1$, $k(t) = t$,

$$\begin{aligned}\widetilde{B_m}(e_1;x) &= \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)}\int_0^1 u^{p+1}(1-u)^{m-p}du \\ &= \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{\beta(p+2,m-p+1)}{\beta(p+1,m-p+1)} \\ &= \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{p+1}{m+2} \\ &= \frac{m}{m+2}B_m^*(e_1;x) + \frac{1}{m+2}B_m^*(e_0;x) \\ &= \frac{m}{m+2}\left(\frac{m-2}{m}x + \frac{1}{m}\right) + \frac{1}{m+2} \\ &= \frac{m-2}{m+2}x + \frac{2}{m+2}.\end{aligned}$$

For $i = 2$, $k(t) = t^2$,

$$\begin{aligned}\widetilde{B_m}(e_2;x) &= \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)}\int_0^1 u^{p+2}(1-u)^{m-p}du \\ &= \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{\beta(p+3,m-p+1)}{\beta(p+1,m-p+1)} \\ &= \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{p^2+3p+2}{(m+3)(m+2)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{m^2}{(m+3)(m+2)} B_m^*(e_2;x) + \frac{3m}{(m+3)(m+2)} B_m^*(e_1;x) + \frac{2}{(m+3)(m+2)} B_m^*(e_0;x) \\
&= \frac{m^2}{(m+3)(m+2)} \left(\frac{m^2-7m+6}{m^2} x^2 + \frac{5m-6}{m^2} x + \frac{1}{m^2} \right) + \frac{3m}{(m+3)(m+2)} \left(\frac{m-2}{m} x + \frac{1}{m} \right) \\
&\quad + \frac{2}{(m+3)(m+2)} \\
&= \frac{(m-6)(m-1)}{(m+3)(m+2)} x^2 + \frac{8m-12}{(m+3)(m+2)} x + \frac{6}{(m+3)(m+2)}.
\end{aligned}
$$

The proofs of $\widetilde{B_m}(e_3;x)$ and $\widetilde{B_m}(e_4;x)$ can be obtained in the same manner.

Lemma 2.2. *For each $x \in (0,1)$, we write*

$$
\begin{aligned}
\widetilde{B_m}(t-x;x) &= \frac{-4}{m+2} x + \frac{2}{m+2}, \\
\widetilde{B_m}((t-x)^2;x) &= \frac{-4(m-6)}{(m+3)(m+2)} x^2 + \frac{4m-24}{(m+3)(m+2)} x + \frac{6}{(m+3)(m+2)}, \\
\widetilde{B_m}((t-x)^4;x) &= \frac{3}{(m+5)(m+4)(m+3)(m+2)} \frac{1}{x-1} \Big\{ 4(160-93m+3m^2)x^5 \\
&\quad -12(160-93m+3m^2)x^4 + 4(560-303m+9m^2)x^3 \\
&\quad -4(320-141m+3m^2)x^2 + (353-89m)x - 40 \Big\}.
\end{aligned} \tag{2.1}
$$

Proof. By using following equalities

$$
\begin{aligned}
\widetilde{B_m}(t-x;x) &= \widetilde{B_m}(e_1;x) - x\widetilde{B_m}(1;x), \\
\widetilde{B_m}((t-x)^2;x) &= \widetilde{B_m}(e_2;x) - 2x\widetilde{B_m}(e_1;x) + x^2\widetilde{B_m}(1;x), \\
\widetilde{B_m}((t-x)^4;x) &= \widetilde{B_m}(e_4;x) - 4x\widetilde{B_m}(e_3;x) + 6x^2\widetilde{B_m}(e_2;x) - 4x^3\widetilde{B_m}(e_1;x) + x^4\widetilde{B_m}(1;x),
\end{aligned}
$$

we finish the proof of the lemma.

Remark 1. We have the following results

$$
\lim_{m\to\infty} m\widetilde{B_m}(t-x;x) = -4x+2, \tag{2.2}
$$

$$
\lim_{m\to\infty} m\widetilde{B_m}((t-x)^2;x) = -4x(1-x), \tag{2.3}
$$

$$
\lim_{m\to\infty} m^2\widetilde{B_m}((t-x)^4;x) = 36(x-1)^2x^2. \tag{2.4}
$$

3. Main results

Let the Banach space of all continuous functions k on $[0,1]$ is denoted by $C[0,1]$ endowed with the norm

$$
\|k\| = \max_{x\in(0,1)} |k(x)|.
$$

Theorem 3.1. *For every $k \in C[0,1]$*

$$
\left\| \widetilde{B_m}(k;x) - k(x) \right\| \to 0, \tag{3.1}
$$

uniformly as $m \to \infty$.

Proof. It can be seen easily from Lemma 2.1 that

$$\lim_{m\to\infty}\widetilde{B_m}(e_i;x)=x^i,\qquad i=0,1,2.$$

Then we apply Korovkin's theorem [19], which concludes the proof. □

4. Rate of convergence

For $k\in C[0,1]$, the modulus of continuity is given by

$$\omega(k,\delta):=\sup_{|t-x|\le\delta}\ \sup_{x\in(0,1)}|k(t)-k(x)|,\quad \delta>0.$$

Additionally, modulus of continuity of the function k has the following property [1]:

$$|k(t)-k(x)|\le\left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(k,\delta),\ \ \delta>0. \tag{4.1}$$

Theorem 4.1. *For each $x\in(0,1)$ and $k\in C[0,1]$, we have*

$$|\widetilde{B_m}(k;x)-k(x)|\le 2\omega(k,\delta_m), \tag{4.2}$$

where

$$\delta_m(x)=\sqrt{\frac{-4(m-6)x^2+4(m-6)x+6}{(m+3)(m+2)}}. \tag{4.3}$$

Proof. By using the linearity of the $\widetilde{B_m}$ operators and Eq (4.1), we obtain

$$\begin{aligned}\left|\widetilde{B_m}(k;x)-k(x)\right| &= \Bigg|\frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)}\\ &\quad\times\int_0^1u^p(1-u)^{m-p}k(u)\,du-k(x)\Bigg|\\ &\le \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)}\\ &\quad\times\int_0^1u^p(1-u)^{m-p}\,|k(u)-k(x)|\,du\\ &\le \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)}\\ &\quad\times\int_0^1u^p(1-u)^{m-p}\left(1+\frac{(u-x)^2}{\delta^2}\right)\omega(k,\delta)du\\ &= \left(1+\frac{1}{\delta^2}\frac{-4(m-6)x^2+4(m-6)x+6}{(m+3)(m+2)}\right)\omega(k,\delta).\end{aligned}$$

If we choose

$$\delta=\delta_m=\sqrt{\frac{-4(m-6)x^2+4(m-6)x+6}{(m+3)(m+2)}}$$

then we arrive at

$$|\widetilde{B_m}(k;x)-k(x)| \leq 2\omega\left(k, \sqrt{\frac{-4(m-6)x^2+4(m-6)x+6}{(m+3)(m+2)}}\right),$$

which is the required result. □

Right now, we show the rate of convergence of $\widetilde{B_m}(k;x)$ by using the function k, which belongs to Lipschitz class. A function k is said to be in the Lipschitz class $k \in Lip_K(c)$ if the inequality

$$|k(t)-k(x)| \leq K|t-x|^c \quad ; \ \forall t, x \in (0,1) \tag{4.4}$$

holds. Hölder inequality [10] is defined as for $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$

$$\sum_{r=0}^{m}|\xi_r\eta_r| \leq \left(\sum_{r=0}^{m}(\xi_r)^p\right)^{\frac{1}{p}}\left(\sum_{r=0}^{m}(\eta_r)^q\right)^{\frac{1}{q}}. \tag{4.5}$$

Theorem 4.2. *Let $k \in Lip_K(c)$ and $0 < c \leq 1$ then we write*

$$\left|\widetilde{B_m}(k;x)-k(x)\right| \leq K\delta_m^c(x),$$

where $\delta_m(x)$ is the same as (4.3).

Proof. Let k belongs to Lipschitz class $Lip_K(c)$ and $0 < c \leq 1$. From (4.4) and by using the linearity and monotonicity of the operators $\widetilde{B_m}$, we get

$$\begin{aligned}\left|\widetilde{B_m}(k;x)-k(x)\right| &\leq \widetilde{B_m}(|k(t)-k(x)|;x)\\ &\leq K\widetilde{B_m}(|t-x|^c;x).\end{aligned}$$

By choosing $p=\frac{2}{c}$, $q=\frac{2}{2-c}$ in the Hölder inequality, we get

$$\begin{aligned}\left|\widetilde{B_m}(k;x)-k(x)\right| &\leq K\left\{\widetilde{B_m}\left((t-x)^2;x\right)\right\}^{\frac{c}{2}}\\ &\leq K\delta_m^c(x).\end{aligned}$$

Here, $\delta_m(x)$ is as given in (4.3). Thus, we write

$$\left|\widetilde{B_m}(k;x)-k(x)\right| \leq K\left(\frac{-4(m-6)x^2+4(m-6)x+6}{(m+3)(m+2)}\right)^{\frac{c}{2}}.$$

□

Now, the rate of convergence of the newly constructed operators $\widetilde{B_m}(k;x)$ is investigated by using the Peetre-$\mathcal{K}$ functionals.

Lemma 4.3. *For $x \in (0,1)$ and $k \in C[0,1]$, we obtain*

$$|\widetilde{B_m}(k;x)| \leq ||k||. \tag{4.6}$$

Proof. From the definition of $\widetilde{B_m}(k;x)$ operators, we have

$$
\begin{aligned}
|\widetilde{B_m}(k;x)| &= \left|\frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)}\int_0^1 u^p(1-u)^{m-p}k(u)\,du\right| \\
&\leq \frac{1}{m}\sum_{p=0}^{m}\binom{m}{p}(p-mx)^2x^{p-1}(1-x)^{m-p-1}\frac{1}{\beta(p+1,m-p+1)}\int_0^1 u^p(1-u)^{m-p}|k(u)|\,du \\
&\leq \|k\|\widetilde{B_m}(1;x) \\
&= \|k\|.
\end{aligned}
$$

□

Let $C^2[0,1]$ be the space of the functions k, for which k, k' and k'' are continuous on $[0,1]$. We write the norm of function k in this space as follows:

$$\|k\|_{C^2[0,1]} = \|k\|_{C[0,1]} + \|k'\|_{C[0,1]} + \|k''\|_{C[0,1]}.$$

The classical Peetre-$\mathcal{K}$ functional is defined as

$$\mathcal{K}(k,\delta) := \inf_{u\in C^2[0,1]}\{\|k-u\|_{C[0,1]} + \delta\,\|u''\|_{C[0,1]}\}$$

and second modulus of smoothness of the function is given by

$$\omega_2(k,\delta) := \sup_{0<h<\delta}\ \sup_{x,x+h\in(0,1)} |k(x+2h) - 2k(x+h) + k(x)|$$

where $\delta > 0$. By DeVore and Lorentz [9], it is known that for $M > 0$

$$\mathcal{K}(k,\delta) \leq M\omega_2(k, \sqrt{\delta}). \tag{4.7}$$

Theorem 4.4. *Let $x \in (0,1)$ and $k \in C[0,1]$. Then for each $m \in \mathbb{N}$, there exists a positive constant M such that*

$$|\widetilde{B_m}(k;x) - k(x)| \leq M\omega_2(k, \alpha_m(x)) + 2\omega(k, \beta_m(x)).$$

Here

$$\alpha_m(x) = \sqrt{\frac{18(1-2x)^2 - 4s^2(-1+x)x + 8s(1-3x+3x^2)}{(2+s)^2(3+s)}}$$

and

$$\beta_m(x) = \left|\frac{-4x+2}{m+2}\right|.$$

Proof. We introduce the proof by defining an auxiliary operator $B_m^{**} : C[0,1] \to C[0,1]$ by

$$B_m^{**}(s;x) = \widetilde{B_m}(s;x) - s\left(\frac{(m-2)x+2}{m+2}\right) + s(x). \tag{4.8}$$

From Lemma 2.1, we have

$$B_m^{**}(1;x) = 1,$$

$$\begin{aligned}
B_m^{**}(t-x;x) &= \widetilde{B_m}((t-x);x) - \left(\frac{(m-2)x+2}{m+2} - x\right) + x - x \\
&= \frac{-4}{m+2}x + \frac{2}{m+2} - \left(\frac{(m-2)x+2}{m+2} - x\right) + x - x \\
&= 0. \qquad (4.9)
\end{aligned}$$

For $s \in C^2[0,1]$, we write by using the Taylor expansion that

$$s(t) = s(x) + (t-x)s'(x) + \int_x^t (t-u)s''(u)du, \quad t \in (0,1). \qquad (4.10)$$

Applying B_m^{**} operator to both sides of the equation (4.10), we obtain

$$\begin{aligned}
B_m^{**}(s;x) &= B_m^{**}\left(s(x) + (t-x)s'(x) + \int_x^t (t-u)s''(u)du\right) \\
&= s(x) + B_m^{**}\left((t-x)s'(x);x\right) + B_m^{**}\left(\int_x^t (t-u)s''(u)du\right).
\end{aligned}$$

So,

$$B_m^{**}(s;x) - s(x) = s'(x)B_m^{**}\left((t-x);x\right) + B_m^{**}\left(\int_x^t (t-u)s''(u)du\right).$$

By using (4.9) and (4.8), we achieve

$$\begin{aligned}
B_m^{**}(s;x) - s(x) &= B_m^{**}\left(\int_x^t (t-u)s''(u)du\right) \\
&= \widetilde{B_m}\left(\int_x^t (t-u)s''(u)du\right) - \int_x^{\frac{(m-2)x+2}{m+2}} \left(\frac{(m-2)x+2}{m+2} - u\right)s''(u)du \\
&\quad + \int_x^x \left(\frac{(m-2)x+2}{m+2} - u\right)s''(u)du. \qquad (4.11)
\end{aligned}$$

Furthermore

$$\begin{aligned}
\left|\int_x^t (t-u)s''(u)du\right| &\leq \int_x^t |t-u||s''(u)|du \leq \|s''\| \int_x^t |t-u|du \\
&\leq (t-x)^2\|s''\|, \qquad (4.12)
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_x^{\frac{(m-2)x+2}{m+2}} \left(\frac{(m-2)x+2}{m+2} - u\right)s''(u)du\right| &\leq \|s''\| \int_x^{\frac{(m-2)x+2}{m+2}} \left(\frac{(m-2)x+2}{m+2} - u\right)du \\
&= \frac{\|s''\|}{2}\left(\frac{(m-2)x+2}{m+2} - x\right)^2 \\
&= \frac{\|s''\|}{2}\left(\frac{(m-2)x+2}{m+2-x}\right)^2. \qquad (4.13)
\end{aligned}$$

When we rewrite (4.12) and (4.13) in the absolute value of (4.11). Then we get

$$
\begin{aligned}
\left|B^{**}_{s-m}(s;x)-s(x)\right| &\le \|s''\|\widetilde{B_m}((t-x)^2;x)+\frac{\|s''\|}{2}\left(\frac{(m-2)x+2}{m+2}-x\right)^2\\
&= \|s''\|\left(\widetilde{B_m}((t-x)^2;x)+\frac{1}{2}\left(\frac{(m-2)x+2}{m+2}-x\right)^2\right)\\
&= \|s''\|\alpha_m^2(x),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_m(x) &= \sqrt{\widetilde{B_m}((t-x)^2;x)+\frac{1}{2}\left(\frac{(m-2)x+2}{m+2}-x\right)^2}\\
&= \sqrt{\frac{18(1-2x)^2-4s^2(-1+x)x+8s(1-3x+3x^2)}{(2+s)^2(3+s)}}.
\end{aligned}
$$

Right now, we will try to find a bound for the auxiliary operator $B^{**}_m(s;x)$. In the light of the Lemma 4.3 and using Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
|B^{**}_m(s;x)| &= \left|\widetilde{B_m}(s;x)-s\left(\frac{(m-2)x+2}{m+2}\right)+s(x)\right|\\
&\le |\widetilde{B_m}(s;x)|+\left|s\left(\frac{(m-2)x+2}{m+2}\right)\right|+|s(x)|\\
&\le 3\|s\|.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
|\widetilde{B_m}(k;x)-k(x)| &= \left|B^{**}_m(k;x)-k(x)+k\left(\frac{(m-2)x+2}{m+2}\right)-k(x)+s(x)-s(x)+B^{**}_m(s;x)-B^{**}_m(s;x)\right|\\
&\le |B^{**}_m(k-s;x)-(k-s)(x)|+|B^{**}_m(s;x)-s(x)|+\left|k\left(\frac{(m-2)x+2}{m+2}\right)-k(x)\right|\\
&\le 4\|k-s\|+\|s''\|\alpha_m^2(x)+\omega(k,\beta_m(x))\left(\frac{\left|\frac{(m-2)x+2}{m+2}-x\right|}{\beta_m(x)}+1\right)\\
&= 4\left(\|k-s\|+\|s''\|\alpha_m^2(x)\right)+2\omega\left(k,\left|\frac{(m-2)x+2}{m+2}-x\right|\right),
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
\beta_m(x) &= \left|\frac{(m-2)x+2}{m+2}-x\right|\\
&= \left|\frac{-4x+2}{m+2}\right|.
\end{aligned}
$$

So, for all $k\in C^2[0,1]$ by taking the infimum of the Eq (4.14), we get

$$
|\widetilde{B_m}(k;x)-k(x)| \le 4\mathcal{K}(s,\alpha_m^2(x))+2\omega(k,\beta_m(x)). \tag{4.15}
$$

As a result, using Eq (4.7), we obtain

$$|\widetilde{B_m}(k;x)-k(x)|\leq M\omega_2(k,\alpha_m(x))+2\omega(k,\beta_m(x)). \tag{4.16}$$

Thusly, the proof is finished. □

5. Voronovskaya type theorem

In 1932, Voronovskaya [20] obtained the convergence rate of the Bernstein operators (1.1) to the function k. In this part, we derive a Voronovskaya-type asymptotic formula for $\widetilde{B_m}(k;x)$ operators.

Theorem 5.1. *Let k be integrable on the interval $(0,1)$, also k' and k'' exist at a fixed point $x\in(0,1)$. Then we have*

$$\lim_{m\to\infty} m\left(\widetilde{B_m}(k;x)-k(x)\right)=(-4x+2)\,k'(x)-2x(1-x)k''(x). \tag{5.1}$$

Proof. By using the well-known Taylor's formula, we write

$$k(t)=k(x)+(t-x)k'(x)+\frac{(t-x)^2}{2}k''(x)+\mathcal{R}(t,x)(t-x)^2. \tag{5.2}$$

Here, $\mathcal{R}(t,x):=\frac{k''(\xi)-k''(x)}{2}$ is the remainder term. ξ is situated between x and t. Also, $\lim\limits_{t\to x}\mathcal{R}(t,x)=0$. When we apply $\widetilde{B_m}$ operators to (5.2), we obtain

$$\widetilde{B_m}(k;x)-k(x)=k'(x)\widetilde{B_m}((t-x);x)+\frac{k''(x)}{2}\widetilde{B_m}((t-x)^2;x)+\widetilde{B_m}(\mathcal{R}(t,x)(t-x)^2;x). \tag{5.3}$$

By multiplying (5.3) by m and take the limit as m goes to infinity, we achieve

$$\begin{aligned}\lim_{m\to\infty} m\left(\widetilde{B_m}(k;x)-k(x)\right) &= \lim_{m\to\infty} mk'(x)\widetilde{B_m}((t-x);x)+\lim_{m\to\infty} m\frac{k''(x)}{2}\widetilde{B_m}((t-x)^2;x)\\ &\quad+\lim_{m\to\infty} m\widetilde{B_m}(\mathcal{R}(t,x)(t-x)^2;x).\end{aligned}$$

By taking into consideration Eqs (2.2) and (2.3), we obtain

$$\begin{aligned}\lim_{m\to\infty} mk'(x)\widetilde{B_m}((t-x);x) &= k'(x)\lim_{m\to\infty} m\widetilde{B_m}((t-x);x)\\ &= k'(x)\,(-4x+2)\end{aligned}$$

and

$$\begin{aligned}\lim_{m\to\infty} m\frac{k''(x)}{2}\widetilde{B_m}((t-x)^2;x) &= \frac{k''(x)}{2}\lim_{m\to\infty} m\widetilde{B_m}((t-x)^2;x)\\ &= \frac{k''(x)}{2}\,(-4x(1-x))\,.\end{aligned}$$

Thus we have

$$\lim_{m\to\infty} m\left(\widetilde{B_m}(k;x)-k(x)\right)=(-4x+2)\,k'(x)-2x(1-x)k''(x)+\lim_{m\to\infty} m\widetilde{B_m}(\mathcal{R}(t,x)(t-x)^2;x). \tag{5.4}$$

By using the Cauchy-Schwarz inequality for the remainder term, we write

$$m\widetilde{B_m}(\mathcal{R}(t,x)(t-x)^2;x) \leq \sqrt{m^2\widetilde{B_m}(\mathcal{R}^2(t,x);x)}\sqrt{\widetilde{B_m}((t-x)^4;x)}. \tag{5.5}$$

We already know the term $\widetilde{B_m}((t-x)^4;x)$ from Eq (2.1). Since $\mathcal{R}^2(.,x)$ is continuous at $t \in (0,1)$ and $\lim_{t\to x}\mathcal{R}(t,x) = 0$, we observe that

$$\lim_{m\to\infty} \widetilde{B_m}(\mathcal{R}^2(t,x);x) = \mathcal{R}^2(x,x) = 0. \tag{5.6}$$

Hence, by using (2.1), (5.5), (5.6) and positivity of the linear operators $\widetilde{B_m}$, we have

$$\lim_{m\to\infty} m\widetilde{B_m}(\mathcal{R}(t,x)(t-x)^2;x) = 0. \tag{5.7}$$

Finally, by substituting (5.7) in (5.4), we achieve

$$\lim_{m\to\infty} m\left(\widetilde{B_m}(k;x) - k(x)\right) = (-4x+2)\,k'(x) - 2x(1-x)k''(x),$$

which is the desired result.

6. Numerical experiments

In last section, we give the convergence behaviour of the newly constructed operators $\widetilde{B_m}$ with function k.

Example 1. *Let the function k be*

$$k(x) = 1 - \cos(4e^x).$$

The convergence behaviour of the operators $\widetilde{B_m}(k;x)$ is illustrated in Figure 1, where $k(x) = 1 - \cos(4e^x)$, $x \in (0,1)$ and $m \in \{100, 300, 500, 1000\}$.

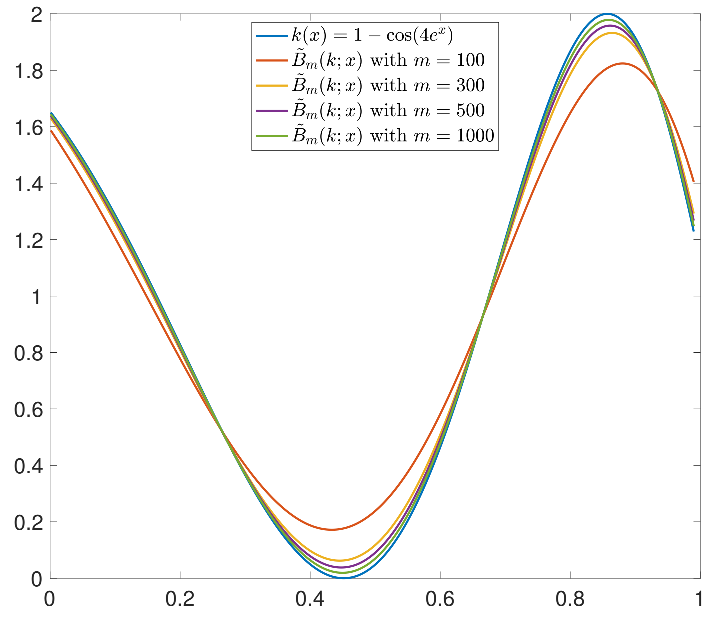

Figure 1. Convergence of $\widetilde{B_m}(k;x)$ for different values of m.

The error estimation for operators $\widetilde{B_m}(k;x)$ *to the function* $k(x) = 1-\cos(4e^x)$ *is presented in Table 1 for different values of m.*

Table 1. Error comparison table of $\widetilde{B_m}(k;x)$.

m	$\max\lvert\widetilde{B_m}(k;x) - k(x)\rvert$
100	0.22148
300	0.082166
500	0.050427
1000	0.025653

Example 2. *Let the function k be chosen as*

$$k(x) = \left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{4}\right).$$

We have shown the convergence behaviour of the $\widetilde{B_m}(k;x)$ *Bernstein-Beta operators to the function k in Figure 2 for* $m \in \{20, 50, 100, 200\}$.

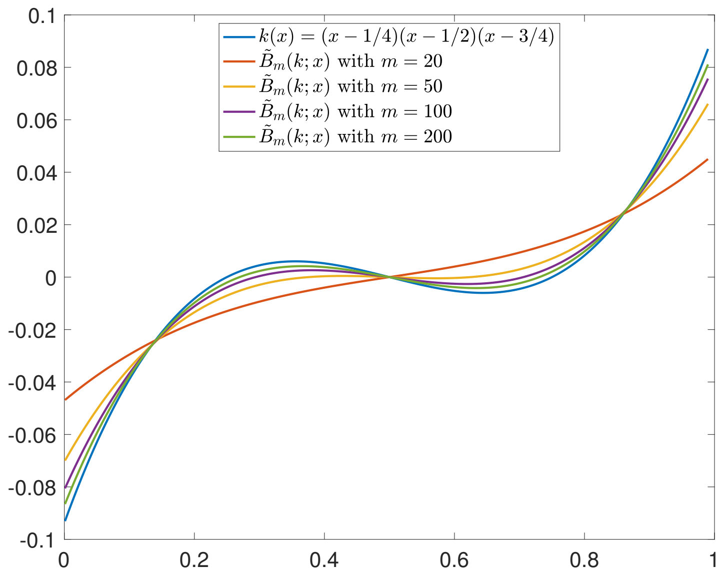

Figure 2. Convergence behaviour of $\widetilde{B_m}(k;x)$.

The error results of the operators $\widetilde{B_m}(k;x)$ *to the function* $k(x) = \left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{3}{4}\right)$ *are given in Table 2 for different values of m.*

Table 2. Error comparison table of $\widetilde{B_m}(k;x)$.

m	$\max\lvert\widetilde{B_m}(k;x) - k(x)\rvert$
20	0.04621
50	0.02309
100	0.01251
200	0.00652

When we investigate these two examples, we understand that for the increasing values of m, the graph of the operators $\widetilde{B_m}(k;x)$ goes to the graph of the function k.

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Conflict of interest

The authors declared there is no conflict of interest associated with this work.

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