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Research article

A discrete-time dual risk model with dependence based on a Poisson INAR(1) process

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Abstract: In this paper, we consider an extension of the classical discrete-time dual risk model, in which the first-order integer-valued autoregressive (INAR(1)) process with Poisson distributed innovations is utilized to fit the temporal dependence between the number of gains for each period. We derive the explicit expression for a function that allows us to find the Lundberg adjustment coefficient and obtain the Lundberg approximation formula for ruin probability. Some numerical examples are provided to illustrate our main results.

Keywords: dual risk model; INAR(1) process; Lundberg adjustment coefficient; ruin probability
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1. Introduction

Actuarial science arises from insurance practice, and risk theory is one of its most active research fields. By modeling the surplus process seriously and analyzing various uncertainties quantitatively, insurance companies can manage their faced risks in an efficient way. The fundamental Lundberg-Cramér risk model is defined as

$$U_t^0 = u + ct - S_t, \quad t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial surplus of an insurance company, $c > 0$ is the rate of premium income, $\{S_t, t \geq 0\}$ is the aggregate claims process and is assumed to be a compound Poisson process, i.e.,

$$S_t = \sum_{i=1}^{N_t^0} Y_i,$$

in which $\{N_t^0, t \geq 0\}$ is a homogeneous Poisson process, representing the total numbers of claims up to time t, and Y_i denotes the amount of the ith claim. Please see Asmussen and Albrecher [1] and the references therein on this well-known model.

Analogous to the classical risk model (1.1), its dual process also plays an important role in risk management of insurance and finance. In the so-called dual risk model, the surplus or equity of a company at time t satisfies the following dynamics:

$$U_t^1 = u - ct + S_t = u - ct + \sum_{i=1}^{N_t^1} Y_i, \quad t \geq 0. \tag{1.2}$$

Similarly, $u \geq 0$ is again the initial surplus. However now, c is the constant rate of expenses, N_t^1 is total number of gains up to time t, and Y_i is the amount of the ith random gain, such that S_t represents the aggregate gains during time 0 to t.

It can be seen from (1.2) that the dual risk model can be used to fit the surplus of a company with fixed expense rate, while stochastic gains arrive occasionally due to some contingent events (e.g., inventions, discoveries and sales). As stated in Avanzi et al. [2], there are many possible interpretations for this model, and prime examples are pharmaceutical, petroleum or R & D companies, as well as whole-of-life annuity insurances. The dual risk model was first named by Mazza and Rullière [3] because of its duality to the Lundberg-Cramér model, and the relevant results can go back to Cramér [4]. Since Avanzi et al. [2], various performance measures of the dual risk model have been extensively studied. We refer to [5–11] and the references therein for details.

On the other hand, some assumptions of independence are usually made in classical risk models (1.1). For example, the claim numbers of different periods are supposed to be an independent and identically distributed (i.i.d.) random variables sequence. However, this condition may not be realistic in practice. To avoid this restriction, more and more actuaries have been paying attention to the modeling of dependent risks during the last decades. Particularly, because of the flexible application in temporal dependence, integer-valued time series have been extensively applied in risk theory. The relevant study topic was initiated by [12], in which the first-order integer-valued autoregressive (INAR(1)) process is applied to analyze car accident count data and update the premiums. Afterwards, [13] uses INAR(1) process and first-order integer-valued moving average (INMA(1)) process to construct the dependence between the numbers of claims for each period. Since then, such discrete-time risk model has been extensively revised by many researchers. Some interesting discussions can be found in [14–19].

To our knowledge, there are few papers concerning the dependent dual risk model at present. Dimitrina et al. [20] and Li et al. [21] consider a dual risk model with dependence between inter-gain times and gain sizes, and they study corresponding ruin problems. The main objective of this paper is to fit the temporal dependence between the number of gains for each period and propose an extension of the classical discrete-time dual risk model based on INAR(1) process with Poisson distributed innovations. Our goal is to examine the Lundberg adjustment coefficient and approximate the ruin probability of the surplus process.

The remainder of the paper is organized as follows. In Sections 2, we introduce the proposed risk model and give some basic properties. In Section 3, we derive an equation satisfied by the Lundberg adjustment coefficient. In Section 4, we establish an explicit asymptotic estimation for ruin probability. In Section 5, we illustrate the main results by some numerical simulations. We conclude the paper in Section 6.

2. Model and properties

In this section, we establish the framework of a discrete-time dual risk model based on the INAR(1) process with Poisson distributed innovations to describe the dependence structure between the numbers of gains for each period.

Let $\{U_t, t = 0, 1, 2, \cdots\}$ be the surplus process of an insurance portfolio, in which U_t corresponds to the surplus level at time t. For $t = 1, 2, \cdots$, the dynamic of the surplus process for the insurer is expressed recursively as

$$U_t = U_{t-1} + W_t - c, \tag{2.1}$$

where $U_0 = u \geq 0$ is the initial surplus, c is the constant rate of expenses, and the aggregate gain amount in period t can be expressed by

$$W_t = \sum_{k=1}^{N_t} X_{t,k}, \tag{2.2}$$

in which N_t and $X_{t,k}$ represent the gain numbers and the kth gain amount for the insurance portfolio in period t, respectively. Moreover, we assume that

(1) $\{X_{t,k}, t = 1, 2, \cdots, k = 1, 2, \cdots\}$ is an array of i.i.d. non-negative random variables, having the same distributions as X.

(2) $\{X_{t,k}, t = 1, 2, \cdots, k = 1, 2, \cdots\}$ and $\{N_t, t = 1, 2, \cdots\}$ are mutually independent.

(3) $\{N_t, t = 1, 2, \cdots\}$ is a Poisson INAR(1) process satisfying

$$N_t = \alpha \circ N_{t-1} + \varepsilon_t, \quad t = 2, 3, \cdots, \tag{2.3}$$

where the binomial thinning operator "$\circ$" is defined by

$$\alpha \circ N_{t-1} = \sum_{m=1}^{N_{t-1}} B_{t,m}, \quad t = 2, 3, \cdots, \tag{2.4}$$

in which

- The thinning parameter $\alpha \in [0, 1)$.
- $\{B_{t,m}, t = 2, 3, \cdots, m = 1, 2, \cdots\}$ is an array of i.i.d. random variables, having a Bernoulli distribution with mean α.
- $\{\varepsilon_t, t = 2, 3, \cdots\}$ is a sequence of i.i.d. random variables, having a Poisson distribution with mean λ.
- N_1, $\{B_{t,m}, t = 2, 3, \cdots, m = 1, 2, \cdots\}$ and $\{\varepsilon_t, t = 2, 3, \cdots\}$ are independent.

Remark 2.1. *As an intuitive interpretation of the proposed model (2.1), the number of gains in period t consists of two parts: One part denoted by ε_t is the new gains arriving between period $t-1$ and t, and the other part denoted by $\alpha \circ N_{t-1}$ is a random portion of the number of gains in the previous period, meaning that each of the gains from the previous period has a probability α to contribute one gain to the number of gains in the next period. For insurance practice, it can be explained that every insured could continue to purchase the product or withdraw from the contract in the next period. If $\alpha = 0$, then N_t which corresponds to number of gains in period t, is totally explained by ε_t, so that $\{N_t, t = 1, 2, \cdots\}$ are independent; and our proposed model (2.1), which is further described by (2.2)–(2.4), will reduce to the classical discrete-time dual risk model.*

According to [22] and [23], the condition $0 \leq \alpha < 1$ implies that the process of gain numbers $\{N_t, t = 1, 2, \cdots\}$ is a stationary Markov chain. Furthermore, it is known that if an INAR(1) process $\{N_t, t = 1, 2, \cdots\}$ is stationary with appropriate distributions for the innovations $\{\varepsilon_t, t = 2, 3, \cdots\}$, then the marginal distribution of $\{N_t, t = 1, 2, \cdots\}$ can be determined by the following equation:

$$P_N(s) = P_N(1 - \alpha + \alpha s) \cdot P_\varepsilon(s), \tag{2.5}$$

where $P_N(s)$ and $P_\varepsilon(s)$ denote the probability generating functions of N_t's and ε_t's, respectively. Therefore, under the assumption of $\varepsilon_t \sim P(\lambda)$, it follows from (2.5) that $\{N_t, t = 1, 2, \cdots\}$ also has Poisson marginals which are distributed by $P(\frac{\lambda}{1-\alpha})$, to keep the process itself stationary. Also, some basic probabilistic properties of the aggregate gain amount process $\{W_t, t = 1, 2, \cdots\}$ indicate that

$$E(W_t) = E(N_t)E(X) = \frac{\lambda}{1-\alpha}E(X),$$

and

$$\mathrm{Var}(W_t) = E(N_t)\mathrm{Var}(Z) + \mathrm{Var}(N_t)[E(Z)]^2 = \frac{\lambda}{1-\alpha}E(X^2).$$

In addition, for $h = 1, 2, \cdots$, it holds that

$$\begin{aligned}
&\mathrm{Cov}(W_{t+h}, W_t)\\
=&E(W_{t+h}W_t) - E(W_{t+h})E(W_t)\\
=&E[E(W_{t+h}W_t|N_{t+h}, N_t)] - E(W_{t+h})E(W_t)\\
=&E\left[E\left(\left(\sum_{k=1}^{N_{t+h}} X_{t+h,k}\right)\left(\sum_{k=1}^{N_t} X_{t,k}\right)|N_{t+h}, N_t\right)\right] - E(W_{t+h})E(W_t)\\
=&E\left[\sum_{k=1}^{N_{t+h}}\sum_{k=1}^{N_t} E(X_{t+h,k}X_{t,k}|N_{t+h}, N_t)\right] - E(W_{t+h})E(W_t)\\
=&E(N_{t+h}N_t)E(X)E(X) - E(N_{t+h})E(X)E(N_t)E(X)\\
=&[E(X)]^2\mathrm{Cov}(N_{t+h}, N_t)\\
=&\frac{\lambda\alpha^h}{1-\alpha}[E(X)]^2.
\end{aligned}$$

3. Lundberg adjustment coefficient

It is well known that the Lundberg adjustment coefficient is one of the most useful measures for dangerousness of an insurance portfolio. In this section, we derive the expression for a function, from which the Lundberg adjustment coefficient in the discrete-time dual risk model based on Poisson INAR(1) process can be defined.

By direct calculation, risk model (2.1) can be rewritten as

$$\begin{aligned}
U_t &= U_{t-1} + W_t - c\\
&= U_{t-1} + \sum_{k=1}^{N_t} X_{t,k} - c
\end{aligned}$$

$$= u + \sum_{i=1}^{t} \sum_{k=1}^{N_i} X_{i,k} - ct$$
$$= u + S_t - ct, \quad t = 1, 2, \cdots, \tag{3.1}$$

in which $S_t = \sum_{i=1}^{t} \sum_{k=1}^{N_i} X_{i,k}$ represents the aggregate gain amount of the surplus process up to time t.

Define

$$c_t(r) = \frac{1}{t} \ln E([e^{r(ct - S_t)}]), \tag{3.2}$$

and let

$$c(r) = \lim_{t \to +\infty} c_t(r). \tag{3.3}$$

Then the Lundberg adjustment coefficient R is defined by the positive solution to the equation $c(r) = 0$. Section 4 will discuss how to use R to approximate the ruin probability of our proposed model.

In what follows, we derive the expression for $c(r)$. First, it is easy to know

$$\lim_{t \to +\infty} \frac{1}{t} \ln E(e^{rct}) = rc. \tag{3.4}$$

On the other hand, denote the moment generating function of S_t by $M_{S_t}(\cdot)$, and then we have

$$\begin{aligned} M_{S_t}(-r) =& E\left(e^{-rS_t}\right) \\ =& E\left\{\exp\left\{-r \sum_{i=1}^{t} \sum_{k=1}^{N_i} X_{i,k}\right\}\right\} \\ =& \sum_{n_1, \cdots, n_t} [E(e^{-rX})]^{n_1 + \cdots + n_t} \times P(N_1 = n_1, \cdots, N_t = n_t) \\ =& E[M_X(-r)^{N_1 + \cdots + N_t}] \\ =& P_{N_1 + \cdots + N_t}(M_X(-r)), \end{aligned} \tag{3.5}$$

in which $M_X(\cdot)$ is the moment generating function of X. Therefore, in order to obtain the expression for (3.5), we need to find $P_{N_1 + \cdots + N_t}(\cdot)$, the probability generating function of the total gain number up to time t of model (2.1).

Lemma 3.1. *Let* $0 \le s \le 1$*, and then the probability generating function of* $N_1 + \cdots + N_t$ *is given by*

$$P_{N_1 + \cdots + N_t}(s) = \exp\left\{\lambda \frac{s-1}{1-\alpha s}\left[t + \frac{1-(\alpha s)^t}{1-\alpha} - \frac{1-(\alpha s)^t}{1-\alpha s}\right]\right\}. \tag{3.6}$$

Proof. By the definition (2.3), when $t = 1$, we have for $0 \le s \le 1$ that

$$P_{N_1}(s) = E(s^{N_1}) = \exp\left\{\frac{\lambda}{1-\alpha}(s-1)\right\} = \exp\left\{\frac{\lambda}{1-\alpha}(h_1(s) - 1)\right\},$$

in which $h_1(s) = s$. Note that

$$N_2 = \alpha \circ N_1 + \varepsilon_2,$$

and

$$\begin{aligned}
P_{N_1+\alpha\circ N_1}(s) &= E(s^{N_1+\alpha\circ N_1}) \\
&= E[E(s^{N_1} s^{\alpha\circ N_1}|N_1)] \\
&= E[s^{N_1} E(s^{\alpha\circ N_1}|N_1)] \\
&= E[s^{N_1}(\alpha s + 1 - \alpha)^{N_1}] \\
&= \exp\left\{\frac{\lambda}{1-\alpha}(s(\alpha h_1(s) + 1 - \alpha) - 1)\right\}.
\end{aligned} \tag{3.7}$$

Hence, for $t = 2$ and $0 \le s \le 1$, it follows that

$$\begin{aligned}
P_{N_1+N_2}(s) &= E(s^{N_1+N_2}) \\
&= E(s^{N_1+\alpha\circ N_1} s^{\varepsilon_2}) \\
&= E(s^{N_1+\alpha\circ N_1})E(s^{\varepsilon_2}) \\
&= \exp\left\{\frac{\lambda}{1-\alpha}(s(\alpha h_1(s) + 1 - \alpha) - 1)\right\} \exp\{\lambda(s-1)\} \\
&= \exp\left\{\frac{\lambda}{1-\alpha}(h_2(s) - 1)\right\} \exp\{\lambda(h_1(s) - 1)\},
\end{aligned}$$

in which $h_2(s) = s(\alpha h_1(s) + 1 - \alpha)$.

Similarly, when $t = 3$, because

$$\begin{aligned}
N_1 + N_2 + N_3 &= N_1 + \alpha \circ N_1 + \varepsilon_2 + \alpha \circ N_2 + \varepsilon_3 \\
&= N_1 + \alpha \circ N_1 + \varepsilon_2 + \alpha \circ \alpha \circ N_1 + \alpha \circ \varepsilon_2 + \varepsilon_3 \\
&= N_1 + \alpha \circ N_1 + +\alpha \circ \alpha \circ N_1 + \varepsilon_2 + \alpha \circ \varepsilon_2 + \varepsilon_3,
\end{aligned}$$

and the same method to derive (3.7) leads to

$$P_{\alpha\circ\varepsilon_2+\varepsilon_2}(s) = \exp\{\lambda(s(\alpha h_1(s) + 1 - \alpha) - 1)\} = \exp\{\lambda(h_2(s) - 1)\},$$

as well as

$$\begin{aligned}
P_{N_1+\alpha\circ N_1++\alpha\circ\alpha\circ N_1}(s) &= E(s^{N_1+\alpha\circ N_1++\alpha\circ\alpha\circ N_1}) \\
&= E\left[s^{N_1} s^{\alpha\circ N_1} E\left(s^{\alpha\circ\alpha\circ N_1}|N_1, \alpha \circ N_1\right)\right] \\
&= E\left[s^{N_1} s^{\alpha\circ N_1}(\alpha s + 1 - \alpha)^{\alpha\circ N_1}\right] \\
&= E\left[s^{N_1}(h_2(s))^{\alpha\circ N_1}\right] \\
&= \exp\left\{\frac{\lambda}{1-\alpha}(s(\alpha h_2(s) + 1 - \alpha) - 1)\right\} \\
&= \exp\left\{\frac{\lambda}{1-\alpha}(h_3(s) - 1)\right\},
\end{aligned}$$

where $h_3(s) = s(\alpha h_2(s) + 1 - \alpha)$, we can get

$$
\begin{aligned}
&P_{N_1+N_2+N_3}(s)\\
=&P_{N_1+\alpha\circ N_1++\alpha\circ\alpha\circ N_1}(s)P_{\alpha\circ\varepsilon_2+\varepsilon_2}(s)P_{\varepsilon_3}(s)\\
=&\exp\left\{\frac{\lambda}{1-\alpha}(h_3(s)-1)\right\}\exp\{\lambda(h_2(s)-1)\}\exp\{\lambda(h_1(s)-1)\}.
\end{aligned}
$$

Consequently, we deduce for $t > 3$ by recursive calculation that

$$
P_{N_1+\cdots+N_t}(s) = \exp\left\{\frac{\lambda}{1-\alpha}(h_t(s)-1)\right\}\prod_{i=1}^{t-1}\exp\{\lambda(h_i(s)-1)\}, \tag{3.8}
$$

in which $h_t(s) = s(\alpha h_{t-1}(s) + 1 - \alpha)$. Furthermore, it is easy to obtain from this relation that

$$
h_t(s) - 1 = s - 1 + \alpha s(h_{t-1}(s) - 1),
$$

and then for any $t \geq 1$, we have

$$
h_t(s) - 1 = (s-1)\frac{1-(\alpha s)^t}{1-\alpha s}. \tag{3.9}
$$

Substituting (3.9) into (3.8) and direct calculation lead to

$$
P_{N_1+\cdots+N_t}(s) = \exp\left\{\lambda\frac{s-1}{1-\alpha s}\left[t + \frac{1-(\alpha s)^t}{1-\alpha} - \frac{1-(\alpha s)^t}{1-\alpha s}\right]\right\}.
$$

This completes the proof. □

Now, we give the expression of $c(r)$.

Theorem 3.1. *For $r \geq 0$, we have*

$$
c(r) = \lambda\frac{M_X(-r)-1}{1-\alpha M_X(-r)} + cr. \tag{3.10}
$$

Proof. Since $r \geq 0$ and X is non-negative, $0 \leq M_X(-r) \leq 1$, $0 \leq \alpha M_X(-r) < 1$, and it follows from Lemma 3.1 that

$$
\begin{aligned}
\lim_{t\to+\infty}\frac{1}{t}\ln E(e^{-rS_t}) &= \lim_{t\to+\infty}\frac{1}{t}\ln P_{N_1+\cdots+N_t}(M_X(-r))\\
&= \lim_{t\to+\infty}\frac{1}{t}\ln\left(\exp\left\{\lambda\frac{M_X(-r)-1}{1-\alpha M_X(-r)}\left[t+\frac{1-(\alpha M_X(-r))^t}{1-\alpha}-\frac{1-(\alpha M_X(-r))^t}{1-\alpha M_X(-r)}\right]\right\}\right)\\
&= \lambda\frac{M_X(-r)-1}{1-\alpha M_X(-r)}.
\end{aligned} \tag{3.11}
$$

Combining (3.2), (3.3), (3.4) and (3.11), we obtain

$$
\begin{aligned}
c(r) &= \lim_{t\to+\infty} c_t(r)\\
&= \lim_{t\to+\infty}\frac{1}{t}\ln E([e^{r(ct-S_t)}])\\
&= \lim_{t\to+\infty}\frac{1}{t}\ln E(e^{-rS_t}) + \lim_{t\to+\infty}\frac{1}{t}\ln E(e^{rct})\\
&= \lambda\frac{M_X(-r)-1}{1-\alpha M_X(-r)} + cr.
\end{aligned}
$$

This completes the proof. □

4. Lundberg approximation for ruin probability

Ruin probability is one of the most important measures to quantify the dangerousness of an insurance portfolio in risk theory. For our proposed model (2.1), we denote by random variable T the time of ruin, i.e.,

$$T = \inf\{t : t = 0, 1, 2, \cdots, U_t \leq 0\},$$

in which $\inf\{\emptyset\} = +\infty$. Then, the infinite-time ruin probability $\psi(u)$ is defined by

$$\psi(u) = P(T < +\infty | U_0 = u).$$

Though the dependence structure makes the risk model more practical, it also creates some difficulties for analysis on ruin probability. In what follows, we give the Lundberg approximation formula for the ruin probability of our model.

Theorem 4.1. *For the discrete-time dual risk model with dependence based on Poisson INAR(1) process, if*

$$\frac{\lambda}{1-\alpha} E(X) > c, \tag{4.1}$$

then the ruin probability $\psi(u)$ can be approximated by the following asymptotic Lundberg-type formula:

$$\lim_{u \to +\infty} -\frac{\ln(\psi(u))}{u} = R, \tag{4.2}$$

where u is the initial surplus, and R is the Lundberg adjustment coefficient.

Proof. By Theorem 2.1 in Müller and Pflug [24], the result (4.2) holds if it is proved that the Lundberg adjustment coefficient R is the unique positive solution to the equation $c(r) = 0$.

To this end, first we have

$$c(0) = \lambda \frac{M_X(0) - 1}{1 - \alpha M_X(0)} + c \cdot 0 = 0. \tag{4.3}$$

Second, direct calculation yields

$$c'(r) = \frac{-\lambda(1-\alpha) M_X'(-r)}{[1 - \alpha M_X(-r)]^2} + c,$$

such that

$$c'(0) = \frac{-\lambda(1-\alpha) M_X'(0)}{[1 - \alpha M_X(0)]^2} + c = c - \frac{\lambda}{1-\alpha} E(X) < 0. \tag{4.4}$$

Next, it is obvious that $c(r)$ is a convex function because of the convexity of $c_t(r)$ and the fact $c(r) = \lim_{t \to +\infty} c_t(r)$.

Finally, note that

$$\lim_{r \to +\infty} c(r) = \lim_{r \to +\infty} \left(\lambda \frac{M_X(-r) - 1}{1 - \alpha M_X(-r)} + cr \right) = +\infty. \tag{4.5}$$

As a result, we conclude that the equation $c(r) = 0$ has a unique positive solution, and then (4.2) follows. □

Remark 4.1. *The assumption (4.1) is also called relative safety loading conditions in risk theory, which means that the expected gain should be greater than the expenses, so that the insurance company could run normally and realize its profits.*

Remark 4.2. *According to approximation formula (4.2), the ruin probability $\psi(u)$ can be asymptotically estimated by*

$$\psi(u) \simeq e^{-Ru}, \tag{4.6}$$

for large values of initial surplus u.

Since the thinning parameter α could be taken as a measure for the degree of the dependence for risk model (2.1), we discuss its impact on the adjustment coefficient at the end of this section.

Proposition 4.1. *As a function of the thinning parameter, the Lundberg adjustment coefficient R is increasing with respect to α.*

Proof. We rewrite $c(r)$ as $c(\alpha, r)$, and then R can be taken as a function of α that is determined by $c(\alpha, R) = 0$. From the properties derived in the proof of Theorem 4.1, it holds that

$$\frac{\partial c(\alpha, R)}{\partial R} > 0.$$

On the other hand, when $R > 0$, we have $0 \leq M_X(-R) < 1$. Then, taking the derivation of $c(\alpha, R)$ with respect to α, we get

$$\frac{\partial c(\alpha, R)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\lambda \frac{M_X(-R) - 1}{1 - \alpha M_X(-R)} + cR\right) = \frac{\lambda[M_X^2(-R) - M_X(-R)]}{[1 - \alpha M_X(-R)]^2} < 0. \tag{4.7}$$

Therefore, by the implicit function theorem, we obtain

$$\frac{\partial R}{\partial \alpha} = -\frac{(\partial/\partial\alpha)c(\alpha, R)}{(\partial/\partial R)c(\alpha, R)} > 0,$$

which implies that R is an increasing function with respect to α. □

Remark 4.3. *Proposition 4.1 shows that for an insurance portfolio, the degree of dangerousness measured by the adjustment coefficient decreases with the dependence parameter α. In our risk model, when α increases, it becomes that every Insured would prefer to renew his insurance contract in the next period, which lowers the risk of the portfolio. As a result, it is very important for insurance companies to keep their customers by various strategies.*

5. Numerical examples

In this section, we present some simulation studies to illustrate the main results obtained in Sections 3 and 4. To achieve this, the gain amount X is assumed to follow an exponential distribution with mean $1/\beta$, i.e., $X \sim Exp(\beta)$. Then, the moment generating function of X is given by

$$M_X(-r) = \frac{\beta}{\beta + r}, \quad r > 0.$$

Substituting the above expression into (3.10) leads to

$$c(r) = -\frac{\lambda r}{r + (1-\alpha)\beta} + cr. \tag{5.1}$$

Solving the equation $c(r) = 0$ gives

$$R = \frac{\lambda}{c} - (1-\alpha)\beta.$$

It is obvious that the Lundberg adjustment coefficient R is increasing with respect to the thinning parameter α.

Next, we consider some numerical results about two risk models with the following parameters.

(1) Risk model I: $\lambda = 1, \beta = 1$ and $c = 0.8$;

(2) Risk model II: $\lambda = 0.5, \beta = 0.4$ and $c = 1$.

Tables 1 and 2 give the computed values of Lundberg adjustment coefficient R and the approximated results of ruin probability e^{-Ru} corresponding to different α and u. It confirms that as α increases, R also increases, so that the insurance portfolio becomes less dangerous because the approximation of the ruin probability decreases. The same conclusion can be made for the initial surplus u.

Table 1. Lundberg adjustment coefficient and approximation for ruin probability of risk model I.

		e^{-Ru}				
α	R	$u = 1$	$u = 2$	$u = 3$	$u = 4$	$u = 5$
0.25	0.5000	0.6065	0.3679	0.2231	0.1353	0.0821
0.5	0.7500	0.4724	0.2231	0.1054	0.0498	0.0235
0.75	1.0000	0.3679	0.1353	0.0498	0.0183	0.0067
0.95	1.2000	0.3012	0.0907	0.0273	0.0082	0.0025

Table 2. Lundberg adjustment coefficient and approximation for ruin probability of risk model II.

		e^{-Ru}				
α	R	$u = 6$	$u = 7$	$u = 8$	$u = 9$	$u = 10$
0.25	0.2000	0.3012	0.2466	0.2019	0.1653	0.1353
0.5	0.3000	0.1653	0.1225	0.0907	0.0672	0.0498
0.75	0.4000	0.0907	0.0608	0.0408	0.0273	0.0183
0.95	0.4800	0.0561	0.0347	0.0215	0.0133	0.0082

To assess the performance of the approximation formula for ruin probability, we fix $\alpha = 0.5$ in the two risk models and calculate the true ruin probabilities via the Monte Carlo method. Specifically, we simulate the surplus process (2.1) by randomly drawing sample paths according to the Poisson INAR(1) process $\{N_t, t = 1, 2, \cdots\}$ for the gain arrivals and according to the given exponential distributions for the gain amount $X_{t,k}$, with each sample path starting at $U_0 = u$. We replicate these simulations $N = 10000$ times, count the trajectories that lead to ruin and divide this number by the total number N of simulated trajectories. Then, the estimator for ruin probability $\psi(u)$ can be obtained by

$$\hat{\psi}(u) = \frac{1}{N} \sum_{i=1}^{N} I_A(W_i), \tag{5.2}$$

where W_i is the ith trajectory, A represents the set of all trajectories that lead to ruin, and $I_A(\cdot)$ denotes the indicator function. As stated in Albrecher and Kantor [25], it would frequently happen that $U_t \to +\infty$ as $t \to +\infty$, without U_t ever becoming negative because of (4.1); and it is therefore necessary for us to choose a suitably large T_{st}, at which the simulated process should be stopped. Consequently, (5.2) is actually the estimator for the finite-time ruin probability $\psi(u, T_{st}) = P(T \leq T_{st} | U_0 = u)$. Here, we set $T_{st} = 1000$, such that the bias in the estimate of $\psi(u)$ becomes negligible.

In Tables 3 and 4, we compare the simulated values with the approximated values of ruin probability $\psi(u)$. It can be seen that the approximation method does not work well for small values of u. There are mainly two reasons: (1) e^{-Ru} is the limit of $\psi(u)$ as $u \to +\infty$, so the values of e^{-Ru} and $\psi(u)$ may be very different at small values of u; (2) we actually use the simulated values for finite-time ruin probability $\psi(u, T_{st})$ to estimate the ultimate ruin probability $\psi(u)$, and there are many factors that can affect the results, such as the total number of simulated trajectories N, the chosen time T_{st} at which the simulated surplus process should be stopped, etc. However, it should be noted that the ratio $\frac{\hat{\psi}(u)}{e^{-Ru}}$ would approach 1 as $u \to +\infty$, which implies that the approximated ruin probabilities become closer and closer to the true ruin probabilities, indicating the asymptotic validity of the approximated result e^{-Ru} for $\psi(u)$. Figures 1 and 2 also show this trend visually.

Table 3. Comparison of the simulated and approximated $\psi(u)$ of risk model I.

u	$\hat{\psi}(u)$	e^{-Ru}	$\frac{\hat{\psi}(u)}{e^{-Ru}}$
4	0.0230	0.0498	0.4618
5	0.0130	0.0235	0.5532
6	0.0058	0.0111	0.5225
7	0.0033	0.0052	0.6346
8	0.0020	0.0025	0.8000
9	0.0009	0.0012	0.7500
10	0.0005	0.0006	0.8333

Table 4. Comparison of the simulated and approximated $\psi(u)$ of risk model II.

u	$\hat{\psi}(u)$	e^{-Ru}	$\frac{\hat{\psi}(u)}{e^{-Ru}}$
9	0.0503	0.0672	0.7485
10	0.0347	0.0498	0.6968
11	0.0261	0.0369	0.7073
12	0.0195	0.0273	0.7142
13	0.0147	0.0202	0.7277
14	0.0113	0.0150	0.7533
15	0.0089	0.0111	0.8018

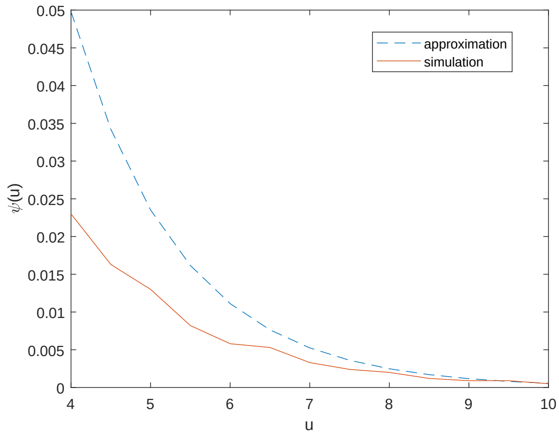

Figure 1. The simulated values and approximated values of $\psi(u)$ for risk model I.

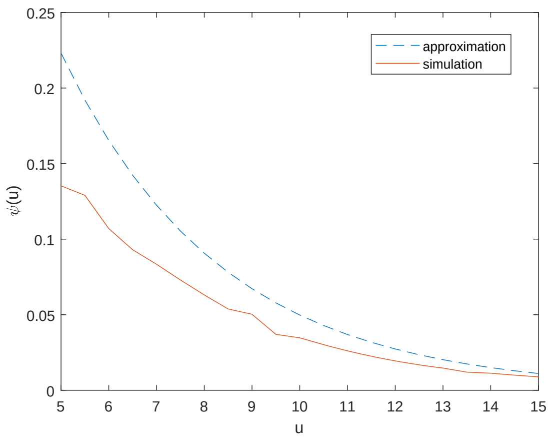

Figure 2. The simulated values and approximated values of $\psi(u)$ for risk model II.

6. Conclusions

In this paper, we utilize a Poisson INAR(1) process to extend the classical discrete-time dual risk model by introducing a temporal dependence between the gain numbers of each period. We derive the explicit expression for a function that allows one to find the Lundberg adjustment coefficient and obtain the Lundberg approximation formula for ruin probability. We also discuss the impact of the thinning parameter on the risk of an insurance portfolio, showing that the dependence structure has significant influence and cannot be ignored in practice. As for future work, we could take the INAR(1) process with other distributed innovations (e.g., compound Poisson distribution studied in [26], zero-and-one inflated Poisson distribution considered in [27], Poisson-BE2 distribution discussed in [28] and so on) into consideration, or we could use a higher-order INAR process to make the risk model more flexible. The main difficulties that we will face when dealing with such kinds of risk models possibly are how to derive the explicit expressions for $P_{N_1+\cdots+N_t}(s)$ and $c(r)$, and proving the expected properties of the solution to the equation $c(r) = 0$. Other potential issues include investigating the Lundberg inequalities for ruin probability and studying some other important ruin quantities, such as the Gerber-Shiu expected discounted penalty function, the duration of ruin, etc. More importantly, applications of our model in practice shall also be focused on. For example, we can price various insurance products that include green building insurance, global weather insurance, renewable energy insurance and green car insurance more reasonably to promote insurance companies' environmental risk management performance ([29, 30]).

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Conflict of interest

The authors declare that they have no competing interests.

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