Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(12): 20838–20849.
DOI:10.3934/math.20221142
Received: 13 July 2022
Revised: 14 October 2022
Accepted: 24 October 2022
Published: 02 November 2022

Research article

Applications of relative statistical convergence and associated approximation theorem

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Abstract: In this work, we investigate a new type of convergence known as relative statistical convergence through the use of the deferred Nörlund and deferred Riesz means. We demonstrate that the idea of deferred Nörlund and deferred Riesz statistically relative uniform convergence is significantly stronger than deferred Nörlund and deferred Riesz statistically uniform convergence. We provide some interesting examples which explain the validity of the theoretical results and effectiveness of constructed sequence spaces. Furthermore, as an application point of view we prove the Korovkin-type approximation theorem in the context of relative equi-statistical convergence for real valued functions and demonstrate that our theorem effectively extends and most of the earlier existing results. Finally, we present an example involving the Meyer-König-Zeller operator of real sequences proving that our theorem is a stronger approach than its classical and statistical version.

Keywords: statistical convergence; deferred Nörlund; deferred Riesz; relative uniform convergence; pointwise convergence
Mathematics Subject Classification: 40A05, 40A30

1. Introduction and preliminaries

The convergence of sequences plays a significant role in sequence spaces. The abrupt extension of sequence spaces has been accompanied by the recent works of many researchers in the field of statistical convergence. The statistical convergence has remarkable significance over conventional convergence. The perception of statistical convergence for a real sequence was established by Fast [13] and Schoenberg [26] independently. The notion of statistical convergence is an effective tool to resolve many problems in Ergodic theory, Fuzzy set theory, Trigonometric series, Fourier analysis and number theory. Currently, the approximation of functions by positive linear operators based on statistical convergence and statistical summability has emerged as an active field of study. Balcerzak et al. [2] established a number of methods for the convergence of sequences of functions such as pointwise, equi-statistical and uniform convergence methods. The notion of pointwise and uniform convergence by using double sequences was examined by Gökhan et al. [14]. The conviction of the uniform convergence of sequences of functions in relation to a scale function was presented by Moore [19]. Later on, Chittenden [4, 5] studied this concept. The classical uniform convergence is a special case of relative uniform convergence. In 2016, Demirci and Orhan [8] examined statistically relative uniform convergence through the use of positive linear operators. Recently, Jena et al. [15] introduced relatively equi-statistical convergence via the deferred Nörlund by using a difference operator of fractional order.

Essentially motivated by the aforementioned investigations and outcomes defined above, we explore the concept of relative pointwise statistical convergence, relative uniform statistical convergence and relative equi-statistical convergence by virtue of the deferred Nörlund and deferred Riesz mean. As an application to our newly formed sequence space, we introduce the Korovkin-type approximation theorem by means of deferred Nörlund and deferred Riesz relative equi-statistical convergence and present examples to illustrate the findings. Finally, we give an illustration using the Meyer-König-Zeller operator on real sequences to demonstrate the superiority of our theory over the classical and statistical versions.

A sequence $\{z_{\mathfrak{n}}\}$ is statistically convergent to z if $\forall\ \epsilon > 0$:

$$\lim_{n\to\infty} \frac{1}{\mathfrak{n}} |\{k \leq \mathfrak{n} : |z_k - z| \geq \epsilon\}| = 0,$$

i.e., $\delta(\{k : |z_k - z| \geq \epsilon\}) = 0$. It is denoted by $st - \lim_{\mathfrak{n}\to\infty} z_{\mathfrak{n}} = z$. For more recent works in this direction see [6, 10–12, 16, 23–25, 27].

Definition 1.1. A sequence $\{g_{\mathfrak{n}}\}$ of functions defined over the interval $I = (a \leq z \leq b)$ converges relatively uniformly to a limit function g if $\exists$ a non zero pointwise scale function $\boldsymbol{\sigma}(z)$ defined over I, s.t. $\forall\ \epsilon > 0$, $\exists$ an integer $\mathfrak{n}_\epsilon$, the inequality

$$\frac{|g_{\mathfrak{n}}(z) - g(z)|}{|\boldsymbol{\sigma}(z)|} < \epsilon$$

holds uniformly in z for $\mathfrak{n} > \mathfrak{n}_\epsilon$.

To know about the significance of relatively uniform convergence over usual uniform convergence, we give an example below:

Example 1. Define $g_{\mathfrak{n}} : [0, 1] \to \mathbb{R}$ by

$$g_{\mathfrak{n}}(z) = \begin{cases} \frac{1}{\mathfrak{n} z^{\mathfrak{n}}}, & z \in (0, 1], \\ 0, & z = 0, \end{cases}$$

where $\mathfrak{n} \in \mathbb{N}$. Clearly, $\{g_{\mathfrak{n}}\}$ is not classically uniform convergent on $[0, 1]$; however, it is converging to $g = 0$ uniformly with respect to the scale function

$$\boldsymbol{\sigma}(z) = \begin{cases} \frac{1}{z^{\mathfrak{n}}}, & z \in (0, 1], \\ 1, & z = 0. \end{cases}$$

Definition 1.2. Assume that $\{r_\mathfrak{n}\}$ and $\{s_\mathfrak{n}\}$ are sequences of non-negative integers satisfying the following criteria:
(i) $r_\mathfrak{n} < s_\mathfrak{n}$, $(\forall \mathfrak{n} \in \mathbb{N})$ and
(ii) $\lim_{\mathfrak{n}\to\infty} s_\mathfrak{n} = \infty$.

Now, suppose u_m and v_m are two sequences of non-negative real numbers s.t.

$$U_\mathfrak{n} = \sum_{j=r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} u_j$$

and

$$V_\mathfrak{n} = \sum_{j=r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} v_j.$$

The convolution of the above sequences is defined as:

$$\mathcal{P}_{r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} = (U * V)_\mathfrak{n} = \sum_{m=r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} u_m v_{s_\mathfrak{n}-m}.$$

Then, the deferred Nörlund mean $\omega_\mathfrak{n}$ is defined as

$$\omega_\mathfrak{n} = \frac{1}{\mathcal{P}_{r_{\mathfrak{n}+1}}^{s_\mathfrak{n}}} \sum_{j=r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} u_{s_\mathfrak{n}-j} v_j z_j.$$

More information on the deffered Nörlund mean can be found in [9].

Definition 1.3. Suppose $\{q_\mathfrak{n}\}$ is the sequence of non-negative real numbers and

$$R_{r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} = \sum_{j=r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} q_j,$$

where the sequences $\{r_\mathfrak{n}\}$ and $\{s_\mathfrak{n}\}$ satisfy conditions (i) and (ii) of Definition 1.2. Then, the deferred Riesz mean $t_\mathfrak{n}$ is defined as

$$t_\mathfrak{n} = \frac{1}{\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_\mathfrak{n}}} \sum_{j=r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} q_j z_j.$$

To know more about the deferred Riesz mean see [22]. Now, we define the product of the deferred Riesz mean and deferred Nörlund mean as follows:

$$\psi_\mathfrak{n} = (\omega t)_\mathfrak{n} = \frac{1}{\mathcal{P}_{r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} \mathcal{R}_{r_{\mathfrak{n}+1}}^{s_\mathfrak{n}}} \sum_{j=r_{\mathfrak{n}+1}}^{s_\mathfrak{n}} u_{s_\mathfrak{n}-j} v_j q_j z_j.$$

Also, the sequence $\{\psi_\mathfrak{n}\}$ is said to be summable to z if

$$\lim_{\mathfrak{n}\to\infty} \psi_\mathfrak{n} = z.$$

Remark. The deferred Nörlund means and deferred Riesz means are transformations of the type $z \mapsto Az$, for some infinite matrix A (see [3] and [20]).

A sequence $\{z_{\mathfrak{n}}\}$ is said be deferred Nörlund and deferred Riesz statistically convergent to z if $\forall$ $\epsilon > 0$:

$$\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}}} \left|\left\{\mathfrak{n} : \mathfrak{n} \leq \mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \text{ and } u_{s_{\mathfrak{n}}-j} v_j q_j |z_{\mathfrak{n}} - z| \geq \epsilon\right\}\right| = 0.$$

It is written as $st - \lim z_{\mathfrak{n}} = z$ or $z_{\mathfrak{n}} \ \rightarrow \ z$ ($\psi_{\mathfrak{n}}-$statistically) as $\mathfrak{n} \to \infty$.

2. Main results

In the current section, let g_n and $g \in C_b(X)$, $\forall$ $n \in \mathbb{N}$, where $C_b(X)$ is the Banach space of continuous real-valued bounded functions defined on a topological space X.

Definition 2.1. A sequence of functions $\{g_{\mathfrak{n}}\}$ is said to be deferred Nörlund and deferred Riesz statistically uniform convergent to g on X if $\forall$ $\epsilon > 0$:

$$\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}}} \left|\left\{\mathfrak{n} : \mathfrak{n} \leq \mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \text{ and } u_{s_{\mathfrak{n}}-j} v_j q_j \left\| g_{\mathfrak{n}}(z) - g(z) \right\|_{C_b(X)} \geq \epsilon\right\}\right| = 0.$$

It can be written as $g_{\mathfrak{n}} \rightrightarrows g$ ($\psi_{\mathfrak{n}}-$statistically uniform) on X.

Definition 2.2. A sequence of functions $\{g_{\mathfrak{n}}\}$ is said to be deferred Nörlund and deferred Riesz statistically relative pointwise convergent to g on X if $\exists$ a non zero pointwise scale function $\boldsymbol{\sigma}(z)$ on X s.t. $\forall$ $\epsilon > 0$ and $z \in X$:

$$\lim_{\mathfrak{n}\to\infty} \frac{\Upsilon_{\mathfrak{n}}(z, \epsilon)}{\mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}}} = 0,$$

where

$$\Upsilon_{\mathfrak{n}}(z, \epsilon) = \left|\left\{\mathfrak{n} : \mathfrak{n} \leq \mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \text{ and } u_{s_{\mathfrak{n}}-j} v_j q_j \left| \frac{g_{\mathfrak{n}}(z) - g(z)}{\boldsymbol{\sigma}(z)} \right| \geq \epsilon\right\}\right|.$$

We can write it as $g_{\mathfrak{n}} \rightarrow g$ ($\psi_{\mathfrak{n}}-$statistically relative pointwise) on X.

Definition 2.3. A sequence of functions $\{g_{\mathfrak{n}}\}$ is said to be deferred Nörlund and deferred Riesz statistically relative uniform convergent to g on X if $\exists$ a non zero pointwise scale function $\boldsymbol{\sigma}(z)$ on X s.t. $\forall$ $\epsilon > 0$:

$$\lim_{\mathfrak{n}\to\infty} \frac{\Phi_{\mathfrak{n}}(z, \epsilon)}{\mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}}} = 0,$$

where

$$\Phi_{\mathfrak{n}}(z, \epsilon) = \left|\left\{\mathfrak{n} : \mathfrak{n} \leq \mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \text{ and } u_{s_{\mathfrak{n}}-j} v_j q_j \sup_{z\in X} \left| \frac{g_{\mathfrak{n}}(z) - g(z)}{\boldsymbol{\sigma}(z)} \right| \geq \epsilon\right\}\right|.$$

It can be written as $g_{\mathfrak{n}} \rightrightarrows g$ ($\psi_{\mathfrak{n}}-$statistically relative uniform) on X.

Definition 2.4. A sequence of functions $\{g_{\mathfrak{n}}\}$ is said to be deferred Nörlund and deferred Riesz relative equi-statistically convergent to g on X if $\exists$ a non zero pointwise scale function $\boldsymbol{\sigma}(z)$ on X s.t. if for every $\epsilon > 0$, we have

$$\lim_{\mathfrak{n}\to\infty} \frac{\Upsilon_{\mathfrak{n}}(z, \epsilon)}{\mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}}} = 0$$

uniformly relatively with respect to $z \in X$, that is

$$\lim_{\mathfrak{n}\to\infty} \frac{\|\Upsilon_{\mathfrak{n}}(z,\epsilon)\|_{C_b(X)}}{\mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}} = 0,$$

where

$$\Upsilon_{\mathfrak{n}}(z,\epsilon) = \left|\left\{\mathfrak{n} : \mathfrak{n} \le \mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}} \text{ and } u_{s_{\mathfrak{n}}-j}v_jq_j\left|\frac{g_{\mathfrak{n}}(z)-g(z)}{\sigma(z)}\right| \ge \epsilon\right\}\right|.$$

We write $g_{\mathfrak{n}} \twoheadrightarrow g$ ($\psi_{\mathfrak{n}}-$relative equi-statistical).

Lemma 1. $g_{\mathfrak{n}} \rightrightarrows g$ ($\psi_{\mathfrak{n}}-$statistically uniform) implies $g_{\mathfrak{n}} \rightrightarrows g$ ($\psi_{\mathfrak{n}}-$statistically relative uniform). But, the converse of this lemma is not always true.

Proof. First suppose $g_{\mathfrak{n}} \rightrightarrows g$ ($\psi_{\mathfrak{n}}-$statistically uniform); then, by definition we have

$$\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}}\left|\left\{\mathfrak{n} : \mathfrak{n} \le \mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}} \text{ and } u_{s_{\mathfrak{n}}-j}v_jq_j\left\|g_{\mathfrak{n}}(z)-g(z)\right\|_{C_b(X)} \ge \epsilon\right\}\right| = 0.$$

Since the scale function is a constant function on X, we have

$$\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}}\left|\left\{\mathfrak{n} : \mathfrak{n} \le \mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}} \text{ and } u_{s_{\mathfrak{n}}-j}v_jq_j\sup_{z\in X}\left|\frac{g_{\mathfrak{n}}(z)-g(z)}{\sigma(z)}\right| \ge \epsilon\right\}\right| = 0.$$

Thus, $g_{\mathfrak{n}} \rightrightarrows g$ ($\psi_{\mathfrak{n}}-$statistically relative uniform). This proves the direct part. To prove the converse part let us consider an example as follows:

Example 2. Let $\{g_{\mathfrak{n}}\}$ be the sequence of continuous real valued functions. For $u_{s_{\mathfrak{n}}-j} = 1$, $v_{\mathfrak{n}} = \frac{\mathfrak{n}}{\mathfrak{n}+1}$, $r_{\mathfrak{n}} = 2\mathfrak{n}-1$ and $s_{\mathfrak{n}} = 4\mathfrak{n}-1$, $\forall n \in \mathbb{N}$. Now we define $g_{\mathfrak{n}} : [0,1] \to \mathbb{R}$ by

$$g_{\mathfrak{n}}(z) = \begin{cases} \frac{nz}{1+\mathfrak{n}^2z^2}, & z \in [0,\frac{1}{\mathfrak{n}}], \\ 0, & z \in (\frac{1}{\mathfrak{n}},1] \end{cases}$$

and $g(z) = 0$ on $[0,1]$. Since $\sup_{z\in[0,1]}|g_{\mathfrak{n}}(z)-g(z)| = \frac{1}{2}$, we have that

$$u_{s_{\mathfrak{n}}-j}v_jq_j\sup_{z\in[0,1]}|g_{\mathfrak{n}}(z)-g(z)| = \frac{\mathfrak{n}}{2(\mathfrak{n}+1)}.$$

Hence, for $\epsilon \in [0,1]$, we get

$$\begin{aligned}
&\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}}\left|\left\{\mathfrak{n} \le \mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}} : u_{s_{\mathfrak{n}}-j}v_jq_j\sup|g_{\mathfrak{n}}(z)-g(z)| \ge \epsilon\right\}\right| \\
&= \lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}}\left|\left\{\mathfrak{n} \le \mathcal{P}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}}\mathcal{R}_{r_{\mathfrak{n}+1}}^{s_{\mathfrak{n}}} : \frac{\mathfrak{n}}{2(\mathfrak{n}+1)} \ge \epsilon\right\}\right| \\
&= \frac{1}{2}.
\end{aligned}$$

So, $\{g_{\mathfrak{n}}\}$ is not $\psi_{\mathfrak{n}}-$statistically uniform convergent. Now, let us consider the scale function $\sigma(z)$ as

$$\sigma(z) = \begin{cases} \frac{1}{z}, & z \in (0,1], \\ 1, & z = 0. \end{cases}$$

Here we can see that $||\frac{g_\mathfrak{n}(z)-g(z)}{\sigma(z)}|| = \frac{1}{\mathfrak{n}}$, so it is clear that

$$\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}\mathcal{R}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}}\left|\left\{\mathfrak{n} \leq \mathcal{P}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}\mathcal{R}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}} : u_{s_\mathfrak{n}-j}v_j q_j \sup\left|\frac{g_\mathfrak{n}(z)-g(z)}{\sigma(z)}\right| \geq \epsilon\right\}\right|$$

$$\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}\mathcal{R}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}}\left|\left\{\mathfrak{n} \leq \mathcal{P}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}\mathcal{R}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}} : \frac{\mathfrak{n}}{2(\mathfrak{n}+1)}.\frac{1}{\mathfrak{n}} \geq \epsilon\right\}\right| \leq \frac{1}{\mathcal{P}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}\mathcal{R}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}} \to 0$$

as $\mathfrak{n}\to\infty$. Thus, $g_\mathfrak{n} \rightrightarrows g$ ($\psi_\mathfrak{n}-$statistically relative uniform) on $[0,1]$. □

Lemma 2. Suppose $\{g_\mathfrak{n}\}$ is a sequence of real valued functions. Then, $g_\mathfrak{n} \rightrightarrows g$ ($\psi_\mathfrak{n}-$statistically relative uniform) implies $g_\mathfrak{n} \twoheadrightarrow g$ ($\psi_\mathfrak{n}-$relative equi-statistical) implies $g_\mathfrak{n} \to g$ ($\psi_\mathfrak{n}-$statistically relative pointwise). But the converse of these implications is not necessarily true. The following examples justifies the strictness of the implications of Lemma 2.

Proof. The following example shows that the converse of the above inclusion need not be true:
Example 3. Consider $u_{s_\mathfrak{n}-j} = 1$, $v_\mathfrak{n} = \sqrt{j}$, $r_\mathfrak{n} = 2\mathfrak{n}-1$, $s_\mathfrak{n} = 4\mathfrak{n}-1$ and $\sigma(z) = 1$.

$$g_\mathfrak{n}(z) = \begin{cases} \frac{1-z^2(\mathfrak{n}+1)^2}{1+z^2}, & z \in [0, \frac{1}{\mathfrak{n}+1}], \\ \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}\mathcal{R}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}}\left|\left\{\mathfrak{n} \leq \mathcal{P}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}\mathcal{R}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}} : u_{s_\mathfrak{n}-j}v_j q_j\left|\frac{g_\mathfrak{n}(z)-g(z)}{\sigma(z)}\right| \geq \epsilon\right\}\right|$$

$$\leq \lim_{\mathfrak{n}\to\infty} \frac{1}{\mathcal{P}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}\mathcal{R}^{s_\mathfrak{n}}_{r_{\mathfrak{n}+1}}} \to 0 \text{ as } \mathfrak{n}\to\infty,$$

uniformly on $[a,b]$. This implies $g_\mathfrak{n} \twoheadrightarrow 0$ ($\psi_\mathfrak{n}-$relative equi-statistically). But

$$\sup_{z\in[0,1]}\left|\frac{g_\mathfrak{n}(z)-g(z)}{\sigma(z)}\right| = 1, \qquad \mathfrak{n}\in\mathbb{N}.$$

Thus, $\{g_\mathfrak{n}\}$ does not converge to 0 ($\psi_\mathfrak{n}-$statistically relative uniform). □

Example 4. Let $u_{s_\mathfrak{n}-j} = 1$, $v_\mathfrak{n} = 1$, $r_n = 0$, $s_n = n$, $g_\mathfrak{n}(z) = z^{\mathfrak{n}-1}$, $\sigma(z) = \frac{1}{z}$ and $g(z)$ be given as

$$g(z) = \begin{cases} 0, & z \in [0,1), \\ 1, & z = 1. \end{cases}$$

Here we can see that $g_\mathfrak{n} \to g$ ($\psi_\mathfrak{n}-$statistically relative pointwise). Now, if we choose $\epsilon = \frac{1}{3}$ then $\forall$ $n\in\mathbb{N}$, $\exists\, r > \mathfrak{n}$ s.t. for every $\mathfrak{m} \in [3^{r-1}, 3^r)$ and $z \in (\sqrt[3r]{\frac{1}{3}}, 1)$; we have

$$\begin{aligned} \left|\frac{g_\mathfrak{n}(z)}{\sigma(z)}\right| &= |z^\mathfrak{m}| \\ &> \left|\left(\sqrt[3r]{\frac{1}{3}}\right)^\mathfrak{m}\right| \\ &> \left|\left(\frac{1}{3}\right)^{\frac{3r}{3r}}\right| = \frac{1}{3}. \end{aligned}$$

Thus, clearly $\{g_\mathfrak{n}\}$ does not converge to 0 ($\psi_\mathfrak{n}-$relative equi-statistical).

3. Korovkin-type approximation theorem

Korovkin-type approximation theorems have been investigated by many mathematicians with various backgrounds involving function spaces, Banach spaces and so on. In 2010, Altomare [1] "studied Korovkin-type theorem and approximation by positive linear operators". In this segment, we proved a Korovkin-type approximation theorem in relation to the conception of $\psi_{\mathfrak{n}}$−relative equi-statistically convergence by using deferred Nörlund and deferred Riesz means that effectively extend most of the existing results. For a detailed study on approximation theorems one may refer to [7, 17, 18, 21].

Suppose $C[a,b]$ is the space of all real-valued continuous functions on $[a,b]$. The space $C[a,b]$ is a Banach space with the norm

$$||g||_{C[a,b]} = \sup_{z\in[a,b]} |g(z)|,$$

$\forall\ g \in C[a,b]$. Let $\mathfrak{F}_{\mathfrak{n}}$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$, that is, $\mathfrak{F}(g) \geq 0$ whenever $g \geq 0$. By $\mathfrak{F}(g,z)$, we represent the value of $\mathfrak{F}g$ at a point z.

Theorem 3.1. *Let $\mathfrak{F}_{\mathfrak{n}} : C[a,b] \to C[a,b]$ be a sequence of positive linear operators. Then $\forall\ g \in C[a,b]$,*

$$\mathfrak{F}_{\mathfrak{n}}(g,z) \twoheadrightarrow g(z) \quad (\psi_{\mathfrak{n}} - \textit{relative equi-statistical}) \tag{3.1}$$

iff

$$\mathfrak{F}_{\mathfrak{n}}(g_i,z) \twoheadrightarrow g_i(z) \quad (\psi_{\mathfrak{n}} - \textit{relative equi-statistical}) \tag{3.2}$$

where $\boldsymbol{\sigma}(z) = \max\{|\boldsymbol{\sigma}_i(z)| : |\boldsymbol{\sigma}_i(z)| > 0\}$ given that $\boldsymbol{\sigma}_i(z)$ is unbounded $(i = 0,1,2)$ and

$$g_0(z) = 1, \quad g_1(z) = z \ \text{ and } g_2(z) = z^2.$$

Proof. As each of the functions given by $g_i(z) = z^i \in C[a,b]$ $(i = 0,1,2)$ is continuous, condition (3.1) implies (3.2). Suppose (3.2) holds and assume $g \in C[a,b]$; then, $\exists$ a constant D such that

$$|g(z)| \leq D \qquad (\forall\ z \in\ [a,b])$$

which implies

$$|g(y) - g(z)| \ \leq \ 2D \qquad (\forall\ y,z \in\ [a,b]). \tag{3.3}$$

For a given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$|g(y) - g(z)| \ \leq \ \epsilon \tag{3.4}$$

whenever

$$|y - z| < \delta.$$

Select $\varphi = \varphi(y,z) = (y-z)^2$. If $|y - z| \geq \delta$, then

$$|g(y) - g(z)| \ < \ \frac{2D}{\delta^2}\varphi(y,z). \tag{3.5}$$

From (3.4) and (3.5)

$$|g(y) - g(z)| < \epsilon + \frac{2D}{\delta^2}\varphi(y, z)$$

$$-\epsilon - \frac{2D}{\delta^2}\varphi(y, z) \leq g(y) - g(z) \leq \epsilon + \frac{2D}{\delta^2}\varphi(y, z). \tag{3.6}$$

By using linearity and monotonicity of the operator $\mathfrak{F}_{\mathfrak{n}}(1, z)$, we get

$$\begin{aligned} \mathfrak{F}_{\mathfrak{n}}(1, z)\Big(- \epsilon - \frac{2D}{\delta^2}\varphi(y, z)\Big) &\leq \mathfrak{F}_{\mathfrak{n}}(1, z)[g(y) - g(z)] \\ &\leq \mathfrak{F}_{\mathfrak{n}}(1, z)\Big(\epsilon + \frac{2D}{\delta^2}\varphi(y, z)\Big). \end{aligned} \tag{3.7}$$

Suppose z is fixed, so $g(z)$ is a constant number. So, we have

$$\begin{aligned} -\epsilon\mathfrak{F}_{\mathfrak{n}}(1, z) - \frac{2D}{\delta^2}\mathfrak{F}_{\mathfrak{n}}(\varphi, z) &\leq \mathfrak{F}_{\mathfrak{n}}(g, z) - g(z)\mathfrak{F}_{\mathfrak{n}}(1, z) \\ &\leq \epsilon\mathfrak{F}_{\mathfrak{n}}(1, z) + \frac{2D}{\delta^2}\mathfrak{F}_{\mathfrak{n}}(\varphi, z). \end{aligned} \tag{3.8}$$

But

$$\mathfrak{F}_{\mathfrak{n}}(g, z) - g(z) = [\mathfrak{F}_{\mathfrak{n}}(g, z) - g(z)\mathfrak{F}_{\mathfrak{n}}(1, z)] + g(z)[\mathfrak{F}_{\mathfrak{n}}(1, z) - 1] \tag{3.9}$$

yields

$$\mathfrak{F}_{\mathfrak{n}}(g, z) - g(z) < \epsilon\mathfrak{F}_{\mathfrak{n}}(1, z) + \frac{2D}{\delta^2}\mathfrak{F}_{\mathfrak{n}}(\varphi, z) + g(z)[\mathfrak{F}_{\mathfrak{n}}(1, z) - 1]. \tag{3.10}$$

Next, computing $\mathfrak{F}_{\mathfrak{n}}(\varphi, z)$ as

$$\begin{aligned} \mathfrak{F}_{\mathfrak{n}}(\varphi, z) &= \mathfrak{F}_{\mathfrak{n}}((y - z)^2, z) = \mathfrak{F}_{\mathfrak{n}}(y^2 - 2yz + z^2, z) \\ &= \mathfrak{F}_{\mathfrak{n}}(y^2, z) - 2z\mathfrak{F}_{\mathfrak{n}}(y, z) + z^2\mathfrak{F}(1, z) \\ &= [\mathfrak{F}_{\mathfrak{n}}(y^2, z) - z^2] - 2z[\mathfrak{F}_{\mathfrak{n}}(y, z) - z] + z^2[\mathfrak{F}_{\mathfrak{n}}(1, z) - 1] \end{aligned}$$

and using (3.10), we get

$$\begin{aligned} \mathfrak{F}_{\mathfrak{n}}(g, z) - g(z) &< \epsilon\mathfrak{F}_{\mathfrak{n}}(1, z) + \frac{2D}{\delta^2}\{[\mathfrak{F}_{\mathfrak{n}}(y^2, z) - z^2] - 2z[\mathfrak{F}_{\mathfrak{n}}(y, z) - z] \\ &+ z^2[\mathfrak{F}_{\mathfrak{n}}(1, z) - 1]\} + g(z)[\mathfrak{F}_{\mathfrak{n}}(1, z) - 1] \\ &= \epsilon[\mathfrak{F}_{\mathfrak{n}}(1, z) - 1] + \epsilon + \frac{2D}{\delta^2}\{[\mathfrak{F}_{\mathfrak{n}}\varphi(y^2, z) - z^2] - 2z[\mathfrak{F}_{\mathfrak{n}}(y, z) - z] \\ &+ z^2[\mathfrak{F}_{\mathfrak{n}}(1, z) - 1]\} + g(z)[\mathfrak{F}_{\mathfrak{n}}(1, z) - 1]. \end{aligned}$$

Then,

$$|\mathfrak{F}_{\mathfrak{n}}(g, z) - g(z)| \leq \epsilon + \Big(\epsilon + \frac{2D}{\delta^2} + D\Big)|\mathfrak{F}_{\mathfrak{n}}(1, z) - 1|$$

$$\begin{aligned} &+ \frac{4D}{\delta^2}|\mathfrak{F}_{\mathfrak{n}}(y,z)-z| + \frac{2D}{\delta^2}|\mathfrak{F}_{\mathfrak{n}}(y^2,z)-z^2| \\ &\leq M\Big(|\mathfrak{F}_{\mathfrak{n}}(1,z)-1| + |\mathfrak{F}_{\mathfrak{n}}(y,z)-z| + |\mathfrak{F}_{\mathfrak{n}}(y^2,z)-z^2|\Big) \end{aligned}$$

where $M = \max\{\epsilon + \frac{2D}{\delta^2} + D, \frac{2D}{\delta^2}, \frac{4D}{\delta^2}\}$. Now, for a given $x > 0$, select $\epsilon > 0$ s.t. $\frac{\epsilon}{\sigma(z)} < x$. Then, by setting

$$\Upsilon_{\mathfrak{n}}(z, x\boldsymbol{\sigma}) = \left|\left\{\mathfrak{n} : \mathfrak{n} \leq \mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \text{ and } u_{s_{\mathfrak{n}}-j} v_j q_j \left|\frac{\mathfrak{F}_{\mathfrak{n}}(g,z) - g(z)}{\boldsymbol{\sigma}(z)}\right| \geq x\right\}\right|$$

and

$$\Upsilon_{i,\mathfrak{n}}(z, x\boldsymbol{\sigma}) = \left|\left\{\mathfrak{n} : \mathfrak{n} \leq \mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \text{ and } u_{s_{\mathfrak{n}}-j} v_j q_j \left|\frac{\mathfrak{F}_{\mathfrak{n}}(g_i,z) - g_i(z)}{\boldsymbol{\sigma}_i(z)}\right| \geq \frac{x - \frac{\epsilon}{\sigma(z)}}{3M}\right\}\right|,$$

for $i = 0, 1, 2$, we get

$$\Upsilon(z, x\boldsymbol{\sigma}) \leq \sum_{i=0}^{2} \Upsilon_{i,\mathfrak{n}}(z, x\boldsymbol{\sigma}).$$

Thus, we have

$$\frac{\|\Upsilon_{i,\mathfrak{n}}(z, x\boldsymbol{\sigma})\|_{C[a,b]}}{\mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}_{r^{s_{\mathfrak{n}}}_{\mathfrak{n}+1}}} \leq \sum_{i=0}^{2} \frac{\|\Upsilon_{i,\mathfrak{n}}(z, x\boldsymbol{\sigma})\|_{C[a,b]}}{\mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}_{r^{s_{\mathfrak{n}}}_{\mathfrak{n}+1}}}. \tag{3.11}$$

Consequently, by the definition of $\psi_{\mathfrak{n}}-$relative equi-statistical and supposition of implication (3.2), the right side of (3.11) approaches zero as $\mathfrak{n} \to \infty$. So, we have

$$\frac{\|\Upsilon_{i,\mathfrak{n}}(z, x\boldsymbol{\sigma})\|_{C[a,b]}}{\mathcal{P}^{s_{\mathfrak{n}}}_{r_{\mathfrak{n}+1}} \mathcal{R}_{r^{s_{\mathfrak{n}}}_{\mathfrak{n}+1}}} = 0 \quad (x > 0).$$

Thus, the implication (3.1) is true. □

Next, we give an example in support of the above theorem.

Example 5. Let $I = [0, 1]$, the Meyer-König and Zeller operator $\mathfrak{m}_{\mathfrak{n}(g,z)}$ on $C[0, 1]$ is given by

$$\mathfrak{m}_{\mathfrak{n}}(g,z) = \sum_{j=0}^{\infty} g\left(\frac{j}{j+\mathfrak{n}+1}\right)\binom{\mathfrak{n}+j}{j} z^j (1-z)^{\mathfrak{n}-j} \quad (z \in [0,1]).$$

Also, we define an operator $\mathfrak{F}_{\mathfrak{n}} : C[0, 1] \to C[0, 1]$ by

$$\mathfrak{F}_{\mathfrak{n}}(g,z) = [1 + g_{\mathfrak{n}}(z)]z(1 + zM)\mathfrak{m}_{\mathfrak{n}}(g,z), \quad (g \in C[0,1]) \tag{3.12}$$

where the sequence $\{g_{\mathfrak{n}}(z)\}$ of functions is as given in Example 1. Thus, we have

$$\begin{aligned} \mathfrak{F}_{\mathfrak{n}}(g_0,z) &= [1 + g_{\mathfrak{n}}(z)]z(1 + zM)g_0(z) = [1 + g_{\mathfrak{n}}(z)]z, \\ \mathfrak{F}_{\mathfrak{n}}(g_1,z) &= [1 + g_{\mathfrak{n}}(z)]z(1 + zM)g_1(z) = [1 + g_{\mathfrak{n}}(z)]z(1 + z) \end{aligned}$$

and

$$\mathfrak{F}_{\mathfrak{n}}(g_2,z) = [1 + g_{\mathfrak{n}}(z)]z(1 + zM)\left\{g_2(z)\left(\frac{\mathfrak{n}+2}{\mathfrak{n}+1}\right) + \frac{z}{\mathfrak{n}+1}\right\}$$

$$= \quad [1 + g_{\mathfrak{n}}(z)]\Big\{(z)^2\Big[\Big(\frac{\mathfrak{n}+2}{\mathfrak{n}+1}\Big)z + 2\Big(\frac{1}{\mathfrak{n}+1}\Big) + 2z\Big(\frac{\mathfrak{n}+2}{\mathfrak{n}+1}\Big)\Big]\Big\}.$$

Then,

$$g_{\mathfrak{n}} \twoheadrightarrow g = 0 \qquad (\psi_{\mathfrak{n}} - \text{relative equi-statistically}) \text{ on } [0,1].$$

Thus, we have

$$\mathfrak{F}_{\mathfrak{n}}(g_i, z) \twoheadrightarrow g_i \quad (\psi_{\mathfrak{n}} - \text{relative equi-statistically}) \text{ on } [0,1],$$

$\forall i = 0, 1, 2$. So, by Theorem 3.1, we have

$$\mathfrak{F}_{\mathfrak{n}}(g, z) \twoheadrightarrow g \qquad (\psi_{\mathfrak{n}} - \text{relative equi-statistically}) \text{ on } [0,1]$$

$\forall\ g \in C[0,1]$. Since $\{g_{\mathfrak{n}}\}$ is not $\psi_{\mathfrak{n}}$−relative equi-statistical to g on [0,1], we can conclude that the work in [17] is not valid for our operator given in (3.12), whereas our Theorem 3.1 still valid for operators defined by (3.12).

4. Conclusions

Over many years, a great deal of work has been done on statistical convergence. In this paper, we investigated deferred Nörlund and deferred Riesz relative equi-statistical convergence, relative uniform statistical convergence and relative pointwise statistical convergence and proved the Korovkin-type approximation theorem by using deferred Nörlund and deferred Riesz relative uniform convergence. For further research one can use deferred Nörlund and deferred Riesz fractional difference statistical probability convergence to prove the approximation theorem.

Acknowledgments

The authors thank the Fujian Province Innovation Strategy Research Project (Grant No. 2020R0153) and the Council of Scientific and Industrial Research (CSIR) of India for partial support under Grant No. 25(0288)/18/EMR-II, dated May 24, 2018.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. F. Altomare, Korovkin-type theorems and approximation by positive linear operator, *Surveys in Approximation Theory*, **5** (2010), 92–164.
2. M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, *J. Math. Anal. Appl.*, **328** (2007), 715–729. https://doi.org/10.1016/j.jmaa.2006.05.040
3. J. Connor, P. Leonetti, A characterization of $(\mathcal{I}, \mathcal{J})$−regular matrices, *J. Math. Anal. Appl.*, **504** (2021), 125374. https://doi.org/10.1016/j.jmaa.2021.125374

4. E. W. Chittenden, On the limit functions of sequences of continuous functions converging relatively uniformly, *Trans. Amer. Math. Soc.*, **20** (1919), 179–184. https://doi.org/10.1090/S0002-9947-1919-1501120-6

5. E. W. Chittenden, Relatively uniform convergence of sequences of functions, *Trans. Amer. Math. Soc.*, **15** (1914), 197–201. https://doi.org/10.1090/S0002-9947-1914-1500972-9

6. J. S. Connor, The statistical and strong p-Cesàro convergence of sequence, *Analysis*, **8** (1988), 47–63. https://doi.org/10.1524/anly.1988.8.12.47

7. H. Dutta, S. K. Paikray, B. B. Jena, On statistical deferred Cesàro summability, In: *Current trends in mathematical analysis and its interdisciplinary applications*, Cham: Birkhäuser, 2019, 885–909. https://doi.org/10.1007/978-3-030-15242-0_23

8. K. Demirci, S. Orhan, Statistically relatively uniform convergence of positive linear operators, *Results Math.*, **69** (2016), 359–367. https://doi.org/10.1007/s00025-015-0484-9

9. K. Demirci, F. Dirik, S. Yildiz, Deferred Nörlund statistical relative uniform convergence and Korovkin-type approximation Theorem, *Commun. Fsc. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **70** (2021), 279–289. https://doi.org/10.31801/CFSUASMAS.807169

10. M. Et, B. C. Tripathy, A. J. Dutta, On pointwise statistical convergence of order α of sequences of fuzzy mappings, *Kuwait J. Sci.*, **41** (2014), 17–30.

11. J. A. Fridy, On statistical convergent, *Analysis*, **5** (1985), 301–313. https://doi.org/10.1524/anly.1985.5.4.301

12. J. A. Fridy, Statistical limit points, *Proc. Amer. Math. Soc.*, **118** (1993), 1187–1192. https://doi.org/10.1090/S0002-9939-1993-1181163-6

13. H. Fast, Sur la convergence statistique, *Colloq. Math.*, **2** (1951), 241–244. https://doi.org/10.4064/CM-2-3-4-241-244

14. A. Gökhan, M. Güngör, M. Et, Statistical convergence of double sequences of real-valued functions, *Int. Math. Forum*, **2** (2007), 365-374. http://doi.org/10.12988/imf.2007.07033

15. B. B. Jena, S. K. Paikray, S. A. Mohiuddine, V. N. Mishra, Relatively equi-statistical convergence via deferred Nörlund mean based on difference operator of fractional-order and related approximation theorems, *AIMS Mathematics*, **5** (2020), 650–672. https://doi.org/10.3934/math.2020044

16. S. Jasrotia, U. P. Singh, K. Raj, Application of statistical convergence of order $(\eta, \delta + \gamma)$ in difference sequence spaces of fuzzy numbers, *J. Intell. Fuzzy Syst.*, **40** (2021), 4695–4703. https://doi.org/10.3233/JIFS-201539

17. S. Karakuş, K. Demirci, O. Duman, Equi-statistical Convergence of positive linear operators, *J. Math. Anal. Appl.*, **339** (2008), 1065–1072. https://doi.org/10.1016/j.jmaa.2007.07.050

18. S. A. Mohiuddine, A. Alotaibi, M. Mursaleen, Statistical summability $(C, 1)$ and a Korovkin type approximation theorem, *J. Inequal. Appl.*, **2012** (2012), 172. https://doi.org/10.1186/1029-242X-2012-172

19. E. H. Moore, *An introduction to a form of general analysis*, New Haven: Yale University Press, 1910.

20. G. M. Petersen, *Regular matrix transformations*, London: McGraw-Hill, 1996.

21. K. Raj, A. Choudhary, Relative modular uniform approximation by means of power series method with applications, *Revista de la Unión Matemática Argentina*, **60** (2019), 187–208. https://doi.org/10.33044/revuma.v60n1a11

22. K. Saini, K. Raj, Application of statistical convergence in complex uncertain sequence via deferred Reisz mean, *Int. J. Uncertain Fuzz. Knowl. Based Syst.*, **29** (2021), 337–351. https://doi.org/10.1142/S021848852150015X

23. E. Savas, M. Mursaleen, On statistically convergent double sequence of fuzzy numbers, *Inform. Sci.*, **162** (2004), 183–192. https://doi.org/10.1016/j.ins.2003.09.005

24. P. D. Srivastava, S. Ojha, $\lambda-$Statistical convergence of fuzzy numbers and fuzzy functions of order θ, *Soft Comput.*, **18** (2014), 1027–1032. https://doi.org/10.1007/s00500-013-1125-4

25. T. Salat, On statistically convergent sequences of real numbers, *Math. Slovaca*, **30** (1980), 139–150.

26. I. J. Schoenberg, The integrability of certain function and related summability methods, *Amer. Math. Mon.*, **66** (1959), 361–375. https://doi.org/10.1080/00029890.1959.11989303

27. B. C. Tripathy, A. Baruah, M. Et, M. Gungor, On almost statistical convergence of new type of generalized difference sequence of fuzzy numbers, *Iran. J. Sci. Technol. A*, **36** (2012), 147–155.

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