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Research article

Least energy sign-changing solution for a fractional p-Laplacian problem with exponential critical growth

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Abstract: In this paper, we consider the following fractional p-Laplacian equation involving Trudinger-Moser nonlinearity:

$$(-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u = f(u) \text{ in } \mathbb{R}^N,$$

where $s \in (0,1), 2 < \frac{N}{s} = p$. The nonlinear function f has exponential critical growth, and potential V is a continuous function. By using the constrained variational methods, quantitative Deformation Lemma and Brouwer degree theory, we prove the existence of least energy sign-changing solutions.

Keywords: sign-changing solution; fractional p-Laplacian; exponential critical growth
Mathematics Subject Classification: 35J60, 35J92, 35R11

1. Introduction and main results

In this paper, we investigate the existence of a least energy sign-changing solution for the following fractional p-Laplacian problem:

$$(-\Delta)^s_{N/s}u + V(x)|u|^{\frac{N}{s}-2}u = f(u) \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $2 < \frac{N}{s} := p$, $(-\Delta)^s_p$ is the fractional p-Laplacian operator which, up to a normalizing constant, may be defined by setting

$$(-\Delta)^s_p u(x) = 2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy,\ x\in\mathbb{R}^N$$

along functions $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$. In addition, the potential $V \in C(\mathbb{R}^N, \mathbb{R})$, the nonlinear f has exponential critical growth, and such nonlinear behavior is motivated by the Trudinger-Moser inequality (Lemma 2.2).

Recently, the study of nonlocal problems driven by fractional operators has piqued the mathematical community's interest, both because of their intriguing theoretical structure and due to concrete applications such as obstacle problems, optimization, finance, phase transition, and so on. We refer to [18] for more details. In fact, when $p = 2$, problem (1.1) appears in the study of standing wave solutions, i.e., solutions of the form $\psi(x,t) = u(x)e^{-i\omega t}$, to the following fractional Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \hbar^2(-\Delta)^s\psi + W(x)\psi - f(|\psi|) \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

where $\hbar$ is the Planck constant, $W : \mathbb{R}^N \to \mathbb{R}$ is an external potential, and f is a suitable nonlinearity. Laskin [25, 26] first introduced the fractional Schrödinger equation due to its fundamental importance in the study of particles on stochastic fields modeled by Lévy processes.

After that, remarkable attention has been devoted to the study of fractional Schrödinger equations, and a lot of interesting results were obtained. For the existence, multiplicity and behavior of standing wave solutions to problem (1.2), we refer to [1, 9, 10, 14, 19, 33] and the references therein.

For general p with $2 < p < \frac{N}{s}$, problem (1.1) becomes the following fractional Laplacian problem:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

We emphasize that the fractional p-Laplacian is appealing because it contains two phenomena: the operator's nonlinearity and its nonlocal character. In fact, for the fractional p-Laplacian operator $(-\Delta)_p^s$ with $p \neq 2$, one cannot obtain a similar equivalent definition of $(-\Delta)_p^s$ by the harmonic extension method in [10]. For those reasons, the study of (1.3) becomes attractive. In [13], the authors obtained infinitely many sign-changing solutions of (1.3) by using the invariant sets of descent flow. Moreover, they also proved (1.3) possesses a least energy sign-changing solution via deformation Lemma and Brouwer degree. We stress that, by using a similar method, Wang and Zhou [35], Ambrosio and Isernia [3] obtained the least energy sign-changing solutions of (1.3) with $p = 2$. For more results involving the fractional p-Laplacian, we refer to [2, 17, 20, 21, 32] and the references therein.

Another motivation to investigate problem (1.1) comes from the fractional Schrödinger equations involving exponential critical growth. Indeed, we shall study the case where the nonlinearity $f(t)$ has the maximum growth that allows us to treat problem (1.1) variationally in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$ (see the definition in (1.5)). If $p < \frac{N}{s}$, Sobolev embedding (Theorem 6.9 in [18]) states $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$, where $p_s^* = \frac{Np}{N-sp}$, and p_s^* is called the critical Sobolev exponent. Moreover, the same result ensures that $W^{s,\frac{N}{s}}(\mathbb{R}^N) \hookrightarrow L^\lambda(\mathbb{R}^N)$ for any $\lambda \in [\frac{N}{s}, +\infty)$. However, $W^{s,\frac{N}{s}}(\mathbb{R}^N)$ is not continuously embedded in $L^\infty(\mathbb{R}^N)$ (for more details, we refer to [18]). On the other hand, in the case $p = \frac{N}{s}$, the maximum growth that allows us to treat problem (1.1) variationally in Sobolev space $W^{s,\frac{N}{s}}(\mathbb{R}^N)$, which is motivated by the fractional Trudinger-Moser inequality proved by Ozawa [30] and improved by Kozono et al. [23] (see Lemma 2.2). More precisely, inspired by [23], we say that $f(t)$ has exponential critical growth if there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{f(t)}{\exp(\alpha|t|^{\frac{N}{N-s}})} = \begin{cases} 0, & \text{for } \alpha > \alpha_0, \\ +\infty, & \text{for } \alpha < \alpha_0. \end{cases}$$

On the basis of this notation of critical, many authors pay their attention to investigating elliptic problems involving the fractional Laplacian operator and nonlinearities with exponential growth. When $N = 1, s = \frac{1}{2}$, $p = 2$ and replacing $\mathbb{R}$ by (a, b), problem (1.1) becomes the following fractional Laplacian equation:

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = f(u) \text{ in } (a, b), \\ u = 0 \text{ in } \mathbb{R} \backslash (a, b). \end{cases} \tag{1.4}$$

When f is $o(|t|)$ at the origin and behaves like $\exp(\alpha t^2)$ as $|t| \to \infty$, by virtue of the Mountain Pass theorem, Iannizzotto and Squassina [22] proved the existence and multiplicity of solutions for (1.4). Utilizing the constrained variational methods and quantitative deformation lemma, Souza et al. [16] considered the least energy sign-changing solution of problem (1.4) involving exponential critical growth.

For problem (1.1) with exponential critical growth nonlinearity f, we would like to mention references [8, 27, 34]. In [27], by applying variational methods together with a suitable Trudinger-Moser inequality for fractional space, the authors obtained at least two positive solutions of (1.1). In [8], the authors considered problem (1.1) with a Choquard logarithmic term and exponential critical growth nonlinearity f. They proved the existence of infinite many solutions via genus theory. In [34], by using Ljusternik-Schnirelmann theory, Thin obtained the existence, multiplicity and concentration of nontrivial nonnegative solutions for problem (1.1). For more recent results on fractional equations involving exponential critical growth, see [8, 15, 16, 22, 27, 31, 34] and the references therein. We also refer the interested readers to [11, 29] for general problems with Trudinger-Moser-type behavior.

Inspired by the works mentioned above, it is natural to ask whether problem (1.1) has sign-changing solutions when the nonlinearity f has exponential critical growth. To our knowledge, there are few works on it except [16]. Souza et al. [16] considered the case when $N = 1, p = 2$ and $s = \frac{1}{2}$. However, compared with [16], the situation when $p > 2$ is quite different. In particular, the decomposition of functional I (see the definition in (1.8)) is more difficult than that in [16]. Therefore, some difficulties arise in studying the existence of a least energy sign-changing solution for problem (1.1), and this makes the study interesting.

In order to study problem (1.1), from now on, we fix $p = \frac{N}{s}$, $p' = \frac{p}{p-1} = \frac{N}{N-s}$ and consider the following assumptions on V and f :

(V) $V(x) \in C\left(\mathbb{R}^N\right)$ and there exists $V_0 > 0$ such that $V(x) \geq V_0$ in $\mathbb{R}^N$. Moreover, $\lim_{|x|\to\infty} V(x) = +\infty$;

(f_1) The function $f \in C^1(\mathbb{R})$ with exponential critical growth at infinity, that is, there exists a constant $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{\exp(\alpha|t|^{p'})} = 0 \ \text{ for } \alpha > \alpha_0 \quad \text{ and } \quad \lim_{|t|\to\infty} \frac{|f(t)|}{\exp(\alpha|t|^{p'})} = \infty \ \text{ for } \alpha < \alpha_0;$$

(f_2) $\displaystyle\lim_{t\to 0} \frac{|f(t)|}{|t|^{p-1}} = 0$;

(f_3) There exists $\theta > p$ such that

$$0 < \theta F(t) \leq t f(t) \text{ for } t \in \mathbb{R} \backslash \{0\},$$

where $F(t) = \int_0^t f(s) ds$;

(f_4) There are two constants $q > p$ and $\gamma > 1$ such that

$$\operatorname{sgn}(t)f(t) \geq \gamma|t|^{q-1} \text{ for } t \in \mathbb{R};$$

(f_5) $\dfrac{f(t)}{|t|^{p-1}}$ is an increasing function of $t \in \mathbb{R}\backslash\{0\}$.

Before stating our results, we recall some useful notations. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, where $p = \frac{N}{s}$, is defined by

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\right\}, \tag{1.5}$$

where $[u]_{s,p}$ denotes Gagliardo seminorm, that is,

$$[u]_{s,p} := \left(\int\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy\right)^{\frac{1}{p}}.$$

$W^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space (see [18]) with norm

$$\|u\|_{W^{s,p}} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p}\right)^{\frac{1}{p}}.$$

Let us consider the work space

$$X := \left\{u \in W^{s,p}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} V(x)|u|^p dx < +\infty\right\}, \tag{1.6}$$

with the norm

$$\|u\|_X := \left([u]^p_{s,p} + \int\limits_{\mathbb{R}^N} V(x)|u|^p dx\right)^{\frac{1}{p}}.$$

X is a uniformly convex Banach space, and thus X is a reflexive space. By the condition (V), we have that the embedding from X into $W^{s,p}(\mathbb{R}^N)$ is continuous.

Definition 1.1. *We say that $u \in X$ is a weak solution of problem* (1.1) *if*

$$\int\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dxdy + \int\limits_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x)dx$$
$$= \int\limits_{\mathbb{R}^N} f(u(x))\varphi(x)dx \tag{1.7}$$

for all $\varphi \in X$.

Define the energy functional $I : X \to \mathbb{R}$ associated with problem (1.1) by

$$I(u) = \frac{1}{p}\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(x)|u|^p dx - \int\limits_{\mathbb{R}^N} F(u)dx. \tag{1.8}$$

By the Trudinger-Moser type inequality in [23] (see Lemma 2.2), we prove that $I(u) \in C^1(X,\mathbb{R})$, and the critical point of functional I is a weak solution of problem (1.1) (see Remark 2.2 in Section 2).

For convenience, we consider the operator $A : X \to X^*$ given by

$$\langle A(u), v\rangle_{X^*,X} = \int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy + \int\limits_{\mathbb{R}^N} V(x)|u|^{p-2}uvdx,\ \forall u, v \in X, \tag{1.9}$$

where X^* is the dual space of X. In the sequel, for simplicity, we denote $\langle\cdot,\cdot\rangle_{X^*,X}$ by $\langle\cdot,\cdot\rangle$. Moreover, we denote the Nehari set $\mathcal{N}$ associated with I by

$$\mathcal{N} = \{u \in X\backslash\{0\} : \langle I'(u), u\rangle = 0\}. \tag{1.10}$$

Clearly, $\mathcal{N}$ contains all the nontrivial solutions of (1.1). Define $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \min\{u(x), 0\}$, and then the sign-changing solution of (1.1) stays on the following set:

$$\mathcal{M} = \{u \in X\backslash\{0\} : u^\pm \neq 0,\ \langle I'(u), u^+\rangle = 0,\ \langle I'(u), u^-\rangle = 0\}. \tag{1.11}$$

Set

$$c := \inf_{u\in\mathcal{N}} I(u), \tag{1.12}$$

and

$$m := \inf_{u\in\mathcal{M}} I(u). \tag{1.13}$$

Now, we can state our main results.

Theorem 1.1. *Assume that* (V) *and* (f_1) *to* (f_5) *hold, and then problem* (1.1) *possesses a least energy sign-changing solution* $u^* \in X$*, provided that*

$$\gamma > \gamma^* := \left[\frac{\theta p m_q}{\theta - p}\left(\frac{\alpha_0}{\alpha^*}\right)^{p-1}\right]^{\frac{q-p}{p}} > 0,$$

where α^* *is a constant defined in Lemma 2.2,*

$$m_q = \inf_{u\in\mathcal{M}_q} I_q(u),\ \mathcal{M}_q = \left\{u \in X : u^\pm \neq 0, \left\langle I_q'(u), u^\pm\right\rangle = 0\right\},$$

and

$$I_q(u) = \frac{1}{p}\|u\|_X^p - \frac{1}{q}\int\limits_{\mathbb{R}^N} |u|^q dx.$$

The proof of Theorem 1.1 is based on the arguments presented in [7]. We first check that the minimum of functional I restricted on set $\mathcal{M}$ can be achieved. Then, by using a suitable variant of the quantitative deformation Lemma, we show that it is a critical point of I. However, due to the nonlocal term

$$\frac{1}{p}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy,$$

the functional I no longer satisfies the decompositions

$$I(u) = I(u^+) + I(u^-) \quad \text{and} \quad \langle I'(u), u^\pm\rangle = \langle I'(u^\pm), u^\pm\rangle, \tag{1.14}$$

which are very useful to get sign-changing solutions of problem (1.1) (see for instance [4–7, 12]). In fact, due to the fractional p-Laplacian operator $(-\Delta)^s_p$ with $s \in (0,1)$ and $p \neq 2$, one cannot obtain a similar equivalent definition of $(-\Delta)^s_p$ by the harmonic extension method (see [10]). In addition, compared with [16], problem (1.1) contains nonlocal operator $(-\Delta)^s_p$ with $p > 2$, which brings some difficulties while studying problem (1.1). In particular, for problem (1.4), one has the following decomposition:

$$\begin{aligned} J(u) &= J(u^+) + J(u^-) + \int_{\mathbb{R}^{2N}} \frac{-u^+(x)u^-(y) - u^-(x)u^+(y)}{|x-y|^{N+2s}}dxdy, \\ \langle J'(u), u^+\rangle &= \langle J'(u^+), u^+\rangle + \int_{\mathbb{R}^{2N}} \frac{-u^+(x)u^-(y) - u^-(x)u^+(y)}{|x-y|^{N+2s}}dxdy, \end{aligned} \tag{1.15}$$

where J is the energy functional of (1.4). However, for the energy functional I, it is impossible to obtain a similar decomposition like (1.15) due to the nonlinearity of the operator $(-\Delta)^s_p$. In order to overcome this difficulty, we divide $\mathbb{R}^{2N}$ into several regions (see Lemma 2.6) and decompose functional I on each region carefully. Furthermore, another difficulty arises in verifying the compactness of the minimizing sequence in X since problem (1.1) involves the exponential critical nonlinearity term. Fortunately, thanks to the Trudinger-Moser inequality in [23], we overcome this difficulty by choosing γ in assumption (f_4) appropriately large to ensure the compactness of the minimizing sequence. To achieve this, a more meticulous calculation is needed in estimating m.

On the other hand, by a similar argument in [13], we also consider the energy behavior of $I(u^*)$ in the following theorem, where u^* is the least energy sign-changing solution obtained in Theorem 1.1.

Theorem 1.2. *Assume that* (V) *and* (f_1) *to* (f_4) *are satisfied. Then,* $c > 0$ *is achieved, and*

$$I(u^*) > 2c,$$

where u^* *is the least energy sign-changing solution obtained in Theorem 1.1.*

The paper is organized as follows: Section 2 contains the variational setting of problem (1.1), and it establishes a version of the Trudinger-Moser inequality for (1.1). In Section 3, we give some technical lemmas which will be crucial in proving the main result. Finally, in Section 4, we combine the minimizing arguments with a variant of the Deformation Lemma and Brouwer degree theory to prove the main result.

Throughout this paper, we will use the following notations: $L^\lambda(\mathbb{R}^N)$ denotes the usual Lebesgue space with norm $|\cdot|_\lambda$; $C, C_1, C_2, \cdots$ will denote different positive constants whose exact values are not essential to the exposition of arguments.

2. Preliminaries

In this section, we outline the variational framework for problem (1.1) and give some preliminary Lemmas. For convenience, we assume that $V_0 = 1$ throughout this paper. Recalling the definition of fractional Sobolev space X in (1.6), we have the following compactness results.

Lemma 2.1. *Suppose that* (V) *holds. Then, for all* $\lambda \in [p, \infty)$*, the embedding* $X \hookrightarrow L^{\lambda}(\mathbb{R}^N)$ *is compact.*

Proof. Define $Y = L^{\lambda}\left(\mathbb{R}^N\right)$ and $B_R = \left\{x \in \mathbb{R}^N : |x| < R\right\}, B_R^c = \mathbb{R}^N \backslash \overline{B_R}$. Let $X(\Omega)$ and $Y(\Omega)$ be the spaces of functions $u \in X, u \in Y$ restricted onto $\Omega \subset \mathbb{R}^N$ respectively. Then, it follows from Theorems 6.9, 6.10 and 7.1 in [18] that $X(B_R) \hookrightarrow Y(B_R)$ is compact for any $R > 0$. Define $V_R = \inf\limits_{x \in B_R^c} V(x)$. By (V), we deduce that $V_R \to \infty$ as $R \to \infty$. Therefore, when $\lambda = \frac{N}{s} = p$, we have

$$\int_{B_R^c} |u|^p dx \leq \frac{1}{V_R} \int_{B_R^c} V(x)|u|^p dx \leq \frac{1}{V_R} \|u\|_X^p,$$

which implies

$$\lim_{R \to +\infty} \sup_{u \in X \backslash \{0\}} \frac{\|u\|_{L^p(B_R^c)}}{\|u\|_X} = 0.$$

By virtue of Theorem 7.9 in [24], we can see that $X \hookrightarrow L^p(\mathbb{R}^N)$ is compact.

When $\lambda > p$, by the interpolation inequality, one can also obtain that $X \hookrightarrow L^{\lambda}(\mathbb{R}^N)$ is compact. This completes the proof. □

To study problems involving exponential critical growth in the fractional Sobolev space, the main tool is the following fractional Trudinger-Moser inequality, and its proof can be found in Zhang [37]. First, to make the notation concise, we set, for $\alpha > 0$ and $t \in \mathbb{R}$,

$$\Phi(\alpha, t) = \exp(\alpha |t|^{p'}) - S_{k_p - 2}(\alpha, t), \tag{2.1}$$

where $S_{k_p-2}(\alpha, t) = \sum_{k=0}^{k_p-2} \frac{\alpha^k}{k!} |t|^{p'k}$ with $k_p = \min\{k \in \mathbb{N}; k \geq p\}$.

Lemma 2.2. *Let* $s \in (0, 1)$ *and sp=N. Then, there exists a positive constant* $\alpha^* > 0$ *such that*

$$\int\limits_{\mathbb{R}^N} \Phi(\alpha, u) dx < +\infty, \ \forall \alpha \in (0, \alpha^*), \tag{2.2}$$

for all $u \in W^{s,p}(\mathbb{R}^N)$ *with* $\|u\|_{W^{s,p}(\mathbb{R}^N)} \leq 1$.

For $\Phi(\alpha, u)$, we also have the following properties, which have been proved in [8, 27]. For the reader's convenience, we sketch the proof here.

Lemma 2.3. *[27] Let* $\alpha > 0$ *and* $r > 1$*. Then, for every* $\beta > r$*, there exists a constant* $C = C(\beta) > 0$ *such that*

$$\left(\exp(\alpha |t|^{p'}) - S_{k_p-2}(\alpha, t)\right)^r \leq C(\beta) \left(\exp(\beta \alpha |t|^{p'}) - S_{k_p-2}(\beta \alpha, t)\right).$$

Proof. Noticing that

$$\frac{\left(\exp(\alpha|t|^{p'}) - S_{k_p-2}(\alpha,t)\right)^r}{\exp(\beta\alpha|t|^{p'}) - S_{k_p-2}(\beta\alpha,t)} = \frac{\left(\displaystyle\sum_{j=k_p-1}^{\infty}\frac{\alpha^j|t|^{jp'}}{j!}\right)^r}{\displaystyle\sum_{j=k_p-1}^{\infty}\frac{(\beta\alpha)^j|t|^{jp'}}{j!}} = \frac{|t|^{r(p-1)p'}\left(\displaystyle\sum_{j=k_p-1}^{\infty}\frac{\alpha^j|t|^{(j-p+1)p'}}{j!}\right)^r}{|t|^{(p-1)p'}\displaystyle\sum_{j=k_p-1}^{\infty}\frac{(\beta\alpha)^j|t|^{(j-p+1)p'}}{j!}},$$

we deduce

$$\lim_{t\to 0}\frac{\left(\exp(\alpha|t|^{p'}) - S_{k_p-2}(\alpha,t)\right)^r}{\exp(\beta\alpha|t|^{p'}) - S_{k_p-2}(\beta\alpha,t)} = 0.$$

On the other hand, it holds that

$$\lim_{|t|\to\infty}\frac{\left(\exp(\alpha|t|^{p'}) - S_{p-2}(\alpha,t)\right)^r}{\exp(\beta\alpha|t|^{p'}) - S_{p-2}(\beta\alpha,t)} = \lim_{|t|\to\infty}\frac{\exp(\alpha r|t|^{p'})\left(1-\frac{S_{p-2}(\alpha,t)}{\left(\exp(\alpha|t|)\right)^{p'}}\right)^r}{\exp(\beta\alpha|t|^{p'})\left(1-\frac{S_{p-2}(\beta\alpha,t)}{\exp(\beta\alpha|t|^{p'})}\right)} = 0, \tag{2.3}$$

and the lemma follows. □

Lemma 2.4. *[8] Let $\alpha > 0$. Then, $\Phi(\alpha,u) \in L^1\left(\mathbb{R}^N\right)$ for all $u \in X$.*

Proof. Let $u \in X\backslash\{0\}$ and $\varepsilon > 0$. Since $C_0^\infty\left(\mathbb{R}^N\right)$ is dense in X, there exists $\phi \in C_0^\infty\left(\mathbb{R}^N\right)$ such that $0 < \|u-\phi\|_X < \varepsilon$. Observe that for each $k \geq k_p - 1$,

$$|u|^{p'k} \leq 2^{p'k}\varepsilon^{p'k}\left|\frac{u-\phi}{\|u-\phi\|_X}\right|^{p'k} + 2^{p'k}|\phi|^{p'k}.$$

Consequently,

$$\Phi(\alpha,u) \leq \Phi\left(\alpha 2^{p'}\varepsilon^{p'}, \left|\frac{u-\phi}{\|u-\phi\|_X}\right|^{p'}\right) + \Phi\left(\alpha 2^{p'}, |\phi|^{p'}\right).$$

From Lemma 2.2, choosing $\varepsilon > 0$ sufficiently small such that $\alpha 2^{p'}\varepsilon^{p'} < \alpha^*$, we have

$$\int_{\mathbb{R}^N}\Phi\left(\alpha 2^{p'}\varepsilon^{p'}, \left|\frac{u-\phi}{\|u-\phi\|_X}\right|^{p'}\right)dx < +\infty.$$

On the other side, since $\exp\left(\alpha 2^{p'}|\phi|^{p'}\right) = \sum_{k=0}^{+\infty}\frac{\alpha^k}{k!}2^{p'k}|\phi|^{p'k}$, there exists $k_0 \in \mathbb{N}$ such that $\sum_{k=k_0}^{+\infty}\frac{\alpha^k}{k!}2^{p'k}|\phi|^{p'k} < \varepsilon$. This fact, combined with the fact that $p'k > 0$ for all $k_p - 1 \leq k \leq k_0$, gives us

$$\int_{\mathbb{R}^N}\Phi\left(\alpha 2^{p'}, |\phi|^{p'}\right)dx = \int_{\mathrm{supp}\phi}\Phi\left(\alpha 2^{p'}, |\phi|^{p'}\right)dx < +\infty.$$

Therefore, $\Phi(\alpha,u) \in L^1\left(\mathbb{R}^N\right)$ for all $u \in X$. □

Remark 2.1. *From Lemmas 2.2–2.4, we deduce that $\Phi(\alpha,u)^l \in L^1(\mathbb{R}^N)$ for all $u \in X$, $\alpha > 0$ and $l \geq 1$.*

The next lemma shows the growth behavior of the nonlinearity f.

Lemma 2.5. *Given $\varepsilon > 0$, $\alpha > \alpha_0$ and $\zeta > p$, for all $u \in X$, it holds that*

$$|f(u)| \leq \varepsilon |u|^{p-1} + C_1(\varepsilon)|u|^{\zeta-1}\Phi(\alpha, u),$$

and

$$|F(u)| \leq \frac{\varepsilon}{p}|u|^p + C_2(\varepsilon)|u|^{\zeta}\Phi(\alpha, u).$$

Proof. We will only prove the first result. Since the second inequality is a direct consequence of the first one due to assumption (f_3), we omit it here.

In fact, by (f_2), for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(u)| \leq \varepsilon|u|^{p-1}$ for all $|u| < \delta$. Now, as $\zeta > p$, there exists $r > 0$ such that $\zeta = p + r$. Hence, once $z^{\zeta-1}, z^{q'k}$ for all $k \geq k_p - 1$ and z^r are increasing functions, if $|u| > \delta$, it follows from (f_1) that

$$|f(u)| \leq \varepsilon|u|^{p-1} + C_\varepsilon \Phi(\alpha, u) \leq \varepsilon|u|^{p-1}\frac{|u|^r\Phi(\alpha, u)}{\delta^r\Phi(\alpha, \delta)} + C_\varepsilon\frac{|u|^{\zeta-1}}{\delta^{\zeta-1}}\Phi(\alpha, u) = C_1(\varepsilon)|u|^{\zeta-1}\Phi(\alpha, u).$$

This completes the proof. □

Remark 2.2. *It follows from Lemmas 2.1–2.3 that I is well-defined on X. Moreover, $I \in C^1(X, \mathbb{R}^N)$, and*

$$\langle I'(u), v\rangle = \int\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}}dxdy + \int\limits_{\mathbb{R}^N} V(x)|u|^{p-2}uvdx - \int\limits_{\mathbb{R}^N} f(u)vdx$$

for all $v \in X$. Consequently, the critical point of I is the weak solution of problem (1.1).

We seek the sign-changing solution of problem (1.1). As we saw in Section 1, one of the difficulties is the fact that the functional I does not possess a decomposition like (1.14). Inspired by [13, 35], we have the following:

Lemma 2.6. *Let $u \in X$ with $u^\pm \neq 0$. Then,*

(i) $I(u) > I(u^+) + I(u^-)$,
(ii) $\langle I'(u), u^\pm\rangle > \langle I'(u^\pm), u^\pm\rangle$.

Proof. Observe that

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_X^p - \int\limits_{\mathbb{R}^N} F(u)dx = \frac{1}{p}\langle A(u), u\rangle - \int\limits_{\mathbb{R}^N} F(u)dx \\ &= \frac{1}{p}\langle A(u), u^+\rangle - \int\limits_{\mathbb{R}^N} F(u^+)dx + \frac{1}{p}\langle A(u), u^-\rangle - \int\limits_{\mathbb{R}^N} F(u^-)dx. \end{aligned} \tag{2.4}$$

By density (see Di Nezza et al. Theorem 2.4 [18]), we can assume that u is continuous. Define

$$(\mathbb{R}^N)_+ = \{x \in \mathbb{R}^N; u^+(x) \geq 0\} \text{ and } (\mathbb{R}^N)_- = \{x \in \mathbb{R}^N; u^-(x) \leq 0\}.$$

Then, for $u \in X$ with $u^{\pm} \neq 0$, by a straightforward computation, one can see that

$$
\begin{aligned}
\langle A(u), u^+ \rangle &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^+(x)-u^+(y))}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u^+|^p dx \\
&= \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_+} \frac{|u^+(x)-u^+(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^+(x)-u^-(y)|^{p-1}u^+(x)}{|x-y|^{N+ps}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x)-u^+(y)|^{p-1}u^+(y)}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u^+|^p dx \\
&= \langle A(u^+), u^+ \rangle + \int_{(\mathbb{R}^N)_+ \times (\mathbb{R}^N)_-} \frac{|u^+(x)-u^-(y)|^{p-1}u^+(x) - |u^+(x)|^p}{|x-y|^{N+ps}} dxdy \\
&\quad + \int_{(\mathbb{R}^N)_- \times (\mathbb{R}^N)_+} \frac{|u^-(x)-u^+(y)|^{p-1}u^+(y) - |u^+(y)|^p}{|x-y|^{N+ps}} dxdy > \langle A(u^+), u^+ \rangle .
\end{aligned} \tag{2.5}
$$

Similarly, we also have

$$\langle A(u), u^- \rangle > \langle A(u^-), u^- \rangle . \tag{2.6}$$

Taking into account (2.4)–(2.6), we deduce that $I(u) > I(u^+) + I(u^-)$. Analogously, one can prove (ii). □

In the last part of this section, we prove the following inequality, which will play an important role in estimating the upper bound for $m := \inf_{u \in \mathcal{M}} I(u)$.

Lemma 2.7. *For all $u \in X$ with $u^{\pm} \neq 0$ and constants $\sigma, \tau > 0$, it holds that*

$$\left\| \sigma u^+ + \tau u^- \right\|_X^p \leq \sigma^p \langle A(u), u^+ \rangle + \tau^p \langle A(u), u^- \rangle . \tag{2.7}$$

Furthermore, the inequality in (2.7) *is an equality if and only if $\sigma = \tau$.*

Proof. First, we claim the following elementary inequality holds true:

$$(\sigma a + \tau b)^p \leq (a+b)^{p-1}(\sigma^p a + \tau^p b), \tag{2.8}$$

where $a, b \geq 0$, $\sigma, \tau > 0$ and $p = \frac{N}{s} > 2$.

Indeed, if $a = 0$ or $b = 0$, one can easily check (2.8). Thus, we can assume that $a, b > 0$. Setting $\kappa = \frac{a}{a+b}$ and $t = \frac{\sigma}{\tau}$, (2.8) becomes

$$(\kappa t + (1-\kappa))^p \leq \kappa t^p + (1-\kappa). \tag{2.9}$$

Let us define $g(t) := \kappa t^p + (1-\kappa) - (\kappa t + (1-\kappa))^p$, and then $g'(t) = \kappa p \left[t^{p-1} - (\kappa t + (1-\kappa))^{p-1} \right]$. Noting that $0 < \kappa < 1$, we can observe that $g'(1) = 0$, $g'(t) < 0$ for $0 < t < 1$, and $g'(t) > 0$ for $t > 1$. Therefore, $g(t) > g(1) = 0$ for all $t > 0$ and $t \neq 1$, which implies (2.9). Consequently, (2.8) holds true.

Now, let us consider the inequality (2.7). By a straightforward computation, one can see from (2.8) that

$$\left\| \sigma u^+ + \tau u^- \right\|_X^p - \sigma^p \langle A(u), u^+ \rangle - \tau^p \langle A(u), u^- \rangle$$

$$= \int\limits_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma u^+(x)-\tau u^-(y)|^p - |u^+(x)-u^-(y)|^{p-1}(\sigma^p u^+(x)-\tau^p u^-(y))}{|x-y|^{N+ps}}dxdy$$
$$+ \int\limits_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau u^-(x)-\sigma u^+(y)|^p - |u^-(x)-u^+(y)|^{p-1}(\tau^p(-u^-(x))+\sigma^p u^+(y))}{|x-y|^{N+ps}}dxdy \le 0,$$

which implies (2.7), and Lemma 2.7 follows. □

3. Some technical lemmas

The aim of this section is to prove some technical lemmas related to the existence of a least energy nodal solution. Firstly, we collect some preliminary lemmas which will be fundamental to prove our main result.

Lemma 3.1. *Under the assumptions of Theorem 1.1, we have*
(i) For all $u \in \mathcal{N}$ such that $\|u\|_X \to \infty$, $I(u) \to \infty$;
(ii) There exist $\rho, \mu > 0$ such that $\|u^\pm\|_X \ge \rho$ for all $u \in \mathcal{M}$ and $\|u\|_X > \mu$ for all $u \in \mathcal{N}$.

Proof. (i) Since $u \in \mathcal{N}$ and (f_3) holds, we see that

$$I(u) = I(u) - \frac{1}{\theta}\langle I'(u), u\rangle \ge \left(\frac{1}{p}-\frac{1}{\theta}\right)\|u\|_X^p. \tag{3.1}$$

Hence, the above inequality ensures that $I(u) \to \infty$ as $\|u\|_X \to \infty$.

(ii) We claim that there exists $\mu > 0$, such that $\|u\|_X > \mu$ for all $u \in \mathcal{N}$. By contradiction, we suppose that there exists a sequence $\{u_n\} \subset \mathcal{N}$ such that $\|u_n\|_X \to 0$ in X.

Then, it follows from Lemmas 2.1, 2.3, 2.5 and Hölder's inequality that

$$\begin{aligned}\|u_n\|_X^p &= \int\limits_{\mathbb{R}^N} f(u_n)u_n dx \le \varepsilon \int\limits_{\mathbb{R}^N} |u_n|^p dx + C_1(\varepsilon)\int\limits_{\mathbb{R}^N} |u_n|^\zeta \Phi(\alpha, u_n)dx \\ &\le \varepsilon\|u_n\|_X^p + C_1(\varepsilon)\left(\int\limits_{\mathbb{R}^N} |u_n|^{\zeta r'}dx\right)^{\frac{1}{r'}}\left(\int\limits_{\mathbb{R}^N}\Phi(\alpha,u_n)^r dx\right)^{\frac{1}{r}} \\ &= \varepsilon\|u_n\|_X^p + C_2(\varepsilon)\left(\int\limits_{\mathbb{R}^N}\Phi(\alpha,u_n)^r dx\right)^{\frac{1}{r}}\|u_n\|_X^\zeta,\end{aligned} \tag{3.2}$$

where $\alpha > \alpha_0$, $\zeta > p$ and $r > 1$ with $\frac{1}{r} + \frac{1}{r'} = 1$.

On the other hand, since $\|u_n\|_X \to 0$, there exists $N_0 \in \mathbb{N}$ and $\vartheta > 0$ such that $\|u_n\|_X^{p'} < \vartheta < \frac{\alpha^*}{\alpha_0}$ for all $n > N_0$. Choosing $\alpha > \alpha_0$, $r > 1$ and $\beta > r$ satisfying $\alpha\vartheta < \alpha^*$ and $\beta\alpha\vartheta < \alpha^*$, for all $n > N_0$, we deduce from Lemmas 2.2 and 2.3 that

$$\int\limits_{\mathbb{R}^N}\Phi(\alpha,u_n)^r dx = \int\limits_{\mathbb{R}^N}\left(\exp(\alpha|u_n|^{p'}) - S_{k_p-2}(\alpha,u_n)\right)^r dx$$

$$\begin{aligned}
&= \int\limits_{\mathbb{R}^N} \left(\exp\left(\alpha \|u_n\|_X^{p'} \left(\frac{u_n}{\|u_n\|_X} \right)^{p'} \right) - S_{k_p-2}\left(\alpha \|u_n\|_X^{p'}, \frac{u_n}{\|u_n\|_X} \right) \right)^r dx \\
&\le C \int\limits_{\mathbb{R}^N} \left(\exp\left(\beta\alpha\vartheta \left(\frac{u_n}{\|u_n\|_X} \right)^{p'} \right) - S_{k_p-2}\left(\beta\alpha\vartheta, \frac{u_n}{\|u_n\|_X} \right) \right) dx \\
&\le C.
\end{aligned} \tag{3.3}$$

Combining (3.3) with (3.2), and choosing $\varepsilon = \frac{1}{2}$ in (3.2), we can see that

$$\|u_n\|_X^p \le C' \|u_n\|_X^{\zeta}, \tag{3.4}$$

where C' is a constant independent of n. Obviously, (3.4) contradicts with $\|u_n\|_X \to 0$, and we have proved the claim.

For $u_n \in \mathcal{M}$, we have $\langle I'(u_n), u_n^{\pm} \rangle = 0$. Hence, it follows from Lemma 2.6 that

$$\langle I'(u_n^{\pm}), u_n^{\pm} \rangle < 0,$$

which implies

$$\left\|u_n^{\pm}\right\|_X^p < \int\limits_{\mathbb{R}^N} f(u_n^{\pm}) u_n^{\pm} dx.$$

By arguments as with (3.2) and (3.3), we deduce that there exist $\rho_1, \rho_2 > 0$ such that $\left\|u_n^+\right\|_X > \rho_1$ and $\left\|u_n^-\right\|_X > \rho_2$. Select $\rho = \min(\rho_1, \rho_2)$, and then $\left\|u_n^{\pm}\right\|_X > \rho$, and this completes the proof. □

Lemma 3.2. *Under the assumptions of Theorem 1.1, for any $u \in X\backslash\{0\}$, there exists a unique $\nu_u \in \mathbb{R}^+$ such that $\nu_u u \in \mathcal{N}$. Moreover, for any $u \in \mathcal{N}$, we have*

$$I(u) = \max_{\nu \in [0,\infty)} I(\nu_u u). \tag{3.5}$$

Proof. Given $u \in X\backslash\{0\}$, we define $g(\nu) = I(\nu u)$ for all $\nu \ge 0$, i.e.,

$$g(\nu) = \frac{1}{p} \nu^p \|u\|_X^p - \int\limits_{\mathbb{R}^N} F(\nu u) dx.$$

It follows from (f_2) and (f_4) that the function $g(\nu)$ possesses a global maximum point ν_u, and $g'(\nu_u) = 0$, i.e.,

$$\nu_u^{p-1} \|u\|_X^p = \int\limits_{\mathbb{R}^N} f(\nu_u u) u dx, \tag{3.6}$$

which implies $\nu_u u \in \mathcal{N}$. Now, we claim the uniqueness of ν_u. Suppose, on the contrary, there exists $\tilde{\nu}_u \neq \nu_u$ such that $\tilde{\nu}_u u \in \mathcal{N}$. Then, it holds that

$$\tilde{\nu}_u^{p-1} \|u\|_X^p = \int\limits_{\mathbb{R}^N} f(\tilde{\nu}_u u) u dx. \tag{3.7}$$

Without loss of generality, we may assume $\tilde{\nu}_u > \nu_u$. Combining (3.6), (3.7) and (f_5), we deduce that

$$0 = \int\limits_{\{x\in\mathbb{R}^N:\ u(x)\neq 0\}} \left(\frac{f(\tilde{\nu}_u u)}{|\tilde{\nu}_u u|^{p-1}} - \frac{f(\nu_u u)}{|\nu_u u|^{p-1}}\right)|u|^{p-1}u dx > 0,$$

which leads to a contradiction. Thus, ν_u is unique. Obviously, for any $u \in \mathcal{N}$, $\nu_u = 1$, and (3.5) follows. This completes our proof. □

Since we are considering the constrained minimization problem on $\mathcal{M}$, in the following, we will show that the set $\mathcal{M}$ is nonempty.

Lemma 3.3. *If $u \in X$ with $u^{\pm} \neq 0$, then there exists a unique pair (σ_u, τ_u) of positive numbers such that*

$$\left(\left\langle I'(\sigma_u u^+ + \tau_u u^-), u^+\right\rangle, \left\langle I'(\sigma_u u^+ + \tau_u u^-), u^-\right\rangle\right) = (0, 0).$$

Consequently, $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}$.

Proof. Let $G : (0, +\infty) \times (0, +\infty) \to \mathbb{R}^2$ be a continuous vector field given by

$$G(\sigma, \tau) = \left(\left\langle I'(\sigma u^+ + \tau u^-), \sigma u^+\right\rangle, \left\langle I'(\sigma u^+ + \tau u^-), \tau u^-\right\rangle\right)$$

for every $(\sigma, \tau) \in (0, +\infty) \times (0, +\infty)$. By virtue of Lemmas 2.5 and 2.6, we deduce that

$$\begin{aligned}\left\langle I'(\sigma u^+ + \tau u^-), \sigma u^+\right\rangle &\geq \left\langle I'(\sigma u^+), \sigma u^+\right\rangle\\ &= \sigma^p \left\|u^+\right\|_X^p - \int\limits_{\mathbb{R}^N} f(\sigma u^+)\sigma u^+ dx\\ &\geq \sigma^p \left\|u^+\right\|_X^p - \varepsilon\sigma^p \int\limits_{\mathbb{R}^N} |u^+|^p dx - C_\varepsilon \int\limits_{\mathbb{R}^N} |\sigma u^+|^\zeta \Phi(\alpha, \sigma u^+) dx,\end{aligned} \tag{3.8}$$

where $\alpha > \alpha_0$ and $\zeta > q$. Choose $\varepsilon = \frac{1}{2}$, σ small enough such that $\|\sigma u^+\|_X^{p'} < \frac{\alpha^*}{\alpha_0}$. Taking into account (3.8) and arguing as with (3.3) in Lemma 3.1, one can see that

$$\left\langle I'(\sigma u^+ + \tau u^-), \sigma u^+\right\rangle \geq \frac{1}{2}\sigma^p \left\|u^+\right\|_X^p - C_1\sigma^\zeta \left\|u^+\right\|_X^\zeta, \tag{3.9}$$

and similarly

$$\left\langle I'(\sigma u^+ + \tau u^-), \tau u^-\right\rangle \geq \frac{1}{2}\tau^p \left\|u^-\right\|_X^p - C_2\tau^\zeta \left\|u^-\right\|_X^\zeta. \tag{3.10}$$

Hence, it follows from (3.9) and (3.10) that there exists $R_1 > 0$ small enough such that

$$\left\langle I'(R_1 u^+ + \tau u^-), R_1 u^+\right\rangle > 0 \text{ for all } \tau > 0, \tag{3.11}$$

and

$$\left\langle I'(\sigma u^+ + R_1 u^-), R_1 u^-\right\rangle > 0 \text{ for all } \sigma > 0. \tag{3.12}$$

On the other hand, by (f_3), there exist constants $D_1, D_2 > 0$ such that

$$F(t) \geq D_1 t^\theta - D_2 \text{ for all } \ t > 0. \tag{3.13}$$

Then, we have

$$\begin{aligned}
&\left\langle I'(\sigma u^+ + \tau u^-), \sigma u^+ \right\rangle \\
&\leq \sigma^p \int\limits_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_+} \frac{|u^+(x) - u^+(y)|^p}{|x-y|^{N+ps}} dxdy + \int\limits_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma u^+(x) - \tau u^-(y)|^{p-1}\sigma u^+(x)}{|x-y|^{N+ps}} dxdy \\
&\quad + \int\limits_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau u^-(x) - \sigma u^+(y)|^{p-1}\sigma u^+(y)}{|x-y|^{N+ps}} dxdy + \sigma^p \int\limits_{\mathbb{R}^N} V(x)|u^+|^p dx \\
&\quad - D_1 \sigma^\theta \int\limits_{A^+} |u^+|^\theta dx + D_2|A^+|,
\end{aligned} \tag{3.14}$$

where $A^+ \subset \operatorname{supp}(u^+)$ is a measurable set with finite and positive measure $|A^+|$. Due to the fact that $\theta > p$, for R_2 sufficiently large, we get

$$\left\langle I'(R_2 u^+ + \tau u^-), R_2 u^+ \right\rangle < 0 \text{ for all } \tau \in [R_1, R_2]. \tag{3.15}$$

Similarly, we get

$$\left\langle I'(\sigma u^+ + R_2 u^-), R_2 u^- \right\rangle < 0 \text{ for all } \sigma \in [R_1, R_2]. \tag{3.16}$$

Hence, taking into account (3.11), (3.12), (3.15), (3.16) and thanks to the Miranda theorem [28], there exists $(\sigma_u, \tau_u) \in [R_1, R_2] \times [R_1, R_2]$ such that $G(\sigma_u, \tau_u) = 0$, which implies $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}$.

Now, we are in the position to prove the uniqueness of the pair (σ_u, τ_u). First, we assume that $u = u^+ + u^- \in \mathcal{M}$ and $(\sigma_u, \tau_u) \in (0, +\infty) \times (0, \infty)$ is another pair such that $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}$. In this case, we just need to prove that $(\sigma_u, \tau_u) = (1, 1)$. Notice that

$$\left\langle A(u), u^+ \right\rangle = \int\limits_{\mathbb{R}^N} f(u^+)u^+ dx, \ \left\langle A(u), u^- \right\rangle = \int\limits_{\mathbb{R}^N} f(u^-)u^- dx, \tag{3.17}$$

and

$$\left\langle A(\sigma_u u^+ + \tau_u u^-), \sigma_u u^+ \right\rangle = \int\limits_{\mathbb{R}^N} f(\sigma_u u^+)\sigma_u u^+ dx, \ \left\langle A(\sigma_u u^+ + \tau_u u^-), \tau_u u^- \right\rangle = \int\limits_{\mathbb{R}^N} f(\tau_u u^-)\tau_u u^- dx. \tag{3.18}$$

Without loss of generality, we may assume $\sigma_u \leq \tau_u$. Then, by a direct computation, one has

$$\begin{aligned}
\left\langle A(\sigma_u u^+ + \tau_u u^-), \sigma_u u^+ \right\rangle = &\int\limits_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_+} \frac{|\sigma_u u^+(x) - \sigma_u u^+(y)|^p}{|x-y|^{N+ps}} dxdy \\
&+ \int\limits_{(\mathbb{R}^N)_+\times(\mathbb{R}^N)_-} \frac{|\sigma_u u^+(x) - \tau_u u^-(y)|^{p-1}\sigma_u u^+(x)}{|x-y|^{N+ps}} dxdy
\end{aligned}$$

$$
\begin{aligned}
&+ \int\limits_{(\mathbb{R}^N)_-\times(\mathbb{R}^N)_+} \frac{|\tau_u u^-(x) - \sigma_u u^+(y)|^{p-1}\sigma_u u^+(y)}{|x-y|^{N+ps}} dxdy \\
&\geq \langle A(\sigma_u u^+ + \sigma_u u^-), \sigma_u u^+ \rangle \\
&= \sigma_u^p \langle A(u), u^+ \rangle, \qquad (3.19)
\end{aligned}
$$

which together with (3.18) implies

$$\sigma_u^p \langle A(u), u^+ \rangle \leq \int\limits_{\mathbb{R}^N} f(\sigma_u u^+)\sigma_u u^+ dx. \tag{3.20}$$

Combining (3.20) with (3.17), we deduce that

$$\int\limits_{\{x\in\mathbb{R}^N:\ u^+(x)\neq 0\}} \left(\frac{f(\sigma_u u^+)}{(\sigma_u u^+)^{p-1}} - \frac{f(u^+)}{(u^+)^{p-1}} \right) (u^+)^p dx \geq 0.$$

Hence, by (f_5) and since $u^+ \neq 0$, we obtain $\sigma_u \geq 1$. Moreover, using similar arguments as in (3.19), we can deduce that

$$\langle A(\sigma_u u^+ + \tau_u u^-), \tau_u u^- \rangle \leq \langle A(\tau_u u^+ + \tau_u u^-), \tau_u u^- \rangle. \tag{3.21}$$

Therefore, it follows from (3.17), (3.18) and (3.21) that

$$\int\limits_{\{x\in\mathbb{R}^N:\ u^-(x)\neq 0\}} \left(\frac{f(\tau_u u^-)}{(\tau_u u^-)^{p-1}} - \frac{f(u^-)}{(u^-)^{p-1}} \right) (u^-)^p dx \leq 0,$$

which together with (f_5) implies $\tau_u \leq 1$. Thus, we conclude the proof of the uniqueness of the pair $(1, 1)$.

For the general case, we suppose that u does not necessarily belong to $\mathcal{M}$. Let (σ_u, τ_u), $(\sigma'_u, \tau'_u) \in (0, +\infty) \times (0, \infty)$. We define $v = v^+ + v^-$ with $v^+ = \sigma_u u^+$ and $v^- = \tau_u u^-$. Therefore, we have $v \in \mathcal{M}$ and $\frac{\sigma'_u}{\sigma_u} v^+ + \frac{\tau'_u}{\tau_u} v^- \in \mathcal{M}$, which implies $(\sigma_u, \tau_u) = (\sigma'_u, \tau'_u)$, and this completes the proof. □

The following two lemmas will be useful in proving Theorem 1.1.

Lemma 3.4. *Under the assumptions of Theorem 1.1, and let $u \in X$ with $u^\pm \neq 0$ such that $\langle I'(u), u^\pm \rangle \leq 0$. Then, the unique pair of positive numbers obtained in Lemma 3.3 satisfies $0 < \sigma_u, \tau_u \leq 1$.*

Proof. Here we will only prove $0 < \sigma_u \leq 1$. The proof of $0 < \tau_u \leq 1$ is analogous, and we omit it here. Since $\langle I'(u), u^+ \rangle \leq 0$, it holds that

$$\langle A(u), u^+ \rangle \leq \int\limits_{\mathbb{R}^N} f(u^+)u^+ dx. \tag{3.22}$$

Without loss of generality, we can assume that $\sigma_u \geq \tau_u > 0$, and $\sigma_u u^+ + \tau_u u^- \in \mathcal{M}$. Therefore, utilizing a similar argument as in (3.19), we deduce that

$$\sigma_u^p \langle A(u), u^+ \rangle = \langle A(\sigma_u u^+ + \sigma_u u^-), \sigma_u u^+ \rangle \geq \langle A(\sigma_u u^+ + \tau_u u^-), \sigma_u u^+ \rangle = \int\limits_{\mathbb{R}^N} f(\sigma_u u^+)\sigma_u u^+ dx. \tag{3.23}$$

Taking into account (3.22) and (3.23), we obtain

$$\int_{\mathbb{R}^N} \left(\frac{f(\sigma_u u^+)}{(\sigma_u u^+)^{p-1}} - \frac{f(u^+)}{(u^+)^{p-1}} \right) (u^+)^p dx \leq 0,$$

which together with assumption (f_5) and the fact $u^+ \neq 0$ shows that $\sigma_u \leq 1$. Hence, we finish the proof. □

Lemma 3.5. *Under the assumptions of Theorem 1.1, let $u \in X$ with $u^\pm \neq 0$ and (σ_u, τ_u) be the unique pair of positive numbers obtained in Lemma 3.3. Then, (σ_u, τ_u) is the unique maximum point of the function $h^u : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ given by*

$$h^u(\sigma, \tau) := I\left(\sigma u^+ + \tau u^-\right).$$

Proof. In the demonstration of Lemma 3.3, we saw that (σ_u, τ_u) is the unique critical point of h^u in $(0, +\infty) \times (0, +\infty)$. In addition, by the definition of h^u and (3.13), we have

$$\begin{aligned} h^u(\sigma, \tau) &= I(\sigma u^+ + \tau u^-) \\ &\leq \frac{1}{p} \left\| \sigma u^+ + \tau u^- \right\|_X^p - D_1 \sigma^\theta \int_{A^+} |u^+|^\theta dx - D_1 \tau^\theta \int_{A^-} |\tau u^-|^\theta dx + D_2(|A^+| + |A^-|), \end{aligned}$$

where $A^+ \subset \operatorname{supp}(u^+)$ and $A^- \subset \operatorname{supp}(u^-)$ are measurable sets with finite and positive measures $|A^+|$ and $|A^-|$. Since $\theta > p$, we conclude that $h^u(\sigma, \tau) \to -\infty$ as $|(\sigma, \tau)| \to \infty$. In particular, one can easily check that there exists $R > 0$ such that $h^u(a, b) < h^u(\sigma_u, \tau_u)$ for all $(a, b) \in (0, \infty) \times (0, \infty) \backslash \overline{B_R(0)}$, where $\overline{B_R(0)}$ is a closure of the ball of radius R in $\mathbb{R}^2$.

To end the proof, we just need to verify that the maximum of h^u does not occur in the boundary of $[0, +\infty) \times [0, +\infty)$. Suppose, by contradiction, that $(0, b)$ is a maximum point of h^u. Then, for $a > 0$ small enough, one can see from (3.9) that $h^u(a, 0) = I(au^+) > 0$. Hence, it follows from Lemma 2.6 that

$$h^u(a, b) = I(au^+ + bu^-) > I(au^+) + I(bu^-) > h^u(0, b),$$

for $a > 0$ small enough. However, this contradicts with the assumption that $(0, b)$ is a maximum point of h^u. The case $(a, 0)$ is similar, and we complete the proof. □

Since Lemma 3.3 shows $\mathcal{M}$ is nonempty, and Lemma 3.1 implies that $I(u) > 0$ for all $u \in \mathcal{M}$, I is bounded below in $\mathcal{M}$, which means that $m := \inf_{u \in \mathcal{M}} I(u)$ is well-defined. Now, we shall prove an upper bound for m to recover the compactness, which urges us to prove that m can be achieved.

Lemma 3.6. *Under the assumptions of Theorem 1.1 and that θ is the constant given by (f_3),*

$$0 < m < \frac{\theta - p}{\theta p} \left(\frac{\alpha^*}{\alpha_0} \right)^{p-1}. \tag{3.24}$$

Proof. Due to $\mathcal{M} \subset \mathcal{N}$, we have $m \geq c := \inf_{u \in \mathcal{N}} I(u)$. Moreover, for all $u \in \mathcal{N}$, by Lemma 3.1 it holds that

$$I(u) \geq \left(\frac{1}{p} - \frac{1}{\theta} \right) \|u\|_X^p \geq \frac{\theta - p}{\theta p} \mu^p > 0. \tag{3.25}$$

On the other hand, by the similar procedure used in [13], there exists $w \in \mathcal{M}_q$ with $w^\pm \neq 0$, such that $I_q(w) = m_q$ and $\left\langle I_q'(w), w^\pm \right\rangle = 0$. Therefore, it holds that

$$\frac{1}{p}\|w\|_X^p - \frac{1}{q}|w|_q^q = m_q \quad \text{and} \quad \langle A(w), w^\pm \rangle = |w^\pm|_q^q. \tag{3.26}$$

In addition, by virtue of Lemma 3.3, there exist $\sigma, \tau > 0$ such that $\sigma w^+ + \tau w^- \in \mathcal{M}$. Therefore, it holds that

$$m \leq I(\sigma w^+ + \tau w^-) = \frac{1}{p}\left\|\sigma w^+ + \tau w^-\right\|_X^p - \int\limits_{\mathbb{R}^N} F(\sigma w^+ + \tau w^-)dx,$$

which together with (f_4) implies that

$$m \leq \frac{1}{p}\left\|\sigma w^+ + \tau w^-\right\|_X^p - \frac{\gamma}{q}\sigma^q|w^+|_q^q - \frac{\gamma}{q}\tau^q|w^-|_q^q.$$

Now, from (3.26) and thanks to Lemma 2.7, we conclude that

$$\begin{aligned}
m &\leq \frac{1}{p}\sigma^p\langle A(w), w^+\rangle + \frac{1}{p}\tau^p\langle A(w), w^-\rangle - \frac{\gamma}{q}\sigma^q|w^+|_q^q - \frac{\gamma}{q}\tau^q|w^-|_q^q \\
&\quad + \frac{1}{p}\left(\left\|\sigma w^+ + \tau w^-\right\|_X^p - \sigma^p\langle A(w), w^+\rangle - \tau^p\langle A(w), w^-\rangle\right) \\
&\leq \left(\frac{1}{p}\sigma^p - \frac{\gamma}{q}\sigma^q\right)|w^+|_q^q + \left(\frac{1}{p}\tau^p - \frac{\gamma}{q}\tau^q\right)|w^-|_q^q \\
&\leq \max_{\xi\geq 0}\left(\frac{1}{p}\xi^p - \frac{\gamma}{q}\xi^q\right)|w|_q^q = \left(\frac{1}{p} - \frac{1}{q}\right)\gamma^{\frac{p}{p-q}}|w|_q^q \\
&= \gamma^{\frac{p}{p-q}}m_q.
\end{aligned}$$

Therefore, by the definition of γ in Theorem 1.1, we obtain (3.24). □

Lemma 3.7. *Under the assumptions of Theorem 1.1, let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence for m, and then*

$$\int\limits_{\mathbb{R}^N} f(u_n^\pm)u_n^\pm dx \to \int\limits_{\mathbb{R}^N} f(u^\pm)u^\pm dx \text{ as } n \to \infty, \tag{3.27}$$

and

$$\int\limits_{\mathbb{R}^N} F(u_n^\pm)dx \to \int\limits_{\mathbb{R}^N} F(u^\pm)dx \text{ as } n \to \infty \tag{3.28}$$

hold for some $u \in X$.

Proof. We will only prove the first result, since the second limit is a direct consequence of the first one. Since $\{u_n\} \subset \mathcal{M}$ is a minimizing sequence for m, $I(u_n) \to m$, and it follows from (3.25) that

$$\|u_n\|_X^p \leq \frac{\theta p}{\theta - p}I(u_n), \tag{3.29}$$

which implies $\{u_n\}$ is bounded in X. Then, by virtue of Lemma 2.1, up to a subsequence, there exists $u \in X$ such that

$$
\begin{aligned}
&u_n \rightharpoonup u \text{ in } X,\\
&u_n \to u \text{ in } L^{\lambda}(\mathbb{R}^N) \text{ for } \lambda \in [p, +\infty),\\
&u_n \to u \text{ a.e. in } \mathbb{R}^N.
\end{aligned} \tag{3.30}
$$

Hence,

$$
\begin{aligned}
&u_n^{\pm} \rightharpoonup u^{\pm} \text{ in } X,\\
&u_n^{\pm} \to u^{\pm} \text{ in } L^{\lambda}(\mathbb{R}^N) \text{ for } \lambda \in [p, +\infty),\\
&u_n^{\pm} \to u^{\pm} \text{ a.e. in } \mathbb{R}^N.
\end{aligned} \tag{3.31}
$$

Moreover, utilizing (3.29) again, we deduce that there exist $n_0 \in \mathbb{N}$ and $\vartheta > 0$ such that $\|u_n\|_X^{p'} < \vartheta < \frac{\alpha^*}{\alpha_0}$ for $n > n_0$. Choose $\alpha > \alpha_0$, $r > 1$ and close to 1, and $\beta > r$ satisfying $\alpha\vartheta < \alpha^*$ and $\beta\alpha\vartheta < \alpha^*$, and then for all $n > n_0$, it follows from Lemmas 2.2 and (2.3) that

$$
\begin{aligned}
\int_{\mathbb{R}^N} \Phi(\alpha, u_n)^r dx &= \int_{\mathbb{R}^N} \Big(\exp(\alpha |u_n|^{p'}) - S_{k_p-2}(\alpha, u_n)\Big)^r dx\\
&= \int_{\mathbb{R}^N} \left(\exp\left(\alpha \|u_n\|_X^{p'} \left(\frac{u_n}{\|u_n\|_X}\right)^{p'}\right) - S_{k_p-2}\left(\alpha \|u_n\|_X^{p'}, \frac{u_n}{\|u_n\|_X}\right)\right)^r dx\\
&\leq C \int_{\mathbb{R}^N} \left(\exp\left(\beta\alpha\vartheta \left(\frac{u_n}{\|u_n\|_X}\right)^{p'}\right) - S_{k_p-2}\left(\beta\alpha\vartheta, \frac{u_n}{\|u_n\|_X}\right)\right)\\
&\leq C,
\end{aligned} \tag{3.32}
$$

where C is a constant independent of n. Thus, by virtue of Lemma 2.5 and Hölder's inequality, for every Lebesgue measurable set $A \subset \mathbb{R}^N$ and $n > n_0$, it holds that

$$
\begin{aligned}
\left|\int_A f(u_n) u_n dx\right| &\leq C_1 \int_A |u_n|^p dx + C_2 \int_A |u_n|^{\zeta} \Phi(\alpha, u_n) dx\\
&\leq C_1 \int_A |u_n|^p dx + C_2 \left(\int_A |u_n|^{\zeta r'} dx\right)^{\frac{1}{r'}} \left(\int_A \Phi(\alpha, u_n)^r dx\right)^{\frac{1}{r}}\\
&= C_1 \int_A |u_n|^p dx + C_2 C^{\frac{1}{r}} \left(\int_A |u_n|^{\zeta r'} dx\right)^{\frac{1}{r'}}.
\end{aligned} \tag{3.33}
$$

Due to (3.33) and the fact that $u_n^{\pm} \to u^{\pm}$ in $L^p(\mathbb{R}^N)$ and $L^{\zeta r'}(\mathbb{R}^N)$, we conclude that for any $\varepsilon > 0$ and $n > n_0$, there exists $\delta > 0$ such that for every Lebesgue measurable set $A \subset \mathbb{R}^N$ with $\mathrm{meas}(A) \leq \delta$, it holds that

$$
\left|\int_A f(u_n^{\pm}) u_n^{\pm} dx\right| < \varepsilon. \tag{3.34}
$$

Similarly, for any $\varepsilon > 0$ and $n > n_0$, there exists $R > 0$ such that

$$\left| \int_{\mathbb{R}^N \setminus B_R(0)} f(u_n^\pm)u_n^\pm dx \right| < \varepsilon. \tag{3.35}$$

Therefore, by (3.31), (3.34), (3.35) and thanks to Vitali's convergence theorem, one can prove

$$\int_{\mathbb{R}^N} f(u_n^\pm)u_n^\pm dx \to \int_{\mathbb{R}^N} f(u^\pm)u^\pm dx \text{ as } n \to \infty. \tag{3.36}$$

Thus, we finish the proof. □

4. Proof of main results

In this section, we will prove Theorems 1.1 and 1.2. To this end, we consider the minimization problem

$$m := \inf_{u \in \mathcal{M}} I(u).$$

Firstly, let us start with the existence of a minimizer $u^* \in \mathcal{M}$ of I.

Lemma 4.1. *Under the assumptions of Theorem 1.1, the infimum m is achieved.*

Proof. By using Lemma 3.1, we know that there exists a minimizing sequence $\{u_n\}_n \subset \mathcal{M}$ bounded in X, such that

$$I(u_n) \to m \ \text{ as } n \to \infty. \tag{4.1}$$

Without loss of generality, we may assume up to a subsequence that there exists u^* such that

$$\begin{aligned} &u_n^\pm \rightharpoonup (u^*)^\pm \text{ in } X, \\ &u_n^\pm \to (u^*)^\pm \text{ in } L^\lambda(\mathbb{R}^N) \text{ for all } \lambda \in [p, +\infty), \\ &u_n^\pm \to (u^*)^\pm \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

Then, by Lemmas 2.6, 3.1 and 3.7, we conclude that

$$\rho^p \le \liminf_{n\to\infty} \left\|u_n^\pm\right\|_X^p \le \liminf_{n\to\infty} \langle A(u_n), u_n^\pm \rangle = \liminf_{n\to\infty} \int_{\mathbb{R}^N} f(u_n^\pm)u_n^\pm dx = \int_{\mathbb{R}^N} f((u^*)^\pm)(u^*)^\pm dx,$$

which implies $(u^*)^\pm \neq 0$, and consequently $u^* = (u^*)^+ + (u^*)^-$ is sign-changing. Hence, by Lemma 3.3, there exist $\sigma, \tau > 0$ such that

$$\langle I'(\sigma(u^*)^+ + \tau(u^*)^-), (u^*)^+ \rangle = 0, \qquad \langle I'(\sigma(u^*)^+ + \tau(u^*)^-), (u^*)^- \rangle = 0. \tag{4.2}$$

Now, we claim that $\sigma = \tau = 1$. Indeed, since $u_n \in \mathcal{M}$, $\langle I'(u_n), u_n^\pm \rangle = 0$, i.e.,

$$\left\|u_n^+\right\|_X^p + \int_{(\mathbb{R}^N)_n^+ \times (\mathbb{R}^N)_n^-} \left(\frac{|u_n^+(x) - u_n^-(y)|^{p-1} u_n^+(x)}{|x-y|^{N+ps}} - \frac{|u_n^+(x)|^p}{|x-y|^{N+ps}} \right) dxdy$$

$$+\int\limits_{(\mathbb{R}^N)_n^-\times(\mathbb{R}^N)_n^+}\left(\frac{|u_n^-(x)-u_n^+(y)|^{p-1}u_n^+(y)}{|x-y|^{N+ps}}-\frac{|u_n^+(y)|^p}{|x-y|^{N+ps}}\right)dxdy=\int\limits_{\mathbb{R}^N} f(u_n^+)u_n^+dx, \tag{4.3}$$

and

$$\begin{aligned}&\left\|u_n^-\right\|_X^p+\int\limits_{(\mathbb{R}^N)_n^-\times(\mathbb{R}^N)_n^+}\left(\frac{|u_n^-(x)-u_n^+(y)|^{p-1}(-u_n^-(x))}{|x-y|^{N+ps}}-\frac{|u_n^-(x)|^p}{|x-y|^{N+ps}}\right)dxdy\\&+\int\limits_{(\mathbb{R}^N)_n^+\times(\mathbb{R}^N)_n^-}\left(\frac{|u_n^+(x)-u_n^-(y)|^{p-1}(-u_n^-(y))}{|x-y|^{N+ps}}-\frac{|u_n^-(y)|^p}{|x-y|^{N+ps}}\right)dxdy=\int\limits_{\mathbb{R}^N} f(u_n^-)u_n^-dx, \end{aligned}\tag{4.4}$$

where $(\mathbb{R}^N)_n^+ := \{x\in\mathbb{R}^N : u_n(x)\geq 0\}$ and $(\mathbb{R}^N)_n^- := \{x\in\mathbb{R}^N : u_n(x)\leq 0\}$. Notice the functional $\|u\|_X^p$ is weakly lower semicontinuous on X, and we see that

$$\left\|(u^*)^\pm\right\|_X^p\leq\liminf_{n\to\infty}\left\|u_n^\pm\right\|_X^p. \tag{4.5}$$

Moreover, it follows from Lemma 3.7 that

$$\int\limits_{\mathbb{R}^N} f(u_n^\pm)u_n^\pm dx\to\int\limits_{\mathbb{R}^N} f((u^*)^\pm)(u^*)^\pm dx \text{ as } n\to\infty. \tag{4.6}$$

Taking into account (4.3)–(4.6) and thanks to Fatou's lemma, we deduce that

$$\left\langle I'(u^*),(u^*)^+\right\rangle\leq 0 \text{ and } \left\langle I'(u^*),(u^*)^-\right\rangle\leq 0, \tag{4.7}$$

which together with Lemma 3.4 implies that $0<\sigma,\tau\leq 1$. In the following, we will show that $\sigma=\tau=1$, and $I(u^*)=m$. In fact, by (4.2), (f_5), Fatou's lemma and the definition of $\mathcal{M}$, one has

$$\begin{aligned}m&\leq I(\sigma(u^*)^++\tau(u^*)^-)=I(\sigma(u^*)^++\tau(u^*)^-)-\frac{1}{p}\left\langle I'(\sigma(u^*)^++\tau(u^*)^-),(\sigma(u^*)^++\tau(u^*)^-)\right\rangle\\&=\int\limits_{\mathbb{R}^N}\left[\frac{1}{p}f(\sigma(u^*)^+)\sigma(u^*)^+-F(\sigma(u^*)^+)\right]dx+\int\limits_{\mathbb{R}^N}\left[\frac{1}{p}f(\tau(u^*)^-)\tau(u^*)^--F(\tau(u^*)^-)\right]dx\\&\leq\int\limits_{\mathbb{R}^N}\left[\frac{1}{p}f((u^*)^+)(u^*)^+-F((u^*)^+)\right]dx+\int\limits_{\mathbb{R}^N}\left[\frac{1}{p}f((u^*)^-)(u^*)^--F((u^*)^-)\right]dx\\&\leq\liminf_{n\to\infty}\left[I(u_n)-\frac{1}{p}\left\langle I'(u_n),u_n\right\rangle\right]\\&=\lim_{n\to\infty}I(u_n)=m.\end{aligned}\tag{4.8}$$

Let us observe that by the above calculation we can infer that $\sigma=\tau=1$. Thus, $u^*\in\mathcal{M}$, and $I(u^*)=m$. □

Now we are ready to prove Theorem 1.1. The proof is based on the quantitative deformation lemma and Brouwer degree theory. For more details, we refer to the arguments used in [7].

Proof of Theorem 1.1. We assume by contradiction that $I'(u^*) \neq 0$. Then, there exist $\delta, \kappa > 0$ such that

$$|I'(v)| \geq \kappa, \text{ for all } \|v - u^*\|_X \leq 3\delta.$$

Define $D := [1-\delta_1, 1+\delta_1] \times [1-\delta_1, 1+\delta_1]$ and a map $\xi : D \to X$ by

$$\xi(\sigma, \tau) := \sigma(u^*)^+ + \tau(u^*)^-,$$

where $\delta_1 \in (0, \frac{1}{2})$ small enough such that $\|\xi(\sigma,\tau) - u^*\|_X \leq 3\delta$ for all $(\sigma, \tau) \in \bar{D}$. Thus, by virtue of Lemma 4.1, we can see that

$$I(\xi(1,1)) = m, \ I(\xi(\sigma,\tau)) < m \text{ for all } (\sigma,\tau) \in D \backslash \{(1,1)\}.$$

Therefore,

$$\beta := \max_{(\sigma,\tau)\in\partial D} I(\xi(\sigma,\tau)) < m.$$

By using [36, Theorem 2.3] with

$$\mathcal{S}_\delta := \{v \in X : \|v - u^*\|_X \leq \delta\}$$

and $c := m$. By choosing $\varepsilon := \min\left\{\frac{m-\beta}{4}, \frac{\kappa\delta}{8}\right\}$, we deduce that there exists a deformation $\eta \in C([0,1] \times X, X)$ such that:

(*i*) $\eta(t,v) = v$ if $v \notin I^{-1}([m-2\varepsilon, m+2\varepsilon])$;
(*ii*) $I(\eta(1,v)) \leq m - \varepsilon$ for each $v \in X$ with $\|v - u^*\|_X \leq \delta$ and $I(v) \leq m + \varepsilon$;
(*iii*) $I(\eta(1,v)) \leq I(v)$ for all $u \in X$.

By (*ii*) and (*iii*) we conclude that

$$\max_{(\sigma,\tau)\in\overline{D}} I(\eta(1,\xi(\sigma,\tau))) < m. \tag{4.9}$$

Therefore, to complete the proof of this Lemma, it suffices to prove that

$$\eta(1, \xi(\overline{D})) \cap \mathcal{M} \neq \emptyset. \tag{4.10}$$

Indeed, if (4.10) holds true, then by the definition of m and (4.9), we get a contradiction.

In the following, we will prove (4.10). To this end, let us define $\mathbf{\Psi}^{u^*} : [0,+\infty) \times [0,+\infty) \to \mathbb{R}^2$ by

$$\begin{aligned}\mathbf{\Psi}^{u^*}(\sigma,\tau) &:= \left(\Psi_1^{u^*}(\sigma,\tau), \Psi_2^{u^*}(\sigma,\tau)\right) \\ &= \left(\langle I'\left(\sigma(u^*)^+ + \tau(u^*)^-\right), (u^*)^+\rangle, \langle I'\left(\sigma(u^*)^+ + \tau(u^*)^-\right), (u^*)^-\rangle\right).\end{aligned}$$

Furthermore, for $(\sigma,\tau) \in \overline{D}$, we define

$$\widetilde{\mathbf{\Psi}}(\sigma,\tau) := \left(\frac{1}{\sigma}\langle I'(\eta(1,\xi(\sigma,\tau))), \eta^+(1,\xi(\sigma,\tau))\rangle, \frac{1}{\tau}\langle I'(\eta(1,\xi(\sigma,\tau))), \eta^-(1,\xi(\sigma,\tau))\rangle\right).$$

Since $\eta(1,\xi(\sigma,\tau)) = \xi(\sigma,\tau)$ on ∂D, by the Brouwer degree theory (see Theorem D.9 [36]), we have

$$\deg(\widetilde{\mathbf{\Psi}}, D, 0) = \deg(\mathbf{\Psi}^{u^*}, D, 0). \tag{4.11}$$

Now, we assert that $\deg(\mathbf{\Psi}^{u^*}, D, 0) = 1$. If this assertion holds true, then $\widetilde{\mathbf{\Psi}}(\sigma_0, \tau_0) = 0$ for some $(\sigma_0, \tau_0) \in D$. Thus, there exists $u_0 := \eta(1, \xi(\sigma_0, \tau_0)) \in \mathcal{M}$ and (4.10) follows.

In fact, let us first define

$$A_p := \int_{\mathbb{R}^{2N}} \frac{|u^*(x) - u^*(y)|^{p-2}|(u^*)^+(x) - (u^*)^+(y)|^2}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|(u^*)^+|^p dx,$$

$$B_p := \int_{\mathbb{R}^{2N}} \frac{|u^*(x) - u^*(y)|^{p-2}|(u^*)^-(x) - (u^*)^-(y)|^2}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|(u^*)^-|^p dx,$$

$$C_p := \int_{\mathbb{R}^{2N}} \frac{|u^*(x) - u^*(y)|^{p-2}((u^*)^-(x) - (u^*)^-(y))((u^*)^+(x) - (u^*)^+(y))}{|x-y|^{N+ps}} dxdy,$$

$$D_p := \int_{\mathbb{R}^{2N}} \frac{|u^*(x) - u^*(y)|^{p-2}((u^*)^+(x) - (u^*)^+(y))((u^*)^-(x) - (u^*)^-(y))}{|x-y|^{N+ps}} dxdy,$$

$$a_1 := \int_{\mathbb{R}^N} f'((u^*)^+)|(u^*)^+|^2 dx, \qquad a_2 := \int_{\mathbb{R}^N} f((u^*)^+)(u^*)^+ dx,$$

$$b_1 := \int_{\mathbb{R}^N} f'((u^*)^-)|(u^*)^-|^2 dx, \qquad b_2 := \int_{\mathbb{R}^N} f((u^*)^-)(u^*)^- dx.$$

Clearly, $C_p = D_p > 0$, $A_p, B_p > 0$. Notice that $u \in \mathcal{M}$, and we can see that

$$A_p + C_p = a_2, \qquad B_p + D_p = b_2. \tag{4.12}$$

Moreover, (f_5) guarantees

$$a_1 > (p-1)a_2, \qquad b_1 > (p-1)b_2. \tag{4.13}$$

Then, by a direct computation, we have

$$\frac{\partial \Psi_1^{u^*}}{\partial \sigma}(1,1) = (p-1)A_p - a_1 < 0, \tag{4.14}$$

and

$$\frac{\partial \Psi_2^{u^*}}{\partial \tau}(1,1) = (p-1)B_p - b_1 < 0. \tag{4.15}$$

In addition,

$$\frac{\partial \Psi_1^{u^*}}{\partial \tau}(1,1) = \frac{\partial \Psi_2^{u^*}}{\partial \sigma}(1,1) = (p-1)C_p = (p-1)D_p. \tag{4.16}$$

Taking advantage of (4.12)–(4.16), we deduce that

$$\begin{aligned} \det\left(\mathbf{\Psi}^{u^*}\right)'(1,1) &= \left[(p-1)A_p - a_1\right]\left[(p-1)B_p - b_1\right] - (p-1)^2 C_p D_p \\ &> \left[(p-1)a_2 - (p-1)A_p\right]\left[(p-1)b_2 - (p-1)B_p\right] - (p-1)^2 C_p D_p \\ &= (p-1)^2 C_p D_p - (p-1)^2 C_p D_p \end{aligned}$$

$$= 0. \tag{4.17}$$

Notice that $u^* \in \mathcal{M}$, and thanks to Lemmas 3.3 and 3.5, it holds that

$$\deg(\mathbf{\Psi}^{u^*}, D, 0) = \operatorname{sgn}(\det(\mathbf{\Psi}^{u^*})'(1,1)) = 1,$$

which together with (4.11) implies $\deg(\mathbf{\Psi}^{u^*}, D, 0) = 1$. This completes our proof. □

Lemma 4.2. *For any $v \in \mathcal{M}$, there exist $\tilde{\sigma}_v, \tilde{\tau}_v \in (0,1)$ such that $\tilde{\sigma}_v v^+, \tilde{\tau}_v v^- \in \mathcal{N}$.*

Proof. We just prove $\tilde{\sigma}_v \in (0,1)$. The other case can be obtained by similar arguments. Since $v \in \mathcal{M}$, i.e., $\langle I'(v), v^+\rangle = 0$, by Lemma 2.6, we obtain

$$\left\|v^+\right\|_X^p < \int\limits_{\mathbb{R}^N} f(v^+)v^+ dx. \tag{4.18}$$

On the other hand, by Lemma 3.2, there exists $\tilde{\sigma}_v > 0$ such that $\tilde{\sigma}_v v^+ \in \mathcal{N}$, which implies that

$$\tilde{\sigma}_v^p \left\|v^+\right\|_X^p = \int\limits_{\mathbb{R}^N} f(\tilde{\sigma}_v v^+)\tilde{\sigma}_v v^+ dx. \tag{4.19}$$

Taking into account (4.18) and (4.19), we deduce that

$$\int\limits_{\mathbb{R}^N} \left[\frac{f(v^+)}{(v^+)^{p-1}} - \frac{f(\tilde{\sigma}_v v^+)}{(\tilde{\sigma}_v v^+)^{p-1}}\right](\tilde{\sigma}_v v^+)^p dx > 0.$$

Thus, it follows from (f_4) that $\tilde{\sigma}_v < 1$. □

Proof of Theorem 1.2. Using a similar idea from the proof of Lemma 4.1, we find $\bar{u} \in \mathcal{N}$ such that $I(\bar{u}) = c > 0$, where $c := \inf\limits_{u\in\mathcal{N}} I(u)$. Furthermore, utilizing the same steps of the proof of Theorem 1.1, we show that $I'(\bar{u}) = 0$. Thus, $\bar{u}$ is a ground state solution of problem (1.1). Hence, to complete the proof of Theorem 1.2, we need to study the energy behavior of $I(u^*)$, where u^* is the sign-changing solution of (1.1) obtained in Theorem 1.1.

In fact, by Lemma 4.2, there exist $0 < \tilde{\sigma}_{u^*}, \tilde{\tau}_{u^*} < 1$ such that $\tilde{\sigma}_{u^*}(u^*)^+, \tilde{\tau}_{u^*}(u^*)^- \in \mathcal{N}$. Therefore, we deduce from (f_5) and Lemma 3.2 that

$$\begin{aligned}
m = I(u^*) &= I(u^*) - \frac{1}{p}\langle I'(u^*), u^*\rangle \\
&= \int\limits_{\mathbb{R}^N} \left(\frac{1}{p} f(u^*)u^* - F(u^*)\right) dx \\
&> \int\limits_{\mathbb{R}^N} \left(\frac{1}{p} f(\tilde{\sigma}_{u^*}(u^*)^+)\tilde{\sigma}_{u^*}(u^*)^+ - F(\tilde{\sigma}_{u^*}(u^*)^+)\right) dx + \int\limits_{\mathbb{R}^N} \left(\frac{1}{p} f(\tilde{\tau}_{u^*}(u^*)^-)\tilde{\tau}_{u^*}(u^*)^- - F(\tilde{\tau}_{u^*}(u^*)^-)\right) dx \\
&= I(\tilde{\sigma}_{u^*}(u^*)^+) - \frac{1}{p}\langle I'(\tilde{\sigma}_{u^*}(u^*)^+), \tilde{\sigma}_{u^*}(u^*)^+\rangle + I(\tilde{\tau}_{u^*}(u^*)^-) - \frac{1}{p}\langle I'(\tilde{\tau}_{u^*}(u^*)^-), \tilde{\tau}_{u^*}(u^*)^-\rangle \\
&= I(\tilde{\sigma}_{u^*}(u^*)^+) + I(\tilde{\tau}_{u^*}(u^*)^-) \geq 2c,
\end{aligned}$$

which completes the proof of Theorem 1.2. □

5. Conclusions

This manuscript has employed the variational method to study the fractional p-Laplacian equation involving Trudinger-Moser nonlinearity. By using the constrained variational methods, quantitative Deformation Lemma and Brouwer degree theory, we prove the existence of least energy sign-changing solutions for the problem.

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Conflict of interest

The authors declare that there is no conflict of interest.

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